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# COMPUTING CONNECTION COEFFICIENTS OF COMPACTLY SUPPORTED WAVELETS ON BOUNDED INTERVALS

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**Abstract**

Daubechies wavelet basis functions have many properties that make them desirable as a basis for a Galerkin approach to solving PDEs: they are orthogonal, with compact support, and their connection coefficients can be computed. The method developed by Latto *et al.* [6] to compute connection coefficients does not provide the correct inner product near the endpoints of a bounded interval, making the implementation of boundary conditions problematic. Moreover, the highly oscillatory nature of the wavelet basis functions makes standard numerical quadrature of integrals near the boundary impractical. We extend the method of Latto *et al.* to construct and solve a linear system of equations whose solution provides the exact computation of the integrals at the boundaries. As a consequence, we provide the correct inner product for wavelet basis functions on a bounded interval.

## 1. Introduction

Wavelets are receiving increased attention not only as a mechanism for constructing filter banks or compressing data, but as a natural basis for multilevel schemes for solving PDEs. Several papers in recent years have described the use of wavelet basis functions in solving PDEs, for example Amaratunga *et al.* [1, 2], Bacry *et al.* [3], Qian and Weiss [8], and Restrepo and Leaf [9].

Wavelet basis functions have many properties that make them desirable as a basis for a Galerkin approach to solving PDEs: they are orthonormal, with compact support, and their connection coefficients (that is, integrals of products of basis functions, with or without derivatives) can be computed [6]. However, these properties rely on the assumption that the PDE is periodic in the computational domain (which is equivalent to the assumption that the domain is unbounded), and do not all carry over when the domain of the PDE is bounded. Orthogonality, for example, is lost when the basis functions are truncated at a boundary because the domain of integration is a finite interval.

Approaches that assume periodicity complicate the treatment of boundary conditions for PDEs in a finite domain. In a Galerkin formulation, the discretized form of the equation involves connection coefficients on bounded intervals. We call these *proper* connection coefficients, since they involve proper integrals. The usual connection coefficients computed in Latto *et al.* with a doubly infinite domain of integration will be called *improper* connection coefficients. Note that when the support of the integrand lies entirely within the interior of the computational domain, corresponding proper and improper connection coefficients are equal.

The highly oscillatory nature of wavelet basis functions makes standard numerical quadrature for computing connection coefficients impractical. Latto *et al.* circumvent this problem for improper connection coefficients by exploiting properties of the wavelet basis functions to derive a linear system of equations whose solution has as its components the exact improper connection coefficients.

As far as we know, no one has previously devised a method for computing

proper connection coefficients. As a result, the natural inner product for Galerkin solution of boundary value problems has been unavailable, and researchers have been restricted to more indirect means of resolving the boundary, *e.g.*, the *capacitance matrix method* of Proskurowski and Widlund [7]; see Amaratunga, *et al.* [2] and Qian and Weiss [8]. Motivated by the need to extend methods for resolving boundary conditions in a new and natural direction, we address the problem of computing proper connection coefficients. We adapt the methology of Latto *et al.*, exploiting the properties of the wavelet basis functions to derive two linear systems whose solutions have as their components the exact proper connection coefficients.

The paper is organized as follows. Section 2 presents background and notation. Section 3 describes our technique for computing proper connection coefficients. In section 4, we use our technique to solve a simple one-dimensional differential equation with Dirichlet boundary conditions. Section 5 provides a few concluding remarks.

## 2. Background and Notation

The first step in developing a basis is to define the underlying scale function. The scale function satisfies the recursive dilation equation

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k)$$

where $N$ is an even integer no smaller than two and $\{a_k\}$ are the *filter coefficients*. Daubechies [5] imposed conditions on the filter coefficients so that the resulting scale functions with Daubechies number $N$ are differentiable and the resulting bases are orthonormal and have $\frac{N}{2} - 1$ vanishing moments (*i.e.*, can be used to exactly represent polynomials of degree $\leq \frac{N}{2} - 1$). We will use all these important properties in our derivations. Throughout we will use the Daubechies scale functions D4 ($N = 4$), D6 ($N = 6$), D8 ($N = 8$), etc. A graph of the scale function $\phi$ for $N = 6$ is given in Figure 2.1. Also, to improve our notation we

will let

$$\phi_k(x) := \phi(x-k).$$

For more about Daubechies wavelets and their properties consult Daubechies [5], Strang [10], Strang and Nguyen [11] or Coddington *et al.* [4].

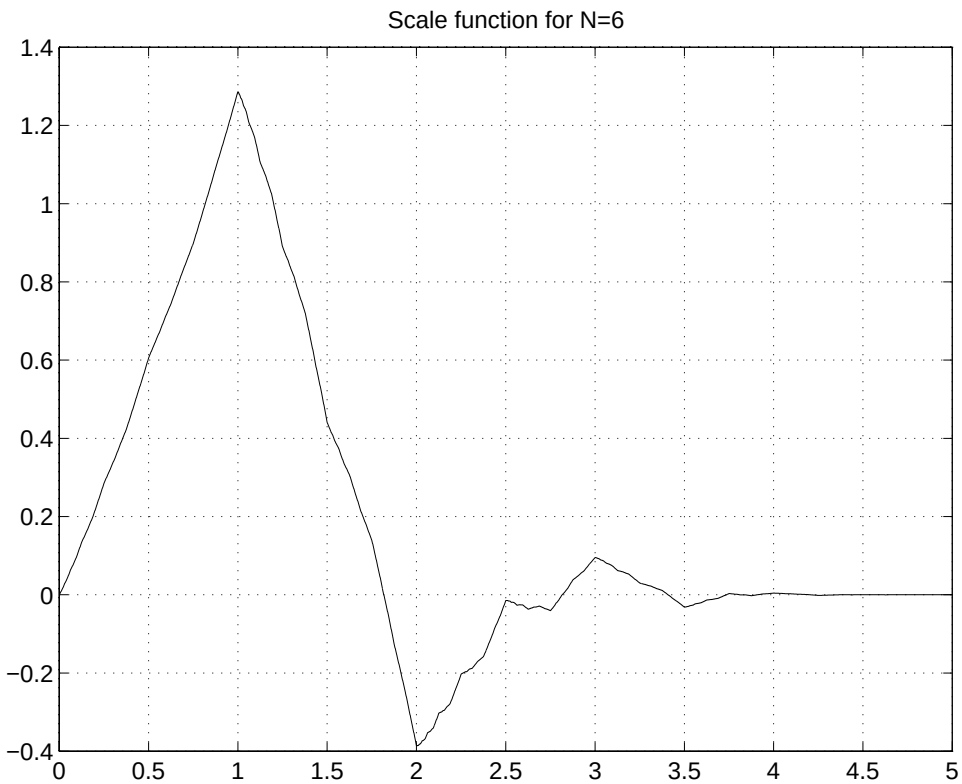


Figure 2.1: The scale function for Daubechies number $N = 6$.

The wavelet-Galerkin method for solving PDEs on an unbounded domain produces *improper* connection coefficients as terms in its equations. If we use the notation $\phi^{(n)} := \frac{d^n\phi}{dx^n}$, then the two-term *improper* connection coefficients are defined (as in Latto, *et al.*) as

$$\Lambda_j^{d_1 d_2} := \int_{-\infty}^{\infty} \phi^{(d_1)}(x)\phi_j^{(d_2)}(x)dx$$

and three-term improper connection coefficients are defined by

$$\Lambda_{j,k}^{d_1 d_2 d_3} := \int_{-\infty}^{\infty} \phi^{(d_1)}(x)\phi_j^{(d_2)}(x)\phi_k^{(d_3)}(x)dx.$$

Only the nonzero coefficients are computed: in the two-term case $2 - N \leq j \leq N - 2$, and in the three term case $2 - N \leq j, k \leq N - 2$ and $|j - k| \leq N - 2$. There is no loss of generality in fixing the shift on the first term at zero because

$$\Lambda_{i,j}^{d_1 d_2} = \int_{-\infty}^{\infty} \phi_i^{(d_1)}(x)\phi_j^{(d_2)}(x)dx = \int_{-\infty}^{\infty} \phi^{(d_1)}(x)\phi_{j-i}^{(d_2)}(x)dx$$

and

$$\Lambda_{i,j,k}^{d_1 d_2 d_3} := \int_{-\infty}^{\infty} \phi_i^{(d_1)}(x)\phi_j^{(d_2)}(x)\phi_k^{(d_3)}(x)dx = \int_{-\infty}^{\infty} \phi^{(d_1)}(x)\phi_{j-i}^{(d_2)}(x)\phi_{k-i}^{(d_3)}(x)dx.$$

The scale function is the foundation upon which the basis is constructed. Each member of the basis at resolution $m$ is of the form

$$\phi_{m,k}(x) := 2^{m/2}\phi(2^m x - k) = 2^{m/2}\phi_k(2^m x).$$

Each member of the basis is thus a scaled, dilated, and translated version of the underlying scale function.

The wavelet-Galerkin method for solving PDEs on a *bounded* domain produces *proper* connection coefficients as terms in its equations. If we assume (in one dimension) that the interval of computation is [0, 1], then for the resolution $m$ basis, the proper two-term connection coefficients will be of the form

$$\int_0^1 \phi_{m,i}^{(d_1)}(x)\phi_{m,j}^{(d_2)}(x)dx.$$

Since the resolution $m$ basis functions are simply scaled, translated, and dilated versions of the underlying scale function, it is enough to compute proper connec-

tion coefficients of the form:

$$, {}^{d_1 d_2}_{i,j} := \int_0^{N-1} \phi_i^{(d_1)}(x)\phi_j^{(d_2)}(x)dx$$

The limits of integration 0 and $N-1$ are a natural choice to ensure that the support of no basis function in the integrand crosses *both* limits of integration. Once the proper connection coefficients $, {}^{d_1 d_2}_{i,j}$ have been tabulated, all connection coefficients at resolution $m$ can be derived. For example, if the computational domain is [0, 1] then assuming $2^m > N-1$,

$$\begin{aligned}
&\int_0^1 \phi_{m,-1}^{(d_1)}\phi_{m,-2}^{(d_2)}dx \\
&\quad = \int_0^{2^m} \phi^{(d_1)}(y+1)\phi^{(d_2)}(y+2)dy, \text{ (where } y = 2^m x) \\
&\quad = \int_0^{N-1} \phi^{(d_1)}(y+1)\phi^{(d_2)}(y+2)dy \\
&\quad = , {}^{d_1 d_2}_{-1,-2},
\end{aligned}$$

since the support of $\phi^{(d_1)}(y+1)\phi^{(d_2)}(y+2)$ is [-1, $N-3$]. Similarly,

$$\begin{aligned}
&\int_0^1 \phi_{m,2^m-1}^{(d_1)}\phi_{m,2^m-2}^{(d_2)}dx \\
&\quad = \int_{-2^m+(N-1)}^{N-1} \phi^{(d_1)}(y-(N-2))\phi^{(d_2)}(y-(N-3))dy, \\
&\quad \text{(where } y = 2^m x - 2^m + (N-1)) \\
&\quad = \int_0^{N-1} \phi^{(d_1)}(y-(N-2))\phi^{(d_2)}(y-(N-3))dy \\
&\quad = , {}^{d_1 d_2}_{N-2,N-3},
\end{aligned}$$

since the support of $\phi^{(d_1)}(y-(N-2))\phi^{(d_2)}(y-(N-3))$ is $[N-2, 2N-4]$. Note that if the support of the integrand lies entirely within the computational domain, the corresponding proper and improper connection coefficients are equal; that is, $, {}^{d_1 d_2}_{i,j} = \Lambda^{d_1 d_2}_{j-i}$ (using the notation in Latto, *et al.*). Two basis functions for a proper two-term connection coefficient that involves a boundary are illustrated in Figure 2.2.

Three-term proper connection coefficients are defined by

$$,_{i,j,k}^{d_1 d_2 d_3} := \int_0^{N-1} \phi_i^{(d_1)}(x)\phi_j^{(d_2)}(x)\phi_k^{(d_3)}(x)dx$$

and similar definitions apply for higher numbers of terms. In the two-term case we can restrict our attention to proper connection coefficients for which either $2-N \leq i,j \leq -1$, or $1 \leq i,j \leq N-2$. All others are either zero, or are equivalent to some improper connection coefficient. Similarly, for the three-term case we restrict our attention to proper connection coefficients for which either $2-N \leq i,j,k \leq -1$, or $1 \leq i,j,k \leq N-2$.

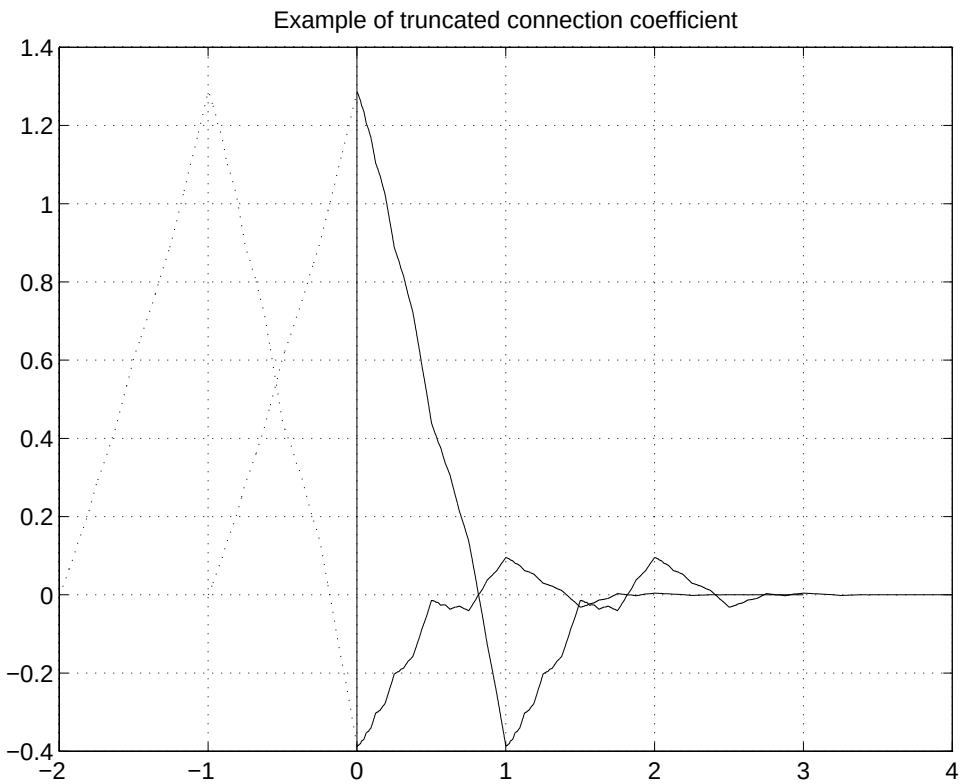


Figure 2.2: Basis functions for a proper connection coefficient

Note that the integrals can no longer be shifted to reduce the total number of distinct proper connection coefficients by assigning a zero shift to the first term.

Moreover, the truncation of a basis function is not arbitrary, but occurs at one of the dyadic points $(i/2^m)$ in the given resolution. Equivalently, the truncation occurs at an integer value at the resolution of the scale function. The interval of integration ($[0,\ N-1]$) combined with all possible integer shifts cover all possible truncations at integer points of shifted products of the scale function.

## 3. Computing Proper Connection Coefficients

We illustrate our technique by computing the three-term proper connection coefficients. The same technique is easily applied to the two-term proper connection coefficients. Our approach is based on suitable modifications of the *scaling equations*, *moment equations*, and the *normalization equation*, described in Latto, *et al.* [6].

### 3.1. Scaling equations

We begin this section by deriving a relationship among the unknowns $,_{i,j,k}^{d_1 d_2 d_3}$ that will eliminate half of them. We note first that the support of $\phi_i^{(d)}$ is $[i,\, i+(N-1)]$. Then for $1 \le i,j,k \le N-2$ we have:

$$,_{i,j,k}^{d_1 d_2 d_3} := \int_0^{N-1} \phi_i^{d_1} \phi_j^{d_2} \phi_k^{d_3} dx$$

and

$$\begin{aligned} ,_{i-(N-1),j-(N-1),k-(N-1)}^{d_1 d_2 d_3} &:= \int_0^{N-1} \phi_{i-(N-1)}^{d_1} \phi_{j-(N-1)}^{d_2} \phi_{k-(N-1)}^{d_3} dx \\ &= \int_{N-1}^{2N-2} \phi_i^{d_1} \phi_j^{d_2} \phi_k^{d_3} dx. \end{aligned}$$

Hence, since the support of $\phi_i^{d_1} \phi_j^{d_2} \phi_k^{d_3}$ is completely contained in the interval $[0,\, 2N-2]$, we have the identity

$$,_{i,j,k}^{d_1 d_2 d_3} + ,_{i-(N-1),j-(N-1),k-(N-1)}^{d_1 d_2 d_3} = \Lambda_{j-i,k-i}^{d_1 d_2 d_3}, \tag{3.1}$$

where $\Lambda_{\ell,m}^{d_1 d_2 d_3}$ denotes the improper connection coefficient described in Latto *et al.* [6].

We now derive the scaling equations. If $2-N \le i \le -1$, then since

$$\phi_i(x) = \phi(x-i) = \sum_{p=0}^{N-1} a_p \phi(2x-(2i+p))$$

we have

$$\phi_i^{(d_1)}(x) = 2^{d_1} \sum_{p=0}^{N-1} a_p \phi^{(d_1)}(2x-(2i+p)).$$

Similarly, with $2-N \le j,k \le -1$ we have

$$\phi_j^{(d_2)}(x) = 2^{d_2} \sum_{q=0}^{N-1} a_q \phi^{(d_2)}(2x-(2j+q))$$

and

$$\phi_k^{(d_3)}(x) = 2^{d_3} \sum_{r=0}^{N-1} a_r \phi^{(d_3)}(2x-(2k+r)).$$

Hence,

$$,_{i,j,k}^{d_1 d_2 d_3} = 2^d \sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\sum_{r=0}^{N-1} a_p a_q a_r \times$$
$$\int_0^{N-1} \phi^{(d_1)}(2x-(2i+p))\phi^{(d_2)}(2x-(2j+q))\phi^{(d_3)}(2x-(2k+r))dx,$$

(where $d = d_1 + d_2 + d_3$), or

$$,_{i,j,k}^{d_1 d_2 d_3} = 2^d \sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\sum_{r=0}^{N-1} a_p a_q a_r \times$$
$$\int_0^{2N-2} \phi^{(d_1)}(y-(2i+p))\phi^{(d_2)}(y-(2j+q))\phi^{(d_3)}(y-(2k+r))dy,$$

or

$$
\begin{aligned}
,_{i,j,k}^{d_1 d_2 d_3} &= 2^d \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} a_p a_q a_r \times \\
&\Big( \int_0^{N-1} \phi^{(d_1)}(y-(2i+p))\phi^{(d_2)}(y-(2j+q))\phi^{(d_3)}(y-(2k+r))dy \\
&+ \int_0^{N-1} \phi^{(d_1)}(y-(2i+p-(N-1)))\phi^{(d_2)}(y-(2j+q-(N-1))) \times \\
&\phi^{(d_3)}(y-(2k+r-(N-1)))dy\Big),
\end{aligned}
$$

or

$$
\begin{aligned}
,_{i,j,k}^{d_1 d_2 d_3} &= 2^{d-1} \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} a_p a_q a_r \times \\
&\Big( ,_{2i+p,2j+q,2k+r}^{d_1 d_2 d_3} + ,_{2i+p-(N-1),2j+q-(N-1),2k+r-(N-1)}^{d_1 d_2 d_3} \Big)
\end{aligned}
\tag{3.2}
$$

Note that some of the terms on the right hand side of (3.2) are equal to *improper* connection coefficients, since the entire support of some of the integrands lies within the bounds of integration. These terms are known (thanks to Latto, *et. al.*), and hence can be moved to the other side. Moreover, we exploit the identity (3.1) to eliminate the unknowns corresponding to $1 \leq i,j,k \leq N-2$, producing a matrix equation of the form:

$$
(I - 2^{d-1}A),^{d_1 d_2 d_3} = R \tag{3.3}
$$

in $(N-2)^2$ unknowns, where $R$ is a vector accumulating the known values in the sum (3.2). A similar treatment holds for the proper connection coefficients $,_{i,j,k}^{d_1 d_2 d_3}$ with $1 \leq i,j,k \leq N-2$, though as noted, these can be computed from (3.1) once the values of $,_{i,j,k}$ for $-(N-2) \leq i,j,k \leq -1$ are known.

The linear system (3.3) is inhomogeneous; however, the coefficient matrix $I - 2^{d-1}A$ in (3.3) may not be of full rank. Indeed, our observations of the spectrum of $I - 2^{d-1}A$ indicate that it has a zero eigenvalue with multiplicity $d$,

which would imply that the rank deficiency of the coefficient matrix in (3.3) is $d$.

### 3.2. Moment Equations

If the coefficient matrix in (3.3) is singular, we augment the matrix by adding $d_1 + d_2 + d_3$ moment equations to the linear system in (3.3), which are derived as follows (see Latto, *et al.* [6]). We assume that the Daubechies number is sufficiently high that we have $\max\{d_1, d_2, d_3\}$ vanishing moments. Then for each $q < d_1$ there exist coefficients $\{M_i^q\}$ such that

$$x^q = \sum_{i=-\infty}^{\infty} M_i^q \phi_i(x).$$

The set $\{M_i^q\}$ are called the *moments* of $\phi$ and its translates, and the reader is referred to Latto *et al.* [6] for details on how to compute them.

On the closed interval $[0,\ N-1]$, we have the identity

$$x^q = \sum_{i=2-N}^{N-2} M_i^q \phi_i(x), \tag{3.4}$$

where we have included all terms in the sum whose support intersects the given interval. Differentiating (3.4) $d_1$ times yields

$$0 = \sum_{i=2-N}^{N-2} M_i^q \phi_i^{d_1}(x). \tag{3.5}$$

If we then multiply both sides of (3.5) by $\phi_j^{d_2}\phi_k^{d_3}$ for some fixed $j$ and $k$ and integrate over $[0,\ N-1]$, we obtain

$$0 = \sum_{i=2-N}^{N-2} M_i^q,\ {}_{i,j,k}^{d_1 d_2 d_3}. \tag{3.6}$$

Again, some of the entries on the right hand side of (3.6) are equal to improper connection coefficients, and hence can be moved to the other side. Others can be eliminated via the identity (3.1). This provides $d_1$ further (inhomogeneous)

linear equations in the unknowns $,\,{}^{d_1 d_2 d_3}_{i,j,k}$, one for each value of $q$. Similarly, we can derive $d_2$ further moment equations

$$0 = \sum_{j=2-N}^{N-2} M_j^q,\,{}^{d_1 d_2 d_3}_{i,j,k}.$$

and $d_3$ moment equations

$$0 = \sum_{k=2-N}^{N-2} M_k^q,\,{}^{d_1 d_2 d_3}_{i,j,k}.$$

### 3.3. Normalization Equation

The rectangular system of linear equations derived from the scaling equations and moment equations for the *improper* connection coefficients is homogeneous, and hence require a nonhomogeneous equation to "normalize" the solution [6]. The rectangular system of equations described above for the *proper* connection coefficients is already nonhomogeneous; however, the matrix may still be rank-deficient. We now derive the *normalization equation* for the proper connection coefficients, and include it in the system of equations. We conjecture (and our tests indicate) that the rectangular system of equations containing the scaling equations, the moment equations and the normalization equation is of full rank, and therefore the nonhomogeneous linear system has a unique solution.

If we assume that the basis functions have $\max\{d_1, d_2, d_3\}$ vanishing moments, then we have the following:

$$x^{d_1} = \sum_{i=-\infty}^{\infty} M_i^{d_1} \phi_i.$$

Differentiating $d_1$ times, we obtain

$$d_1! = \sum_{i=-\infty}^{\infty} M_i^{d_1} \phi_i^{d_1}.$$

Including only those basis functions whose support intersects the interval $[0,\ N-1]$,

we have the identity:

$$d_1! = \sum_{i=2-N}^{N-2} M_i^{d_1} \phi_i^{d_1} \text{ on } [0,\ N-1]. \tag{3.7}$$

Similarly,

$$d_2! = \sum_{j=2-N}^{N-2} M_j^{d_2} \phi_j^{d_2} \text{ on } [0,\ N-1] \tag{3.8}$$

and

$$d_3! = \sum_{k=2-N}^{N-2} M_k^{d_3} \phi_k^{d_3} \text{ on } [0,\ N-1]. \tag{3.9}$$

Multiplying equations (3.7), (3.8) and (3.9) together, and integrating over $[0,\ N-1]$ we obtain

$$(N-1)d_1!d_2!d_3! = \sum_{i=2-N}^{N-2} \sum_{j=2-N}^{N-2} \sum_{k=2-N}^{N-2} M_i^{d_1} M_j^{d_2} M_k^{d_3}\, ,_{i,j,k}^{d_1 d_2 d_3}. \tag{3.10}$$

Again, some of the terms in (3.10) are either known, or can be eliminated via the identity (3.1).

We provide tables of proper connection coefficients in the appendix for the two cases ($N = 6$, $d_1 = 0$, $d_2 = 2$) and ($N = 6$, $d_1 = 1$, $d_2 = 0$ and $d_3 = 0$), the same two cases provided by Latto *et al.* in [6]. Although the values of $,_{i,j,k}^{d_1 d_2 d_3}$ for $1 \leq i, j, k \leq N-2$ can be computed indirectly using the identity (3.1) we compute both tables of proper connection coefficients independently and use 3.1 as an accuracy check. For the tables provided in the appendix, the identity (3.1) is satisfied to within approximately $10^{-15}$.

Since the multiresolution wavelet basis $\{\psi_{m,k}\}$ is defined in terms of the scale function, the proper connection coefficients derived here can also be used for multiresolution analysis. Specifically, if the multiresolution wavelet basis is used in a Galerkin formulation for the solution of PDEs, the necessary *wavelet* connection coefficients can be computed directly from the proper connection coefficients given here.

## 4. Results

In this section, we demonstrate the applicability of proper connection coefficients by solving a simple one-dimensional differential equation on the bounded domain $[0, 1]$. We use the 1-D Poisson problem

$$u_{xx} - f = 0, \tag{4.1}$$

and impose Dirichlet boundary conditions at the endpoints:

$$u(0) = \alpha \text{ and } u(1) = \beta.$$

The Galerkin approach approximates $u$ and $f$ with linear combinations of basis functions:

$$u \approx \sum_i u_i \phi_{m,i}, \text{ and } f \approx \sum_i f_i \phi_{m,i}.$$

The left hand side of (4.1) is now approximated by

$$u_{xx} - f \approx \sum u_i \phi''_{m,i} - \sum f_i \phi_{m,i}. \tag{4.2}$$

In general, these approximations will not satisfy the differential equation exactly; however, we can find the orthogonal projection onto the space spanned by $\{\eta_j\}$ by ensuring that (4.2) is orthogonal to each so-called "test function" $\eta_j$. That is, we define an inner product

$$\langle \phi_{m,i}, \eta_j \rangle = \int_0^1 \phi_{m,i} \eta_j \, dx$$

and solve the following linear equations for the unknowns $u_i$:

$$\sum u_i \langle \phi''_i, \eta_j \rangle - f_i \langle \phi_i, \eta_j \rangle = 0. \tag{4.3}$$

If the test functions and the basis functions coincide (the choice for Galerkin test functions), the linear system will have a square coefficient matrix.

One approach to imposing Dirichlet boundary conditions in a Galerkin formulation is to replace two of the rows in (4.3) with the two linear equations derived from the two boundary conditions, preserving a square coefficient matrix. An alternative, and the approach we chose, is to append the two boundary equations to the matrix, and solve the resulting rectangular system.

Our test problem is

$$u_{xx} = -4\pi^2 \sin(2\pi x), \quad u(0) = 1, \quad u(1) = 2$$

whose exact solution is $u = \sin(2\pi x) + x + 1$. Figure 4.1 is a graph of the error as a function of resolution on a log-log scale for Daubechies number $N = 6$. The slope of the resulting line demonstrates cubic convergence with increase in resolution.

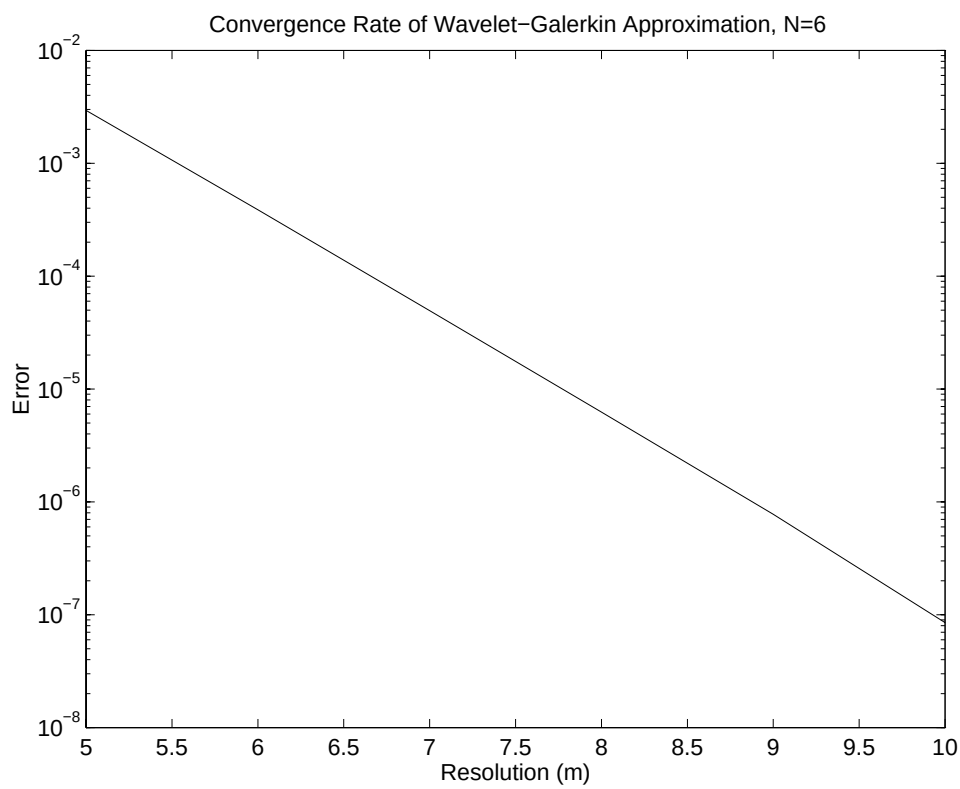


Figure 4.1: Illustration of convergence rate

## 5. Summary

Proper connection coefficients are important for the solution of nonperiodic PDEs. We have demonstrated a technique for deriving a linear system whose solution is the set of proper connection coefficients needed to compute the natural inner product on bounded intervals. The ability to compute proper connection coefficients provides a natural mechanism for imposing boundary conditions. We exhibited a simple one-dimensional test problem that illustrates the use of proper connection coefficients for PDE's on bounded domains with Dirichlet boundary conditions. We showed that convergence of the solution using Daubechies D6 basis functions was cubic with increasing resolution.

The wavelet-Galerkin approach for solving PDEs has suffered from the inability to properly set PDE problems on bounded domains and to impose boundary conditions in a straightforward way. We have shown that this drawback can be eliminated when the proper connection coefficients can be computed. Moreover, these proper connection coefficients can also be used to compute the proper *wavelet* connection coefficients arising from a multiresolution analysis of PDEs.

## 6. Acknowledgements

We are grateful to John Drake and Bill Lawkins for our introduction to wavelets for solving PDEs and for many helpful conversations.

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# A. Tables of Proper Connection Coefficients

| | $j=-4$ | $-3$ | $-2$ | $-1$ |
|---|---|---|---|---|
| $i=-4$ | -0.00038529074526 | -0.00386035962858 | 0.01423373435011 | -0.01534522683341 |
| $-3$ | -0.00573652265285 | -0.09863430110098 | 0.33947817016054 | -0.35475020354956 |
| $-2$ | 0.02181281253982 | 0.33904059142268 | -1.37506723951242 | 1.77076145459755 |
| $-1$ | -0.06519463330387 | -0.90634814610999 | 4.01066114451254 | -5.67304693652727 |

Table A.1: Table of , $^{02}_{i,j} := \int_0^{N-1} \phi_i(x)\phi_j^{(2)}(x)dx$ for $N=6$

| | | | | |
|---|---|---|---|---|
| | | $i = -4$ | | |
| | $k = -4$ | $-3$ | $-2$ | $-1$ |
| $j = -4$ | -0.00000002530682 | -0.00000068992720 | 0.00000306963647 | -0.00001055718019 |
| $-3$ | -0.00000068992720 | -0.00002337260152 | 0.00010672670559 | -0.00037345654025 |
| $-2$ | 0.00000306963647 | 0.00010672670559 | -0.00048341871765 | 0.00168470385967 |
| $-1$ | -0.00001055718019 | -0.00037345654025 | 0.00168470385967 | -0.00585973132675 |
| | | $i = -3$ | | |
| | $k = -4$ | $-3$ | $-2$ | $-1$ |
| $j = -4$ | -0.00000032826247 | -0.00000752895920 | 0.00003138211915 | -0.00010322628046 |
| $-3$ | -0.00000752895920 | -0.00028821307592 | 0.00136993458233 | -0.00488523455350 |
| $-2$ | 0.00003138211915 | 0.00136993458233 | -0.00714394837814 | 0.02677812707263 |
| $-1$ | -0.00010322628046 | -0.00488523455350 | 0.02677812707263 | -0.10280576120976 |
| | | $i = -2$ | | |
| | $k = -4$ | $-3$ | $-2$ | $-1$ |
| $j = -4$ | 0.00000077866139 | 0.00001753615235 | -0.00007347449696 | 0.00024195517558 |
| $-3$ | 0.00001753615235 | 0.00076195056336 | -0.00351927324338 | 0.01231761138106 |
| $-2$ | -0.00007347449696 | -0.00351927324338 | 0.01914653677375 | -0.07281061430283 |
| $-1$ | 0.00024195517558 | 0.01231761138106 | -0.07281061430283 | 0.28689868099225 |
| | | $i = -1$ | | |
| | $k = -4$ | $-3$ | $-2$ | $-1$ |
| $j = -4$ | -0.00000194921939 | -0.00004221923539 | 0.00017491280955 | -0.00057333107231 |
| $-3$ | -0.00004221923539 | -0.00190418619493 | 0.00818703214249 | -0.02741471871558 |
| $-2$ | 0.00017491280955 | 0.00818703214249 | -0.04587567964823 | 0.17576474944728 |
| $-1$ | -0.00057333107231 | -0.02741471871558 | 0.17576474944728 | -0.70948149953627 |

Table A.2: Table of , $^{100}_{i,j,k} := \int_0^{N-1} \phi_i^{(1)}(x)\phi_j(x)\phi_k(x)dx$ for $N = 6$