Sensor Fusion Methods in Physical Systems Based on Physical Laws

Nageswara S.V. Rao
(Nagi Rao)
Computer Science and Mathematics Division
Oak Ridge National Laboratory
Oak Ridge, Tennessee 37831-6364

ONR/GTRI Workshop on
Target Tracking and Sensor Fusion
Atlanta, Georgia
May 17-18, 2000

Research Sponsored by
Office of Naval Research
Ballistic Missile Defense Organization
U. S. Department of Energy

Outline of the Talk

1. **Introduction**

   1.1 Physical System

   1.1 Fusion Rule Estimation Problem

   1.2 Finite Sample Solutions

2. **Fuser Design**

   2.1 Physical Laws

   2.2 Fuser for Smooth Laws

   2.3 Fuser for Non-Smooth Laws

3. **Conclusions**

References:

1. N. S. V. Rao, D. B. Reister, J. Barhen, Fusion Method For Physical Systems Based On Physical Laws, FUSION 2000, Paris.
2. N. S. V. Rao, D. B. Reister, J. Barhen, Information Fusion in Physical Systems Using Physical Laws, Symposium on Energy Engineering Sciences, 2000, Argonne, IL.

Physical System

System Parameters: $P(z) = (p_1, p_2, \ldots, p_n)$

Parameter: $p_i(z)$ is measured and/or estimated:

(i) a_i **measurements**: measured by an instrument

$$m_i(z) = \{m_{i,1}(z), m_{i,2}(z), \ldots, m_{i,a_i}(z)\}$$

(ii) b_i **estimators**: computational method based on measurements

$$e_i(z) = \{e_{i,1}(z), e_{i,2}(z), \ldots, e_{i,b_i}(z)\}$$

In a nutshell:

Every parameter is either measured or estimated

– It's real values may not be known

Fusion Problem

For each parameter p_i, obtain a fuser f_i such that
$f_i(m_i(z), e_i(z))$ is "close" to $p_i(z)$.

Fuser for all parameters: $f = (f_1, \ldots, f_n)$

Expected error of fused is

$$E(f) = \sum_z \int C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n},$$

where $C[.]$ is a cost function.

Example: Classifiers:
Parameter $p_i \in \{0, 1\}$ is the actual class,
Fuser outputs Boolean vectors: $f = (f_1, \ldots, f_n) \in \{0, 1\}^n$
Fuser Error is

$$\begin{aligned} &C[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))] \\ &\quad = [f_1(m_1(z), e_1(z)) \oplus p_1] + \ldots + [f_n(m_n(z), e_n(z)) \oplus p_n] \end{aligned}$$

– simply, the number of parameters misclassified by fuser.

Best Fuser

Choose f_i from class $\mathcal{F}_i$ such as linear space, neural network.

Best Expected Fuser:
$f^* \in \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ one with least $E(.)$.

Present Formulation: Error distributions are not known
Measurements are given

$$\{<(m_1(z), e_1(z)), \ldots, (m_n(z), e_n(z))>: z = 1, \ldots, s\}.$$

Question: can we compute a close approximation to f^* ?

Note: Similar problems have been solved
but this problem is not amenable to conventional approaches

Traditional Paradigm

Cost Function is Direct:
Typically, training data is given for p_i:
– can employ a cost function such as

$$C[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))]$$
$$= [f_1(m_1(z), e_1(z)) - p_1]^2 + \ldots + [f_n(m_n(z), e_n(z) - p_n)]^2$$

Method of Empirical Risk Minimization:
Minimize empirical error by $\hat{f}$

$$\hat{E}(f) = \sum_z C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right],$$

instead of

$$E(f) = \sum_z \int C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n},$$

Result: $E(\hat{f})$ can be shown to be close to $E(f^*)$ either
(i) as sample size goes to infinity (Asymptotic consistency), or
(ii) with high probability $E(\hat{f}) \leq E(f^*) + \epsilon$.

Difficulty: Traditional paradigm is not applicable:
every parameter p_i is either measured or estimated
– No actual parameter values are known

Physical Laws

Physical Law: relates the actual parameters
Consider the form

$$L[p_1(z), p_2(z), \dots p_n(z)] = 0$$

Monotonicity Condition:
for any y_1, y_2, $|y_1| \leq |y_2|$, we have

$$\begin{aligned} &|L[p_1(z), \dots, p_i(z) + y_1, \dots, p_n(z)]| \\ &\quad \leq |L[p_1(z), \dots, p_i(z) + y_2, \dots, p_n(z)]|. \end{aligned}$$

Informally, accurate estimators have lesser or equal error compared to less-accurate estimators.

Note: we only consider time-invariant or position-invariant laws
$L[.]$ itself does not dependent on z

Example

Scenario: Known mass m subjected to a constant force f in a friction-free environment.

Physical Law: in this case is $f = ma$, and employ

$$L[f, a] = (f - ma)^2 = 0$$

Sensors: given
one force sensor, and two acceleration sensor.

Force measurements: $m_{1,1}(z) = f + \epsilon$, for some ϵ,
Acceleration measurements:
$m_{2,1} = a + \delta$
$m_{2,2} = 0.7a$
where δ is a small normally distributed error.

Physical Laws are not Exactly Satisfied:

$$L[m_{1,1}, m_{2,1}] = (\epsilon - m\delta)^2$$

$$L[m_{1,1}, m_{2,2}] = (\epsilon + 0.3ma)^2$$

Physical Law Violations:

For each parameter,
– choose a single estimator or measurement $\hat{p}_i$:
closeness of $L[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)]$ to 0
determines how closely the law is satisfied.

Observation:
Violation of physical law can be considered an error measure

Best Set of Measurements/Estimators

Basic Set: S

For each parameter p_i, choose one measurement or estimator $\hat{p}_i$: The error due to S is

$$\hat{E}(S) = \sum_z L[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)]$$

Random error of S:

$\hat{S}$: has least random error:

$$\hat{E}(\hat{S}) = \min_S \hat{E}(S).$$

Expected error of S:

$$E(S) = \sum_z \int L\left[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n}$$

Best Sensor Set:

S^* has least expected error: $E(S^*) = \min_S E(S^*)$.

– among $\prod_{i=1}^{n}(a_i + b_i)$ possible basic sets

Example – Cntd.

Physical Laws not Exactly Satisfied:

$$L[m_{1,1}, m_{2,1}] = (\epsilon - m\delta)^2$$

$$L[m_{1,1}, m_{2,2}] = (\epsilon + 0.3ma)^2$$

Condition: $a > 0$, and $\epsilon \geq m\delta$; $|\delta| \leq |0.3a|$,
we have

$$L[m_{1,1}, m_{2,1}] \leq L[m_{1,1}, m_{2,2}],$$

As a result, for large values of a: better choice is $m_{2,1}$
otherwise $m_{2,2}$ is a better

Fusion Rule Estimation

Expected error of fuser f:

$$E(f) = \sum_z \int L\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n}$$

Optimal Expected Fuser: f^* has least expected error:
– $E(f)$ cannot be computed if the error distributions are not known,
– hence f^* is not computable.

Solution: compute $\hat{f}$ that minimizes the empirical cost

$$\hat{E}(f) = \sum_z L\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right],$$

based on of measurements

$$\{<(m_1(z), e_1(z)), \ldots, (m_n(z), e_n(z))>: z = 1, \ldots, s\}.$$

Summary of Performance:
We provide methods to ensure $E(f^*) \leq E(S^*)$,

$$E(\hat{f}) < E(\hat{S}),$$

– with a specified probability
– based entirely on the measurements, and
– without any knowledge of underlying distributions.

Isolation Property

Definition: Fuser class

$$\mathcal{F}_i = \{f_i(y) : \Re^{a_i+b_i} \mapsto \Re\},$$

for $y = (y_1, \ldots, y_{(a_i+b_i)})$, has the *isolation property* if it contains the function $\tau_j(y) = y_j$ for all $j = 1, 2, \ldots, (a_i + b_i)$.

Consequences of Isolation Property:
If each $\mathcal{F}_i$ satisfies the isolation property, we have

$$E(f^*) \leq E(S^*)$$

and

$$\hat{E}(\hat{f}) \leq \hat{E}(\hat{S}).$$

Examples:
1. Linear combinations
2. Linearized feedforward networks

Lipschitz Property

For any $g : [-A, A]^d \mapsto \Re$, let

$$\| g(r) \|_\infty = \sup_{r \in [-A,A]^d} |g(r)|.$$

Definition:
A $g(y) : [-A, A]^d \mapsto \Re^a$ is *Lipschitz* with constant k_g if for all $y_1, y_2 \in [-A, A]^d$, we have

$$\| g(y_1) - g(y_2) \|_\infty \leq k_g \| y_1 - y_2 \|_\infty .$$

Examples:
1. sigmoid neural networks are Lipschitz with the constant specified by the parameters of the network;
2. linear combination are Lipschitz

Example - Cntd

Fuser Class:: $f_i(y) = w_1 y_1 + \ldots + w_k y_k$, for $w_i \in [-W, W]$
– isolation property is trivially satisfied
– Lipschitz constant of f_i is W.

Linear Fuser

$$f_2(m_{2,1}, m_{2,2}) = w_1 m_{2,1} + w_2 m_{2,2}.$$

For the choice $w_1 = 0.3$ and $w_2 = 1$, we have $f_2(m_{2,1}, m_{2,2}) = a + 0.3\delta$, and

$$L[m_{1,1}, f_2(m_{2,1}, m_{2,2})] = (\epsilon - 0.3m\delta)^2,$$

– always smaller than $L[m_{1,1}, m_{2,1}]$, and
– is smaller than $L[m_{1,1}, m_{2,2}]$ for small $\delta \leq a$ with probability one

Asymptotic Convergence Result
Smooth Laws and Fusers

Physical Law: is Lipschitz

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: Lipschitz

General Result: Irrespective of the sensor distributions

$$E(\hat{f}) \to E(f^*),$$

as $s \to \infty$.

Informal Statement: error of $\hat{f}$ approaches optimal error

Furthermore:
If $\mathcal{F}_i$ satisfies the isolation property for all i, we have

$$E(\hat{f}) \to E(f^*) \leq E(S^*)$$

as $s \to \infty$.

Informal Statement: $\hat{f}$ approaches f^* which can have much smaller error than S^*.

Finite Sample Result
Smooth Laws and Fusers

Physical Law: Lipschitz

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: Lipschitz

General Result: Given sufficiently large sample,
we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon\right] \leq \delta,$$

irrespective of the sensor distributions.

Informally, with probability $1 - \delta$, the cost of the sample-based solution is within ϵ of the lowest achievable cost

Comments on Result
Smooth Laws and Fusers

Distribution-free finite sample result:

– depends only on measurements and
– does not depend on the sensor distributions.

Best possible result:

– stronger results such as showing $\delta = 0$ is not possible,
since f^* depends on a distribution (which is not finite-dimensional) and
$\hat{f}$ depends on a *finite* sample.

Analytically justifies the method even for small sample sizes

Comments on Conditions
Smooth Laws and Fusers

Physical Law:

– Smoothness conditions are reasonable

– Lipschitz condition is satisfied for a number of physical laws, although not always guaranteed.

Fuser Class:

Isolation and Lipschitz properties are satisfied in

– linear combinations with bounded coefficients and

– piecewise linear feedforward networks

Finite Sample Result – Better than Best Sensor set Smooth Laws and Fusers

Under isolation property for all $\mathcal{F}_i$

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$
$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally speaking, error of the computed fuser $\hat{f}$
(i) is not likely to be much higher than that of the best basic set, and
(ii) could be much smaller.

<u>Theorem</u>

Theorem 1 *Consider that the physical law is Lipschitz with constant* k_L*, and parameters, estimators and measurements are bounded such that* $p \in [-C, C]^n$*,* $m_i \in [-A, A]^{a_i}$*, and* $e_i \in [-B, B]^{b_i}$*. Let each fuser class* $\mathcal{F}_i$ *be Lipschitz with constant* k_{f_i}*. Let*

$$d = \sum_{i=1}^{n} (a_i + b_i)$$

and

$$k = k_L \max(k_{f_1}, k_{f_2}, \ldots, k_{f_n}).$$

Then given a sample size

$$s = \frac{512k(A+B)}{\epsilon^2} \left[d \ln \left(\frac{32k(A+B)}{\epsilon} \right) + \ln(8/\delta) \right],$$

we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon \right] \leq \delta,$$

irrespective of the sensor distributions. Furthermore, $E(\hat{f}) \to E(f^*)$*, as* $s \to \infty$*.*

Corollary

Corollary 1 *Let $\mathcal{F}_i$ satisfy the isolation property for all $i = 1, 2, \ldots, n$. Under the same conditions as Theorem 1, we have following conditions satisfied.*

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$

$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Bounded Variation Property

One-dimensional function $h : [-A, A] \mapsto \Re$.
Partition of $[-A, A]$: Set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$-A = x_0 < x_1 < \ldots < x_n = A$$

$\mathcal{P}[-A, A]$: all possible partitions

Bounded-Variation: $g : [-A, A] \mapsto \Re$ is of bounded variation:
if there exists M such that for any $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$\sum(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M.$$

Multiple Dimensions:
A function $g : [-A, A]^d \mapsto \Re$ is of bounded variation if it is so in each of its input variable for every value of the other input variables.

Useful facts:
(i) not all continuous functions are of bounded variation,
e.g. $g(x) = x \cos(\pi/(2x))$ for $x \neq 0$ and $g(0) = 0$;
(ii) differentiable functions on compact domains are of bounded variation;
(iii) absolutely continuous functions, which include Lipschitz functions, are of bounded variation.

Example

Scenario: of H_2O heated in a container:
p_1: temperature
$p_2 \in \{0, 1\}$: state, $p_2 = 0$: liquid
$p_2 = 1$: steam

T_0: boiling temperature under this condition.

Physical law: : $p_2 = 0$ if $p_1 < T_0$ and $p_2 = 1$ otherwise.

$$L[p_1, p_2] = p_2 1_{\{p_1 < T_0\}} + (p_2 - 1) 1_{\{p_1 \geq T_0\}} = 0,$$

where the indicator function 1_C is:
1 if condition C is true
0 if condition C is false
– Not Lipschitz in p_1

<u>**Motivation:**</u> For non-smooth physical laws
To address the cases typified by the above $L[.]$
we consider the class of functions with bounded variation

Pseudo-Dimension

Fuser Class: With finite pseudo-dimension

For $\mathcal{G} = \{g : X \mapsto \Re\}$ and $S = \{x_1, x_2, \ldots, x_m\} \subseteq X$:

Pseudo-shattering: S is pseudo-shattered by F
if there are real numbers r_1, r_2, $\ldots$, r_m
such that for each $b \in \{0, 1\}^m$ there is a function g_0 in $\mathcal{G}$ with

$$\operatorname{sgn}(f_b(x_i) - r_i) = b_i$$

for $1 \leq i \leq m$.

Pseudo-Dimension: $\mathcal{G}$ has pseudo-dimension d: maximum cardinality of a subset S of X that is pseudo-shattered by $\mathcal{G}$

Examples: Function classes with known pseudo-dimension
1. Feedforward sigmoidal neural networks
2. Vector spaces

Asymptotic Convergence Result

Physical Law: bounded variation

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: bounded pseudo-dimension

General Result: Irrespective of the sensor distributions

$$E(\hat{f}) \to E(f^*),$$

as $s \to \infty$.

Informal Statement: error of $\hat{f}$ approaches optimal error

Furthermore:
If $\mathcal{F}_i$ satisfies the isolation property for all i, we have

$$E(\hat{f}) \to E(f^*) \leq E(S^*)$$

as $s \to \infty$.

Informal Statement: $\hat{f}$ approaches f^* which can have much smaller error than S^*.

Finite Sample Result

Physical Law: Bounded Variation
– includes Lipschitz laws as subclass

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: finite psuedo-dimension

General Result: Given sufficiently large sample,
we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon\right] \leq \delta,$$

irrespective of the sensor distributions.

Informally, with probability $1 - \delta$, the cost of the sample-based solution is within ϵ of the lowest achievable cost

<u>Comments on Result</u>

Distribution-free finite sample result:

– depends only on measurements and
– does not depend on the sensor distributions.

Best possible result:

– stronger results such as showing $\delta = 0$ is not possible,
since f^* depends on a distribution (which is not finite-dimensional) and
$\hat{f}$ depends on a *finite* sample.

Analytically justifies the method even for small sample sizes

Comments on Conditions

Physical Law:
– Allows for discontinuities
– Allows for discrete-valued parameters

Fuser Class:
Isolation and pseudo-dimension properties are satisfied in
– linear combinations with bounded coefficients and
– piecewise linear feedforward networks

Finite Sample Result – Better than Best Sensor set

Under isolation property for all $\mathcal{F}_i$

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$
$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally speaking, error of the computed fuser $\hat{f}$
(i) is not likely to be much higher than that of the best basic set, and
(ii) could be much smaller.

Theorem

Theorem 2 *Consider that the physical law is of bounded variation such that* $|L(p)| \leq M_L$ *for all* p. *Let parameters, estimators and measurements are bounded. Let each fuser class* $\mathcal{F}_i$ *have finite pseudo-dimension* d_i, *and each fuser function* g *be bounded such that* $|g(.)| \leq M$ *for all parameters. Let* $d = \sum_{i=1}^{n} d_i$. *Then given a sample of size*

$$s = \frac{256 M_L^2}{\epsilon^2} \left[4d \ln\left(\frac{128eM}{\epsilon}\right) + (n+1)\ln(4/\delta) \right],$$

we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon \right] \leq \delta,$$

irrespective of the sensor distributions.
Furthermore, $E(\hat{f}) \rightarrow E(f^*)$, *as* $s \rightarrow \infty$.

Corollary

Corollary 2 *Let $\mathcal{F}_i$ satisfy the isolation property for all $i = 1, 2, \ldots, n$. Under the same conditions as Theorem 2, we have following conditions satisfied.*

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta \quad \text{and} \quad \mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally, error of the fuser $\hat{f}$ is not likely to be much higher than that of the best basic set, and could be much smaller.

Methane Hydrates Problem

Gas hydrates: crystalline substances composed of water and gas, in which gas molecules are contained in cage-like lattices formed by solid water.

Gas hydrates are present in permafrost regions and beneath the sea in sediment of outer continental margins.

Alternative to conventional gas as a fuel:
– methane hydrates
– expected to be available in volumes exceeding known estimates for conventional gas reserves.

Challenging Problem: predict prescence of hydrates using measurements collected at wells

Methane Hydrates Problem - Measurements

Measurements:
At each well, using a suite of sensors
– density,
– neutron porosity,
– acoustic transit-time,
– electric resistivity
collected at various depths in the well

Our focus: estimation of the *porosity* at various depths.

Data:
3045 sets of measurements each collected at different depths in a single well.

Methods to estimate porosity: based on
– neutron measurements ($\hat{\phi}_1$),
– density measurements ($\hat{\phi}_2$),
– fluid velocity equation ($\hat{\phi}_3$),
– acoustic travel time based on S-wave ($\hat{\phi}_4$),
– time-average equation based on P-wave ($\hat{\phi}_5$),
– Wood's equation ($\hat{\phi}_6$)

Methane Hydrates Problem - Physical Law

Well-established physical law: relates:

ϕ: porosity
ρ: density
ψ: hydrate concentration

$$L[\phi, \psi, \rho] = (\phi[\rho_m - (1-\psi)\rho_w + \psi\rho_h] - \rho + \rho_m)^2 = 0,$$

where ρ_m, ρ_w, and ρ_h are known constants.

We employ:
– one measurement for density $\hat{\rho}$
– a single estimator $\hat{\psi}$ for the hydrate concentration using the Archie's equation – six estimates of porosity ϕ

Problem Addressed:
– we assume that the estimators $\hat{\rho}$ and $\hat{\psi}$ are reasonably accurate, and
– objective is to fuse the porosity estimates.

Methane Hydrates Problem - Estimators

Error due to a set of estimators $\hat{\phi}$, $\hat{\psi}$, and $\hat{\rho}$ is given by

$$\begin{aligned}\hat{E}(\hat{\phi}, \hat{\psi}, \hat{\rho}) &= \sum_{z=1}^{3045} L[\hat{\phi}, \hat{\psi}, \hat{\rho}] \\ &= \left(\hat{\phi}(z)[\rho_m - (1-\hat{\psi}(z))\rho_w + \hat{\psi}(z)\rho_h] - \hat{\rho}(z) + \rho_m\right)^2.\end{aligned}$$

Error estimates for individual estimators:

estimator	$\hat{E}(.)$
$\hat{\phi}_1$	1.454920
$\hat{\phi}_2$	1.407088
$\hat{\phi}_3$	1.400968
$\hat{\phi}_4$	1.400967
$\hat{\phi}_5$	1.728479
$\hat{\phi}_6$	1.410965
fused estimate	0.079374

In terms of $\hat{E}(.)$, the best estimator for porosity is $\hat{\phi}_4$.

Fuser

Fuser based on linear combination

$$\hat{\phi}_F = w_7 + \sum_{i=1}^{6} w_i \hat{\phi}_i,$$

where $(w_1, \ldots, w_7) \in \Re^7$ is the weight vector.

Computation: Weight vector to minimize the error

$$\hat{E}(\hat{\phi}_F) \;=\; \sum_{z=1}^{3045} L\left[w_7 + \sum_{i=1}^{6} w_i \hat{\phi}_i(z), \hat{\psi}(z), \hat{\rho}(z)\right].$$

Solution:

w_1	1.198060
w_2	2.871261
w_3	-0.264511
w_4	-7.243817
w_5	2.254488
w_6	-6.330654
w_7	2.458406

Results

– Error achieved by $\hat{\phi}_F$ is about 20 times better than best estimator $\hat{\phi}_4$.

– More dramatic performance of the fuser was observed when smaller subsets of well log data were utilized.

Perspective:
Since the magnitude of the error depends critically on the exact form of $L[.]$, the improvement must be interpreted in the context,

Computation:
The computation is performed using the netlib library routines for solving the least square problems.

Conclusions

- New Paradigm for Sensor Fusion:
  - Physical systems - physical laws can be used for fusion
  - Each parameter is sensed and/or estimated by different sensors and methods
  - Provable performance - finite sample and asymptotic guarantees
  - Fused system better than best set
  - Physical laws considered - smooth and non-smooth
- Work in progress
  - Projective fusers — better than best subset of sensors ?
    — beyond isolation property.
  - Time-varying physical laws - can they be used for fusion ?
  - More general physical laws and fusion rules
  - Applications - methane hydrates, ballistic objects.