Approximate Identification of Boundary Properties in a Wave Equation by Optimal Control Techniques [1]

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ABSTRACT: We apply optimal control techniques to find approximate solutions to an inverse problem for the acoustic wave equation. The inverse problem is to determine the shape and reflection coefficient of a part of the boundary from partial measurements of the acoustic signal. The sought functions are treated as controls and the goal is to drive the model solution close to the experimental data by adjusting these functions.

§1. An Identification Problem

Let $G \subset \mathbb{R}^2$ be a bounded domain with a C^1 boundary, ∂G. Define

$$\begin{cases} \Omega = G \times (0,2) \subset \mathbb{R}^3, \\ \Gamma = G \times \{2\}. \end{cases} \tag{1.1}$$

Thus, Γ is the *top* portion of the boundary $\partial\Omega$. In what follows, we denote a generic point in $\mathbb{R}^2$ by $x = (x_1, x_2)$. Suppose $u_0 : \overline{G} \to [1,2]$ is continuous. Let $\Omega(u_0) = \big\{(x,z) \in \Omega \;\big|\; x \in G,\ 0 < z < u_0(x)\big\}$,

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whose boundary is denoted by $\partial\Omega(u_0)$ with the *top* portion, $\Gamma(u_0)$, given by $\Gamma(u_0) = \{(x, u_0(x)) \mid x \in G\}$. We suppose that the region $\Omega(u_0)$ has a known *local velocity tensor* $E \in C(\overline{\Omega}; \mathbb{R}^{3\times 3})$ satisfying

$$E(x,z) \geq \delta I, \qquad \forall (x,z) \in \overline{\Omega}, \tag{1.2}$$

for some $\delta > 0$. Let $\sigma_0 : G \to [0,\infty)$ be a function called the *reflection rate*. We assume that u_0 and σ_0 are unknown. Our goal is to *estimate* both u_0 and σ_0 from observations of the actual solution. Specifically, we send some *signal* f to the region $\Omega(u_0)$ and receive a corresponding observation h from the region ω such that $\omega \subset G \times (0,1) \subseteq \overline{\Omega(u_0)} \subset \Omega$. It is reasonable to expect that based on the information contained in f, h, ω, etc., we can approximately determine u_0 and σ_0.

Define

$$\begin{cases} \mathcal{U} \triangleq \{u \in H^2(G) \mid 1 \leq u(x) \leq 2, \ \forall x \in G\}, \\ \Sigma \triangleq \{\sigma \in \bigcup_{q>2} L^q(G) \mid \sigma(x) \geq 0, \text{ a.e. } x \in G\}. \end{cases} \tag{1.3}$$

By Sobolev's embedding theorem, one has $H^2(G) \hookrightarrow W^{1,p}(G) \bigcap C^\alpha(\overline{G}), \quad \forall 1 \leq p < \infty, \ \alpha \in (0,1)$, since G is a bounded domain in $\mathbb{R}^2$ with a C^1 boundary. We assume that $(u_0, \sigma_0) \in \mathcal{U} \times \Sigma$.

Next, for any $u \in \mathcal{U}$, we introduce a domain contained in Ω, $\Omega(u) = \{(x,z) \in \Omega \mid x \in G, 0 < z < u(x)\}$, whose boundary is denoted by $\partial\Omega(u)$, with the *top* portion, $\Gamma(u)$, given by $\Gamma(u) = \{(x, u(x)) \mid x \in G\}$. Now, for any $u \in \mathcal{U}$ and $\sigma \in \Sigma$, we consider the following wave equation:

$$\begin{cases} W_{tt} - \nabla\cdot(E\nabla W) = f, & \text{in } (0,T) \times \Omega(u), \\ W\big|_{\partial G} = 0, \\ \dfrac{\partial W}{\partial n_E}\Big|_{z=0} = 0, \\ \Big(\dfrac{\partial W}{\partial n_E} + \sigma W\Big)\Big|_{\Gamma(u)} = 0, \\ W\big|_{t=0} = \varphi, \qquad W_t\big|_{t=0} = \psi, \end{cases} \tag{1.4}$$

where $f \in L^2((0,T) \times \Omega)$ is a given *source* term (representing the signal that we send to the domain). The *conormal derivative* $\frac{\partial W}{\partial n_E}$ is defined by

$$\frac{\partial W}{\partial n_E} = \langle E\nabla W, n \rangle, \qquad \text{on } \partial\Omega(u), \tag{1.5}$$

with n being the outward normal of the boundary $\partial\Omega(u)$. We note that $\partial\Omega(u)$ is only C^α for any $\alpha \in (0,1)$ and thus, standard theory for well-posedness does not apply directly to (1.4). We will show that (1.4) admits a unique weak solution in a sense that will be made precise in Section 3.

Our identification problem is to find u and σ in $\mathcal{U}$ and Σ, respectively, such that the (weak) solution W of (1.4) is close to the observation h on ω. Specifically, we want to minimize the functional

$$\mathcal{J}(u,\sigma) = \frac{1}{2}\int_0^T \int_\omega |W(t,x,z) - h(t,x,z)|^2 dz dx dt \tag{1.6}$$

over all possible pairs $(u, \sigma) \in \mathcal{U} \times \Sigma$.

To regularize this ill-posed identification problem, we introduce the following optimal control problem. We consider that u and σ are controls to be adjusted such that - at optimality - they minimize the following penalty functional $(a, b > 0)$:

$$
\begin{aligned}
J(u,\sigma) =& \frac{1}{2}\int_0^T \int_\omega |W(t,x,z) - h(t,x,z)|^2 dz dx dt \\
&+ \frac{a}{2}\int_G \operatorname{tr}\,[u_{xx}(x)^2] dx + \frac{b}{2}\int_G \sqrt{1+|u_x(x)|^2}\,\sigma(x)^2 dx.
\end{aligned} \tag{1.7}
$$

Here, as before, W is the solution of (1.4) corresponding to the chosen u and σ. The form of the penalty terms indicates that we look for functions u and σ that are smooth enough. The functions u_x and u_{xx} denote the gradient and the Hessian of u, respectively. Note that for any $u \in \mathcal{U}$, $u_x \in L^p(G; \mathbb{R}^2)$ for all $p \in [1, \infty)$ (since $G \subset \mathbb{R}^2$ is a bounded region). Thus, we can see that the last integral in (1.7) is well-defined for any $u \in \mathcal{U}$ if and only if $\sigma \in \Sigma$. The optimal control problem is formulated as follows:

Problem (OC). Find $(\overline{u}, \overline{\sigma}) \in \mathcal{U} \times \Sigma$, such that

$$
J(\overline{u}, \overline{\sigma}) = \inf_{\mathcal{U}\times\Sigma} J(u, \sigma). \tag{1.8}
$$

Assuming the inverse problem (1.6) has a unique solution, then for small a, b, the optimal control pair $(\bar{u}, \bar{\sigma})$ that satisfies (1.8), should approximate this solution reasonably well. To obtain a solution to the original identification problem (1.6) we let the parameters a, b tend to zero.

The novelty and difficulty of the problem considered in this paper comes from the combination of shape optimization and bilinear boundary control. The bilinear boundary control alone has been addressed in [2]. Shape optimization for the wave equation or related PDEs has been considered in [1,5]. The results contained here further the application of our general formalism [3] to the parameter identification problem for the acoustic wave equation [2]. A characterization of the optimal pairs can be found in the full treatment of this problem in [4].

fixed domain. In section 3 we prove the well-posedness of the weak formulation of the direct problem for any $(u, \sigma) \in \mathcal{U} \times \Sigma$. The existence of optimal controls $(\overline{u}, \overline{\sigma})$ is proven in section 4. Finally, in section 5 we connect these results to the identification problem (1.6).

§2. An Optimal Control Problem in a Fixed Domain

In this section, we introduce an equivalent formulation of the original problem. For given $(u, \sigma) \in \mathcal{U} \times \Sigma$, suppose W is a solution of (1.4). Let y and z be related by

$$
z = \frac{u(x)y}{2}, \qquad \text{or} \qquad y = \frac{2z}{u(x)}, \qquad x \in G. \tag{2.1}
$$

Thus, $y \in [0,2]$ and $z \in [0, u(x)]$. Define $w(t,x,y) = W(t,x,\frac{u(x)y}{2}), \forall (t,x,y) \in [0,T]\times\Omega$. Then, equivalently, we have $W(t,x,z) = w(t,x,\frac{2z}{u(x)}), \forall (t,x,z) \in [0,T]\times\Omega(u)$. Clearly,

$$
\begin{cases}
W_x(t,x,z) = w_x(t,x,\frac{2z}{u(x)}) - \dfrac{2u_x(x)z}{u(x)^2} w_y(t,x,\frac{2z}{u(x)}) \\
\qquad\qquad = w_x(t,x,y) - \dfrac{u_x(x)y}{u(x)} w_y(t,x,y), \\
W_z(t,x,z) = \dfrac{2}{u(x)} w_y(t,x,\frac{2z}{u(x)}) = \dfrac{2}{u(x)} w_y(t,x,y).
\end{cases} \tag{2.2}
$$

If we let

$$
\begin{cases}
U(x,y) \equiv U(y,u(x),u_x(x)), \quad (x,y)\in\Omega, \\
U(y,u,p) \triangleq \begin{pmatrix} I & -\frac{y}{u}p \\ 0 & \frac{2}{u} \end{pmatrix}, \quad y\in[0,2],\ u\in[1,2],\ p\in\mathbb{R}^2,
\end{cases} \tag{2.3}
$$

then it follows from (2.2) that

$$
\nabla_{(x,z)} W(t,x,z) = U(x,y)\nabla_{(x,y)} w(t,x,y). \tag{2.4}
$$

On the other hand, for any $\Theta(t,x,z) \equiv (\Theta^1(t,x,z), \Theta^2(t,x,z), \Theta^3(t,x,z))^T$, we let $\theta(t,x,y) \equiv (\theta^1(t,x,y), \theta^2(t,x,y), \theta^3(t,x,y))^T = \Theta(t,x,\frac{u(x)y}{2})$. Then, similar to (2.2), one has

$$
\begin{aligned}
\nabla_{(x,z)}\cdot\Theta(t,x,z) &= \Theta^1_{x_1}(t,x,z) + \Theta^2_{x_2}(t,x,z) + \Theta^3_z(t,x,z) \\
= \theta^1_{x_1}(t,x,y) - \frac{u_{x_1}(x)y}{u(x)}\theta^1_y(t,x,y) &+ \theta^2_{x_2}(t,x,y) - \frac{u_{x_2}(x)y}{u(x)}\theta^2_y(t,x,y) + \frac{2}{u(x)}\theta^3_y(t,x,y) \\
&= \nabla_{(x,y)}\cdot\Big(U(x,y)^T\theta(t,x,y)\Big) + \frac{1}{u(x)}\left\langle U(x,y)^T\theta(t,x,y), \nabla_{(x,y)}u\right\rangle.
\end{aligned} \tag{2.5}
$$

Set

$$
\widetilde{E}(x,y) = E(x,\tfrac{u(x)y}{2}), \tag{2.6}
$$

and

$$
\Theta(t,x,z) = E(x,z)\nabla_{(x,z)} W(t,x,z). \tag{2.7}
$$

Then, by (2.4) and (2.6), we have $\theta(t,x,y) = \widetilde{E}(x,y)U(x,y)\nabla_{(x,y)}w(t,x,y)$. Thus, by taking $\Theta(t,x,z)$ as (2.7) in (2.5), we obtain

$$
\begin{aligned}
\nabla_{(x,z)}\cdot\left[E(x,z)\nabla_{(x,z)}W(t,x,z)\right] &= \nabla_{(x,y)}\cdot\Big(U(x,y)^T\widetilde{E}(x,y)U(x,y)\nabla_{(x,y)}w(t,x,y)\Big) \\
&+ \frac{1}{u(x)}\left\langle U(x,y)^T\widetilde{E}(x,y)U(x,y)\nabla_{(x,y)}w(t,x,y), \nabla_{(x,y)}u(x)\right\rangle.
\end{aligned} \tag{2.8}
$$

Let us define

$$
\begin{cases}
F(x,y) \equiv F(x,y,u(x),u_x(x)), \qquad (x,y)\in\Omega, \\
F(x,y,u,p) \triangleq \begin{pmatrix} I & 0 \\ -\frac{y}{u}p^T & \frac{2}{u} \end{pmatrix} E(x,\frac{uy}{2}) \begin{pmatrix} I & -\frac{y}{u}p \\ 0 & \frac{2}{u} \end{pmatrix}, \\
\qquad\qquad (x,y)\in\Omega,\ u\in[1,2],\ p\in\mathbb{R}^2.
\end{cases} \tag{2.9}
$$

Then (2.8) can be rewritten as follows:

$$\nabla_{(x,z)} \cdot \left[E(x,z)\nabla_{(x,z)}W(t,x,z)\right] = \nabla_{(x,y)} \cdot \left[F(x,y)\nabla_{(x,y)}w(t,x,y)\right] + \frac{1}{u(x)} \langle F(x,y)\nabla_{(x,y)}w(t,x,y), \nabla_{(x,y)}u(x) \rangle .$$

Next, we consider the boundary conditions. First, (see (2.4))

$$\begin{aligned} 0 &= \Big(\frac{\partial W}{\partial n_E} + \sigma W\Big)\Big|_{\Gamma(u)} \\ &= \Big(\langle E\nabla_{(x,z)}W, \frac{1}{\sqrt{1+|u_x(x)|^2}}\begin{pmatrix} -u_x(x) \\ 1 \end{pmatrix}\rangle + \sigma W\Big)\Big|_{z=u(x)} \\ &= \Big(\frac{1}{\sqrt{1+|u_x(x)|^2}}\langle U^T\widetilde{E}U\nabla_{(x,y)}w, U^{-1}\begin{pmatrix} -u_x(x) \\ 1 \end{pmatrix}\rangle + \sigma w\Big)\Big|_{y=2} \\ &= \frac{u(x)}{2\sqrt{1+|u_x(x)|^2}}\Big(\langle F\nabla w, e_3\rangle + \frac{2\sqrt{1+|u_x(x)|^2}}{u(x)}\sigma(x)w\Big)\Big|_{y=2}, \end{aligned} \tag{2.10}$$

with $e_3 = (0,0,1)^T$ and hereafter $\nabla = \nabla_{(x,y)}$. Then, by defining the conormal with respect to F as $\frac{\partial w}{\partial n_F} = \langle F\nabla w, n\rangle$, on $\partial\Omega$, with n being the outward normal of $\partial\Omega$, we have from (2.10) that

$$\Big(\frac{\partial w}{\partial n_F} + \frac{2\sqrt{1+|u_x(x)|^2}}{u(x)}\sigma(x)w\Big)\Big|_{y=2} = 0. \tag{2.11}$$

Similarly, we have $\frac{\partial w}{\partial n_F}\Big|_{y=0} = 0$. Thus, we obtain the following equation for $w(t,x,y)$, over the fixed domain $[0,T]\times\Omega$:

$$\begin{cases} w_{tt} - \nabla\cdot(F\nabla w) = g, \\ w\big|_{\partial G} = 0, \\ \dfrac{\partial w}{\partial n_F}\Big|_{y=0} = 0, \\ \Big(\dfrac{\partial w}{\partial n_F} + \mu\sigma w\Big)\Big|_{y=2} = 0, \\ w\big|_{t=0} = w_0, \qquad w_t\big|_{t=0} = w_1, \end{cases} \tag{2.12}$$

with

$$\begin{cases} g(x,y) = \dfrac{1}{u(x)}\langle F(x,y)\nabla w, \nabla u(x)\rangle + f(t,x,\frac{u(x)y}{2}), \quad (t,x,y)\in[0,T]\times\Omega, \\ \mu(x) = \dfrac{2\sqrt{1+|u_x(x)|^2}}{u(x)}, \quad x\in G, \\ w_0(x,y,u(x)) = \varphi(x,\frac{u(x)y}{2}), \qquad (x,y)\in\Omega, \\ w_1(x,y,u(x)) = \psi(x,\frac{u(x)y}{2}), \qquad (x,y)\in\Omega. \end{cases} \tag{2.13}$$

The cost functional (1.7) becomes

$$\begin{aligned} J(u,\sigma) = &\frac{1}{2}\int_0^T\int_\omega \left|w(t,x,\tfrac{2z}{u(x)}) - h(t,x,z)\right|^2 dzdxdt \\ &+ \frac{a}{2}\int_G \operatorname{tr}[u_{xx}(x)^2]dx + \frac{b}{2}\int_G \sqrt{1+|u_x(x)|^2}\,\sigma(x)^2dx. \end{aligned} \tag{2.14}$$

The advantage of the new formulation is that the equation is considered in a fixed domain. However, as a result, the control enters in the leading coefficients of the state equation and the cost functional is somewhat non-standard in the first term.

§3. Weak Formulation and Well-posedness of the State Equation

Let H be the space $L^2(\Omega)$ with the usual L^2 inner product and norm denoted by $|\cdot|_H$, or simply by $|\cdot|$ if there is no ambiguity. For $p \geq 1$, let

$$W^{1,p}_{\partial G}(\Omega) \triangleq \{v \in W^{1,p}(\Omega) \mid v|_{(\partial G)\times[0,2]} = 0\}, \tag{3.1}$$

with the norm

$$\|v\|_{W^{1,p}_{\partial G}(\Omega)} = \Big\{\int_\Omega |\nabla v|^p dxdy\Big\}^{1/p}, \qquad \forall v \in W^{1,p}_{\partial G}(\Omega). \tag{3.2}$$

It is clear that $\|\cdot\|_{W^{1,p}_{\partial G}(\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ are equivalent (for all $p \in [1,\infty)$) and the embedding $W^{1,p}_{\partial G}(\Omega) \hookrightarrow H$ is continuous for $p > \frac{6}{5}$. Now, for fixed $(u,\sigma) \in \mathcal{U} \times \Sigma$, we define (recall (2.3))

$$\|v\|^2_{V(u,\sigma)} \triangleq \int_\Omega \Big|U(y,u(x),u_x(x))\nabla v(x,y)\Big|^2 dxdy + \int_G |u_x(x)\sigma(x)|v(x,2)^2 dx. \tag{3.3}$$

Note that for $(u,\sigma) \in \mathcal{U} \times \Sigma$, $|u_x\sigma| \in \Sigma$. Clearly, $\|\cdot\|_{V(u,\sigma)}$ is a norm defined on $C^\infty(\overline{\Omega})$. We let $V(u,\sigma)$ be the completion of $\{v \in C^\infty(\overline{\Omega}) \mid v|_{(\partial G)\times[0,2]} = 0\}$, under the norm $\|\cdot\|_{V(u,\sigma)}$. It is clear that $V(u,\sigma)$ is a Hilbert space under the inner product whose induced norm is $\|\cdot\|_{V(u,\sigma)}$. We have the following result.

Proposition 3.1. (i) *For any $p > 2$, there exists a constant $C_p > 0$, such that*

$$\begin{aligned} \|v\|_{V(u,\sigma)} \leq & C_p\big(1 + \|u\|_{W^{1,2p/(p-2)}(G)} + \|u_x\sigma\|^{1/2}_{L^q(G)}\big)\|v\|_{W^{1,p}_{\partial G}(\Omega)}, \\ & \forall (u,\sigma) \in \mathcal{U}\times\Sigma,\ v \in W^{1,p}_{\partial G}(\Omega),\ q > 2. \end{aligned} \tag{3.4}$$

Consequently,

$$\bigcup_{p>2} W^{1,p}_{\partial G}(\Omega) \subseteq \bigcap_{\substack{u\in\mathcal{U}\\ \sigma\in\Sigma}} V(u,\sigma). \tag{3.5}$$

Moreover, $W^{1,p}(\Omega)$ is dense in $V(u,\sigma)$ for any $p > 2$ and $(u,\sigma) \in \mathcal{U}\times\Sigma$.

(ii) *There exists a constant $C > 0$, such that for any $r \in (6/5, 2)$,*

$$\|v\|_{W^{1,r}_{\partial G}(\Omega)} \leq C\big(1 + \|u\|_{W^{1,2r/(2-r)}(G)}\big)\|v\|_{V(u,\sigma)}, \quad \forall (u,\sigma) \in \mathcal{U}\times\Sigma,\ v \in V(u,\sigma). \tag{3.6}$$

Consequently,

$$\bigcup_{\substack{u\in\mathcal{U}\\ \sigma\in\Sigma}} V(u,\sigma) \subseteq \bigcap_{r\in(6/5,2)} W^{1,r}_{\partial G}(\Omega) \hookrightarrow H. \tag{3.7}$$

To prove the above result, we need the following lemma.

Lemma 3.2. *For any $p > 1$, there exists a constant $C > 0$, such that*

$$\|v\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\|v\|_{W^{1,p}(\Omega)}, \qquad \forall v \in W^{1,p}(\Omega). \tag{3.8}$$

We note that in the case $\partial\Omega$ is C^1, the above result is standard (see [6]). In our case, $\partial\Omega$ is just Lipschitz and the result can be extended to this case.

Proof of Proposition 3.1. (i) First of all, by definition of $\mathcal{U}$, we have

$$\mathcal{U} \subset W^{1,p}(G), \qquad \forall p \in [1, \infty), \tag{3.9}$$

since $G \subset \mathbb{R}^2$. Let $(u, \sigma) \in \mathcal{U} \times \Sigma$, $v \in V(u, \sigma)$ and $p > 2$. Let us look at the first term on the right-hand side of (3.3).

$$\begin{aligned} &\Big\{\int_\Omega \Big|U(y, u(x), u_x(x))\nabla v(x, y)\Big|^2 dxdy\Big\}^{1/2} = \Big\{\int_\Omega \Big(\Big|v_x - \frac{y}{u}u_x v_y\Big|^2 + \frac{4}{u^2}v_y^2\Big)dxdy\Big\}^{1/2} \\ &\leq C\Big\{\int_\Omega \Big(|v_x|^2 + (1 + |u_x|^2)v_y^2\Big)dxdy\Big\}^{1/2} \leq C(1 + \|u\|_{W^{1,2p/(p-2)}(G)})\|v\|_{W^{1,p}(\Omega)}, \end{aligned} \tag{3.10}$$

with a uniform constant $C > 0$. Here, we have used the fact (3.9).

Next, we look at the second term on the right-hand side of (3.3). We claim that

$$\|v\|_{L^{2q/(q-1)}(\Gamma)} \leq C_p\|v\|_{W^{1,p}(\Omega)}, \quad \forall v \in W^{1,p}(\Omega),\ p, q > 2. \tag{3.11}$$

In fact, for any $q > 2$ and $2 < p < 3$, we have $2q/(q-1) < 2p/(3-p)$. Thus, by Sobolev's embedding theorem and Lemma 3.2, we have

$$\|v\|_{L^{2q/(q-1)}(\Gamma)} \leq C\|v\|_{L^{2p/(3-p)}(\Gamma)} \leq C\|v\|_{W^{1-1/p,p}(\Gamma)} \leq C_p\|v\|_{W^{1,p}(\Omega)}. \tag{3.12}$$

For the case $p \geq 3$, we let $\varepsilon > 0$ such that $p - \varepsilon \in (2, 3)$. Then, by (3.12), one has $\|v\|_{L^{2q/(q-1)}(\Gamma)} \leq C\|v\|_{W^{1,p-\varepsilon}(\Omega)} \leq C\|v\|_{W^{1,p}(\Omega)}$, which proves our claim (3.11). Consequently,

$$\Big\{\int_G |u_x(x)\sigma(x)|v(x, 2)^2 dx\Big\}^{1/2} \leq \|u_x\sigma\|_{L^q(G)}^{1/2}\|v\|_{L^{2q/(q-1)}(\Gamma)} \leq C_p\|u_x\sigma\|_{L^q(G)}^{1/2}\|v\|_{W^{1,p}(\Omega)}. \tag{3.13}$$

Combining (3.10) and (3.13), we obtain (3.4). Then, it follows that $W^{1,p}_{\partial G}(\Omega) \subseteq V(u, \sigma)$, $\forall p > 2$, $(u, \sigma) \in \mathcal{U} \times \Sigma$, which yields (3.5). The conclusion concerning the density is clear.

(ii) For any $(u, \sigma) \in \mathcal{U} \times \Sigma$ and $v \in V(u, \sigma)$, we have (see (3.3))

$$\|v\|^2_{V(u,\sigma)} \geq \int_\Omega |U\nabla v|^2 dxdy \geq \int_\Omega \Big[\Big|v_x - \frac{y}{u}u_x v_y\Big|^2 + |v_y|^2\Big]dxdy. \tag{3.14}$$

Here, we have used the fact that $u \in [1, 2]$. Now, for any $r \in [1, 2)$, we have

$$\begin{aligned} \|v_x\|_{L^r(\Omega)} &\leq \|v_x - \frac{y}{u}u_x v_y\|_{L^r(\Omega)} + \|\frac{y}{u}u_x v_y\|_{L^r(\Omega)} \\ &\leq \|v_x\|_{L^r(\Omega)} + C\|u_x\|_{L^{2r/(2-r)}(\Omega)}\|v_y\|_{L^{2r/(2-r)}(\Omega)} \leq C(1 + \|u_x\|_{L^{2r/(2-r)}(\Omega)})\|v_y\|_{L^{2r/(2-r)}(\Omega)} \end{aligned} \tag{3.15}$$

Then (3.6) follows, which leads to the first inclusion in (3.7). The last embedding follows immediately from the continuity of the embedding $W^{1,r}_{\partial G}(\Omega) \hookrightarrow H$. $\square$

Next, for any $(u,\sigma) \in \mathcal{U} \times \Sigma$, we introduce a bilinear form $a : V(u,\sigma) \times V(u,\sigma) \to \mathbb{R}$ as follows:

$$a(v,w) = \int_\Omega \langle F\nabla v, \nabla w \rangle \, dxdy + \int_\Gamma \mu\sigma v w dx, \quad \forall v,w \in V(u,\sigma), \tag{3.16}$$

Note that

$$|u_x(x)| \le \mu(x) \le 2(1+|u_x(x)|). \tag{3.17}$$

Thus, the second term on the right hand side of (3.16) is well-defined. It is clear that a is symmetric, i.e., $a(v,w) = a(w,v), \forall v,w \in V(u,\sigma)$ and bounded (see (3.3) and (3.22)), i.e. $|a(v,w)| \le C\|v\|_{V(u,\sigma)}\|w\|_{V(u,\sigma)}$, $\forall v,w \in V(u,\sigma)$. Moreover, by (1.2), (2.9), (3.3) and (3.17), we have the coercivity of a, namely for some $\widetilde{\delta} > 0$,

$$a(v,v) = \int_\Omega \langle F\nabla v, \nabla v \rangle \, dxdy + \int_\Gamma \mu\sigma v^2 dx \ge \widetilde{\delta}\|v\|^2_{V(u,\sigma)}, \qquad \forall v \in V(u,\sigma). \tag{3.18}$$

Then, we can define an $A \in \mathcal{L}\big(V(u,\sigma), V(u,\sigma)'\big)$ by

$$\langle Av, w \rangle_{u,\sigma} = a(v,w), \qquad \forall v,w \in V(u,\sigma), \tag{3.19}$$

where $\langle \cdot,\cdot \rangle_{u,\sigma}$ is the duality pairing between $V(u,\sigma)$ and $V(u,\sigma)'$. Since a depends on $(u,\sigma) \in \mathcal{U} \times \Sigma$, we also use the notation

$$a(v,w;u,\sigma) \equiv a(u,v) = \langle Av, w \rangle_{u,\sigma} \equiv \langle A(u,\sigma)v, w \rangle_{u,\sigma}. \tag{3.20}$$

Now consider the following:

$$\begin{cases} w''(t) + Aw(t) = g(t), & \text{in } V(u,\sigma)', \\ w(0) = w_0, \qquad w'(0) = w_1. \end{cases} \tag{3.21}$$

The following well-posedness result holds for (3.21).

Proposition 3.3. *Let $(u,\sigma) \in \mathcal{U} \times \Sigma$. For any $g \in L^2(0,T;V(u,\sigma)')$, $w_0 \in V(u,\sigma)$ and $w_1 \in H$, problem (3.21) admits a unique solution $w \in C([0,T];V(u,\sigma)) \bigcap C^1([0,T];H)$ satisfying the following estimate:*

$$\|w(\cdot)\|_{C([0,T];V(u,\sigma))} + \|w(\cdot)\|_{C^1([0,T];H)} \le C\big\{\|w_0\|_{V(u,\sigma)} + |w_1| + \int_0^T \|g(s)\|_{V(u,\sigma)'} ds\big\}. \tag{3.22}$$

We can put (3.21) into a more convenient equivalent form:

$$\int_0^T \gamma(t)(g(t),v)dt = -\gamma(0)(w_1,v) + \dot{\gamma}(0)(w_0,v) + \int_0^T \big\{\ddot{\gamma}(t)(w(t),v) + \gamma(t)a(w(s),v;u,\sigma)\big\}dt, \tag{3.23}$$
$$\forall v \in V(u,\sigma),\ \gamma \in C^2([0,T]),\ \gamma(T) = \dot{\gamma}(T) = 0.$$

More precisely, by taking into account the density of $W^{1,p}(\Omega)$ in $V(u,\sigma)$, we can write (3.23) as follows: For $p>2$,

(3.24)
$$\begin{aligned}\int_0^T\int_\Omega \gamma(t)g(t,x,y)v(x,y)dxdydt &= -\gamma(0)\int_\Omega w_1(x,y)v(x,y)dxdy\\ &+\dot\gamma(0)\int_\Omega w_0(x,y)v(x,y)dxdy+\int_0^T\int_\Omega \ddot\gamma(t)w(t,x,y)v(x,y)dxdydt\\ &+\int_0^T\int_\Omega \gamma(t)\,\langle F(x,y)\nabla w(t,x,y),\nabla v(x,y)\rangle\,dxdydt+\int_0^T\int_G \gamma(t)\mu(x)\sigma(x)w(t,x,2)v(x,2)dxdt,\\ &\forall v\in W^{1,p}_{\partial G}(\Omega),\ \gamma\in C^2([0,T]),\ \gamma(T)=\dot\gamma(T)=0.\end{aligned}$$

We see that replacing $v\in V(u,\sigma)$ by $v\in W^{1,p}_{\partial G}(\Omega)$ makes the test function in (3.24) independent of (u,σ).

Lemma 3.4. *Let $(u,\sigma)\in\mathcal{U}\times\Sigma$, and*

$$\begin{cases} f\in L^2(0,T;H)\equiv L^2((0,T)\times\Omega),\\ \varphi\in W^{1,4}_{\partial G}(\Omega),\quad \psi\in H\equiv L^2(\Omega).\end{cases}\tag{3.25}$$

Let g, w_0 and w_1 be defined by (2.13) with $w\in V(u,\sigma)$. Then

$$\begin{cases}\|g(t)\|_{V(u,\sigma)'}\le C\big(1+\|u\|^2_{W^{1,2r/(r-1)}(G)}\|w\|_{V(u,\sigma)}+|f(t)|\big),\\ \qquad\qquad \text{a.e. } t\in(0,T),\ 1<r<\dfrac{3}{2},\\ \|w_0\|_{V(u,\sigma)}\le C\|\varphi\|_{W^{1,4}(\Omega)}\big(1+\|\sigma\|^{1/2}_{L^q(G)}\big),\\ |w_1|_H\le C|\psi|_H.\end{cases}\tag{3.26}$$

Proof. For any $1<r<\frac{3}{2}$, we denote $\rho=\frac{6r}{3+2r}\in(\frac{6}{5},\frac{3}{2})\subset(1,2)$. Then $2r=\frac{3\rho}{3-\rho}$, $\frac{2\rho}{2-\rho}\le\frac{2r}{r-1}$. Thus, $W^{1,\rho}(\Omega)\hookrightarrow L^{\frac{3\rho}{3-\rho}}(\Omega)=L^{2r}(\Omega)$. Consequently, by (3.6), for any $v\in V(u,\sigma)$,

$$\begin{aligned}\Big|\int_\Omega\frac{1}{u}\langle F\nabla w,\nabla u\rangle\,vdxdy\Big| &\le C\int_\Omega|U\nabla w|\,|u_x|\,|v|dxdy\\ &\le C\|U\nabla w\|_{L^2(\Omega)}\Big(\int_\Omega|u_x|^2v^2dxdy\Big)^{1/2}\quad\le C\|w\|_{V(u,\sigma)}\|v\|_{L^{2r}(\Omega)}\|u\|_{W^{1,2r/(r-1)}(G)}\\ &\le C\|w\|_{V(u,\sigma)}\|v\|_{W^{1,\rho}(\Omega)}\|u\|_{W^{1,2r/(r-1)}(G)}\\ &\le C\|w\|_{V(u,\sigma)}\|v\|_{V(u,\sigma)}\big(1+\|u\|_{W^{1,2\rho/(2-\rho)}(G)}\big)\|u\|_{W^{1,2r/(r-1)}(G)}\\ &\le C\big(1+\|u\|^2_{W^{1,2r/(r-1)}(G)}\big)\|w\|_{V(u,\sigma)}\|v\|_{V(u,\sigma)}.\end{aligned}\tag{3.27}$$

On the other hand,

$$\Big|\int_\Omega f(t,x,\tfrac{u(x)y}{2})v(x,y)dxdy\Big|\le\Big(\int_{\Omega(u)}|f(t,x,z)|^2\frac{2}{u(x)}dzdx\Big)^{1/2}|v|\le C|f(t)|_H|v|_H.\tag{3.28}$$

Thus, the first inequality in (3.26) follows. To prove the second inequality in (3.26), we first let $q>2$, $\frac{2q}{q-1}<4$. For φ smooth,

$$\begin{aligned}\Big\{\int_G|\varphi(x,u(x))|^{\frac{2q}{q-1}}dx\Big\}^{\frac{q-1}{2q}} &= \Big\{\int_G\Big|\varphi(x,2)-\int_{u(x)}^2\varphi_y(x,y)dy\Big|^{\frac{2q}{q-1}}dx\Big\}^{\frac{q-1}{2q}}\\ &\le C\|\varphi\|_{L^4(\Gamma)}+C\|\nabla\varphi\|_{L^{2q/(q-1)}(\Omega)}\le C\|\varphi\|_{W^{1,4}(\Omega)}.\end{aligned}\tag{3.29}$$

For general $\varphi \in W^{1,4}(\Omega) \hookrightarrow C(\overline{\Omega})$, we can make an approximation. Hence, (3.29) holds for all $q > 2$ and $\varphi \in W^{1,4}(\Omega)$. Next, similar to (2.4), we have $U\nabla_{(x,y)}w_0 = \nabla_{(x,z)}\varphi$. Thus,

$$
\begin{aligned}
\|w_0\|^2_{V(u,\sigma)} &= \int_\Omega |U\nabla w_0|^2 dxdy + \int_G |u_x\sigma| w_0(x,2)^2 dx \\
&\le \int_{\Omega(u)} |\nabla\varphi(x,z)|^2 \frac{2}{u(x)} dzdx + \|u_x\sigma\|_{L^q(\Gamma)} \Big(\int_G |\varphi(x,u(x))|^{\frac{2q}{q-1}} dx \Big)^{\frac{q-1}{q}} \\
&\le 2\|\varphi\|^2_{W^{1,2}(\Omega)} + \|u_x\sigma\|^2_{L^q(\Omega)} \|\varphi\|^2_{W^{1,4}(\Omega)} \le C(1 + \|u_x\sigma\|_{L^q(\Omega)}) \|\varphi\|^2_{W^{1,4}(\Omega)}.
\end{aligned} \tag{3.30}
$$

This gives the second inequality in (3.26). Finally, the third inequality in (3.26) can be proved similar to (3.28). □

Definition 3.5. Let $(u,\sigma) \in \mathcal{U} \times \Sigma$. Let $(\varphi,\psi) \in W^{1,4}_{\partial G}(\Omega) \times H$. A function $w \in C([0,T]; V(u,\sigma)) \cap C^1([0,T]; H)$ is called a *weak solution* of (2.19) if (3.21), (3.23) or (3.24) holds with w_0, w_1 and g given by (2.13).

We note that for any $t \in [0,T]$, (3.22) holds with T replaced by t and with the constant $C > 0$ being uniform in $t \in [0,T]$. Thus, a standard contraction mapping argument provides the following result.

Proposition 3.6. *Let (1.2) hold. Then, for any $(\varphi,\psi) \in W^{1,4}_{\partial G}(\Omega) \times H$, $f \in L^2(0,T;H)$, $(u,\sigma) \in \mathcal{U} \times \Sigma$, there exists a unique weak solution w of (2.12). Moreover, there exists a constant depending on $\|u\|_{H^1(G)}$, such that*

$$
\|w(\cdot)\|_{C([0,T];V(u,\sigma))} + \|w(\cdot)\|_{C^1([0,T];H)} \le C\big\{\|\varphi\|_{W^{1,4}(\Omega)} + |\psi| + \|f(\cdot)\|_{L^2(0,T;H)}\big\}. \tag{3.31}
$$

§4. Existence of Optimal Controls

In this section, we prove the following result.

Theorem 4.1. *Let $(\varphi,\psi) \in W^{1,4}_{\partial G}(\Omega) \times H$ and $f \in L^2(0,T;H)$. Then Problem (OC) admits an optimal triple $(\overline{w},\overline{u},\overline{\sigma})$.*

Proof. We let $(u^k,\sigma^k) \in \mathcal{U} \times \Sigma$ be a minimizing sequence of Problem (OC). Let $w^k \in C([0,T]; V(u^k,\sigma^k)) \bigcap C^1([0,T]; H)$ be the corresponding state. Then,

$$
\begin{aligned}
J(u_k,\sigma_k) \equiv& \frac{1}{2}\int_0^T \int_\omega |w^k(t,x,\tfrac{2z}{u^k(x)}) - h(t,x,z)|^2 dzdxdt + \frac{a}{2}\int_G \mathrm{tr}\,[u^k_{xx}(x)^2] dx \\
&+ \frac{b}{2}\int_G \sqrt{1 + |u^k_x(x)|^2}\,\sigma^k(x)^2 dx \to \inf_{\mathcal{U}\times\Sigma} J(u,\sigma).
\end{aligned} \tag{4.1}
$$

Thus, we may assume that

$$
\begin{cases}
u^k \overset{w}{\rightharpoonup} \overline{u}, & \text{weakly in } H^2(G), \\
u^k \overset{s}{\to} \overline{u}, & \text{strongly in } W^{1,p}(G),\ \forall p \in [1,\infty), \ \text{ and in } C^\alpha(\overline{G}),\ \forall \alpha \in (0,1), \\
\sigma^k \overset{w}{\rightharpoonup} \overline{\sigma}, & \text{weakly in } L^2(G), \\
\big(1 + |u^k_x|^2\big)^{1/4} \sigma^k \overset{w}{\rightharpoonup} \big(1 + |\overline{u}_x|^2\big)^{1/4} \overline{\sigma}, & \text{weakly in } L^2(G).
\end{cases} \tag{4.2}
$$

On the other hand, from (3.31), we see that

$$
\begin{aligned}
&\|w^k(\cdot)\|_{C([0,T];V(u^k,\sigma^k))} + \|w^k(\cdot)\|_{C^1([0,T];H)} \\
&\qquad \le C\big\{\|\varphi\|_{W^{1,4}(\Omega)} + |\psi|_H + \|f(\cdot)\|_{L^2(0,T;H)}\big\}, \quad \forall k \ge 1.
\end{aligned} \tag{4.3}
$$

This yields the following (note the definition of $\mathcal{U}$):

$$
\begin{cases}
\|w_x^k - \dfrac{y}{u^k} u_x^k w_y^k\|_{L^2((0,T)\times\Omega)} + \|w_y^k\|_{L^2((0,T)\times\Omega)} \le C, \\
\|w_t^k\|_{L^2((0,T)\times\Omega)} \le C.
\end{cases} \tag{4.4}
$$

Thus, we may assume that for some $\overline{w}$, we have

$$
\begin{cases}
w_x^k - \dfrac{y}{u^k} u_x^k w_y^k \overset{w}{\rightharpoonup} \overline{w}_x - \dfrac{y}{\overline{u}} \nabla\overline{u}\, \overline{w}_y, \quad \text{weakly in } L^2((0,T)\times\Omega), \\
w_y^k \overset{w}{\rightharpoonup} \overline{w}_y, \quad \text{weakly in } L^2((0,T)\times\Omega), \\
w_t^k \overset{w}{\rightharpoonup} \overline{w}_t, \quad \text{weakly in } L^2((0,T)\times\Omega), \\
w_x^k \overset{w}{\rightharpoonup} \overline{w}_x, \quad \text{weakly in } L^r((0,T)\times\Omega), \\
w^k \overset{s}{\to} \overline{w}, \quad \text{strongly in } L^p((0,T)\times\Omega),
\end{cases} \tag{4.5}
$$

where $r \in (1,2]$ and $p \in (4/3, 4)$.To prove that $(\overline{w}, \overline{u}, \overline{\sigma})$ is an optimal triple, note that

$$
\begin{aligned}
\int_0^T \int_\Omega \gamma(t) g^k(t,x,y) v(x,y)\,dxdydt &= -\gamma(0) \int_\Omega w_1^k(x,y) v(x,y)\,dxdy \\
&\quad + \dot\gamma(0) \int_\Omega w_0^k(x,y) v(x,y)\,dxdy + \int_0^T \int_\Omega \ddot\gamma(t) w^k(t,x,y) v(x,y)\,dxdydt \\
&\quad + \int_0^T \int_\Omega \gamma(t) \langle F^k(x,y) \nabla w^k(t,x,y), \nabla v(x,y) \rangle\,dxdydt, \\
&\quad + \int_0^T \int_G \gamma(t) \mu^k(x) \sigma^k(x) w^k(t,x,2) v(x,2)\,dxdt, \\
&\qquad \forall v \in W^{1,p}_{\partial G}(\Omega),\ \gamma \in C^2([0,T]),\ \gamma(T) = \dot\gamma(T) = 0,
\end{aligned} \tag{4.6}
$$

where

$$
\begin{cases}
g^k(t,x,y) = \dfrac{1}{u^k(x)} \langle F^k(x,y) \nabla w^k(t,x,y), \nabla u^k(x) \rangle + f(t,x,\frac{u^k(x)y}{2}), \\
\qquad\qquad (t,x,y) \in [0,T]\times\Omega, \\
F^k(x,y) = \begin{pmatrix} I & 0 \\ -\frac{y}{u^k(x)} u_x^k(x)^T & \frac{2}{u^k(x)} \end{pmatrix} E(x, \frac{u^k(x)y}{2}) \begin{pmatrix} I & -\frac{y}{u^k(x)} u_x^k(x) \\ 0 & \frac{2}{u^k(x)} \end{pmatrix}, \\
\qquad\qquad (x,y) \in \Omega, \\
\mu^k(x) = \dfrac{2\sqrt{1 + |u_x^k(x)|^2}}{u^k(x)}, \quad x \in G, \\
w_0^k(x,y) = \varphi(x, \frac{u^k(x)y}{2}), \qquad (x,y) \in \Omega, \\
w_1^k(x,y) = \psi(x, \frac{u^k(x)y}{2}), \qquad (x,y) \in \Omega.
\end{cases} \tag{4.7}
$$

We define $\overline{g}$, $\overline{F}$, $\overline{\mu}$, $\overline{w}_0$ and $\overline{w}_1$ the same as (4.7) with w^k, u^k, σ^k replaced by $\overline{w}$, $\overline{u}$ and $\overline{\sigma}$. By (4.2) and (4.5), we see that

$$
\int_0^T \int_\Omega \gamma(t) g^k(t,x,y) v(x,y)\,dxdydt \to \int_0^T \int_\Omega \gamma(t) \overline{g}(t,x,y) v(x,y)\,dxdydt.
$$

Similarly, we have the convergence of the first four terms on the right-hand side of (4.6). For the last term on the right-hand side of (4.6), we note that $W^{1,r}((0,T)\times\Omega)\hookrightarrow W^{1-\frac{1}{r},r}((0,T)\times G)\hookrightarrow L^s((0,T)\times G)$, with the last embedding being compact for $s<\frac{3r}{4-r}$. Thus, we may assume that $w^k(t,x,2)\to\overline{w}(t,x,2)$, in $L^s((0,T)\times G)$, a.s. $(t,x)\in(0,T)\times G$. By taking $r=\frac{9}{5}\in(1,2)$, we have $\frac{3r}{4-r}=\frac{27}{11}>s=2$. Thus, for $p>3$, which implies that $v(x,2)\in C^\alpha(\overline{G})$ for some $\alpha\in(0,1)$. The boundedness of v ensures the convergence of the last term on the right-hand side of (4.6). Consequently, we see that $(\overline{u},\overline{\sigma})$ is an optimal control. □

§5. Solution to the Identification Problem

The last two terms in (1.7), which represent the "cost" of the controls in the optimal control problem are spurious for the identification problem (1.6). Thus, we would like to let a,b tend to zero and recover the solution(s) to the identification problem (1.7). Consider two sequences of positive numbers $\{a_n\},\{b_n\}$ both converging to zero. For each pair (a_n,b_n) we denote the optimal triple by $(\bar{w}_n,\bar{u}_n,\bar{\sigma}_n)$; this triple minimizes the objective functional $J_n=J(a_n,b_n,\bar{w}_n,\bar{u}_n,\bar{\sigma}_n)$. We can prove

Theorem 5.1. *Assume there exist $(u^*,\sigma^*)\in\mathcal{U}\times\Sigma$ such that $w^*=w(u^*,\sigma^*)$ satisfies $w^*|_{\omega\times(0,T)}=h$ a.e. Then there exist $(u_0,\sigma_0)\in\mathcal{U}\times\Sigma$ such that for a pair of sequences $\{a_n,b_n\}\to\{0,0\}$, we have $(\bar{u}_n,\bar{\sigma}_n)\rightharpoonup(u_0,\sigma_0)$ in $H^2(G)\times L^2(G)$, $\bar{w}_n\rightharpoonup w_0$ in $L^2((0,T)\times\Omega)$ and $w_0|_{\omega\times(0,T)}=h$.*

Proof. Since (u^*,σ^*) is an admissible control pair and $(\bar{u}_n,\bar{\sigma}_n)$ minimizes J_n, we have

$$J_n(\bar{u}_n,\bar{\sigma}_n)\le J_n(u^*,\sigma^*)$$

and

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_\omega|\bar{w}_n-h|^2dxdzdt+\frac{a_n}{2}\int_G\operatorname{tr}[(\bar{u}_n)_{xx}]^2dx+\frac{b_n}{2}\int_G\sqrt{1+|\bar{u}_x|^2}\bar{\sigma}_ndx\\&\le\frac{a_n}{2}\int_G\operatorname{tr}[u^*_{xx}]^2dx+\frac{b_n}{2}\int_G\sqrt{1+|u^*_x|^2}\sigma^*dx.\end{aligned}\tag{5.1}$$

This bound together with the a priori estimates on $\bar{w}_n$ (independent of n), imply, as in Theorem 4.1, that there exist $(u_0,\sigma_0)\in\mathcal{U}\times\Sigma$ such that on a subsequence we have

$$\begin{aligned}\bar{u}_n&\rightharpoonup u_0\quad\text{weakly in } L^2(G)\\\bar{\sigma}_n&\rightharpoonup\sigma_0\quad\text{weakly in } L^2(G)\\\bar{w}_n&\to w_0\quad\text{strongly in } L^2((0,T)\times\Omega)\end{aligned}$$

and $\bar{w}_n(\bar{u}_n,\bar{\sigma}_n)\rightharpoonup w_0(u_0,\sigma_0)$. Letting $n\to\infty$ in (5.1) we also obtain

$$\int_0^T\int_\omega|w_0-h|^2dxdzdt=0.$$

Thus, $w_0|_{\omega\times(0,T)}=h$ and therefore, u_0 and σ_0 are the identified boundary shape and reflectivity. □

Note: The identified pair (u_0,σ_0) is not necessarily equal to (u^*,σ^*). If the inverse problem has a unique solution, then $(u_0,\sigma_0)=(u^*,\sigma^*)$.

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