Determining the Potential in the Schrödinger Equation from the Dirichlet to Neumann Map by the Boundary Control Method [1]

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Abstract We consider the problem of identifying the potential in the one-dimensional Schrödinger equation with input Dirichlet data, from measured Neumann data. Knowledge of the Dirichlet to Neumann map together with spectral controllability results for the Schrödinger equation obtained using properties of exponential Riesz bases allow recovery of the spectral data. Once the the spectral data is recovered, we use the Boundary Control method to solve the identification problem.

Key Words: identification, inverse problems, Schrödinger equation, wave equation, controllability, boundary control.

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1. Introduction

The Boundary Control (BC) method was introduced by Belishev [8, 10] as a new method for identification problems. **Let us mention here a method developed by A. S. Blagoveshchenskii for one-dimensional inverse problems for the wave equation (see, e.g.,** [12, 13]) **which was in some sense a predecessor of the BC method.** The BC method, which uses the connection between inverse problems of mathematical physics and functional analysis and control theory for partial differential equations, offers an interesting and powerful alternative to previous identification techniques based on spectral or scattering methods [18, 23, 24] or optimal

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control [7, 20, 21]. The BC method has been successfully applied to treat identification problems for the **wave and heat equations (see, e.g.,** [2, 4, 9, 17]). In [6] we reported the first application of this method to the Schrödinger equation in the one-dimensional case by determining the potential by using the Neumann to Dirichlet (ND) map. We note that, at the same time, this also provided the first use of dynamical boundary data in an inverse problems for the Schrödinger equation, since the traditional Gelfand-Levitan-Marchenko approach reconstructs the potential from either spectral or scattering data [1, 14, 16]. The multidimensional dynamical (ND) inverse problem for the Schrödinger equation has been solved in [3].

In this paper, we present a new quantum application of the BC method and determine the potential in a Schrödinger equation from the Dirichlet to Neumann (DN) map. Using Dirichlet data as opposed to Neumann data in [6] poses an additional technical difficulty, caused by the loss of smoothness. **This in turn requires introducing of bilinear forms instead of operators (Sec. 5)**. First, we recover the spectral data from the DN map using controllability results and properties of exponential Riesz bases (nonharmonic Fourier series), following the ideas from [5]. Second, we recover the potential from an auxiliary wave equation with the same spectral data via the BC method.

The outline of the paper is as follows. In Section 2 we define the Schrödinger problem and the response operator (i.e. the time dependent DN map). In Section 3 we derive a priori regularity results and in Section 4 we prove controllability for the Schrödinger equation. We define the connecting form and express it in terms of the DN map in the fifth section. The connecting form has an obvious interpretation in terms of systems theory, namely it relates the outer space of the controls and the inner space of the states. In Section 6, a variational algorithm is applied to the connecting **and** l^T forms to compute the spectral data for the Schrödinger equation thereby completing the solution of the first inverse sub-problem. In Section 7 we introduce an auxiliary wave equation that has precisely the same spectral data as those computed for the Schrödinger equation. This simply means that the two equations (Schrödinger and wave) share the same time-independent part. Finally, in the last section we use the propagation of singularities for the wave equation and the BC method to find an explicit formula for the sought potential.

2. Statement of the problem

Let T, $\ell > 0$ and let q be a real valued function[2] from $\mathcal{H} := L^2(0, \ell)$. We consider a one-dimensional quantum system described by the Schrödinger equation

$$i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial x^2} - q(x)u(x,t) = 0, \quad x \in (0,\ell)\,,\ t \in (0,T), \tag{1}$$

[2]With some modifications our approach allows to recover complex potentials.

with the boundary conditions

$$u(0,t) = f(t)\,, \quad u(\ell,t) = 0\,, \quad t \in (0,T) \tag{2}$$

and zero initial condition

$$u(x,0) = 0. \tag{3}$$

We denote the solution of this initial boundary value problem by $u^f(x,t)$. We suppose that $f \in \mathcal{F}^T := L^2(0,T)$ and introduce in this space the response operator R^T, defined by:

$$(R^T f)(t) = \frac{\partial u^f(0,t)}{\partial x}. \tag{4}$$

The dynamical inverse problem consists of recovering the unknown potential $q(x)$ from the given operator R^T which is the DN map for the problem (1)–(3).

First we recover the spectral data for the dynamical response operator by using controllability results for the Schrödinger equation. These results are **new: in comparison with those from** [15, 19, 22, 28] **we consider a non–smooth potential** $q \in \mathcal{H}$. For our purposes it is necessary to study controllability for controls from $L^2(0,T)$ and also from $H_0^1(0,T)$.

3. Regularity of solution to the initial boundary value problem

The Fourier method is used to obtain sharp regularity results for the solution of the initial boundary value problem (1)–(3) with $f \in \mathcal{F}^T$ in distributional sense.

We introduce the operator $\mathcal{L}$, defined by

$$(\mathcal{L}\varphi)(x) = -\varphi''(x) + q(x)\varphi(x), \quad x \in (0,\ell), \tag{5}$$

with domain

$$\mathcal{D}(\mathcal{L}) = H^2(0,\ell)$$

and the operator $\mathcal{L}_0$ acting by the same rule (5) with domain

$$\mathcal{D}(\mathcal{L}_0) = \{\varphi \in H^2(0,\ell) : \varphi(0) = \varphi(\ell) = 0\}.$$

The operator $\mathcal{L}_0$ is a self adjoint operator in $\mathcal{H}$ [25, Ch.X]. As an operator with compact resolvent, $\mathcal{L}_0$ has only discrete spectrum. Its eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ satisfy the relations (see e.g. [23])

$$\sqrt{\lambda_n} = \frac{\pi n}{\ell} + o(1), \quad n = 1, 2, \dots, \tag{6}$$

$$0 < \inf_{n\in\mathbb{N}} \frac{|\varphi_n'(0)|}{n} \le \sup_{n\in\mathbb{N}} \frac{|\varphi_n'(0)|}{n} < \infty \tag{7}$$

(we suppose that $\|\varphi_n\|_{\mathcal{H}} = 1$). The set $\{\lambda_n, \chi_n\}_{n\in\mathbb{N}}$ where $\chi_n := \varphi_n'(0)$ represents the spectral data for the problem (1)–(3).

We look for the solution of the problem (1)–(3) in the form

$$u^f(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x). \tag{8}$$

Starting with

$$0 = \int_0^\ell \int_0^t (iu_t^f + u_{xx}^f - qu^f)\varphi_n(x)e^{-i\lambda_n(t-\tau)}\, d\tau\, dx$$

for $f \in C_0^\infty[0,T]$ and using integration by parts, (2), (3), and (8), we obtain

$$a_n(t) = i\chi_n \int_0^t f(\tau)e^{-i\lambda_n(t-\tau)}\, d\tau, \quad n \in \mathbb{N}. \tag{9}$$

These equalities are evidently true for $f \in \mathcal{F}^T$ as well.

In Section 4 we prove using (6) that the family $\mathcal{E} = \{e^{i\lambda_n t}\}$ forms an $\mathcal{L}$-basis, i.e. a Riesz basis in the closure of its linear span in $\mathcal{F}^T$, for any $T > 0$. This means (see e.g. [5, Sec. I.1] for the definition and properties of Riesz bases) that for some positive m and M and any finite sequence $\{\alpha_n\}$ we have

$$m \sum |\alpha_n|^2 \le \left\| \sum \alpha_n e^{i\lambda_n t} \right\|_{\mathcal{F}^T}^2 \le M \sum |\alpha_n|^2 .$$

The $\mathcal{L}$-basis property of the family $\mathcal{E}$ and equality (9) imply that the sequence $\{a_n(t)\chi_n^{-1}\}$ belongs ℓ^2 for any t. Moreover, it can be easily shown (see [5, Sec. III.2] for details) that

$$\sum_{n=1}^{\infty} \left| \frac{a_n(\cdot)}{\chi_n} \right|^2 \in C[0,T] .$$

From (7)–(8) it follows now that

$$u^f \in C([0,T];\mathcal{H}_{-1}) , \ \mathcal{H}_{-1} := H^{-1}(0,\ell) .$$

Moreover, the control operator U^T,

$$U^T : \mathcal{F}^T \mapsto \mathcal{H}_{-1}, \ U^T f = u^f(\cdot,T),$$

is bounded.

Integrating by parts in (9) we see that operator U^T is bounded from $\mathcal{F}_0^T$ to $\mathcal{H}_1$, where $\mathcal{F}_0^T := H_0^1(0,T)$ and $\mathcal{H}_1 := H_0^1(0,\ell)$.

4. Controllability of the Schrödinger equation

Controllability of the system (1)–(3) is the first step in our approach.

Proposition 1. *Let the potential $q \in \mathcal{H}$ be known. Then:*

(i) for any $T > 0$ and any function $z \in \mathcal{H}_{-1}$, there exists a control $f \in \mathcal{F}^T$ such that

$$u^f(x,T) = z(x) \ \ in \ \ \mathcal{H}_{-1}, \tag{10}$$

in other words, $U^T\mathcal{F}^T = \mathcal{H}_{-1}$*;*

(ii) $U^T\mathcal{F}_0^T = \mathcal{H}_1$ *, in particular, for any* $n \in \mathbb{N}$*, there exists a control* $f_n \in \mathcal{F}_0^T := H_0^1(0,T)$ *such that*

$$u^{f_n}(x,T) = \varphi_n(x)\,. \tag{11}$$

Proof. (i) From (8) it follows that for $f \in \mathcal{F}^T$, we have

$$u^f(x,T) = \sum_{n=1}^{\infty} a_n(T)\varphi_n(x),$$

where coefficients a_n are determined by (9). Given an expansion of z in the form

$$z(x) = \sum_{n=1}^{\infty} \hat{\alpha}_n \varphi_n(x)$$

we should choose f such that

$$a_n(T) = i\chi_n \int_0^T f(t)e^{-i\lambda_n(T-t)}\,dt = \hat{\alpha}_n\,. \tag{12}$$

Thus, taking into account (7), we see that (i) is equivalent to the following statement: "For any $T > 0$ and any $\{\alpha_n\} \in \ell^2$, the problem of moments

$$\alpha_n = \int_0^T f(t)e^{i\lambda_n t}\,dt, \quad n \in \mathbb{N}\,, \tag{13}$$

has a solution $f \in \mathcal{F}^T$". By Theorem I.2.1 in [5], the problem of moments is solvable if and only if $\mathcal{E} = \{e^{i\lambda_n t}\}$ is an $\mathcal{L}$-basis in $\mathcal{F}^T$. An $\mathcal{L}$-basis is a Riesz basis in the closure of its linear span. **We show that it can be extended to a Riesz basis, and then, as a subset of a Riesz basis it is, by definition, an $\mathcal{L}$-basis.**

We denote $\Lambda := \{\lambda_n\}$ and

$$n^+(r) := \sup_{x\in\mathbb{R}} \#\{\Lambda \cap [x, x+r)\},$$

where $\#\mathcal{A}$ is the number of elements in the set $\mathcal{A}$, and define in a standard way (see e.g. [11, p. 346]) the *upper uniform density* of Λ to be

$$\mathcal{D}^+(\Lambda) := \lim_{r\to\infty} \frac{n^+(r)}{r}.$$

The limit exists due to the subadditivity of $n^+(r)$.

It is easy to prove, using (6), that $\mathcal{D}^+(\Lambda) = 0$. Indeed, we use the fact that $n^+(k^2) = O(k)$ and $n^+(k^2)/k^2$ tends to zero as k tends to infinity. Proposition 1(i) follows now from the result of Seip [26, Theorem 2.3]:

For any $T > 2\pi\mathcal{D}^+(\Lambda)$ *the family* $\mathcal{E}$ *can be extended to a family of exponentials which forms a Riesz basis in* $\mathcal{F}^T$*.*

(ii) Let $f \in \mathcal{F}_0^T$. Denote $f'(t) = g(t)$, integrate by parts in (12) and recall that $f(0) = f(T) = 0$. We get

$$a_n(T) = -\frac{\chi_n}{\lambda_n} \int_0^T g(t) e^{-i\lambda_n (T-t)}\, dt \quad \text{for} \quad \lambda_n \neq 0\,, \tag{14}$$

$$a_n(T) = i\chi_n \int_0^T g(t)\,(T-t)\, dt \quad \text{for} \quad \lambda_n = 0\,. \tag{15}$$

Taking into account (6) and (7) we see that (ii) is equivalent to solvability of the problem of moments

$$\alpha_n = \int_0^T g(t) e^{i\lambda_n t}\, dt, \quad n \in \mathbb{N}\,, \lambda_n \neq 0\,, \tag{16}$$

$$\alpha_n = \int_0^T g(t)\, t\, dt \quad \text{for} \quad \lambda_n = 0\,. \tag{17}$$

$$\int_0^T g(t)\, dt \; = 0. \tag{18}$$

This means that for any $\{\alpha_n\} \in \ell^2$ there exists $g \in \mathcal{F}^T$ satisfying the equations (16), (17), (18). This is equivalent to the fact that the family $\mathcal{E} \bigcup \{1\}$ (in the case $\lambda_n \neq 0\ \forall n \in \mathbb{N}$) or the family $\mathcal{E} \bigcup \{t\}$ (in the case $\lambda_n = 0$ for some $n \in \mathbb{N}$) forms an $\mathcal{L}$-basis in $\mathcal{F}^T$. This statement is true and can be proved exactly in the same way as we proved the $\mathcal{L}$-basis property of $\mathcal{E}$ in (i).

5. Forms c^T and l^T

In this section we introduce two bilinear forms which play a central role in our approach to the inverse problem. The form c^T connects the (metrics of the) control space $\mathcal{F}^T$ and space of solutions $\mathcal{H}$. **We note that in** [6] **the connection between the spaces of controls and solutions was realized by a connecting operator. We cannot apply the same approach here, due to the lack of smoothness of the solutions**. The form l^T is a bilinear form corresponding to the operator $\mathcal{L}_0$. We prove the very important fact that the value of the both forms can be explicitly expressed in terms of the response operator, R^T.

In the language of system theory, $\mathcal{F}^T$ is the outer space of the system (1)–(3), $\mathcal{H}$ is the inner space, and R^T is the input–output map. We shall define the form c^T in the outer space through the bilinear form of the elements of the inner space, $\mathcal{H}$; however, its action can be expressed via the action of the operator R^T on the outer space, $\mathcal{F}^T$.

More exactly, we define the form c^T in $\mathcal{F}^T$ with the domain $\mathcal{F}_0^T \times \mathcal{F}_0^T$ by the equality

$$c^T[f,g] := (u^f(\cdot,T), u^g(\cdot,T))_{\mathcal{H}}\,, \quad f,g \in \mathcal{F}_0^T\,. \tag{19}$$

Here u^f and u^g are solutions of (1)–(3) corresponding to the boundary controls f and g, respectively. Since $u^f(\cdot,T)=U^Tf$, we can also write c^T in the form:

$$c^T[f,g]=(U^Tf,U^Tg)_{\mathcal{H}}\,,\quad f,g\in\mathcal{F}_0^T\,. \tag{20}$$

To obtain an expression for c^T through R^T we use the equations (1)–(4) for $f,g\in C_0^\infty[0,T]$ and write:

$$\begin{aligned}
0 &= \int_0^\ell\int_0^T\left[\Big(iu_t^f(x,t)+u_{xx}^f(x,t)-q(x)u^f(x,t)\Big)\,\overline{u^g(x,t)}\right]dx\,dt\\
&= \int_0^\ell\int_0^T\left[u^f(x,t)\left(\overline{iu_t^g(x,t)+u_{xx}^g(x,t)-q(x)u^g(x,t)}\right)\right]dx\,dt\\
&+\ i\int_0^\ell\Big[u^f(x,t)\overline{u^g(x,t)}\Big]_{t=0}^{t=T}dx\\
&+\ \int_0^T\Big[u_x^f(x,t)\overline{u^g(x,t)}-u^f(x,t)\overline{u_x^g(x,t)}\Big]_{x=0}^{x=\ell}dt.
\end{aligned} \tag{21}$$

Finally, we have

$$-i\left(u^f(\cdot,T),u^g(\cdot,T)\right)_{\mathcal{H}}=\left(R^Tf,g\right)_{\mathcal{F}^T}-\left(f,R^Tg\right)_{\mathcal{F}^T}\,. \tag{22}$$

Comparing (19) and (22) and using a standard density argument, we get:

$$c^T[f,g]=i[\left(R^Tf,g\right)_{\mathcal{F}^T}-\left(f,R^Tg\right)_{\mathcal{F}^T}]. \tag{23}$$

Formula (23) shows that the form c^T is symmetric,

$$c^T[f,g]=\overline{c^T[g,f]}$$

which is consistent with the representations (22) and (20).

Let us define the form $l^T:=\mathcal{F}^T\times\mathcal{F}^T\mapsto\mathbb{C}$, $\operatorname{Dom}l^T=\mathcal{F}_0^T\times\mathcal{F}_0^T$,

$$l^T[f,g]=\int_0^\ell[u_x^f(\cdot,T)\,\overline{u_x^g(\cdot,T)}+q\,u^f(\cdot,T)\overline{u^g(\cdot,T)}]\,dx \tag{24}$$

and show that its values are determined by the response operator.

Proposition 2. *The following representations are valid:*

$$l^T[f,g]=c^T[if_t,g]=\left(R^Tf,g_t\right)_{\mathcal{F}^T}+\left(f_t,R^Tg\right)_{\mathcal{F}^T}\,. \tag{25}$$

Substituting $f,g\in C_0^\infty[0,T]$ into (24) and integrating by parts we have

$$\begin{aligned}
l^T[f,g] &= \int_0^\ell\left[\left(-u_{xx}^f(x,T)+q(x)u^f(x,T)\right)\overline{u^g(x,T)}\right]dx\\
&= \int_0^\ell iu_t^f(x,T)\overline{u^g(x,T)}\,dx\\
&= \int_0^\ell u^{if_t}(x,T)\overline{u^g(x,T)}\,dx=c^T[if_t,g]
\end{aligned}$$

$$= i\left[\left(R^T(if_t), g\right)_{\mathcal{F}^T} - \left(if_t, R^T g\right)_{\mathcal{F}^T}\right] . \tag{26}$$

Since the coefficients of the system (1)–(2) are time independent, the operator R^T commutes with $\frac{d}{dt}$. Using this fact and integrating by parts in (26), we obtain

$$l^T[f,g] = i\left[\left(R^T f, ig_t\right)_{\mathcal{F}^T} - \left(if_t, R^T g\right)_{\mathcal{F}^T}\right] ,$$

which implies (25).

6. Variational principle

In this we shall recover the spectral inverse data $(\{\lambda_n, \chi_n\}_{n\in\mathbb{N}})$ from the dynamical inverse data (the operator R^T) using the forms c^T and l^T and the controllability of the system (1)–(3).

In general, the spectral data of the operator $\mathcal{L}_0$ may be obtained by applying the well–known variational principle:

$$\lambda_1 = \min_{\varphi\in\mathcal{H}_1,\, \|\varphi\|^2_{\mathcal{H}}=1} \int_0^\ell \left[|\varphi'(x)|^2 + q(x)|\varphi(x)|^2\right] dx ,$$

$$\varphi_1 : \int_0^\ell \left[|\varphi_1'(x)|^2 + q(x)|\varphi_1(x)|^2\right] dx = \lambda_1 ,$$

$$\lambda_n = \min_{\varphi\in\mathcal{H}_1,\, \|\varphi\|^2_{\mathcal{H}}=1,\, (\varphi,\varphi_j)_{\mathcal{H}}=0,\, j=1,\dots,n-1} \int_0^\ell \left[|\varphi'(x)|^2 + q(x)|\varphi(x)|^2\right] dx ,$$

$$\varphi_n : \int_0^\ell \left[|\varphi_n'(x)|^2 + q(x)|\varphi_n(x)|^2\right] dx = \lambda_n ,\ n\in\mathbb{N} .$$

We note that these eigenfunctions are uniquely determined if χ_n are positive. Indeed, since the eigenvalues are simple, the eigenfunctions are uniquely determined by the normalization constant and the sign of χ_n. We will choose all χ_n to be positive.

An alternative way to recover the spectral data is based on using the forms l^T and c^T. **From Proposition 1 (ii) it follows** that given an eigenfunction ϕ and corresponding eigenvalue λ of the operator $\mathcal{L}_0$, one can find $f\in\mathcal{F}_0^T$ such that $\phi = U^T f$ and $l^T[f,g] = \lambda(\phi,\psi)_{\mathcal{H}} = \lambda c^T[f,g]$, where $U^T g = \psi, g\in\mathcal{F}_0^T$. Then we can realize the variational principle using the forms c^T and l^T :

$$\lambda_1 = \min\ l^T[f,f] ,$$

where the minimum is taken over

$$f\in\mathcal{F}_0^T, \quad c^T[f,f] = 1$$

and

$$f_1 : l^T[f_1,f_1] = \lambda_1 .$$

Further,

$$\lambda_n = \min\ l^T[f,f]\,,$$

where the minimum is taken over the functions f that satisfy:

$$f \in \mathcal{F}_0^T, \quad c^T[f,f]=1, \quad c^T[f,f_j]=0, \quad j=1,\dots,n-1\,,$$

and

$$f_n:\ l^T[f_n,f_n]=\lambda_n\,,\ n\in\mathbb{N}.$$

Thus we find λ_n and f_n.

By the definitions of R^T and f_n we have

$$(R^T f_n)(T) = u_x^{f_n}(0,T) = \varphi_n'(0) = \chi_n\,. \tag{27}$$

These relations allow us to find χ_n from f_n, and so we can recover the complete spectral data, $\{\lambda_n,\chi_n\}_{n\in\mathbb{N}}$.

From the spectral data, $\{\lambda_n,\chi_n\}_{n\in\mathbb{N}}$, (or their approximation, $\{\hat{\lambda_n},\hat{\chi_n}\}_{n\in\mathbb{N}}$), one could recover the potential by using the Gelfand–Levitan–Marchenko [1] and Krein [16] methods. Instead, we choose to solve the inverse problem using the extension of the BC method. Besides offering yet another alternative to identification methods based on control and optimization [7, 20, 21], the BC formalism is entirely linear and is essentially independent on dimensionality.

7. Wave equation: regularity and controllability

The application of the BC method to solve the dynamical inverse problem for the system (1)–(3) relies on regularity and controllability results for a closely related boundary value problem for the wave equation, which we briefly discuss in the next section. These results are new since they are obtained for nonsmooth coefficients.

We consider the related initial boundary value problem for wave equation:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} + q(x)w(x,t) = 0, \quad x\in(0,\ell),\ t\in(0,T), \tag{28}$$

with the boundary conditions

$$w(0,t)=f(t), \quad w(\ell,t)=0,\ t\in(0,T), \tag{29}$$

and zero initial conditions

$$w(x,0)=\frac{\partial w}{\partial t}(x,0)=0. \tag{30}$$

We denote the solution of (28)–(30) by w^f. Here and below we assume all functions are real.

To represent the solution w^f and study its properties, we consider the Goursat problem [27]:

$$k_{tt}-k_{xx}+q(x)k=0, \quad 0<x<t<T\,, \tag{31}$$

$$k(0,t)=0\,,\quad k(x,x)=-\frac{1}{2}\int_0^x q(s)\,ds\,. \tag{32}$$

Using the standard successive approximation method, one can prove the following:

Proposition 3. *For $q\in\mathcal{H}$ problem (31), (32) has a unique solution such that $k\in H^1(\Omega^T)$, $\Omega^T:=\{(x,t):0<x<t<T\le\ell\}$, $k_{tt}-k_{xx}\in L^2(\Omega^T)$ and equation (31) is satisfied almost everywhere.*

Similar proposition is proved e.g. in [27, Sec. II.4] for smooth q, but the method works as well for $q\in L^2(0,\ell)$.

The following proposition can be checked by direct calculations.

Proposition 4. *(i) If $f\in C^2[0,T]$ and $f(0)=f'(0)=0$, the problem (28)–(30) has the unique solution $w=w^f(x,t)$,*

$$w^f(x,t)=\begin{cases} f(t-x)+\int_x^t k(x,s)f(t-s)\,ds & \text{for } x<t\\ 0 & \text{for } x\ge t\,,\end{cases} \tag{33}$$

$w^f\in H^2(Q^T)$, $Q^T:=(0,\ell)\times(0,T)$, equation (28) is satisfied almost everywhere, and the boundary and initial conditions are satisfied in a classical sense.

(ii) For $f\in\mathcal{F}^T$ the function $w^f(x,t)$ defined by (33) gives a generalized solution of problem (28)–(30) such that $w^f\in C([0,T];L^2(0,\ell))$.

Let

$$\mathcal{H}^T:=\{f\in\mathcal{H}:\ \operatorname{supp} f\subset[0,T]\,\}.$$

From Proposition 3 it also follows that the control operator $\mathcal{W}^T$,

$$\mathcal{W}^T:\mathcal{F}^T\mapsto\mathcal{H}^T,\ \mathcal{W}^Tf=w^f(\cdot,T),$$

is bounded.

We can prove a controllability result which shows that the operator $\mathcal{W}^T$ is boundedly invertible:

Proposition 5. *When the potential $q\in\mathcal{H}$ is known and $T\in(0,\ell]$, then for any function $z\in\mathcal{H}^T$, there exists a unique control $f\in\mathcal{F}^T$ such that*

$$w^f(x,T)=z(x)\ \text{ in } \mathcal{H}^T. \tag{34}$$

Proof. According to (33), condition (34) is equivalent to the equation

$$z(x)=-f(T-x)+\int_x^T k(x,s)f(T-s)\,ds\ \ x\in(0,T)\,. \tag{35}$$

This is the Volterra equation of the second kind with respect to f. Its solvability proves Proposition 5.

8. Solution of the spectral inverse problem

Now we show how to recover the potential function $q(x)$ from the known spectral data, $\{\lambda_n, \chi_n\}_{n\in\mathbb{N}}$.

Introduce a connecting operator $\mathcal{C}^T : \mathcal{F}^T \to \mathcal{F}^T$ for the wave equation (28)–(30):

$$\left(\mathcal{C}^T f, g\right)_{\mathcal{F}^T} = \left(w^f(\cdot, T), w^g(\cdot, T)\right)_{\mathcal{H}}.$$

The operator $\mathcal{C}^T$ is bounded and boundedly invertible, since $\mathcal{C}^T = \left(\mathcal{W}^T\right)^* \mathcal{W}^T$. Note that this definition of the connecting operator differs from the one we used in a previous paper [6] which included the inner product of w_t instead of w. Indeed, the definition involving w_t is more convenient in problems with Neumann spectral data and ND maps. The definition involving w is more convenient in problems with Dirichlet data and DN maps.

The solution $w^f(x,t)$ admits the (Fourier series) representation

$$w^f(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x),$$

wherefrom we obtain the equalities

$$b_n(t) = -\chi_n \int_0^t f(\tau) \frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\, d\tau, \quad n \in \mathbb{N}. \tag{36}$$

From (36) we compute

$$\begin{aligned} \left(\mathcal{C}^T f, g\right)_{\mathcal{F}^T} &= \sum_{n=1}^{\infty} \chi_n^2 \int_0^T f(t) \frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\, d\tau \cdot \\ &\quad \cdot \int_0^T g(s) \frac{\sin\sqrt{\lambda_n}(T-s)}{\sqrt{\lambda_n}}\, ds, \end{aligned} \tag{37}$$

which shows that the spectral data completely determine the operator $\mathcal{C}^T$.

Let $y(x)$ be a solution to the boundary value problem

$$y''(x) - q(x)y(x) = 0,\ y(0) = 0,\ y'(0) = 1, x \in (0,\, \ell) \tag{38}$$

and let us find a control $p^T \in \mathcal{F}^T$ such that

$$w^{p^T}(x,T) = \begin{cases} y(x), & x \le T, \\ 0, & x > T. \end{cases} \tag{39}$$

Note that, since $q(x)$ is unknown, both $y(x)$ and $p^T(t)$ are unknown functions at this point. For any $g \in C_0^\infty[0,T]$, we have

$$\begin{aligned}
(\mathcal{C}^T p^T, g)_{\mathcal{F}^T} &= \Big(w^{p^T}(\cdot,T), w^g(\cdot,T)\Big)_{\mathcal{H}} \\
&= \int_0^T y(x) w^g(x,T)\,dx \\
&= \int_0^T (T-t)\,dt \int_0^T y(x) w^g_{tt}(x,t)\,dx \\
&= \int_0^T (T-t)\,dt \int_0^T y(x)\left[w^g_{xx}(x,t) - q(x)w^g(x,t)\right]\,dx \\
&= \int_0^T (T-t)[(y(x)w^g_x(x,t) - y'(x)w^g(x,t)]|_{x=0}^T\,dt \\
&= \int_0^T (T-t)g(t)\,dt
\end{aligned}$$

(we used that for $g \in C_0^\infty[0,T]$, the function w^g and its derivatives are equal to zero at $x=T$). Hence the function p^T satisfies the equation

$$\left(\mathcal{C}^T p^T\right)(t) = T - t, \quad t \in [0,T].$$

Since $\mathcal{C}^T$ is boundedly invertible this equation has a unique solution, $p^T \in \mathcal{F}^T$, for any $T \le \ell$.

Moreover, it can be proved that $p^T \in H^1(0,T)$. Indeed, the right hand side of (35) gives the representation of $(\mathcal{W}^T f)(x)$. Therefore, the operator $\mathcal{C}^T$, $\mathcal{C}^T = \left(\mathcal{W}^T\right)^* \mathcal{W}^T$, can be represented in the form

$$\mathcal{C}^T = I^T + \mathcal{C}_0^T\,,$$

where I^T is the identity operator in $\mathcal{F}^T$ and $\mathcal{C}_0^T$ is bounded from $\mathcal{F}^T$ to $H^1(0,T)$. Hence,

$$p^T = -\mathcal{C}_0^T p^T + T - t \in H^1(0,T)\,.$$

Formula (33) implies that solution of the initial boundary value problem (28)–(30) satisfies the relation

$$w^f(x,t) = -f(t-x) + \int_x^t k(x,s) f(t-s)\,ds \ \text{ for } x < t\,.$$

Therefore,

$$w^f(t-0,t) = -f(+0)$$

provided the limit $f(+0)$ exists. Applying the propagation of singularities property for $f = p^T$, we obtain

$$w^{p^T}(T-0,T) = -p^T(+0).$$

Denote $p^T(+0)$ by $\mu(T)$. From (39), $w^{p^T}(T-0,T) = y(T)$, thus $\mu(T)$ is twice differentiable with respect to T, and (38) gives

$$q(T) = \frac{y''(T)}{y(T)} = \frac{\mu''(T)}{\mu(T)}.$$

By varying T in $(0,\ell)$, we obtain $q(\cdot)$ in that interval. Since the function $y(T)$ may have only a finite number of zeroes in $(0,\ell)$, this completes the solution of the identification problem.

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