FAST GALERKIN BEM

BY A

PRECORRECTED-FFT

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Motivation

Large simulations with Complex geometry

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- Electrospray Van Berkel G. J. et al. (1999)

  3D Laplace Equation

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- **Cristal Growth** Phan A-V et al. (2001)

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LARGE SIMULATIONS WITH COMPLEX GEOMETRY

- ELECTROSPRAY Van Berkel G. J. et al. (1999)

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- CRISTAL GROWTH Phan A-V et al. (2001)

  Level Set/Boundary integral technique

- ELECTRONIC APPLICATIONS Masters N. and Ye W. (2004)

  MicroElectroMechanical Systems (MEMS)

PROBLEM FORMULATION

- FIELD EQUATION $\nabla^2 u = 0$ in D

- BOUNDARY CONDITIONS

Dirichlet: $u = u^p$ on S

Neumann: $t \equiv \frac{\partial u}{\partial n} = t^p$ on S

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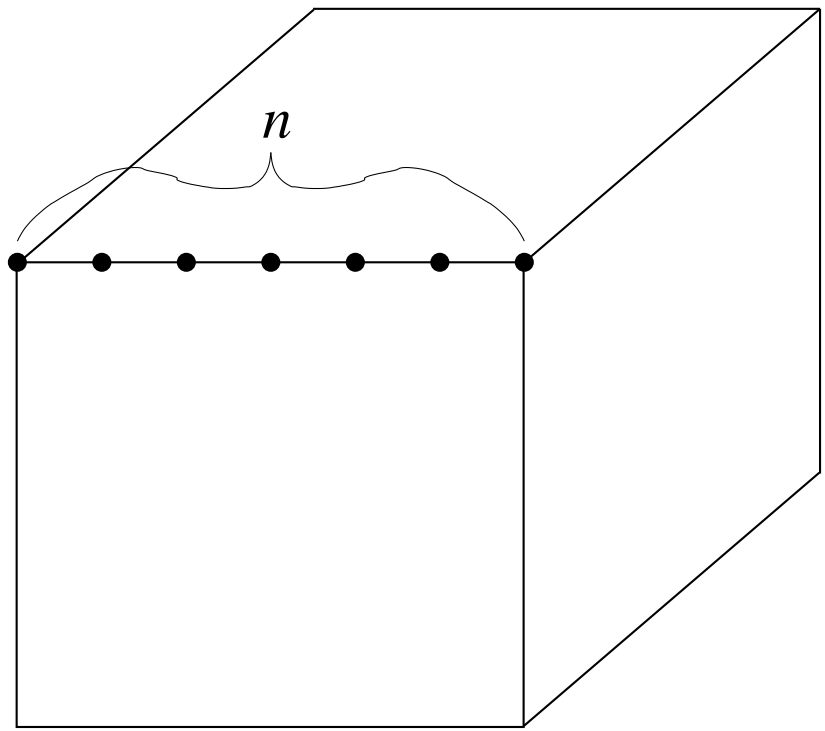
- SINGULAR BIE

$$\int_S G(\boldsymbol{x},\boldsymbol{y})\, t(\boldsymbol{y})\, \mathrm{d}s_{\boldsymbol{y}} - \int_S \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\, u(\boldsymbol{y})\, \mathrm{d}s_{\boldsymbol{y}} = 0 \qquad \boldsymbol{x}\in S,$$

Gray L. J. et al. (2004)

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi\|\boldsymbol{x}-\boldsymbol{y}\|}, \qquad \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^3}, \quad \boldsymbol{x}\neq\boldsymbol{y}$$

BIEM vs Domain methods



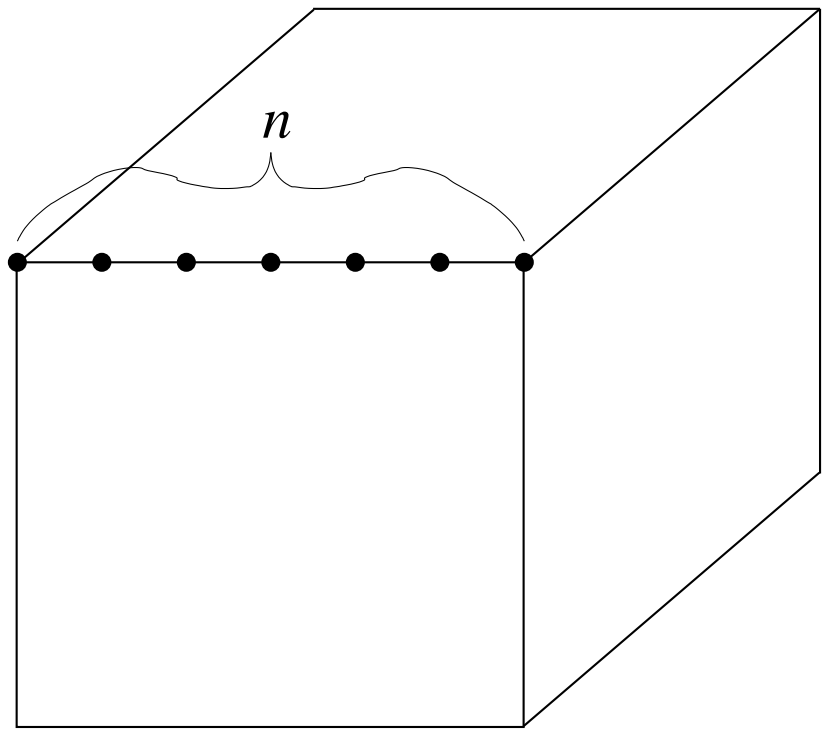
BIEM vs Domain methods

- $N = O(n^2)$
- $M = O(n^3)$

- Computational cost

  BIEM: $O(N^2) = O(n^4)$

  FEM or FDM: $O(M) = O(n^3)$



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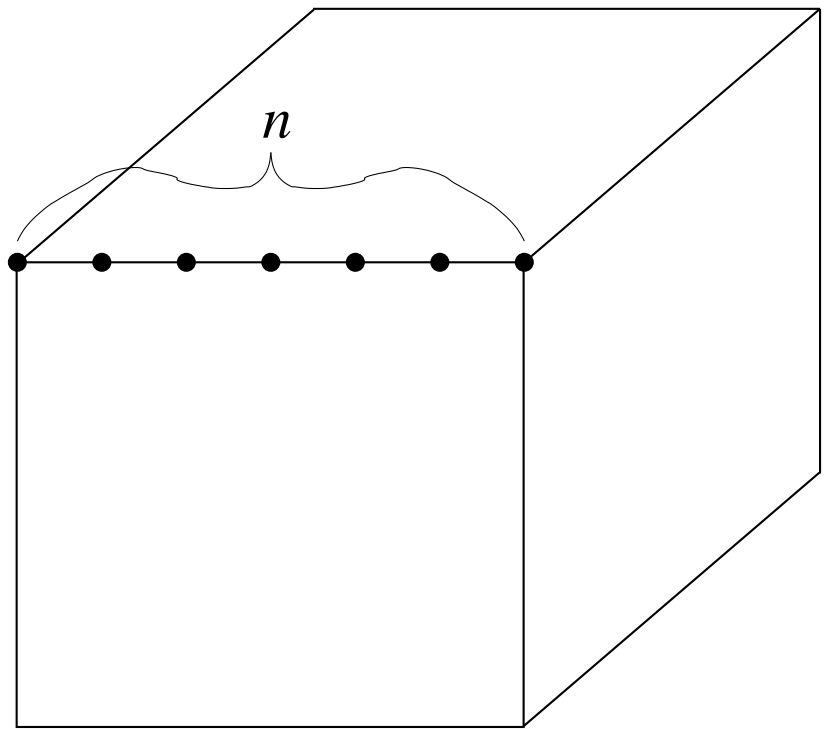


- Computational cost

  BIEM: $O(N^2) = O(n^4)$

  FEM or FDM: $O(M) = O(n^3)$

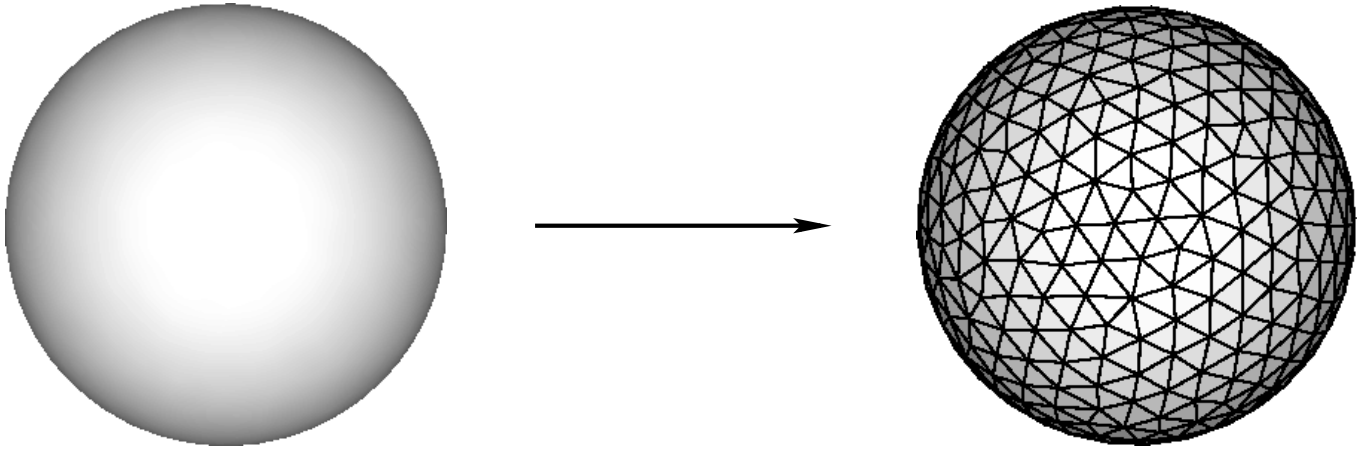
- Acceleration Techniques

  Greengard L. and Rokhlin V. (1987) $\Longrightarrow$ FMM

  Phillips J. R. and White J. K. (1997) $\Longrightarrow$ Precorrected-FFT

  Bespalov A. N. (2000) $\Longrightarrow$ RGM $\quad O(N^{1+\alpha} \log^{\beta} N)$, $0 \leq \alpha < 1$, $\beta \geq 0$

GALERKIN APPROXIMATION



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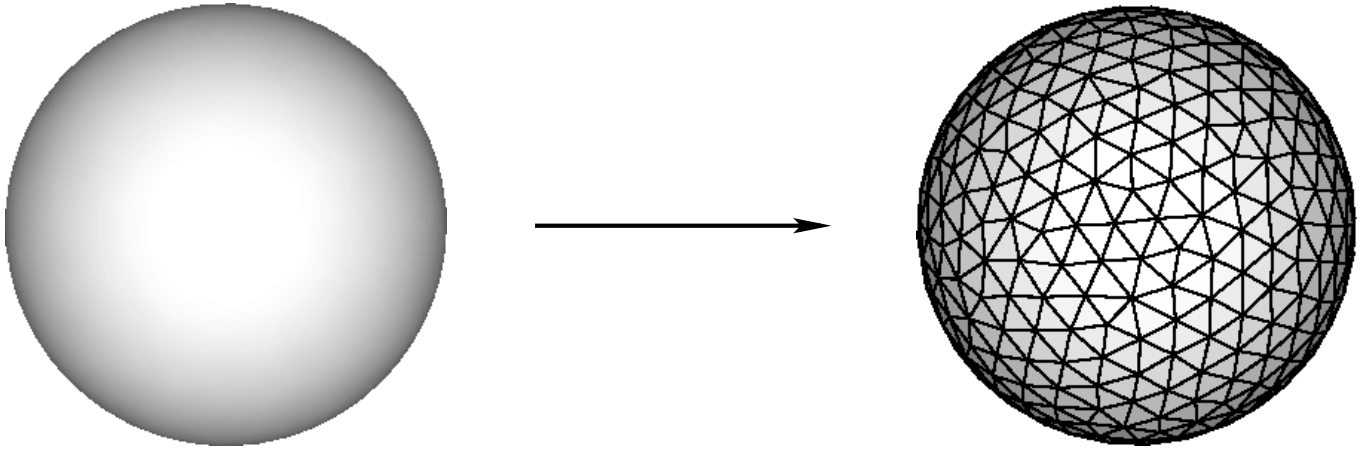

- DECOMPOSITION $\quad u(\boldsymbol{y}) = \sum_j u(\boldsymbol{y}^j)\,\psi_j(\boldsymbol{y}), \qquad t(\boldsymbol{y}) = \sum_j t(\boldsymbol{y}^j)\,\psi_j(\boldsymbol{y})$
- LINEAR SYSTEM $\quad \mathbf{G}\{t\} = \mathbf{H}\{u\}$

$$\mathsf{G}_{ij} = \int_S \int_S \psi_i(\boldsymbol{x})\, G(\boldsymbol{x},\boldsymbol{y})\, \psi_j(\boldsymbol{y})\, \mathrm{d}s_{\boldsymbol{y}} \mathrm{d}s_{\boldsymbol{x}}, \quad \mathsf{H}_{ij} = \int_S \int_S \psi_i(\boldsymbol{x})\, \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, \psi_j(\boldsymbol{y})\, \mathrm{d}s_{\boldsymbol{y}} \mathrm{d}s_{\boldsymbol{x}}$$

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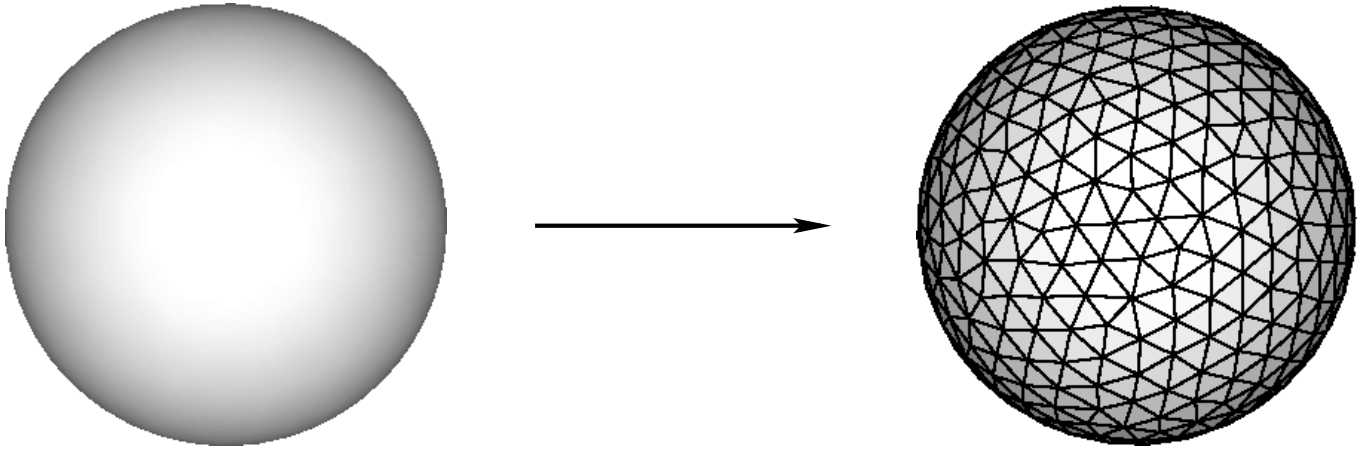

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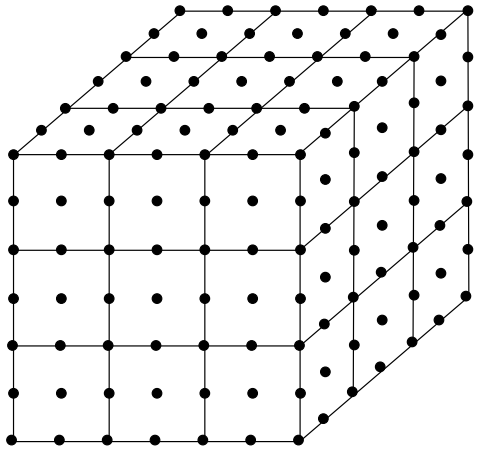
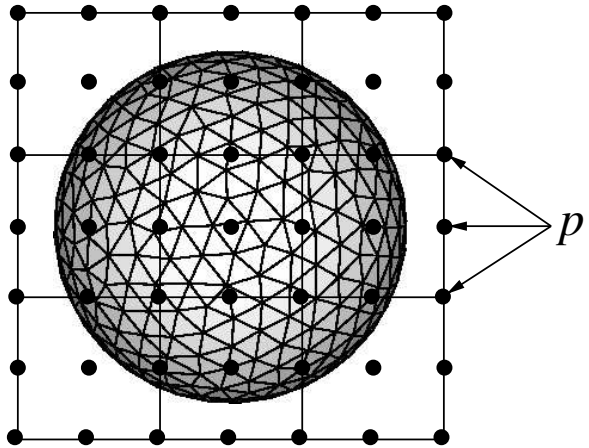
DRAWBACK $\quad \mathbf{G},\ \mathbf{H} \Longrightarrow$ Dense Matrices $\Longrightarrow O(N^2)$

REGULAR GRID APPROXIMATION





- GALERKIN INTEGRAL $$\mathsf{H}_{ij}^{k} = \sum_{s=1}^{N_{E_k}} \sum_{t=1}^{N_{E_k}} \int_{E_s} \int_{E_t} \psi_i(\boldsymbol{x})\, \boldsymbol{H}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, \psi_j(\boldsymbol{y})\, \mathrm{d}s_{\boldsymbol{y}} \mathrm{d}s_{\boldsymbol{x}}$$

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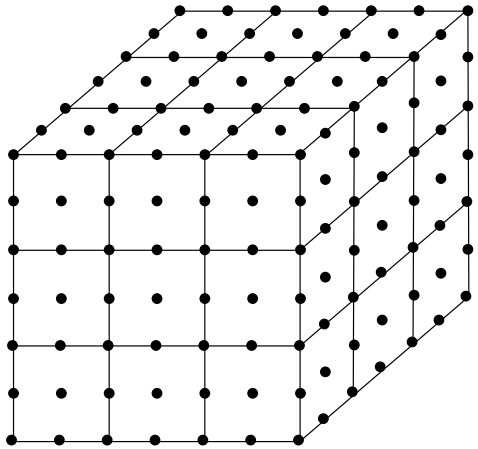

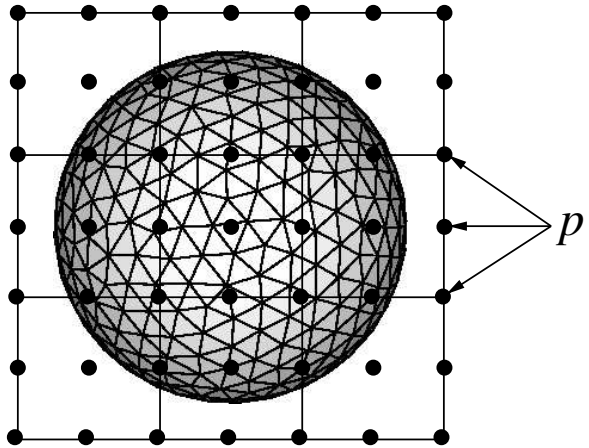

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- QUADRATURE ESTIMATE $$\mathsf{H}_{ij}^{k(s,t)} = \sum_{\beta=1}^{N_G} \sum_{\alpha=1}^{N_G} C_{\beta i}^{s} C_{\alpha j}^{t} n_l^{t\alpha} H_l(\boldsymbol{x}^{s,\beta}, \boldsymbol{y}^{t,\alpha})$$

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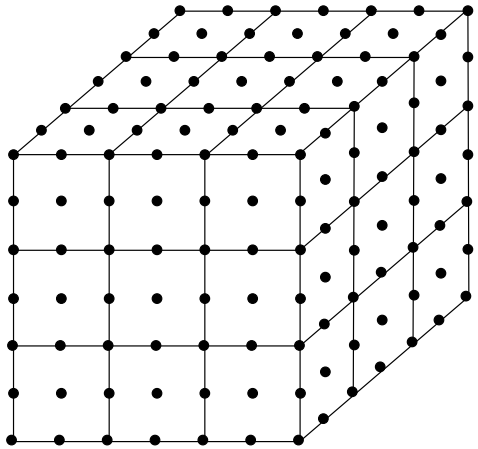

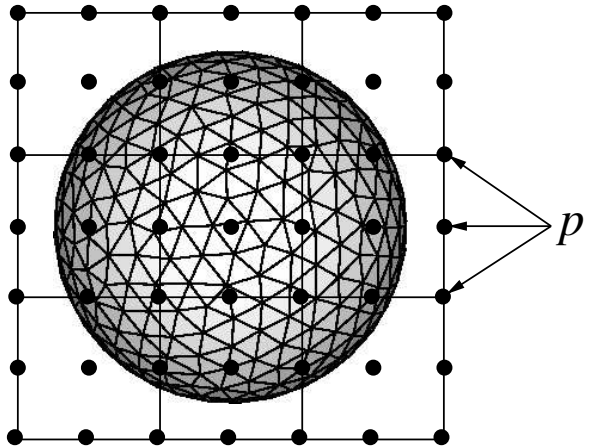

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- LOCAL INTERPOLATION $$H_l(\boldsymbol{x}^{s,\beta}, \boldsymbol{y}^{t,\alpha}) = \sum_{r=1}^{p^3} \sum_{q=1}^{p^3} d_{r\beta}^{s}\, d_{q\alpha}^{t}\, H_l(\widehat{\boldsymbol{x}}^{r}, \widehat{\boldsymbol{y}}^{q})$$

Regular grid approximation

Far-field

- Grid approximation $\quad \mathsf{H}_{ij}^{k} = \sum_{r=1}^{p^3} \sum_{q=1}^{p^3} W_{ri}^{k} H_l(\widehat{\boldsymbol{x}}^r, \widehat{\boldsymbol{y}}^q) V_{qj}^{k,l} \Longrightarrow \mathbf{H}^k = \sum_{l=1}^{3} \mathbf{W}^{k^\top} \widehat{\mathbf{H}}_l^k \mathbf{V}^{k,l}$

- Mapping Coefficients $\quad W_{ri}^{k} = \sum_{s=1}^{N_{E_k}} \sum_{\beta=1}^{N_G} d_{r\beta}^{s} C_{\beta i}^{s}, \quad V_{qj}^{k,l} = \sum_{t=1}^{N_{E_k}} \sum_{\alpha=1}^{N_G} d_{q\alpha}^{t} C_{\alpha j}^{t} n_l^{t\alpha}$

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- Mappings $\quad \mathbf{H} = \sum_{i=1}^{3} \mathbf{W}^{\top} \widehat{\mathbf{H}}_i \mathbf{V}^i, \qquad \mathbf{G} = \mathbf{W}^{\top} \widehat{\mathbf{G}} \mathbf{W}$

REGULAR GRID APPROXIMATION

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- GRID APPROXIMATION $\quad \mathsf{H}^k_{ij} = \sum_{r=1}^{p^3} \sum_{q=1}^{p^3} W^k_{ri} H_l(\widehat{\boldsymbol{x}}^r, \widehat{\boldsymbol{y}}^q) V^{k,l}_{qj} \Longrightarrow \mathbf{H}^k = \sum_{l=1}^{3} \mathbf{W}^{k\top} \widehat{\mathbf{H}}^k_l \mathbf{V}^{k,l}$

- MAPPING COEFFICIENTS $\quad W^k_{ri} = \sum_{s=1}^{N_{E_k}} \sum_{\beta=1}^{N_G} d^s_{r\beta} C^s_{\beta i}, \quad V^{k,l}_{qj} = \sum_{t=1}^{N_{E_k}} \sum_{\alpha=1}^{N_G} d^t_{q\alpha} C^t_{\alpha j} n^{t\alpha}_l$

- MAPPINGS $\quad \mathbf{H} = \sum_{i=1}^{3} \mathbf{W}^\top \widehat{\mathbf{H}}_i \mathbf{V}^i, \qquad \mathbf{G} = \mathbf{W}^\top \widehat{\mathbf{G}} \mathbf{W}$

Feature: $\mathbf{W}, \quad \mathbf{V}^i \Longrightarrow$ Sparse $\Longrightarrow O(N)$

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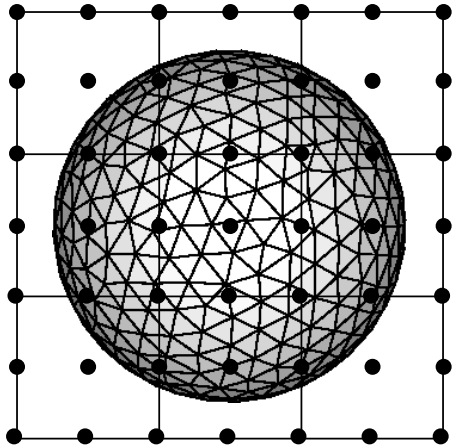
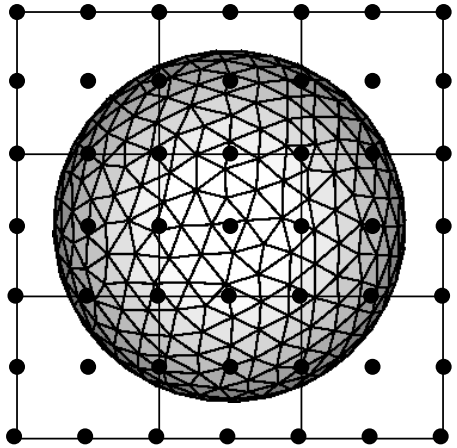
- PRODUCTS $\quad \widehat{\mathbf{G}}\{\hat{z}\}, \quad \widehat{\mathbf{H}}_i\{\hat{z}\} \Longrightarrow$ FFT $\Longrightarrow O(N \log N)$

PRECORRECTION



PRECORRECTION

SINGLE-LAYER KERNEL



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- DECOMPOSITION $\mathbf{G} = \mathbf{P} + \mathbf{W}^\top \widehat{\mathbf{G}} \mathbf{W}$

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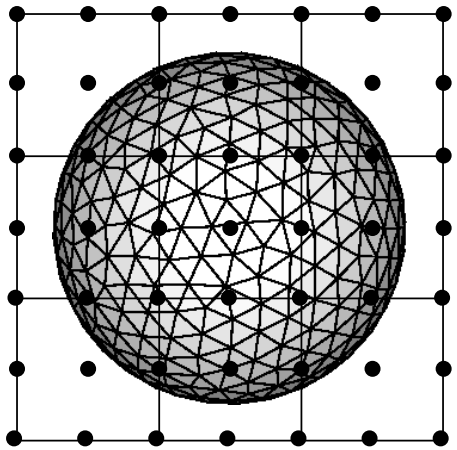

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DOUBLE-LAYER KERNEL

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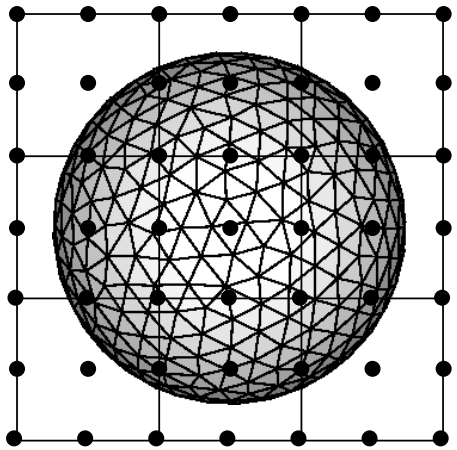

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Feature: $\mathbf{P} \Longrightarrow$ Sparse

RESULTS

CONVENTIONAL APPROACH VS PFFT(3): NEUMANN (time in seconds)

$Nodes$	T_{MAT}	T_{LU}	$Total$	$Accuracy$	$T_{\mathsf{PFFT(3)}}$	T_{it}	$Total$	$Accuracy$
400	111	0	111	2.850×10^{-3}	110	0	110	2.850×10^{-3}
1024	300	1	301	9.024×10^{-4}	297	0	297	9.024×10^{-4}
2500	804	14	818	1.813×10^{-4}	689	7	696	5.421×10^{-4}
4096	1442	69	1515	1.049×10^{-4}	1143	7	1150	5.477×10^{-4}
6400	2624	274	2912	1.887×10^{-4}	1819	8	1827	5.849×10^{-4}
8281	3700	557	4279	3.222×10^{-4}	2417	29	2447	5.925×10^{-4}
10000	4592	980	5605	2.534×10^{-4}	2924	9	2934	6.158×10^{-4}
$*$ 16384	9283	4139	13422	$--$	5043	15	5059	6.426×10^{-4}

$$Total = T_{\mathsf{MAT}} + T_{\mathsf{LU}} + T_{\mathsf{SOLVE}} + T_{\mathsf{RHS}}$$

$$Total = T_{\mathsf{PFFT(3)}} + T_{\mathsf{it}} + T_{\mathsf{RHS}}$$

$$T_{\mathsf{BICGSTAB(4)}} = T_{it}$$

Intel EM64T 3.2GHz/1MB L2 cache/2GB DDR2-400

RESULTS

CONVENTIONAL APPROACH VS PFFT(3): DIRICHLET (time in seconds)

$Nodes$	T_{MAT}	T_{LU}	$Total$	$Accuracy$	$T_{\mathsf{PFFT(3)}}$	T_{it}	$Total$	$Accuracy$
400	111	0	111	1.425×10^{-2}	110	0	110	1.425×10^{-2}
1024	301	1	302	8.459×10^{-3}	298	1	299	8.475×10^{-3}
2500	801	14	815	6.002×10^{-3}	705	9	714	8.629×10^{-3}
4096	1444	70	1517	5.689×10^{-3}	1186	14	1201	8.263×10^{-3}
6400	2623	267	2901	5.948×10^{-3}	1923	20	1944	8.449×10^{-3}
8281	3683	560	4264	1.390×10^{-2}	2536	50	2587	1.680×10^{-2}
10000	4583	979	5595	6.727×10^{-3}	3178	40	3219	9.044×10^{-3}
$*$ 16384	9246	4169	13415	$--$	5750	109	5861	1.091×10^{-2}

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$$T_{\mathsf{BICGSTAB(4)}} = T_{it}$$

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RESULTS

COMPUTER MEMORY (in megabytes)

Nodes	*Dirichlet/Neumann*: LU	*Dirichlet*: PFFT(3)	*Neumann*: PFFT(3)
400	3.77	6.24	6.37
1024	17	26	26
2500	97	78	52
4096	258	146	89
6400	628	286	157
8281	1024	400	237
10000	1536	601	306
$*$ 16384	4096	1433.6	695

SUMMARY

PRECORRECTED-FFT

- ACCELERATION TECHNIQUE
- BOUNDARY INTEGRAL EQUATION
- POTENTIAL THEORY
- DENSE INTO SPARSE MATRICES
- REGULAR GRID
- FFT
- COMPUTATIONALLY EFFICIENT $O(N \log N)$ operations $\Longrightarrow O(N)$ memory