Optimal Control for a Standard CPR Model

Suzanne Lenhart[a], Vladimir Protopopescu [b], Eunok Jung [c], and Charles Babbs [d]

[a]University of Tennessee
Department of Mathematics
Knoxville, TN 37996-1300

[b]Oak Ridge National Laboratory
Computer Science and Mathematics Division
Oak Ridge, TN 37831-6355

[c]Konkuk University
Department of Mathematics
Seoul 143-701, Korea

[d]Purdue University
Basic Medical Sciences
West Lafayette, IN 47907

Optimal control techniques are applied here for the first time to a validated blood circulation model of cardiopulmonary resuscitation (CPR), consisting of a system of seven difference equations. In this system, the non-homogeneous forcing term is the externally applied chest pressure acting as the "control". The optimal control technique seeks to maximize the blood flow as measured by the pressure differences between the thoracic aorta and the right head superior vena cava. As a result, we provide a new CPR strategy, with improved resuscitation rates. The optimal control is characterized in terms of the solutions of the circulation model and of the corresponding adjoint system. the calculated optimal control gives the pattern of the external pressure to be applied on the chest to obtain optimal blood flow and higher resuscitation rates.

Key Words: Cardiopulmonary resuscitation, Difference equation, Optimal Control

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1. Introduction

Each year, more than 250,000 people die from cardiac arrest in the USA alone. Despite widespread use of cardiopulmonary resuscitation, the survival of patients recovering from cardiac arrest remains poor. Indeed, the rate of survival for CPR performed out of the hospital is 3%, while for patients who have cardiac arrest in the hospital, the rate of survival is 10-15%. [4–7] One of the reasons for this situation is that the practical technique of CPR has changed little since the 1960's. The goal of this paper is to reconsider the traditional CPR technique and attempt to improving it by using the optimal control methodology.

The standard and various alternative CPR techniques such as interposed abdominal compression, active compression-decompression, and Lifestick CPR have been represented in various mathematical models. Here, we consider a model for *standard* CPR.

We apply the optimal control strategy for improving resuscitation rates to a validated circulation model developed by Babbs. [1] In his model, heart and blood vessels are represented as resistance-capacitive networks, pressures in the chest and in the vascular components as voltages, blood flow as electric current, and cardiac and venous valves as diodes (electrical devices that permit flow in only one direction).

The chosen CPR model consists of seven difference equations, with time as the discrete underlying variable, which describe the adult human circulation (hemodynamics). For the optimal control application, the circulation model is extended to include the control functions as discrete inputs. As a control, we choose the the pattern of the external pressure on the chest. The pressure state variables are as follows:

P_1 pressure in abdominal aorta
P_2 pressure in inferior vena aorta
P_3 pressure in carotid
P_4 pressure in jugular
P_5 pressure in thoracic aorta
P_6 pressure in right heart and superior vena cava
P_7 pressure in thoracic pump.

At the step n, when time is $n\Delta t$, the pressure vector is denoted by:

$$P(n) = (P_1(n), P_2(n), ..., P_7(n)).$$

We assume that the initial pressure values are known: $P(0) = (P_1(0), ..., P_7(0))$.

To make the chest pressure profiles medically reasonable, we assume that the admissible controls are equal at the beginning and the end of the time interval, i.e.,

$u(0) = u(N-1)$. Using the control vector $u = (u(0), u(1), ..., u(N-2), u(0))$, the difference equations (in vector notation) representing the circulation model are as follows:

$$P(1) = P(0) + T(u(0)) + \Delta t F(P(0)) \tag{1.1}$$
$$P(n+1) = P(n) + T(u(n) - u(n-1)) + \Delta t F(P(n)), n = 1, 2, ..., N-1 \tag{1.2}$$

where T represents the linear map,

$$T(u(n)) = (0, 0, 0, 0, t_p u(n), t_p u(n), u(n)).$$

Thus, at time step n, the control terms in the 5th and 6th equations are $t_p(u(n) - u(n-1))$, while in the 7th equation, the control term is $u(n) - u(n-1)$. We use N time steps, and the initial data for pressures is entered at $n = 0$.

Note that the pressure vector depends on the control, $P = P(u)$, and the calculation of the pressures at the next time step requires the values of the controls at the current and previous time steps.

We define the function $F(P(n))$ by listing its seven components:

$$\frac{1}{c_{aa}} \left[\frac{1}{R_a}(P_5(n) - P_1(n)) - \frac{1}{R_s}(P_1(n) - P_2(n)) \right]$$

$$\frac{1}{c_{ivc}} \left[\frac{1}{R_s}(P_1(n) - P_2(n)) - \frac{1}{R_v}(P_2(n) - P_6(n) \right]$$

$$\frac{1}{c_{car}} \left[\frac{1}{R_c}(P_5(n) - P_3(n)) - \frac{1}{R_h}(P_3(n) - P_4(n)) \right]$$

$$\frac{1}{c_{jug}} \left[\frac{1}{R_h}(P_3(n) - P_4(n)) - \frac{1}{R_j}V(P_4(n) - P_6(n)) \right]$$

$$\frac{1}{c_{ao}} \left[\frac{1}{R_o}V(P_7(n) - P_5(n)) - \frac{1}{R_c}(P_5(n) - P_3(n)) \right]$$
$$+ \frac{1}{R_a}(P_5(n) - P_1(n)) - \frac{1}{R_{ht}}V(P_5(n) - P_6(n)) \Bigg]$$

$$\frac{1}{c_{rh}} \left[\frac{1}{R_j}V(P_4(n) - P_6(n)) + \frac{1}{R_v}(P_2(n) - P_6(n)) \right.$$
$$\left. + \frac{1}{R_{ht}}(P_5(n) - P_6(n)) - \frac{1}{R_i}V(P_6(n) - P_7(n)) \right]$$

$$\frac{1}{c_p}\left[\frac{1}{R_i}V(P_6(n)-P_7(n))-\frac{1}{R_o}V(P_7(n)-P_5(n))\right]$$

where the valve function is defined by
$V(s)=s$ if $s\geq 0$
$V(s)=0$ if $s\leq 0$.
Note that F is a linear function except for the valve function.

To be rigorous mathematically, one should approximate the valve function by a smooth function that is differentiable at zero.

We assume $-K\leq u(n)\leq K$ for all $n=0,1,\ldots,N-2$ and choose the control set

$$U=\{(u(0),u(1),...,u(N-2),u(0))|-K\leq u(n)\leq K,n=0,1,\ldots,N-2\}.$$

We define the objective functional

$$J(u)=\sum_{n=1}^{N}[P_5(n)-P_6(n)]-\sum_{n=0}^{N-2}\frac{B}{2}u^2(n) \tag{1.3}$$

where the first term represents the pressure differences between the thoracic aorta and the right head superior vena cava, called the systemic perfusion pressure. The second term represents the cost of implementing the control and has the double effect of stabilizing the control problem and yielding an explicit characterization for the optimal control. Our goal is to maximize $J(u)$, i.e., to find an u^* such that

$$J(u^*)=\max_u J(u).$$

Optimal control of discrete difference equations has been used for various physical and engineering models, but it has never been applied to a CPR model. In most applications, the control at the current time step only feeds into the states at the next time step. The fact that the controls enter the system at two time levels (current and immediate past time steps) to give input to the pressure at the next time is also a novel feature [2,10]. This new feature requires an innovative adaptation of the discrete version of Pontryagin's Maximum Principle. [3,8,9] The characterization of the optimal control in terms of the solutions of the optimality system, which is the pressure system and an adjoint system, is given in the next section.

2. Characterization of an Optimal Control

The existence of an optimal control u^* in U that maximizes the objective functional J is standard, since we have compactness, due to the finite number of state variables with continuous functions in the equations and the finite number of time steps.

To characterize an optimal control, we must differentiate the map $u \to J(u)$, which requires the differentiation of the solution map $u \longrightarrow P = P(u)$. [3,10]

Theorem 2.1. *The mapping $u \in U \to P$ is differentiable in the following sense:*

$$\frac{P(u+\epsilon l)(n) - P(u)(n)}{\epsilon} \rightharpoonup \psi(n)$$

as $\epsilon \to 0$ for any $u \in U$ and l such that $(u+\epsilon l) \in U$ for ϵ small, for $n = 1, \ldots, N$. Also ψ satisfies the discrete system:

$$\begin{aligned} \psi(n+1) &= \psi(n) + \Delta t M(n)\psi(n) + T(l(n) - l(n-1)) && (2.1)\\ \psi(N) &= \psi(N-1) + \Delta t M(N-1)\psi(N-1) + T(l(0) - l(N-2)) && (2.2)\\ \psi(0) &= 0 && (2.3)\\ \psi(1) &= T(l(0)) && (2.4) \end{aligned}$$

for $n = 1, \ldots, N-2$, where $M(n) = \frac{\partial F(P(n))}{\partial P}$.

Proof: This follows from the component-wise calculation of the difference quotient and passage to the limit in each component, using the differentiability of the function F. Note that, in order to compute the derivative rigorously, one should use here a differentiable approximation to the valve function. □

Note: To illustrate the elements in the matrix M, we write the first row:

$$-\frac{1}{c_{aa}}(\frac{1}{R_a} + \frac{1}{R_s}), \frac{1}{c_{aa}R_s}, 0, 0, \frac{1}{c_{aa}}, 0, 0$$

, and a row with a valve term, like the fourth row:

$$0, 0, \frac{1}{c_{jug}R_h}, -\frac{1}{c_{jug}}(\frac{1}{R_h} + \frac{1}{R_j}V'(P_4 - P_6)), 0, -\frac{1}{c_{jug}R_j}V'(P_4 - P_6)$$

.

Theorem 2.2. *Given an optimal control u^* and the corresponding state solution, $P^* = P(u^*)$, there exists a solution satisfying the adjoint system:*

$$\begin{aligned} \lambda(n-1) &= \lambda(n) + \Delta t M^{\tau}(n-1)\lambda(n) + (0,0,0,0,1,-1,0) && (2.5)\\ \lambda(N) &= (0,0,0,0,1,-1,0), && (2.6) \end{aligned}$$

where $n = N, \ldots 2$. Furthermore, for $n = 1, 2, \ldots, N-2$,

$$u^*(n) = \frac{1}{B}\left(t_p(\lambda_5(n+1) + \lambda_6(n+1) - \lambda_5(n+2) - \lambda_6(n+2)\right)$$

$$+\lambda_7(n+1) - \lambda_7(n+2)) \tag{2.7}$$

and for $n = 0$,

$$u^*(0) = \frac{1}{B}\left(t_p(\lambda_5(N) + \lambda_6(N) + \lambda_5(1) + \lambda_6(1) - \lambda_5(2) - \lambda_6(2)) \right. \\ \left. +\lambda_7(N) + \lambda_7(1) - \lambda_7(2)\right), \tag{2.8}$$

where the controls are subject to the prescribed bounds, M^τ *is the transpose of the matrix* M*, which depends on the state* P*.*

Proof: Let u^* be an optimal control and P its corresponding state. Let $(u^* + \epsilon l) \in U$ for $\epsilon > 0$, and P^ϵ be the corresponding solution of the state system (1.1)-(1.2). Since the adjoint system is linear, there exists a solution λ satisfying (2.5). We compute the directional derivative of the functional $J(u)$ with respect to u in the direction l. Since $J(u^*)$ is the maximum value, we have

$$\begin{aligned}
0 &\le \lim_{\epsilon \to 0^+} \frac{J(u^* + \epsilon l) - J(u^*)}{\epsilon} \\
&= \sum_{n=1}^{N} [\psi_5(n) - \psi_6(n)] - \sum_{n=0}^{N-2} Bu^*(n) l(n) \\
&= \sum_{n=1}^{N-1} \psi(n) \cdot [\lambda(n) - \lambda(n+1) - \Delta t M^\tau(n)\lambda(n+1)] - \sum_{n=0}^{N-2} Bu^*(n) l(n) \\
&\quad + \psi(N) \cdot \lambda(N) \\
&= \sum_{n=1}^{N-2} \lambda(n+1) \cdot [\psi(n+1) - \psi(n) - \Delta t M(n)\psi(n)] - \sum_{n=0}^{N-2} Bu^*(n) l(n) \\
&\quad + \lambda(N) \cdot [\psi(N) - \psi(N-1) - \Delta t M(N-1)\psi(N-1)] + \lambda(1) \cdot \psi(1) \\
&= \sum_{n=1}^{N-2} \lambda(n+1) \cdot T(l(n) - l(n-1)) + \lambda(1) \cdot \psi(1) - \sum_{n=0}^{N-2} Bu^*(n) l(n) \\
&\quad + \lambda(N) \cdot T(l(0) - l(N-2)) \\
&= \sum_{n=1}^{N-3} l(n)[(\lambda_7 + t_p(\lambda_5 + \lambda_6))(n+1) - (\lambda_7 + t_p(\lambda_5 + \lambda_6))(n+2) - Bu^*(n)] \\
&\quad + l(N-2)[t_p(\lambda_5 + \lambda_6)(N-1) + \lambda_7(N-1) - Bu^*(N-2)] + \lambda(1) \cdot \psi(1) \\
&\quad + \lambda(N) \cdot T(l(0) - l(N-2)) - l(0)[t_p((\lambda_5 + \lambda_6)(2)) + \lambda_7(2) - Bu^*(0)]
\end{aligned}$$

Using the equality $\psi(1) = T(l(0))$, we can group together terms with coefficients $l(0)$. Since $l(0)$ is arbitrary within the constraint that $u^*(0)+\epsilon l(0)$ satisfies the control bounds, we can solve for $u^*(0)$. From the summation above with $n = 1$ to $N-3$, we can solve for $u^*(n)$ and then for $u^*(N-2)$. Note that the controls are subject to the control bounds. The representation (2.7)-(2.8) is obtained by choosing appropriate variations l. $\square$

Thus, the optimal control is completely and explicitly characterized in terms of the solution of the optimality system involving the optimal state and adjoint variables. The solution of the optimality system is carried out iteratively. After an initial control guess, the iterative method uses forward sweeps of the state system followed by backward sweeps of the adjoint system with control updates between. See [11] for similar iteration techniques. The numerical solution yields the optimal control and thereby the strategy for improving the standard CPR technique. The details of the numerical algorithm and results are reported elsewhere. [12] The optimal controls yield explicit patterns for the external chest pressure. Our results indicate that more rapid changes in the external pressure levels than those currently performed within standard CPR may yield up to 40% increase in the systemic perfusion pressure. For many people who undergo cardiac arrest, this may represent the difference between life and death. Eventually, the improved CPR strategy suggested by our work could be implemented either by an emergency helper or by a portable pressure device.

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REFERENCES

1. C. F. Babbs, "CPR Techniques that Combine Chest and Abdominal Compression and Decompression: Hemodynamic Insights from a Spreadsheet Model", Circulation 1999, 2146-2152.
2. X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhauser, 1995
3. M. I. Kamien and N. L. Schwarz, *Dynamic Optimization*, North-Holland, Amsterdam 1991.
4. American Heart Association, "1992 Heart and Stroke Facts," Dallas 1991.
5. R. O. Cummins, M. S. Eisenberg, A. P. Hallstrom, and P. E. Litwin, "Survival of Out-of-Hospital Cardiac Arrest with Early Initiation of Cardiopulmonary Resus-

citation," Amer. J. Emerg. Medicine 3 (1985), 114-118.
6. K. J. Tucker, M . A. Savitt, A. Idris, and R. F. Redberg, "Cardiopulmonary Resuscitation: Historical Perpectives, Physiology, and Future Directions, " Achives of Internal Medicine 154 (1994), 2141-2150.
7. S. A. Meador, "Computer Simulatin of Cardiopulmonary Resuscitation: Computer Analysis of a Simple Electrical Model of the Circulation," Resuscitation 13 (1986) 145-157.
8. L. Fan and C. Wang, *The Discrete Maximum Principle*, John Wiley, 1964.
9. S. Sethi and G. L. Thompson, *Optimal Control Theory: Applications to Management Science and Economics*, Kluwer Academic, 2000.
10. J.- L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York, 1971.
11. E. Jung, S. Lenhart, and Z.Feng, "Optimal Control of Treatments in a Two Strain Tuberculosis Model," Discrete and Continuous Dynamical Systems 2 (2002), 473-482.
12. E. Jung, S. Lenhart, V. Protopopescu, and C. Babbs, "Optimal Control Applied to a CPR DifFerence Equation Model," to be published.