Local post-processing for the discontinuous Galerkin method over a nonuniform mesh

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Outline

- Review of post-processing for linear hyperbolic equations
- Post-processing for smoothly varying mesh
- Numerical Examples
- Summary

Discontinuous Galerkin Approximation

$$u_t + (a(x)u)_x = 0$$

Numerical scheme:

$$\int_{I_i} u_t v dx = (\hat{au})_{i-1/2} v^+_{i-1/2} - (\hat{au})_{i+1/2} v^-_{i+1/2} - \int_{I_i} a u v_x dx$$

where

$$v \in V_h = \text{span}\{1, \xi_i, \xi_i^2, \cdots, \xi_i^k, i = 1, \cdots, N\}$$

where $\xi_i = \frac{x - x_i}{\triangle x_i}$ on $I_i = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2})$

$$u_h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) \xi_i^l \quad \text{if} \quad x \in I_i$$

Post-Processor

Cockburn, Luskin, Shu, & Süli (2003)

- Discontinuous Galerkin approximation allows us to use negative order error estimates:

$$||u_h - u||_{-l} = \mathcal{O}(h^{2k+1}).$$

- Post-processor extracts this information.

$$u^*(x) = K * u_h$$

- Works for a locally uniform mesh:
  $\longrightarrow$ Translation invariant
  $\longrightarrow$ Post-Processor is local

Post-Processor Kernel

- Independent of the partial differential equation.
- Applied only at the final time.
- Filters out oscillations in the error.

Kernel Properties

Bramble & Schatz (1977)

Mock & Lax (1978)

- Compact Support.
- Reproduces polynomials of degree $2k$ by convolution.
- Linear combination of B-splines.

Post-Processed Solution

Using Symmetric Kernel

Post-processed solution: $u^*(x) = K_h^{2(k+1),k+1} * u_h$.

$$K_h^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \psi^{(k+1)} \left(\frac{x}{h} - \gamma \right)$$

$h = \triangle x_i$ for all i, and $c_\gamma^{2(k+1),k+1} \in \mathbb{R}$.

$\psi^{(0)} = \delta_0$, $\psi^{(n)} = \psi^{(n-1)} * \chi$ for $n \geq 2$, where

$$\chi(x) = \begin{cases} 1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{else.} \end{cases}$$

Implementation

Know form of approximation and kernel $\Rightarrow$

$$u^*(x) = \frac{1}{h}\int_{-\infty}^{\infty} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right) u_h(y)\,dy$$

$$= \frac{1}{h}\sum_{j=-2k}^{2k}\int_{I_{i+j}} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right)\sum_{l=0}^{k} u_{i+j}^{(l)}\left(\frac{y-x_{i+j}}{h}\right)^l dy$$

$$= \sum_{j=-2k}^{2k}\sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x)$$

$C(j,l,k,x) =$

$\frac{1}{h}\sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1}\int_{I_{i+j}} \psi^{(k+1)}\left(\frac{y-x}{h}-\gamma\right)\left(\frac{y-x_{i+j}}{h}\right)^l dy \in \mathbb{P}^{2k+1}$

$k' = \lceil (3k+1)/2 \rceil \leq 2k$

1-D Variable Coefficient Equation

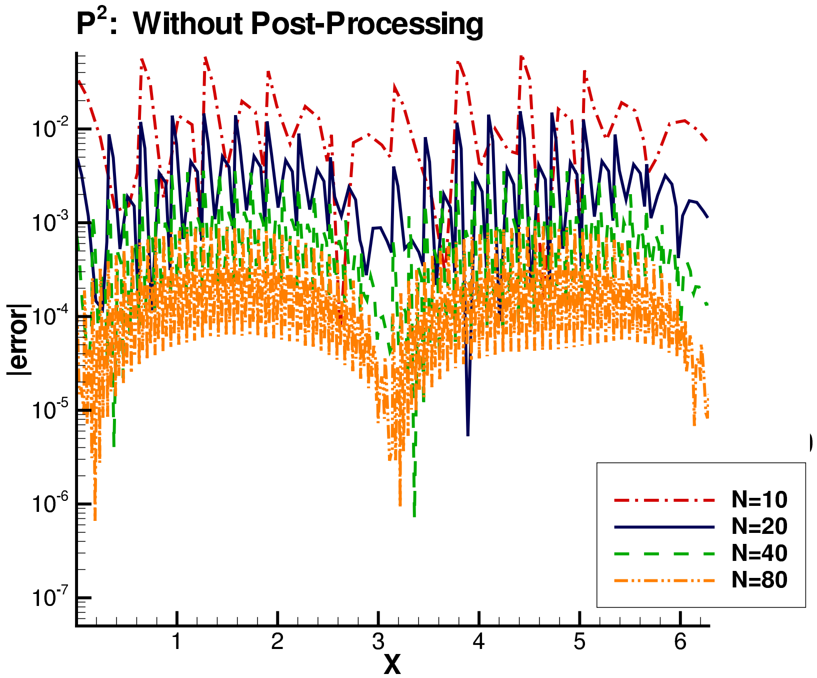
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.83E-02	—	7.82E-02	—
20	4.35E-03	2.07	1.08E-03	2.86
40	1.07E-03	2.03	1.39E-04	2.96
80	2.66E-04	2.01	1.75E-05	2.99
	$\mathbb{P}^2$			
10	8.61E-04	—	1.34E-04	—
20	1.07E-04	3.01	2.34E-06	5.84
40	1.34E-05	3.00	4.55E-08	5.69
80	1.67E-06	3.00	1.09E-09	5.38

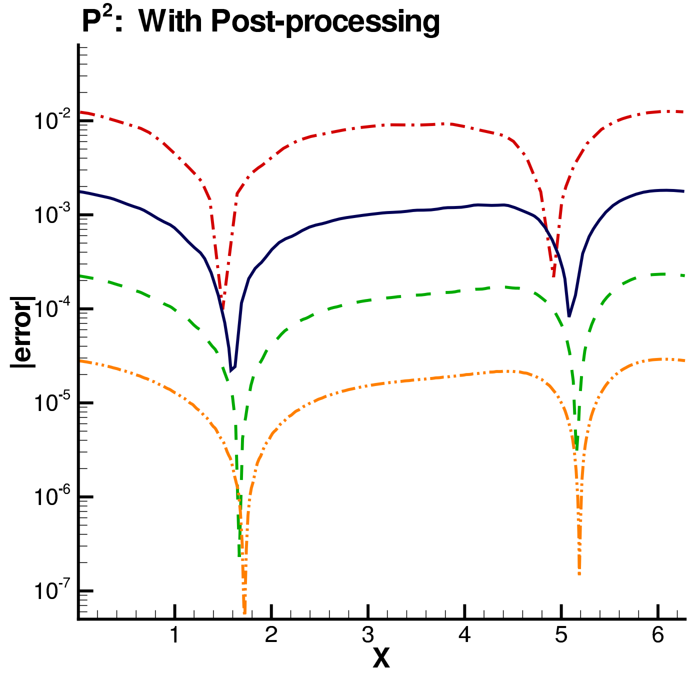
Uniform mesh: $h = \frac{2\pi}{N}$

$$u_t + (au)_x = f$$

$$\begin{aligned} a(x,t) &= 2 + \sin(x - t) \\ u(x,0) &= \sin(3x) \\ u(0,t) &= u(2\pi, t) \\ T &= 12.5 \end{aligned}$$

1-D Variable Coefficient





$u_t+(a(x)u)_x=f(x,t)$
$a(x)=2+\sin(x)$
$u(x,0)=\sin(3x)$
x in $(0,2\pi)$, $T=12.5$

Derivatives

	Approximation		Post-Processed	
N	L^2 error	order	L^2 error	order
	Errors in First Derivative for $\mathbb{P}^2$			
20	3.48E-03	—	6.24E-06	—
40	8.72E-04	2.00	1.61E-07	5.28
80	2.18E-04	2.00	4.51E-09	5.16
160	5.45E-05	2.00	1.39E-10	5.02
	Errors in Second Derivative for $\mathbb{P}^2$			
20	6.78E-02	—	3.64E-05	—
40	3.39E-02	1.00	2.15E-06	4.08
80	1.70E-02	1.00	1.32E-07	4.03
160	8.48E-03	1.00	8.19E-09	4.01

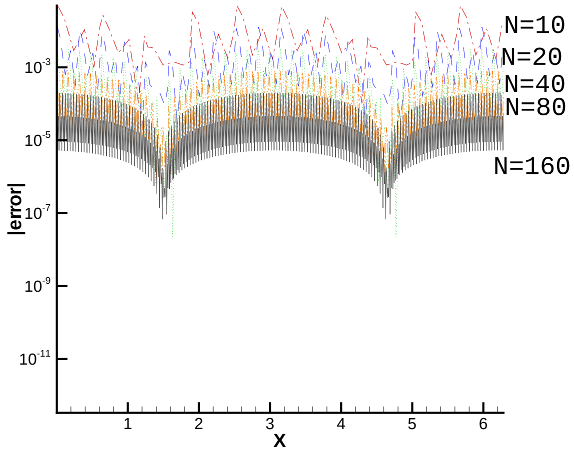
$$\frac{du^*(x)}{dx} \in \mathbb{P}^{2k}$$

Uniform mesh: $h = \frac{2\pi}{N}$

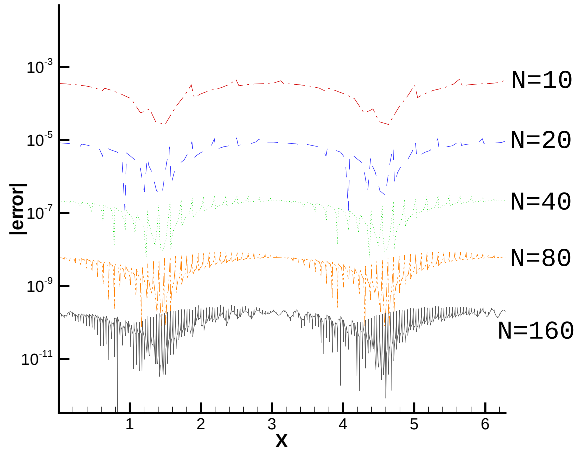
$$u_t + u_x = 0$$
$$u(x,0) = \sin(x)$$
$$x \in (0, 2\pi)$$
$$T = 12.5$$

Derivatives of Post-Processed Solution

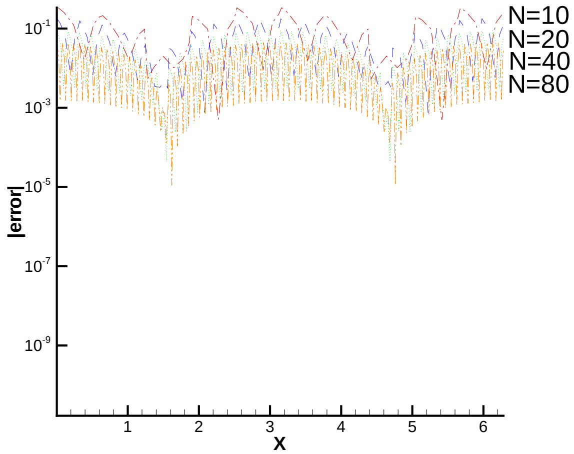
P^2: $|d/dx(u-u_h)|$, Before Post-Processing



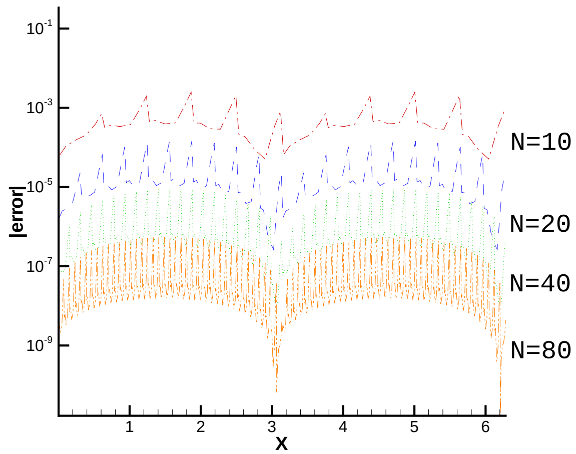
P^2: $|d/dx(u-u_h)|$, After Post-Processing



P^2: $|d^2/dx^2(u-u_h)|$, Before Post-Processing



P^2: $|d^2/dx^2(u-u_h)|$, After Post-Processing

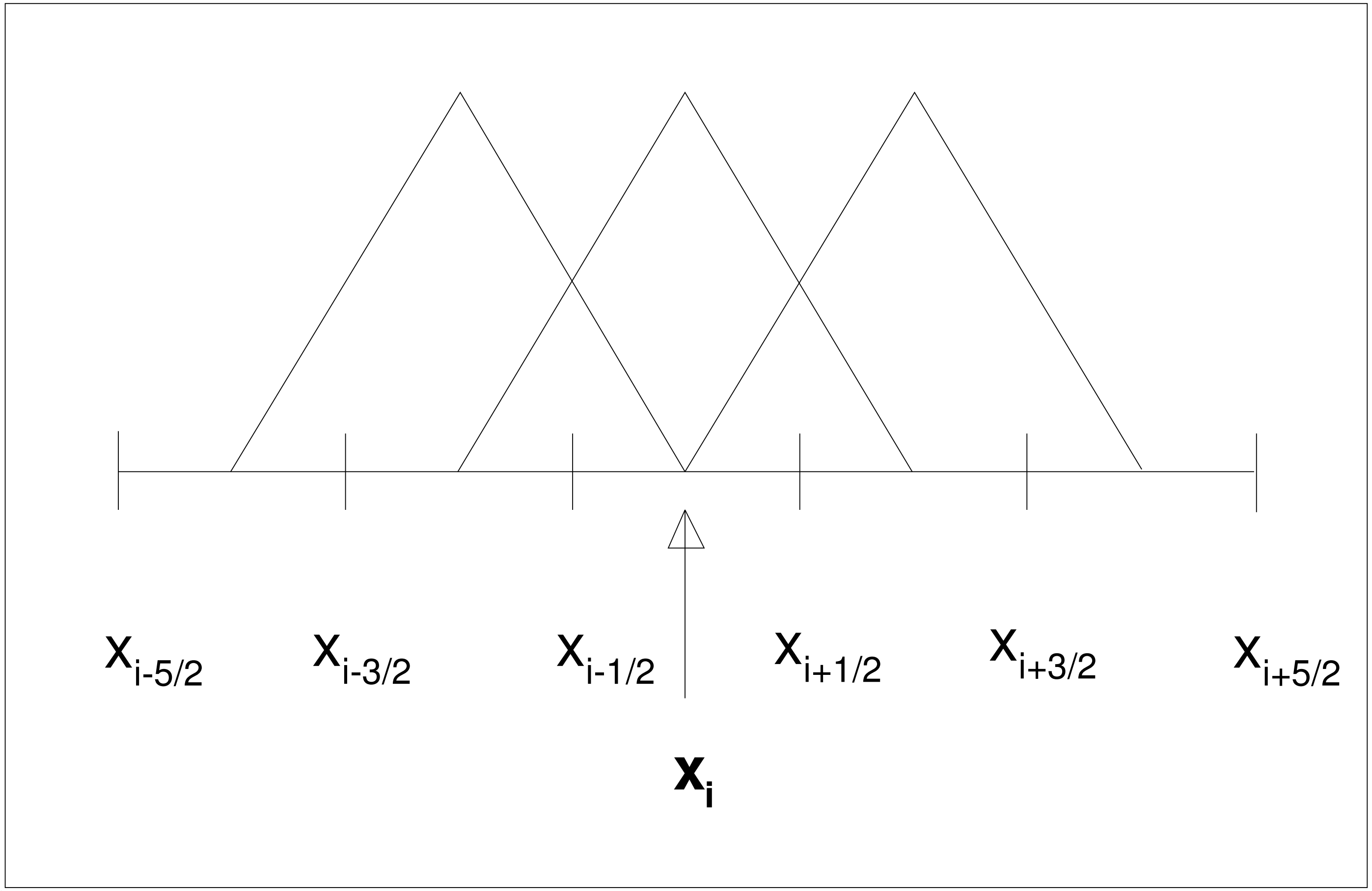


Post-Processor

- Works for a locally uniform mesh:
  $\longrightarrow$ Translation invariant
  $\longrightarrow$ Post-Processor is local

Example

Second Order Approximation



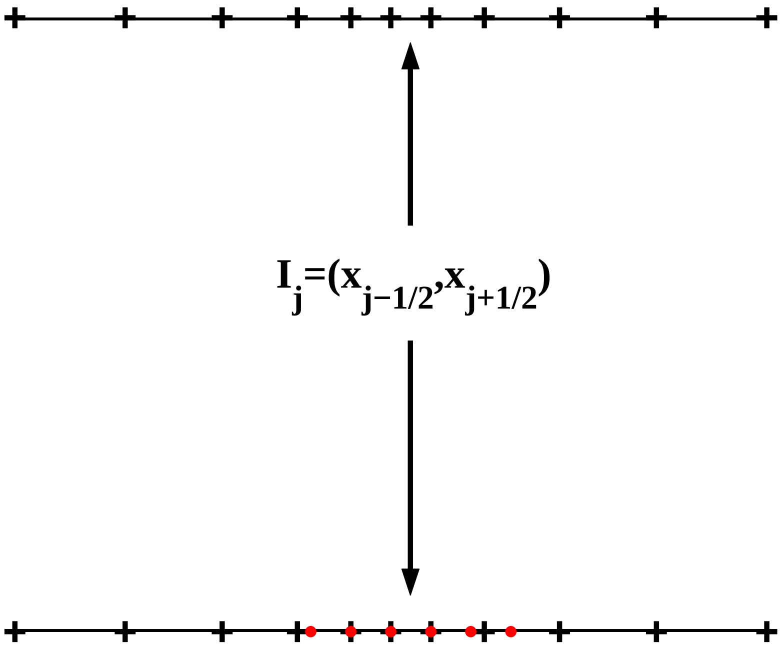
Smoothly Varying Mesh

Mesh defined by $x = x(\xi)$ where $\xi = \frac{\text{interval length}}{N} i,\ i = 1, \cdots, N$ is the uniform mesh variable.

$$u^*(x) = \frac{1}{\triangle x_i} \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \int_{I_{i+j}} \psi^{(k+1)} \left(\frac{y-x}{\triangle x_i} - \gamma \right) \left(\frac{y - x_{i+j}}{\triangle x_{i+j}} \right)^l dy$$

$k' = \lceil (3k+1)/2 \rceil$

Smoothly Varying Mesh



Post-processing solution on cell I_j.

- Create locally uniform mesh of mesh size: $h = \triangle x_j$
- Project $u_h(x, T)$ to locally uniform mesh for all x in the post-processing region.
- Use $u_n(x, T)$ to find post-processed solution on I_j : $u^*(x) = \sum_{i=-2k'}^{2k'} \sum_{l=0}^{k} u_{n_{(i+j)}}^{(l)} C(i, l, k, x)$

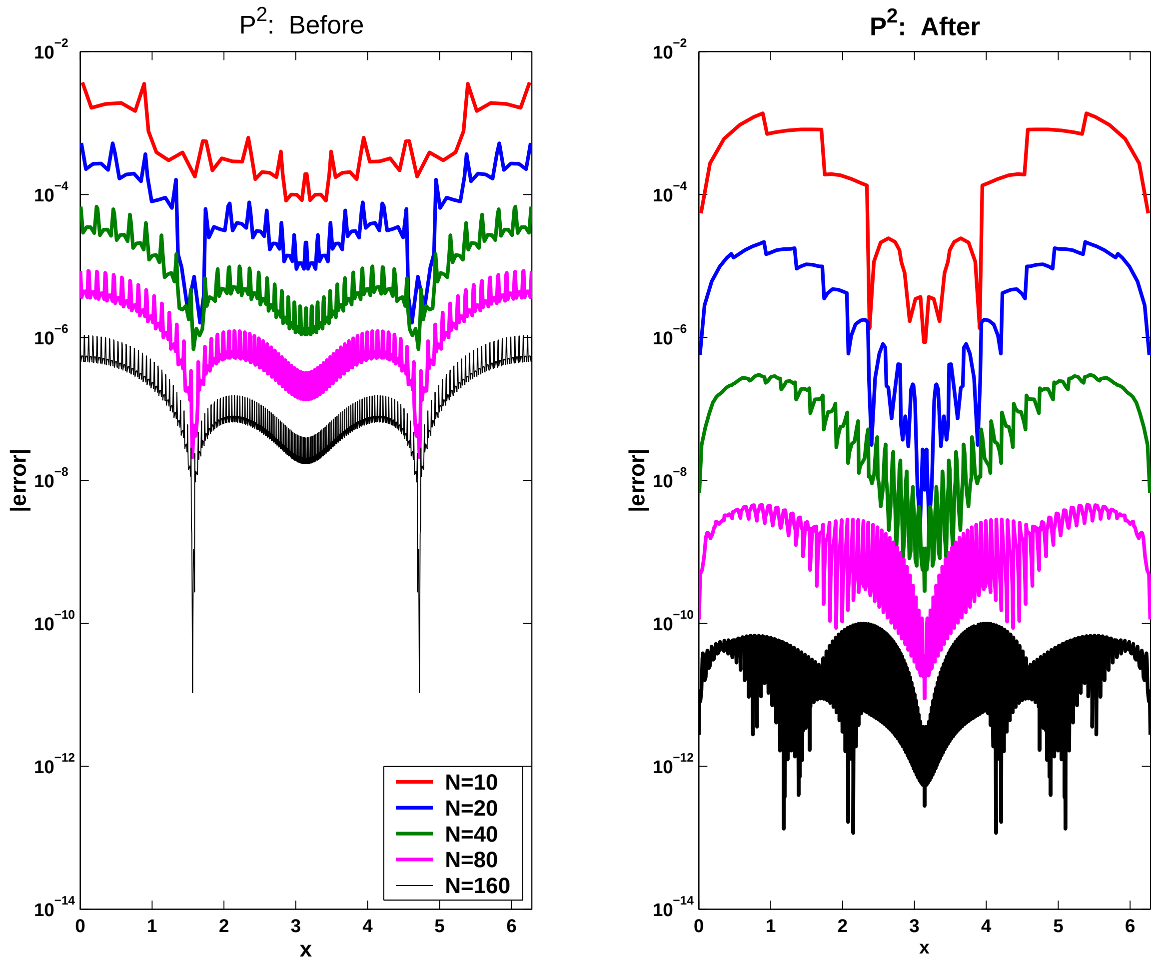
Approximation Level Errors

	Approximation		**Post-Processed**	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.5358E-02	—	3.2149E-03	—
20	3.9029E-03	1.98	2.2069E-04	3.86
40	9.7975E-04	1.99	1.5756E-05	3.81
80	2.4519E-04	2.00	1.2682E-06	3.64
	$\mathbb{P}^2$			
10	1.2175E-03	—	6.1679E-04	—
20	1.5490E-04	2.97	1.0484E-05	5.88
40	1.9448E-05	2.99	1.6048E-07	6.03
80	2.4337E-06	3.00	2.3281E-09	6.11

$u(x) = \sin(x)$

Mesh defined by
$x = \xi + \frac{1}{2}\sin(\xi)$

Approximation Level Errors: $x = \xi + \frac{1}{2}\sin(\xi)$



Approximation Level Errors for the Derivative

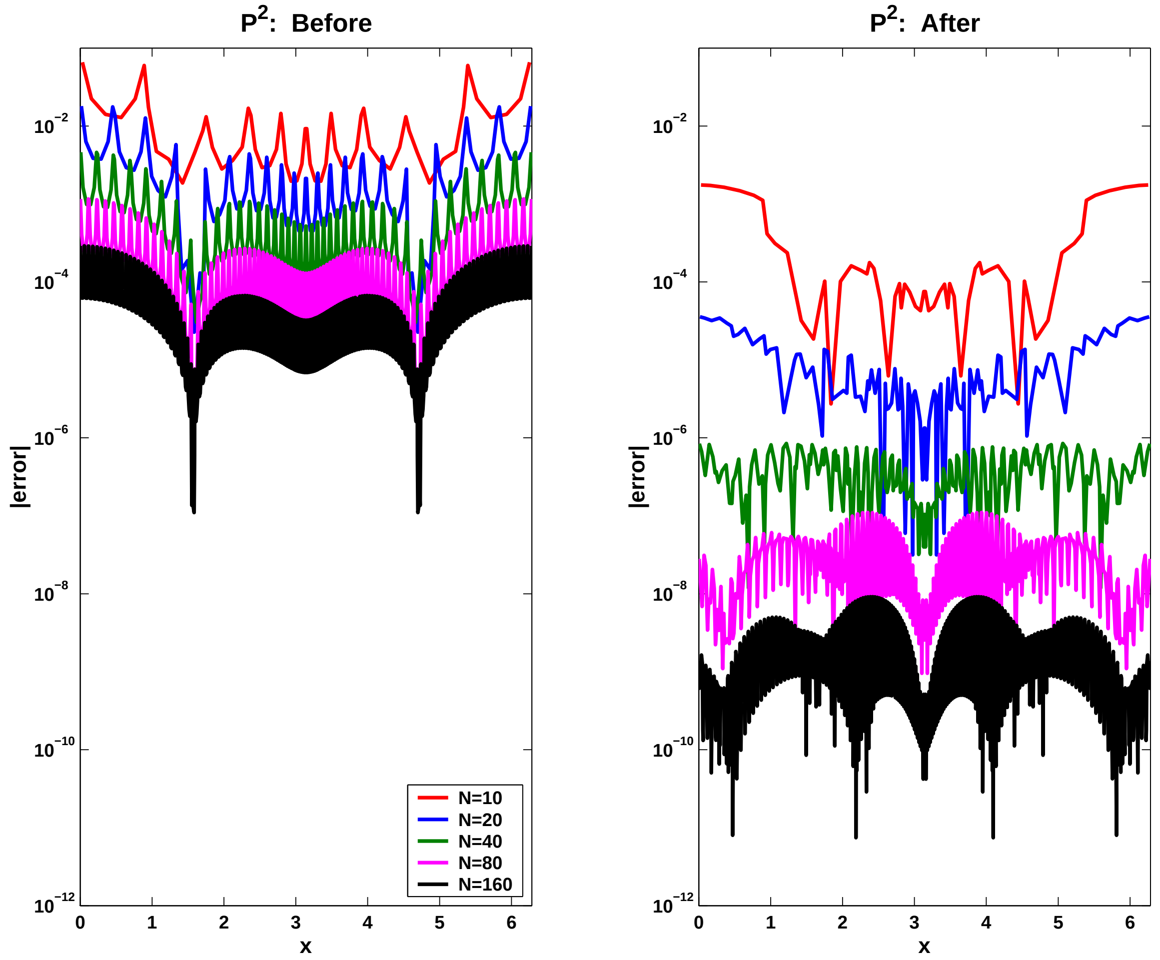
	Approximation		**Post-Processed**	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^2$: Errors in 1^{st} Derivative			
10	1.7523E-02	—	8.4040E-04	—
20	4.4448E-03	1.98	1.6000E-05	5.72
40	1.1153E-03	1.99	4.6767E-07	5.10
80	2.7907E-04	2.00	3.8042E-08	3.62
	$\mathbb{P}^2$: Errors in 2^{nd} Derivative			
10	1.4529E-01	—	7.5038E-04	—
20	7.3365E-02	0.99	7.4545E-05	3.33
40	3.6774E-02	1.00	1.2806E-05	2.54
80	1.8398E-02	1.00	2.3907E-06	2.42

Approximation level Errors in 1^{st} & 2^{nd} Derivatives

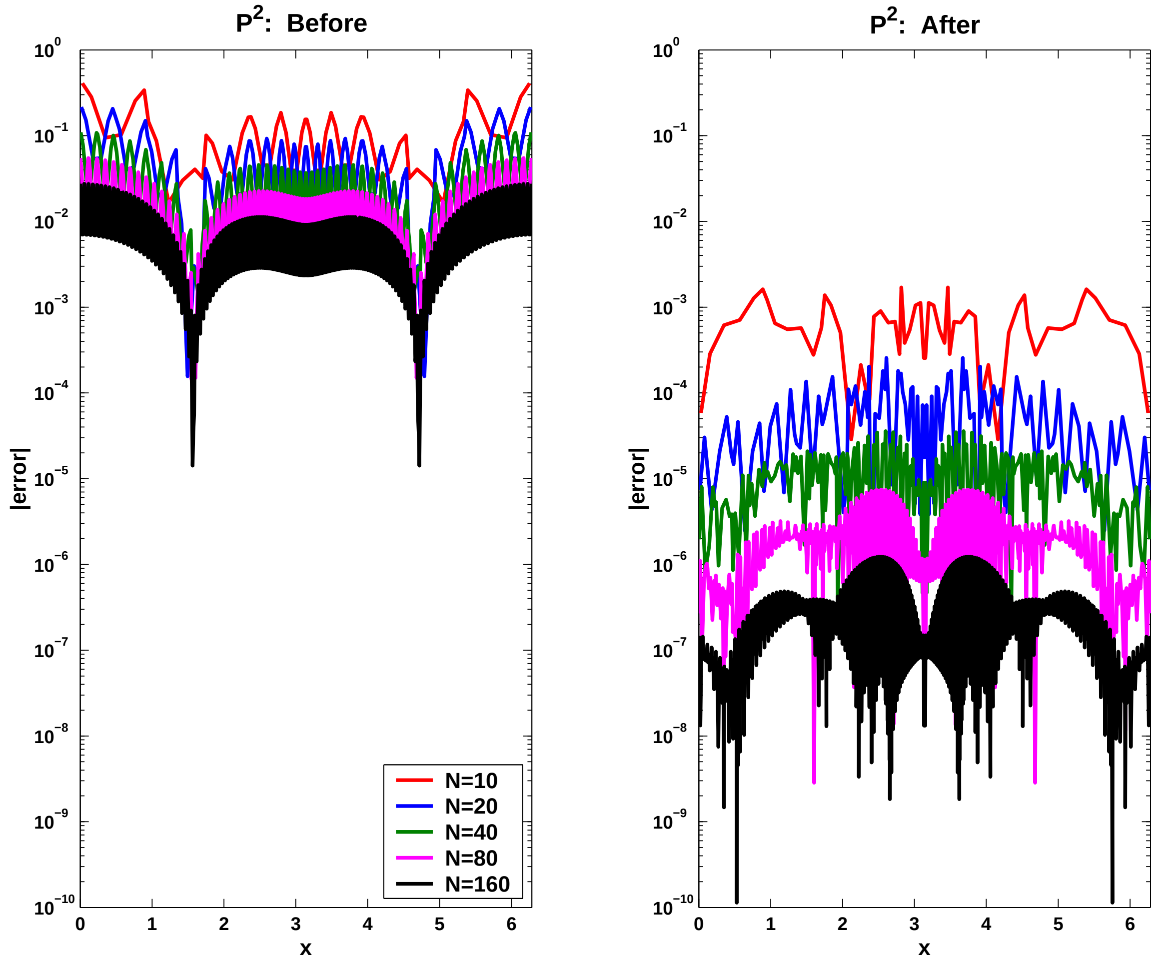
$$u(x) = \sin(x)$$

$$x = \xi + \tfrac{1}{2}\sin(\xi)$$

1^{st} Derivative Approximation Level Errors



2^{nd} Derivative Approximation Level Errors



Linear Convection Equation: Solution Errors

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
	$b = 0.5$			
10	1.9174E-03	—	9.2350E-04	—
20	2.4012E-04	3.00	2.1027E-05	5.46
40	3.0109E-05	3.00	5.1216E-07	5.36
80	3.7677E-06	3.00	1.3750E-08	5.22
	$b = 0.9$			
10	0.3.9694E-03	—	3.4799E-03	—
20	4.9133E-04	3.01	8.0229E-05	5.44
40	6.1623E-05	3.00	1.8864E-06	5.41
80	7.7140E-06	3.00	4.8987E-08	5.27

using $\mathbb{P}^2$

$$u_t + u_x = 0$$

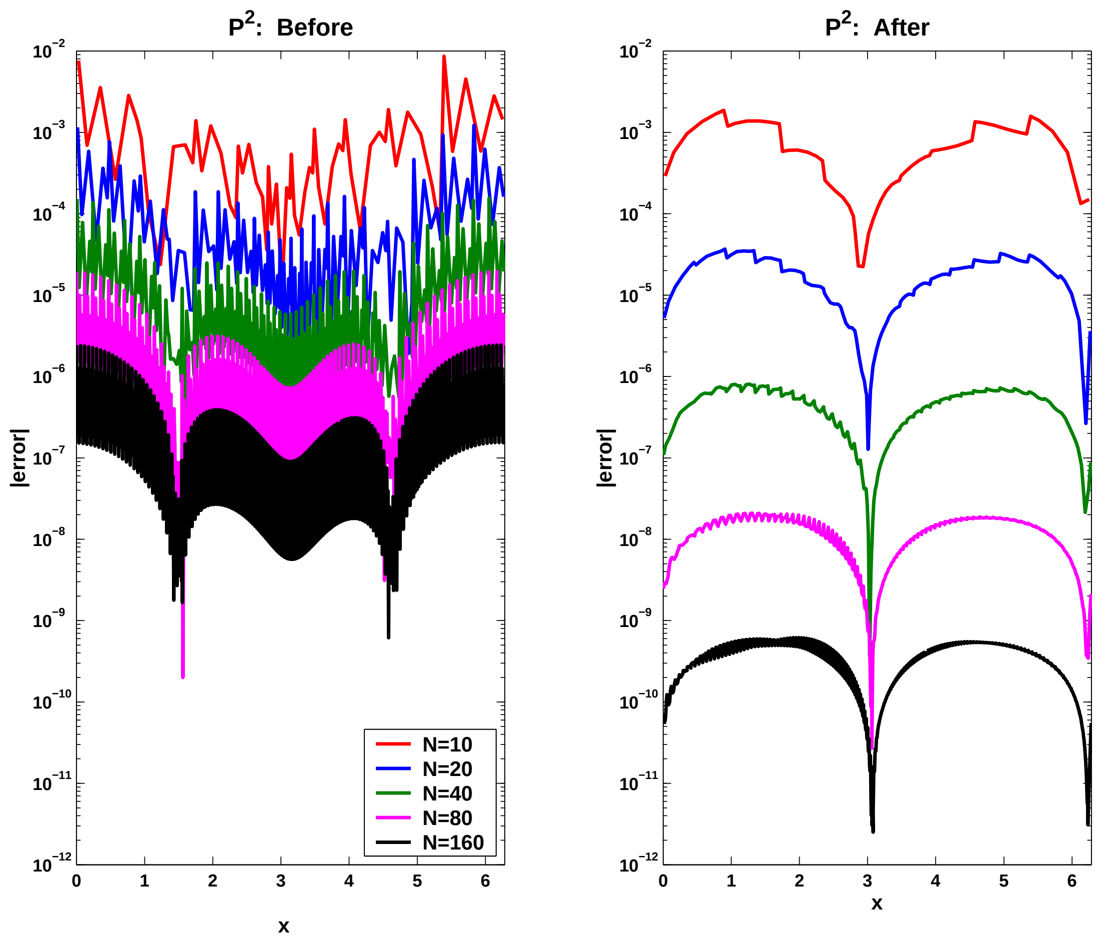
$$u(x, 0) = \sin(x)$$

$$x \in (0, 2\pi)$$

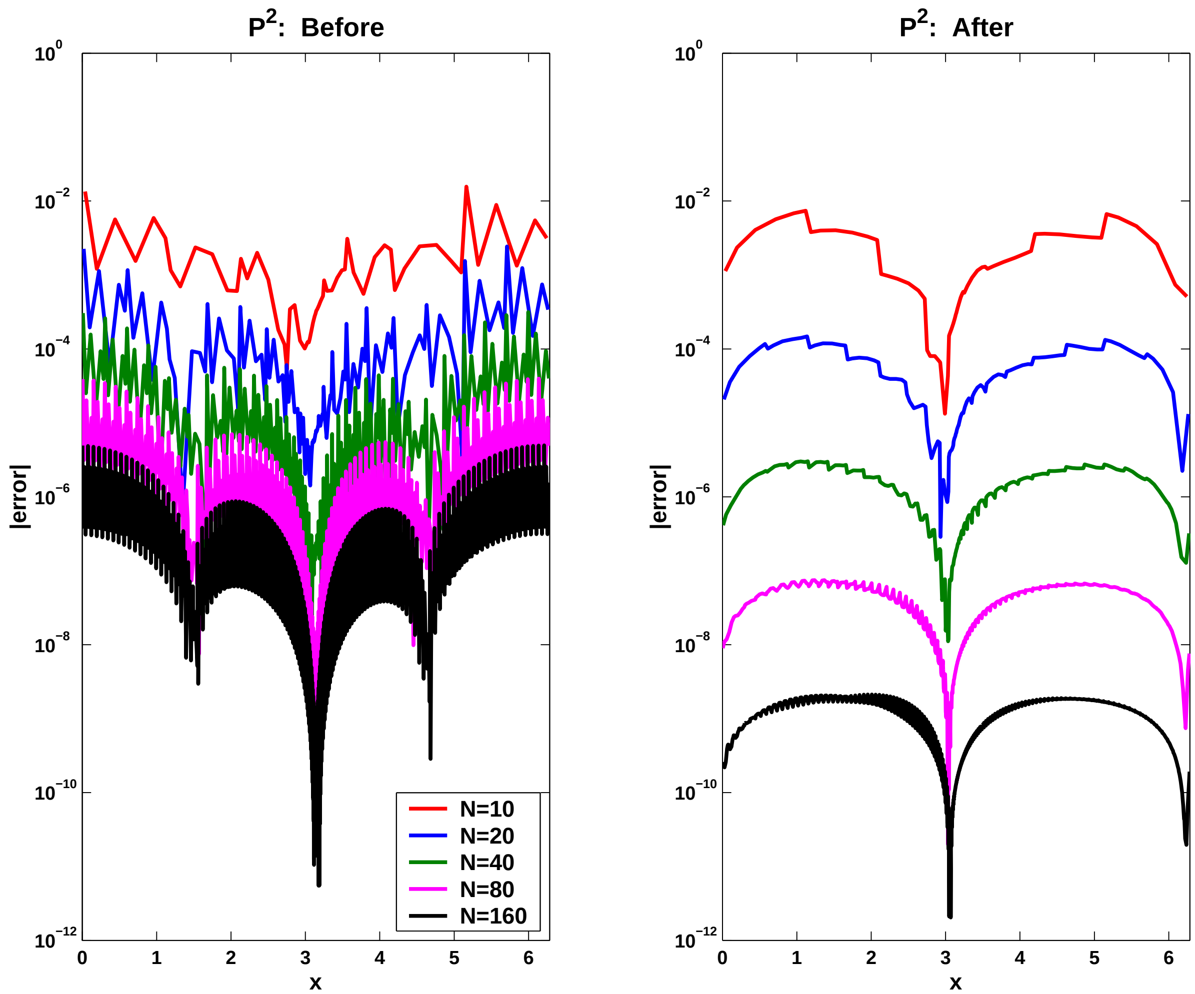
$$T = 12.5$$

$$x = \xi + b\sin(\xi)$$

Linear Convection Equation: $b = 0.5$



Linear Convection Equation: $b = 0.9$



Linear Convection Equation: 1^{st} Derivative Errors

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
$b = 0.5$				
10	2.1277E-02	—	1.1446E-03	—
20	5.4354E-03	1.97	6.5090E-05	4.14
40	1.3653E-03	1.99	4.8212E-06	3.76
80	3.4171E-04	2.00	3.0857E-07	3.97
$b = 0.9$				
10	3.3529E-02	—	4.0396E-03	—
20	8.7117E-03	1.94	9.8900E-05	5.35
40	2.1952E-03	1.99	2.9503E-06	5.07
80	5.4981E-04	2.00	1.6134E-07	4.19

Errors in
first derivative
using $\mathbb{P}^2$

$u_t + u_x = 0$

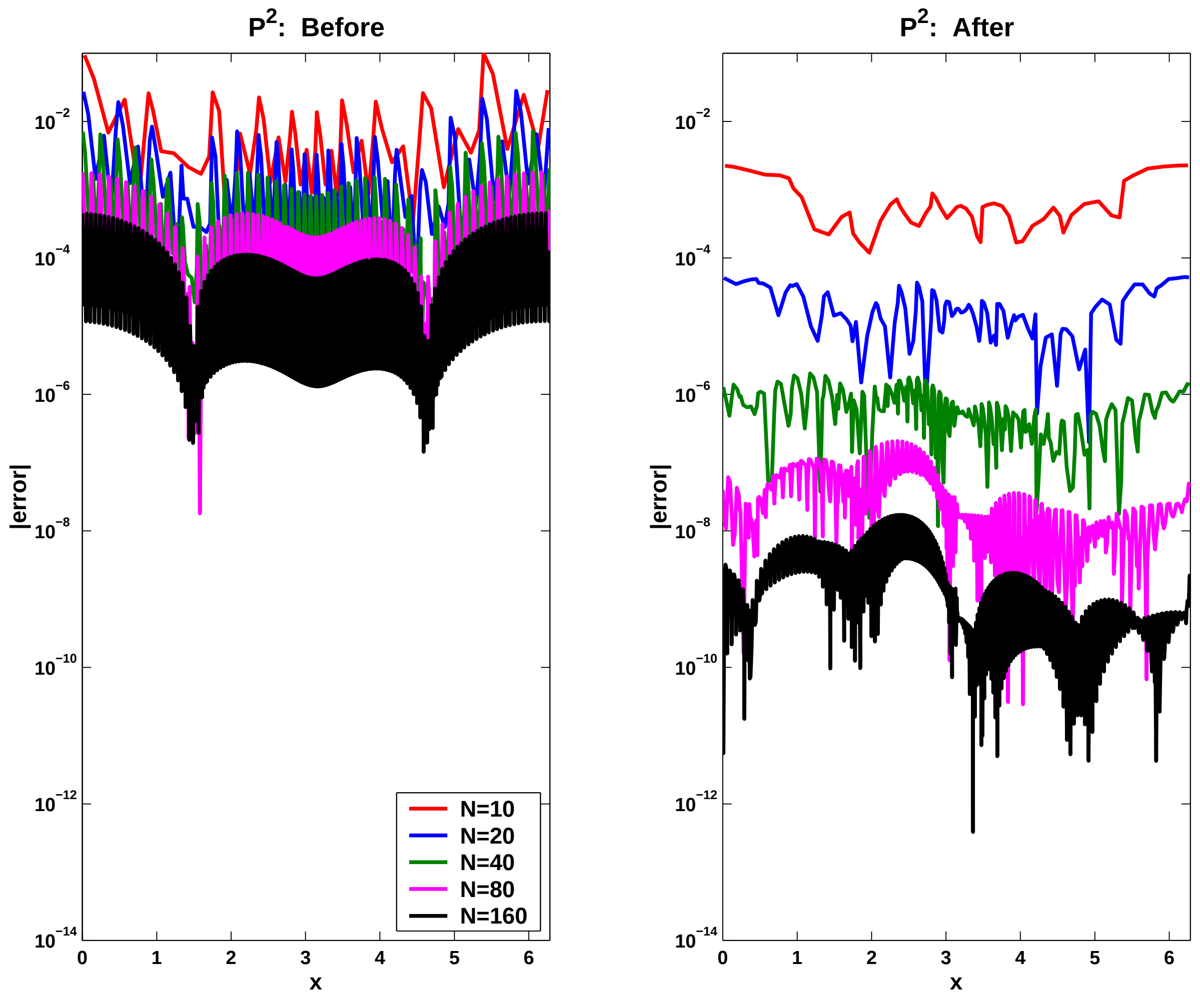
$u(x, 0) = \sin(x)$

$x \in (0, 2\pi)$

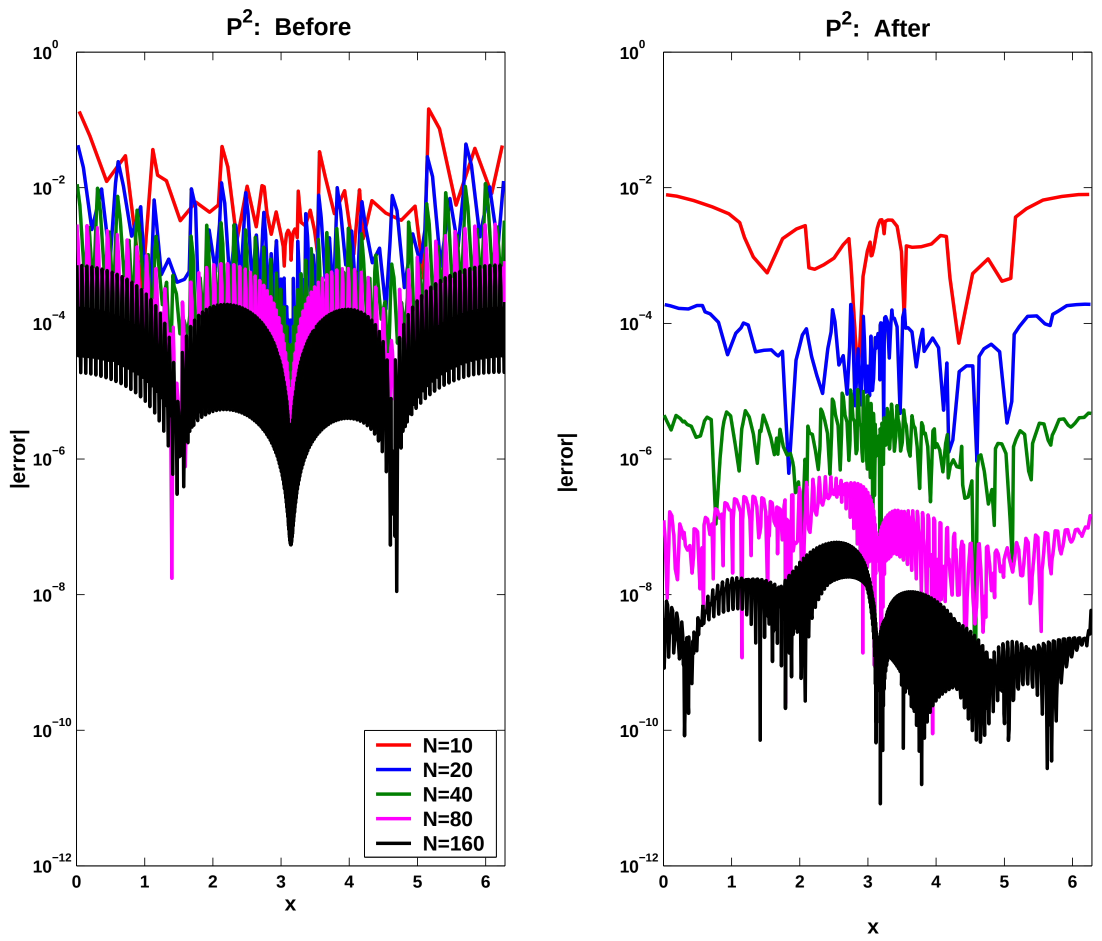
$T = 12.5$

$x = \xi + b\sin(\xi)$

Linear Convection Equation: $b = 0.5$



Linear Convection Equation: $b = 0.9$



Linear Convection Equation: 2^{nd} Derivative Errors

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
$b = 0.5$				
10	1.5337E-01	—	5.6329E-03	—
20	7.7611E-02	0.98	9.9806E-04	2.50
40	3.8919E-02	1.00	1.5234E-04	2.71
80	1.9474E-02	1.00	1.9755E-05	2.95
$b = 0.9$				
10	1.8919E-01	—	7.0755E-03	—
20	9.6780E-02	0.97	9.5435E-04	2.89
40	4.8654E-02	0.99	9.0905E-05	3.39
80	2.4360E-02	1.00	1.0211E-05	3.15

Errors in second derivative using $\mathbb{P}^2$

$u_t + u_x = 0$

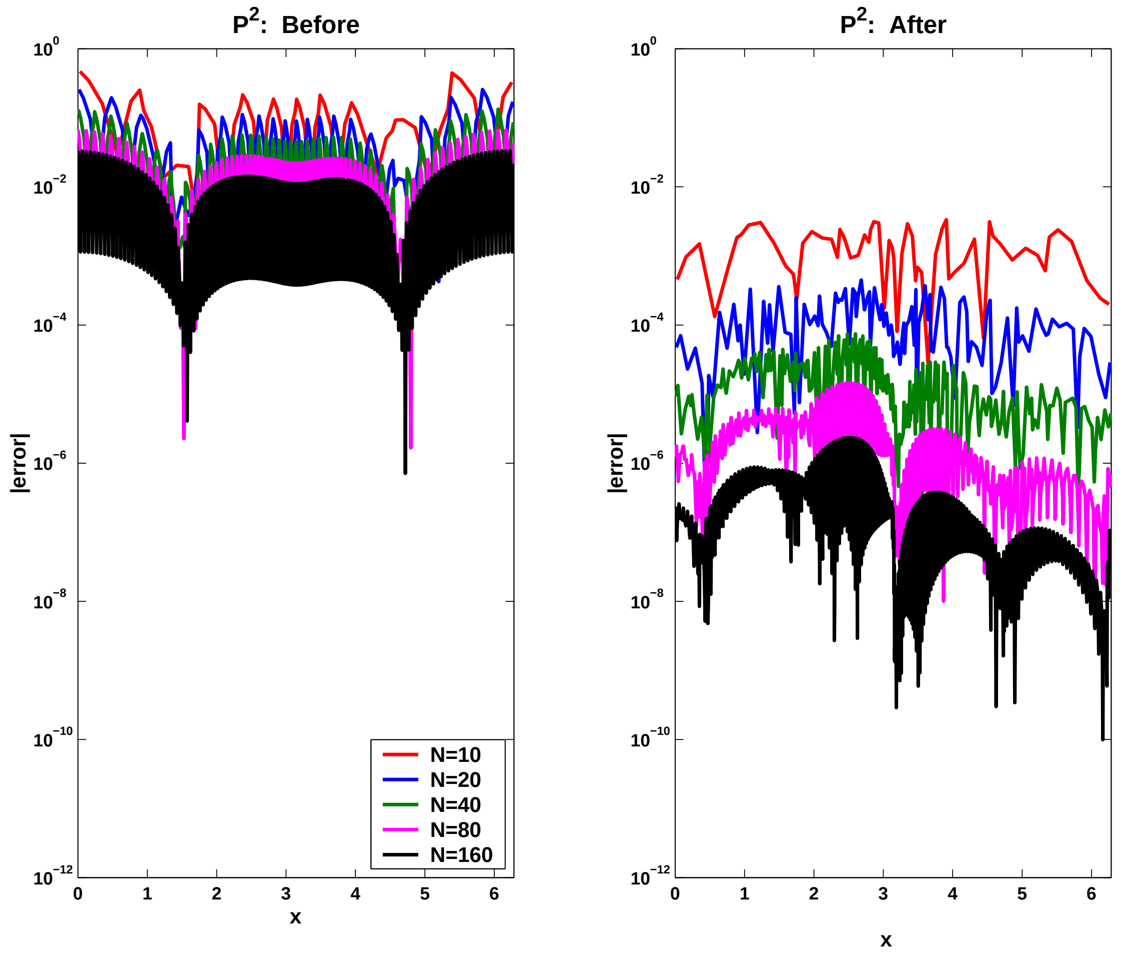
$u(x, 0) = \sin(x)$

$x \in (0, 2\pi)$

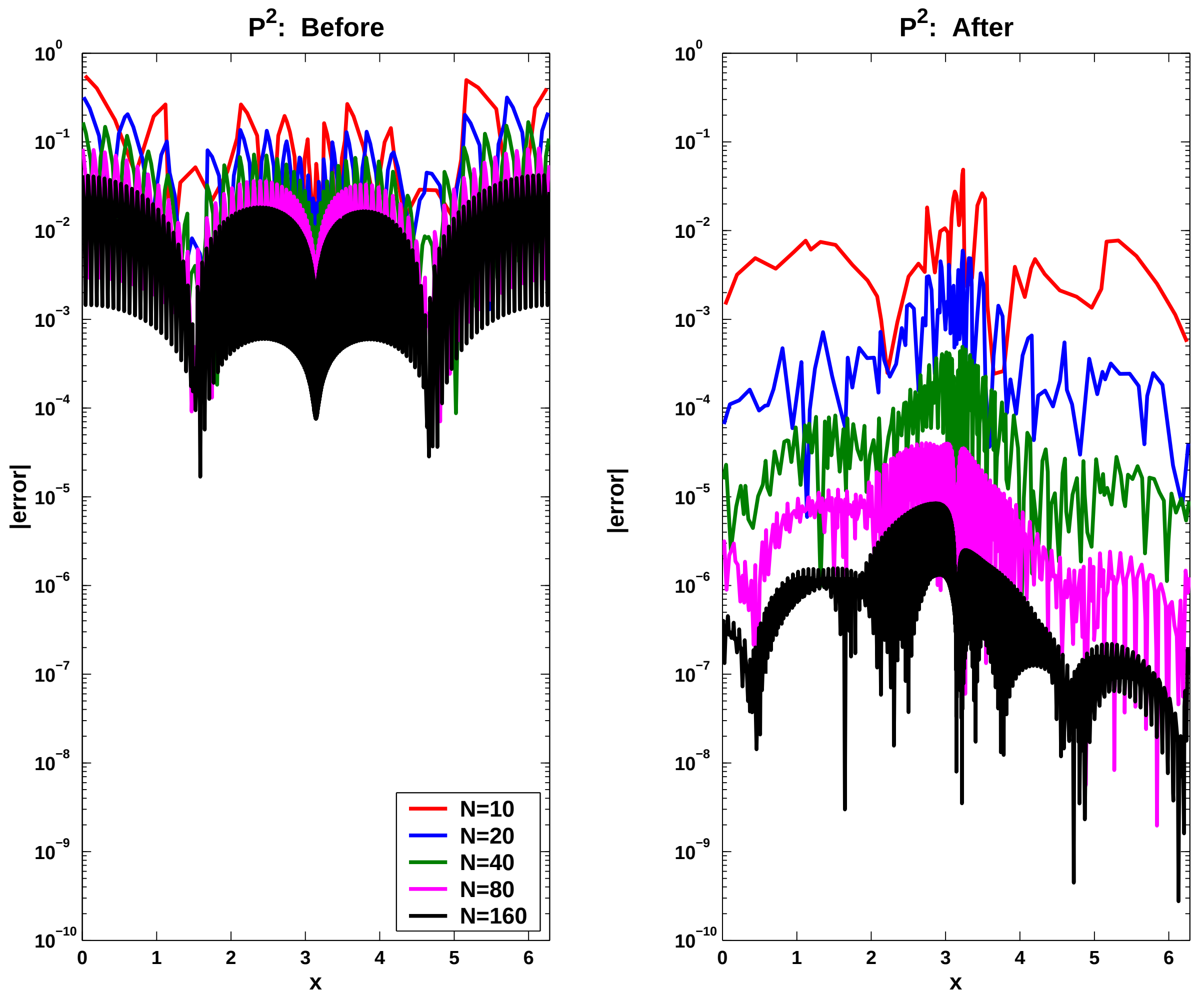
$T = 12.5$

$x = \xi + b\sin(\xi)$

Linear Convection Equation: $b = 0.5$



Linear Convection Equation: $b = 0.9$



Variable Coefficient Equation: $x = \xi + 0.5\sin(\xi)$

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
$\mathbb{P}^1$				
10	2.9105E-02	—	1.8280E-02	—
20	6.7749E-03	2.10	2.6034E-03	2.81
40	1.6318E-03	2.01	3.4310E-04	2.99
80	4.0276E-04	2.02	4.3689E-05	2.97
$\mathbb{P}^2$				
10	2.0195E-03	—	8.9665E-04	—
20	2.4683E-04	3.03	2.0051E-05	5.48
40	3.0528E-05	3.02	4.8284E-07	5.38
80	3.7938E-06	3.01	1.2890E-08	5.23

Errors in solution

$$u_t + (au)_x = f$$

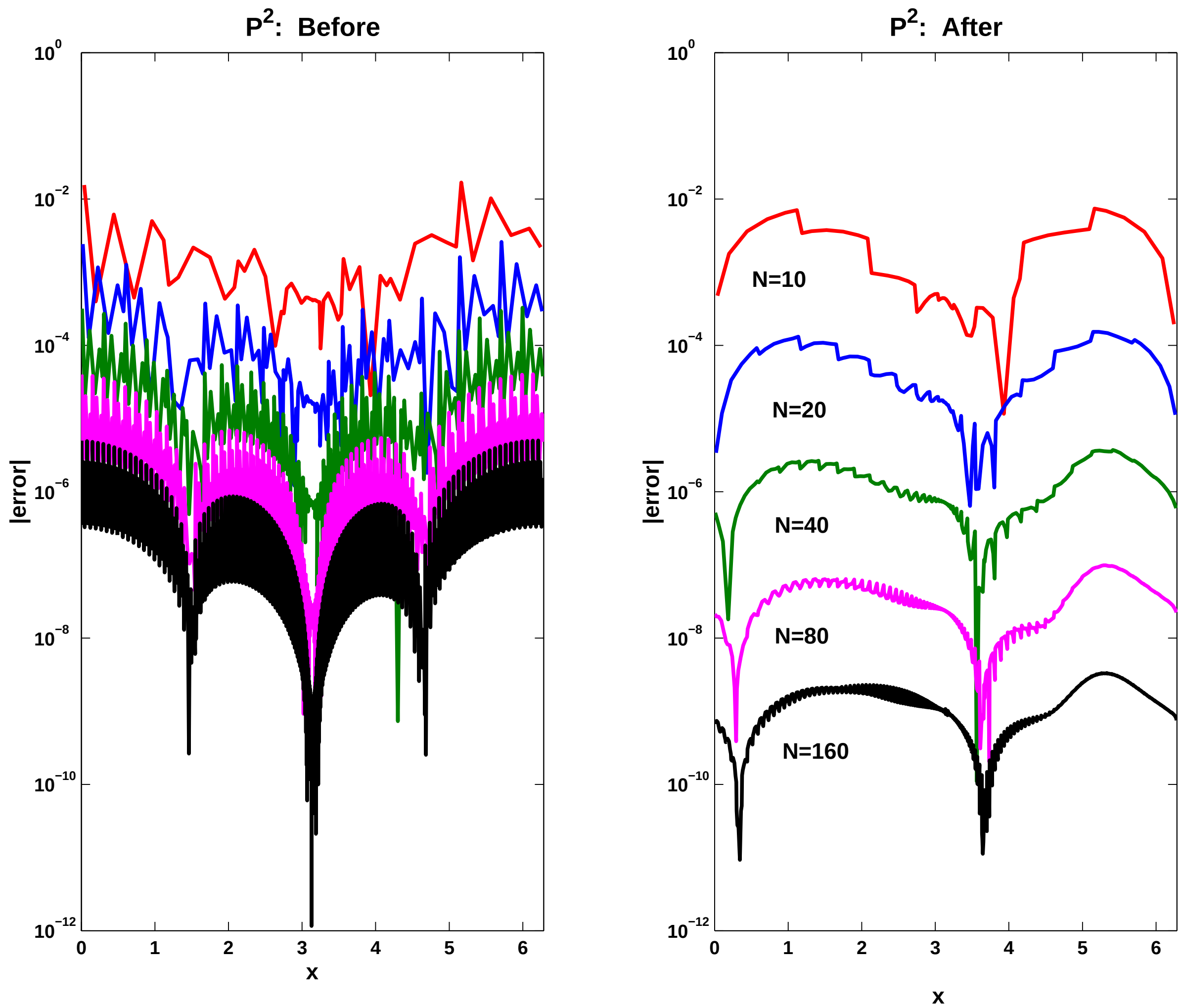
$$a = 2 + \sin(x - t)$$

$$u(x, 0) = \sin(3x)$$

$$x \in (0, 2\pi)$$

$$T = 12.5$$

Variable Coefficient Equation



1-D System: $x = \xi + 0.9\sin(\xi)$

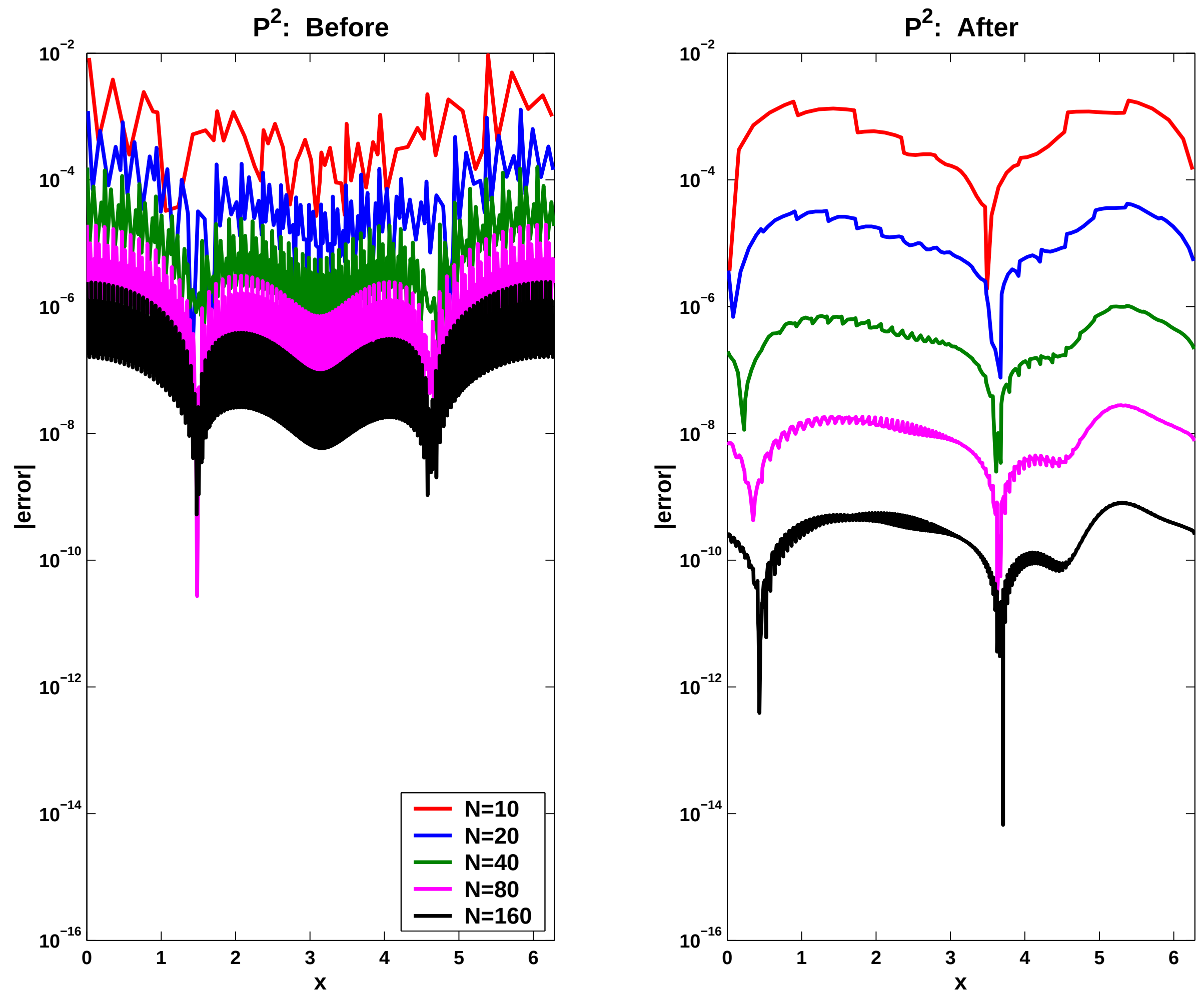
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
$\mathbb{P}^2$: Errors in Solution				
10	1.9174E-03	—	9.2350E-04	—
20	2.4012E-04	3.00	2.1027E-05	5.46
40	3.0109E-05	3.00	5.1216E-07	5.36
80	3.7677E-06	3.00	1.3750E-08	5.22
$\mathbb{P}^2$: Errors in 1^{st} Derivative				
10	2.1277E-02	—	1.1249E-03	—
20	5.4354E-03	1.97	2.6490E-05	5.41
40	1.3653E-03	1.99	8.3662E-07	4.98
80	3.4171E-04	2.00	5.4173E-08	3.95

Errors in solution & 1^{st} derivative

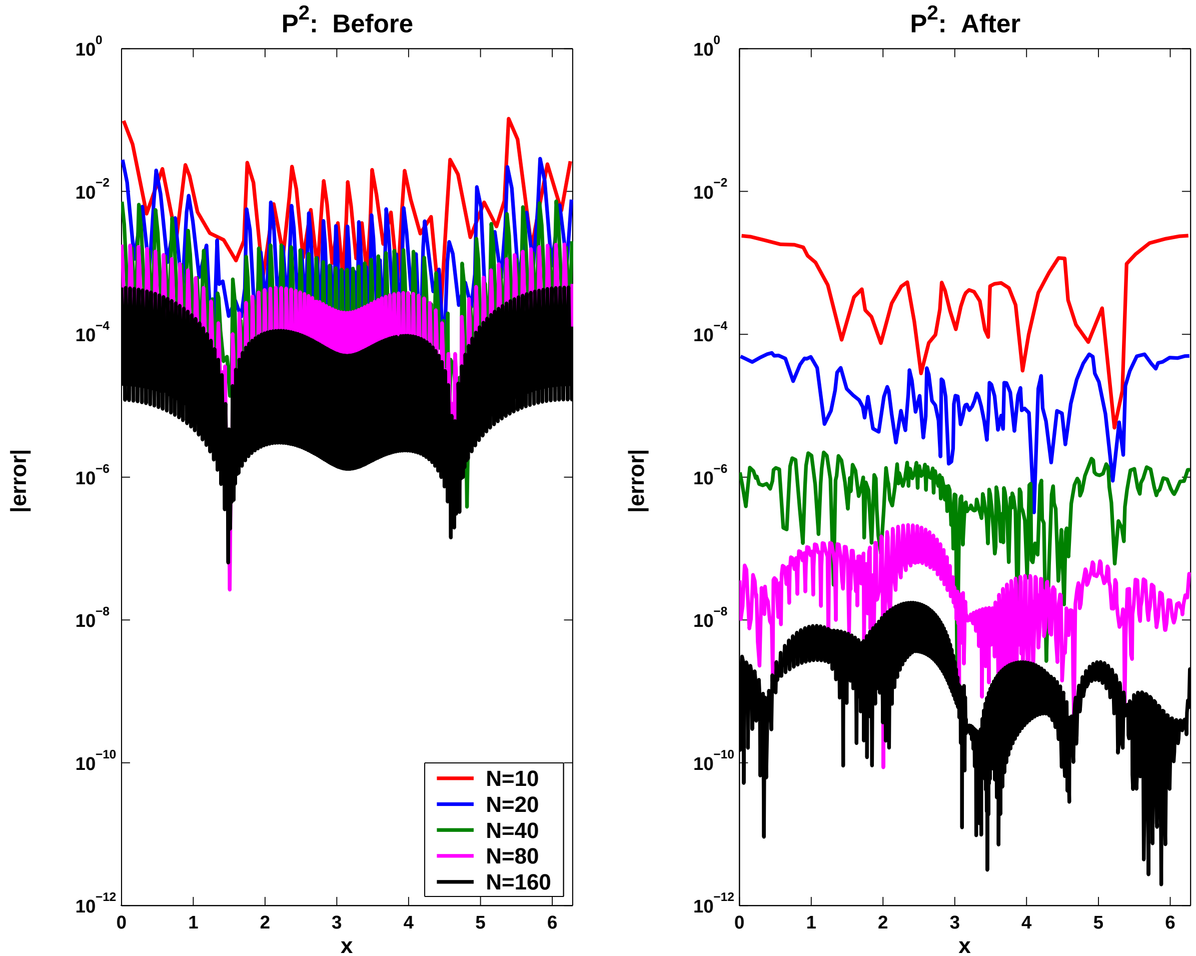
$$u_t + v_x = 0$$
$$v_t + u_x = 0$$

$$u(x,0) = \sin(x)$$
$$v(x,0) = 0$$
$$x \in (0, 2\pi)$$
$$T = 12.5$$

1-D System: Errors in Solution



1-D System: Errors in 1^{st} Derivative



Approximation Level Errors

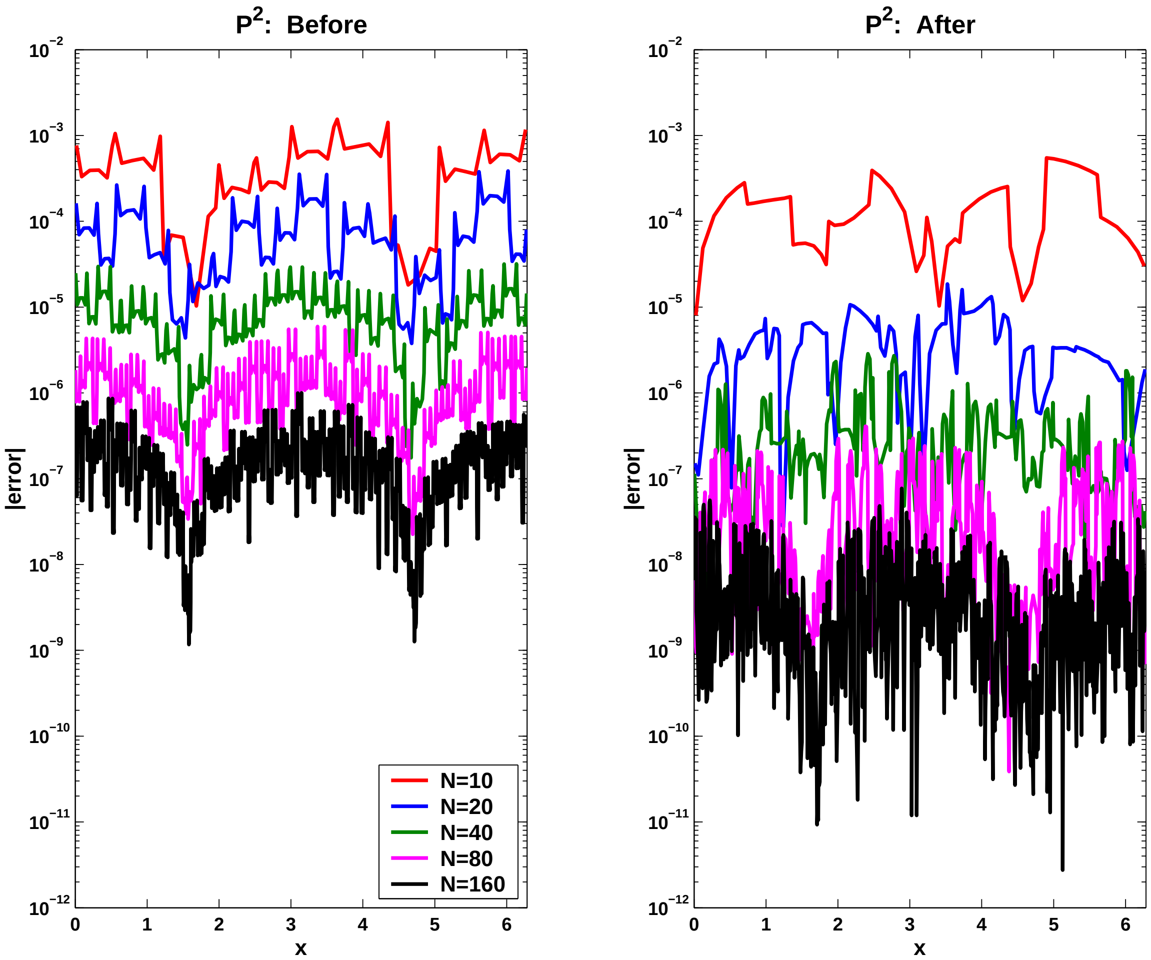
	Approximation		**Post-Processed**	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.1976E-02	—	2.3477E-03	—
20	2.9531E-03	2.02	2.8558E-04	3.04
40	7.9218E-04	1.90	1.1763E-04	1.28
80	1.9461E-04	2.03	2.7815E-05	2.08
	$\mathbb{P}^2$			
10	5.4472E-04	—	2.9433E-04	—
20	1.0699E-04	2.35	7.0259E-06	5.39
40	1.0115E-05	3.40	2.2693E-07	4.95
80	1.4937E-06	2.76	6.1925E-08	1.87

$u(x) = \sin(x)$

Random Mesh

50% mesh variation

Approximation Level Errors: Random Mesh



Approximation Level Errors for the Derivative

	Approximation		Post-Processed	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^2$: Errors in 1^{st} Derivative			
10	1.0906E-02	—	1.7789E-04	—
20	3.5904E-03	1.60	7.7810E-05	1.19
40	7.7520E-04	2.21	5.8255E-06	3.74
80	2.1037E-04	1.88	3.8472E-06	0.60
	$\mathbb{P}^2$: Errors in 2^{nd} Derivative			
10	1.2310E-01	—	4.4819E-04	—
20	6.9232E-02	0.83	8.5565E-04	—
40	3.3065E-02	1.07	1.1823E-04	2.86
80	1.6913E-02	0.97	1.8282E-04	—

Approximation level
Errors in
1^{st} & 2^{nd} Derivatives

$u(x) = \sin(x)$

Random Mesh
50% mesh variation

Linear Convection Equation: Solution Errors

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
	$b = 0.5$			
10	1.3383E-03	—	3.5373E-04	—
20	1.4342E-04	3.22	2.4677E-05	3.04
40	1.8324E-05	2.97	2.4925E-06	3.31
80	2.4217E-06	2.92	3.4897E-07	2.84
	$b = 0.9$			
10	1.3848E-03	—	8.8454E-04	—
20	2.8559E-04	2.28	6.5683E-05	3.75
40	2.6247E-05	3.44	4.8857E-06	3.75
80	3.7041E-06	2.83	7.4199E-07	2.72

using $\mathbb{P}^2$

$u_t + u_x = 0$

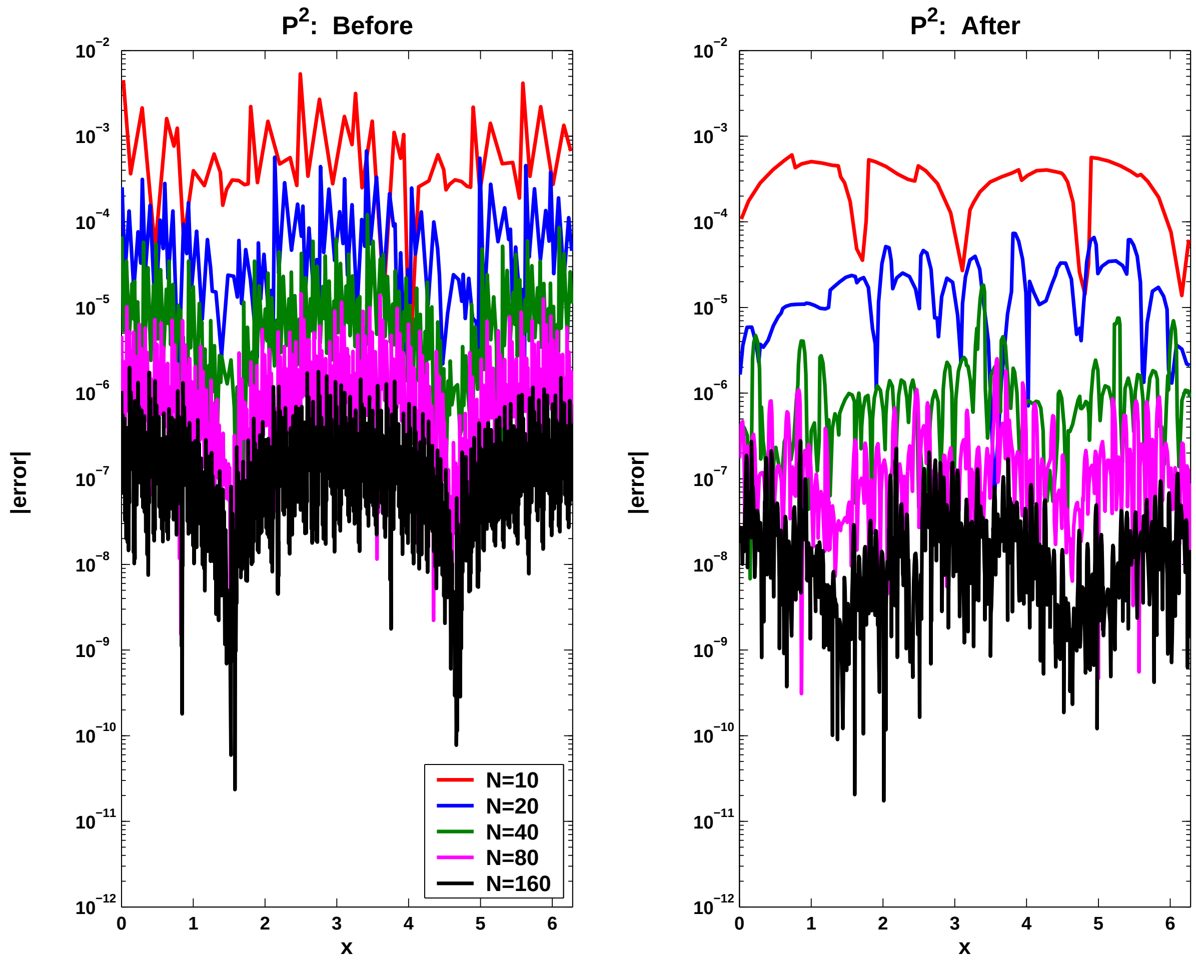
$u(x, 0) = \sin(x)$

$x \in (0, 2\pi)$

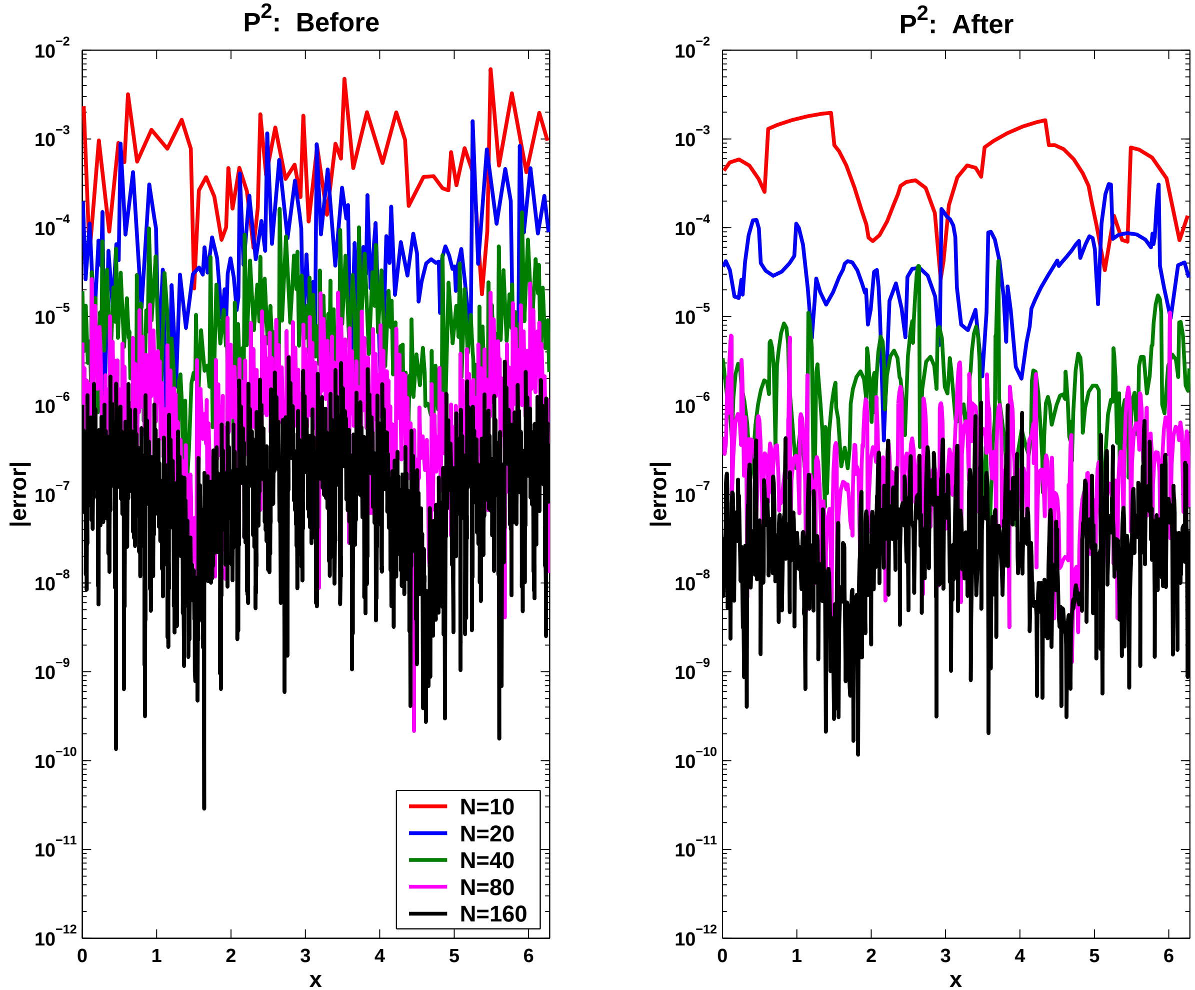
$T = 12.5$

Random mesh

Linear Convection Equation: $b = 0.5$



Linear Convection Equation: $b = 0.9$



Summary

Using a local L^2-projection we are able to:

- Smoothly varying mesh:
  - Obtain $(2k+1)$-th order accuracy for the post-processed solution.
  - Obtain order improvement for the derivatives.
- Random mesh:
  - Obtain improvement in the errors for the post-processed solution.

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