

# Robust physics from random interactions

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Using random two-body interactions between valence space nucleons, I calculate various statistical measures of the many-body wave functions obtained from diagonalization. These measures include the entropy, spectral rigidity, and the inverse participation ratio. From these calculations, I find that the various random ensembles exhibit quite similar spectral characteristics. These characteristics are similar to those obtained from realistic interactions.

## 1. Introduction

Nuclei exhibit a rich variety of behavior including shape-coexistence phenomena, paired ground-states (for even-even nuclei), odd-even mass staggering, and the appearance of well-deformed pairing and vibrational bands. This richness within nuclear structure stems from the underlying effective nucleon-nucleon interaction. Of course, for specific comparisons to experiment, generating an appropriate effective nucleon-nucleon interaction in the model space of interest remains an important and challenging problem.

Let us suppose that instead of trying to understand detailed experimental spectra, we address a different question. Let us assume that we know little about the effective two-body interaction. The nuclear shell model is defined by a set of spin-orbit coupled single-particle states with quantum numbers  $ljm$  denoting the orbital angular momentum ( $l$ ) and the total angular momenta ( $j$ ) and its  $z$ -component,  $m$ . The shell model interactions produce many-body states with good spin total  $\mathcal{J}$  and isospin  $\mathcal{T}$ . Now, we ask “which generic nuclear properties remain when one employs randomized effective interactions and calculates average nuclear properties from an ensemble of these interactions”? This is precisely the line of research pursued in a number of recent papers in the framework of both the shell model [1,2] and the interacting boson model [3].

Using a spherical shell-model Hamiltonian in the  $sd$ -shell, these studies indicated that for a large range of random interaction, a dominance (roughly 60-70%) of  $\mathcal{J}^\pi = 0^+$  ground states exists despite the random nature of the interactions. In addition, evidence was found for the occurrence of pairing properties. For example, pair-transfer amplitudes are strongly enhanced for interactions that give  $\mathcal{J} = 0^+$  ground states. IBM calculations indicated the likelihood of finding vibrational and rotational band structures.

In this paper, I will discuss the statistical nature of the  $0^+$  states for four different random ensembles. These ensembles include the random quasiparticle ensemble (RQE,

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RQE-SPE) and the two-body random ensemble (TBRE, TBRE-SPE). The RQE and TBRE do not include single particle energy splitting, while the RQE-SPE and TBRE-SPE include single particle energies taken from the realistic  $sd$ -shell interaction. I use several well-known statistical quantities to characterize the general behavior of the different ensembles. These include the entropy, inverse participation ratio, and spectral rigidity. I find that while the different random interactions can behave differently in ground-state properties, in general the excitation spectra are statistically quite similar.

## 2. Two-body random interactions

The random interactions employed in shell-model studies preserve rotational and isospin invariance as well as particle number conservation. I use a typical shell-model basis for calculations. The single-particle states of the shell model are oscillator states classified by the quantum numbers  $\{nljmt_z\}$  for the principal quantum number, the orbital angular momentum, the total momentum and its projection, and the isospin projection. The Hamiltonian is given by

$$H = \sum_{\alpha} \varepsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \sum_{JT} V_{JT}^A(\alpha\beta, \gamma\delta) \sum_{MT_z} A_{JT;MT_z}^{\dagger}(ab) A_{JT;MT_z}(cd), \quad (1)$$

where the pair operator is

$$A_{JT;MT_z}^{\dagger}(ab) = \sum_{m_a m_b, t_a t_b} (j_a m_a j_b m_b | JM) \left( \frac{1}{2} t_a \frac{1}{2} t_b | TT_z \right) a_{j_b m_b t_b}^{\dagger} a_{j_a m_a t_a}^{\dagger}. \quad (2)$$

I define the two-body matrix elements  $V_{JT}(ab, cd)$  through an ensemble of two-particle Hamiltonians. This is achieved by taking the matrix elements to be Gaussian distributed with zero mean and with the widths possibly depending on  $J$  and  $T$  such that

$$\langle V_{JT;\alpha,\alpha'}^2 \rangle = c_{JT} (1 + \delta_{\alpha,\alpha'}) \bar{v}^2. \quad (3)$$

$$\langle V_{JT;\alpha,\alpha'} V_{JT;\beta,\beta'} \rangle = 0, \quad (\alpha, \alpha') \neq (\beta, \beta'). \quad (4)$$

Here  $\bar{v}$  is an overall energy scale. The coefficients  $c_{JT}$  then define the ensemble. Note that  $JT$  refer to quantum numbers of *two-body* states and not of the final many-body states (typically consisting of 4-10 particles).

In order to study the statistical nature of the random interactions, I employ four basic ensembles that may be specified by the  $c_{JT}$  coefficients and the single-particle Hamiltonian, if present. The first of these is the Random Quasiparticle Ensemble (RQE). In this case  $c_{JT} = [(2T+1)(2J+1)]^{-1}$ . This relation for the  $c_{JT}$ , which was discussed in [1], follows from imposing on the ensemble the constraint that it should remain the same in the particle-particle and particle-hole channels. Our second ensemble is the two-body random ensemble (TBRE) for which  $c_{JT} = \text{constant}$ . Historically, this was the first two-particle random ensemble to be employed in studying statistical properties of many-particle spectra [7]. These two ensembles assume degenerate single-particle energies. Realistic interactions do have nonzero single-particle energies, and these will, in principle, affect various spectral properties. For these calculations in the  $sd$  shell, I take single-particle energies from the Wildenthal interaction [8], scaling  $\bar{v} = 3.84$  MeV so as

to best match the widths of the two-particle matrix elements. The resulting interactions with the single-particle splitting included are called the RQE-SPE and TBRE-SPE. The RQE and TBRE are also scaled by this  $\bar{v}$ .

### 3. Statistical characterization of random interactions

I diagonalized approximately 1000 random interactions generated from each of the ensembles described in the previous section. In this paper, I concentrate on the system  $^{24}\text{Mg}$  which is comprised of 4 neutrons and 4 protons in the  $1s-0d$  shell-model space. Thus, the maximum spin state that I study in these systems is  $\mathcal{J} = 12$ .

In Table 1, I indicate the relative abundances of  $\mathcal{J}$  spin of ground states that are found in the four ensembles for the  $^{24}\text{Mg}$  system. In each case  $\mathcal{J} = 0$  dominates. Interestingly, the  $\mathcal{J} = 0$  and  $\mathcal{J} = \mathcal{J}_{max}$  are most often the ground states. The preponderance of the  $\mathcal{J} = 0$  states for several ensembles was discussed in some detail in [1,2]. I note that the introduction of single-particle energies reduces the number of  $\mathcal{J} = 0$  states in all cases studied, both in previous work [2] and in this work. Another interesting feature is the presence of high-spin ground states and their enhancement when single-particle splitting is included.

Once the Hamiltonian is diagonalized, a given many-body state  $|\alpha\rangle$  is a superposition of the (normalized) many-body basis states  $|k\rangle$

$$\phi^\alpha = \sum_k A_k^\alpha |k\rangle = \sum_k \langle k | \alpha \rangle |k\rangle, \quad (5)$$

where the coefficients of expansion obey the relation  $\sum_k |A_k^\alpha|^2 = 1$ . I define the entropy within this basis as [4]

$$S = - \sum_k W_k^\alpha \ln W_k^\alpha, \quad (6)$$

where  $W_k^\alpha = |A_k^\alpha|^2$  are the overlap intensities. Thus,  $S = 0$  if only one many-body basis state contributes to the sum (i.e., for a particular  $k$   $W_k^\alpha = 1$ ), whereas if all states equally contribute, we obtain  $S = \ln D$  where  $D$  is the total number of many-body basis states. Typically, instead of plotting  $S$  directly, we discuss for a given eigenstate of the Hamiltonian  $P(\alpha) = \exp(S^\alpha)$ . For the two extreme cases  $P(\alpha) = 1$  when the wave function is equivalent to a single basis state, and  $P(\alpha) = D$  when all basis states equally contribute to the sum.

I now limit the discussion to those interactions that give  $\mathcal{J} = 0$  ground states. I show in Fig. 1a the averaged entropy for the first 150  $\mathcal{J} = 0^+$ ,  $\mathcal{T} = 0$  wavefunctions for the four ensembles. No statistical difference exists among the four ensembles. I also show the entropy of the first 150  $0^+$  states in the USD interaction. Since there is no ensemble average, the line is much less smooth, but the general agreement between the USD and the random interactions is obvious. None of these interactions reach the GOE limit which is  $0.48D = 8415$  in this case.

I also use the inverse participation ratio (IPR) to quantify the random ensembles [5]. The IPR is

$$IPR_\alpha = D \sum_{k=1}^D (W_k^\alpha)^2, \quad (7)$$

Table 1

Number of ground states with a given spin in the  $^{24}\text{Mg}$  system for approximately 1000 random-interaction samples. The maximum spin obtainable by  $^{24}\text{Mg}$  in this model space is  $\mathcal{J}_{\text{max}} = 12$ . Results are shown for the four random ensembles discussed in this paper.

Spin	$^{24}\text{Mg}$			
	RQE	TBRE	RQE-SPE	TBRE-SPE
0	703	548	612	494
1	39	30	41	35
2	164	135	231	193
3	12	20	7	18
4	38	85	42	96
5	4	3	5	7
6	8	21	13	20
7	5	7	2	9
8	22	85	42	91
9	1	8	0	9
10	4	17	10	22
11	0	2	0	0
12	0	3	0	6

and measures the inverse fraction of Fock states that participate in forming the full wavefunction  $|\alpha\rangle$ . This measure emphasises the contribution of the large components of the wave function. The extreme cases correspond to  $\text{IPR}_\alpha = 1$  when all Fock states equally contribute to the wave function  $|\alpha\rangle$ , while  $\text{IPR}_\alpha = D$  when only one Fock state contributes. As a point of reference, I calculated the IPR for the first  $\mathcal{J} = 0^+$  state using the USD interaction and find  $\text{IPR}_1 = 64.18$  and  $13.48$  with and without single-particle energy splitting, respectively. As one would expect, the single-particle energy splitting enhances the  $d_{5/2}$  occupation and hence acts as a filter for choosing many-body basis states. I show in Fig. 1b the IPR for the first 150  $0^+$  states. I see no statistically significant difference in the character of the random ensembles beyond the first ( $\alpha = 1$ ) state. The inset shows in more detail the first state. Both the RQE-SPE and TBRE-SPE show depressed IPRs compared to the RQE and TBRE. This property is somewhat different than I anticipated from the findings using the realistic two-body interaction. The RQE shows some enhancement of order when compared to the RQE-SPE. Furthermore, the IPR for the random interactions is somewhat larger than for the USD interaction.

Another quantity widely used to characterize the spectrum of states is the spectral rigidity  $\Delta_3(L)$  [6] which is defined as

$$\Delta_3(L) = \frac{1}{L} \left\langle \min(a, b) \int_L^{L+x} d\alpha (N(\alpha) - a\alpha - b)^2 \right\rangle_x, \quad (8)$$

where  $\tilde{N}(\alpha)$  is the cumulative sum of states at a given energy  $\alpha$  of the unfolded excitation energy spectra. Unfolding removes the smooth part of the excitation spectrum, leaving only the fluctuating part.

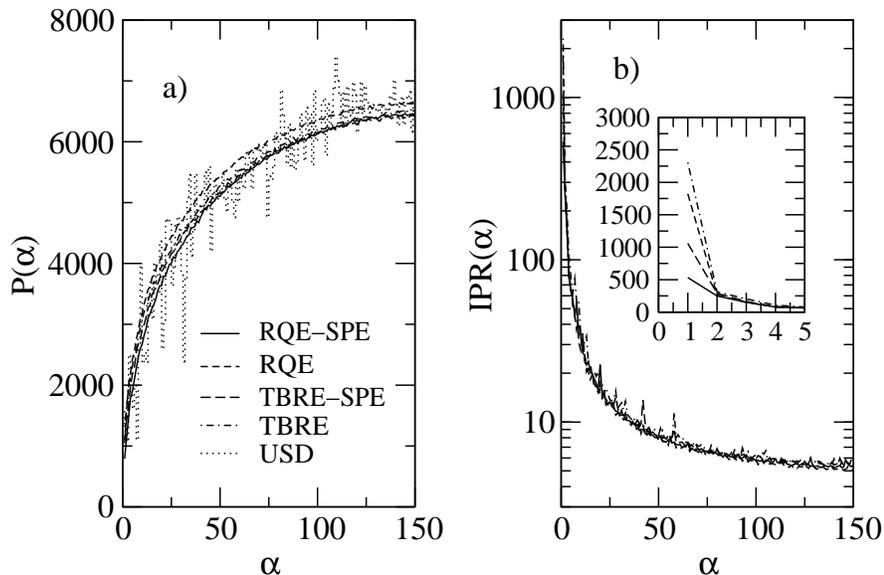


Figure 1. a) Exponential of the entropy as a function of state number; b) the IPR as a function of state number; inset: the IPR for the first few states.

After obtaining the unfolded energies, we proceed to calculate  $\Delta_3(L)$ . This measure characterizes the amount of chaos within a given spectrum of states with the same spin and parity. Shown in Fig. 2 are the results for calculations of  $\Delta_3(L)$  using 200  $0^+$  states in the  $^{24}\text{Mg}$  nucleus. A Poisson-distributed spectrum yields  $\Delta_3(L) = \frac{L}{15}$  as shown in the figure. All distributions follow this line for small values of  $L \leq 5$  which is the expected behavior since one cannot distinguish random or non-random effects with just a small window of levels. In the next region (from  $5 \leq L \leq 18$ ) we see a GOE character. The deviation from GOE occurs at approximately  $L = L_d = 20$  and is independent of the single-particle Hamiltonian. I note that the deviation from a GOE distribution, starting at  $L_d$ , is approximately equal to the ratio of the major shell structure,  $\delta\varepsilon$  and the average level spacing  $D$ . The average level spacing is  $D = 0.22, 0.20, 0.25, 0.25$  MeV for the RQE-SPE, RQE, TBRE-SPE, and TBRE, respectively. These differences are statistically insignificant as the statistical error in each case is roughly 0.05 MeV. One surmises from the figure that in each case  $L_d \approx 18$ . Thus  $\delta\varepsilon = 4.2 \pm 0.9$  MeV and is independent of the random interaction used. Thus I find no qualitative difference of the overall spectral rigidity for the four interactions studied at lower values of  $L$ . As  $L$  reaches larger values, we see the systematic trend that  $\Delta_3(L)$  is smaller when the single-particle energies are included.

#### 4. Conclusions

In this brief description, I have discussed some of the interesting aspects of the statistical properties of random interactions, including the entropy, inverse participation ratio, and the spectral rigidity. I have shown that the various random interactions behave very

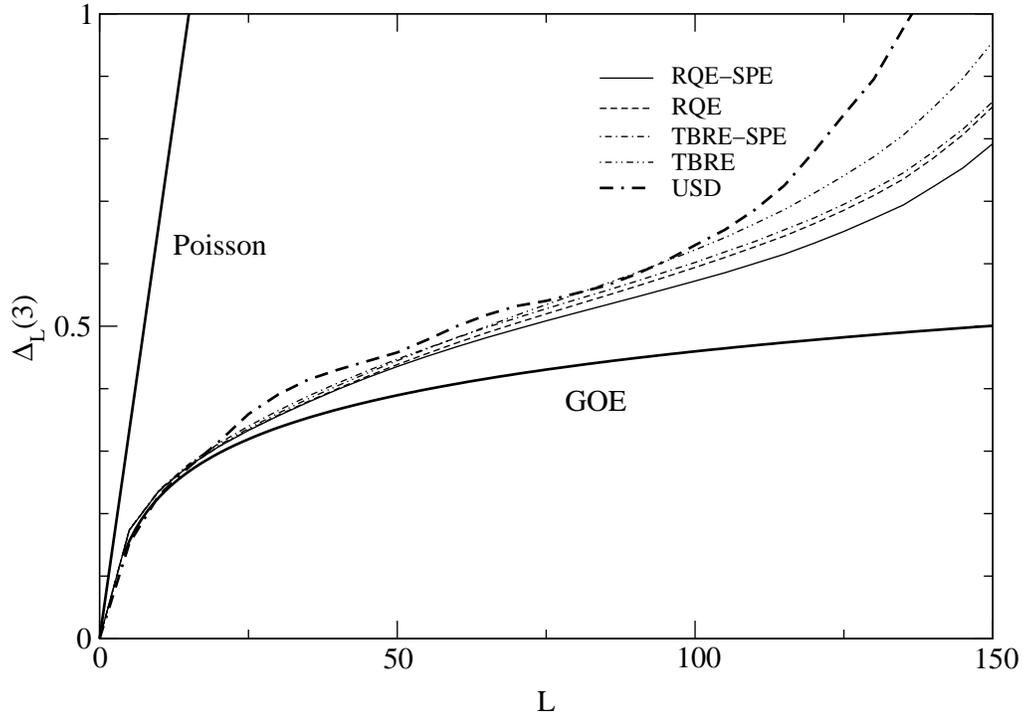


Figure 2.  $\Delta_3(L)$  as a function of  $L$  for the ensembles studied in this work.

similarly, and much like the realistic interactions, when the wave functions are viewed through this type of analysis. Only in ground-state properties are there significant differences. Indeed, the IPR analysis indicates that the random interactions appear to give a more correlated ground state when compared to realistic interactions.

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