ENO type stencil choosing for one-sided post-processing for the discontinuous Galerkin method

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Outline

- Review
  - Discontinuous Galerkin approximation
  - Symmetric and one-sided post-processing for linear hyperbolic equations
- ENO-type post-processing stencil choosing techniques
- Numerical Examples
- Summary

Discontinuous Galerkin Approximation

$$u_t + (a(x)u)_x = 0$$

Numerical scheme:

$$\int_{I_i} u_t v dx = (\hat{au})_{i-1/2} v^+_{i-1/2} - (\hat{au})_{i+1/2} v^-_{i+1/2} - \int_{I_i} a u v_x dx$$

where

$$v \in V_h = \mathrm{span}\{1, \xi_i, \xi_i^2, \cdots, \xi_i^k, i = 1, \cdots, N\}$$

where $\xi_i = \frac{x - x_i}{\triangle x_i}$ on $I_i = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2})$

$$u_h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) \xi_i^l \quad \text{if} \quad x \in I_i$$

Background: Post-Processor

- Bramble and Schatz, **Higher order local accuracy by averaging in the finite element method**, *Mathematics of Computation,* 31 (1977), 94-111.
- Mock and Lax, **The computation of discontinuous solutions of linear hyperbolic equations**, *Communications on Pure and Applied Mathematics,* 31 (1978), 423-430.
- Cockburn, Luskin, Shu, and Süli, **Enhanced accuracy by post-processing for finite element methods for hyperbolic equations,** *Mathematics of Computation,* 72 (2003), pp. 577-606.
- Ryan, Shu, and Atkins, **Extension of a post-processing technique for the discontinuous Galerkin method for hyperbolic equations with applications to an aeroacoustic problem,** *SIAM Journal on Scientific Computing,* to appear.

Post-Processed Solution

Using Symmetric Kernel

Post-processed solution: $u^*(x) = K_h^{2(k+1),k+1} * u_h$.

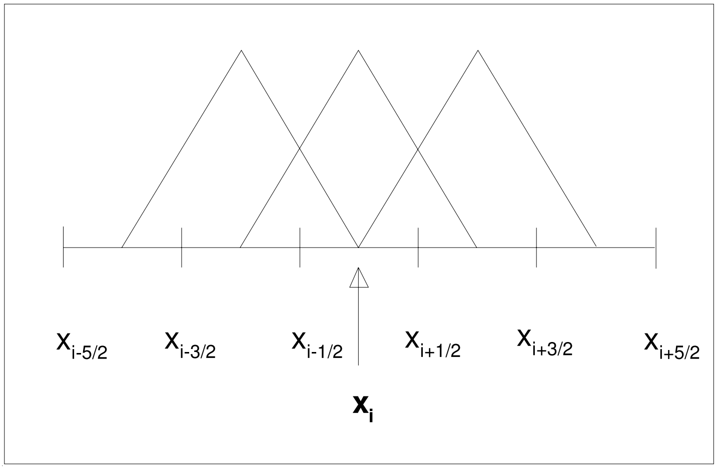
$$K_h^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \psi^{(k+1)}\left(\frac{x}{h} - \gamma\right)$$

$h = \triangle x_i$ for all i, and $c_\gamma^{2(k+1),k+1} \in \mathbb{R}$.

$\psi^{(0)} = \delta_0$, $\psi^{(n)} = \psi^{(n-1)} * \chi$ for $n \geq 2$, where

$$\chi(x) = \begin{cases} 1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{else.} \end{cases}$$

Symmetric Post-Processor



$$u^*(x) = \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)} \, C(j,l,k,x)$$

where $k' = \lceil \frac{3k+1}{2} \rceil$.

$$u^*(x) \in \mathbb{P}^{2k+1}$$

$$K_h^{4,2}(y) = \frac{1}{h}\left(\frac{-1}{12}\psi^{(2)}(\frac{y}{h}-1) + \frac{7}{6}\psi^{(2)}(\frac{y}{h}) - \frac{1}{12}\psi^{(2)}(\frac{y}{h}+1) \right)$$

$$C(j,l,k,x) = \frac{1}{h}\sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \int_{I_{i+j}} \psi^{(k+1)}\left(\frac{y-x}{h} - \gamma\right)\left(\frac{y-x_{i+j}}{h}\right)^l \, dy$$

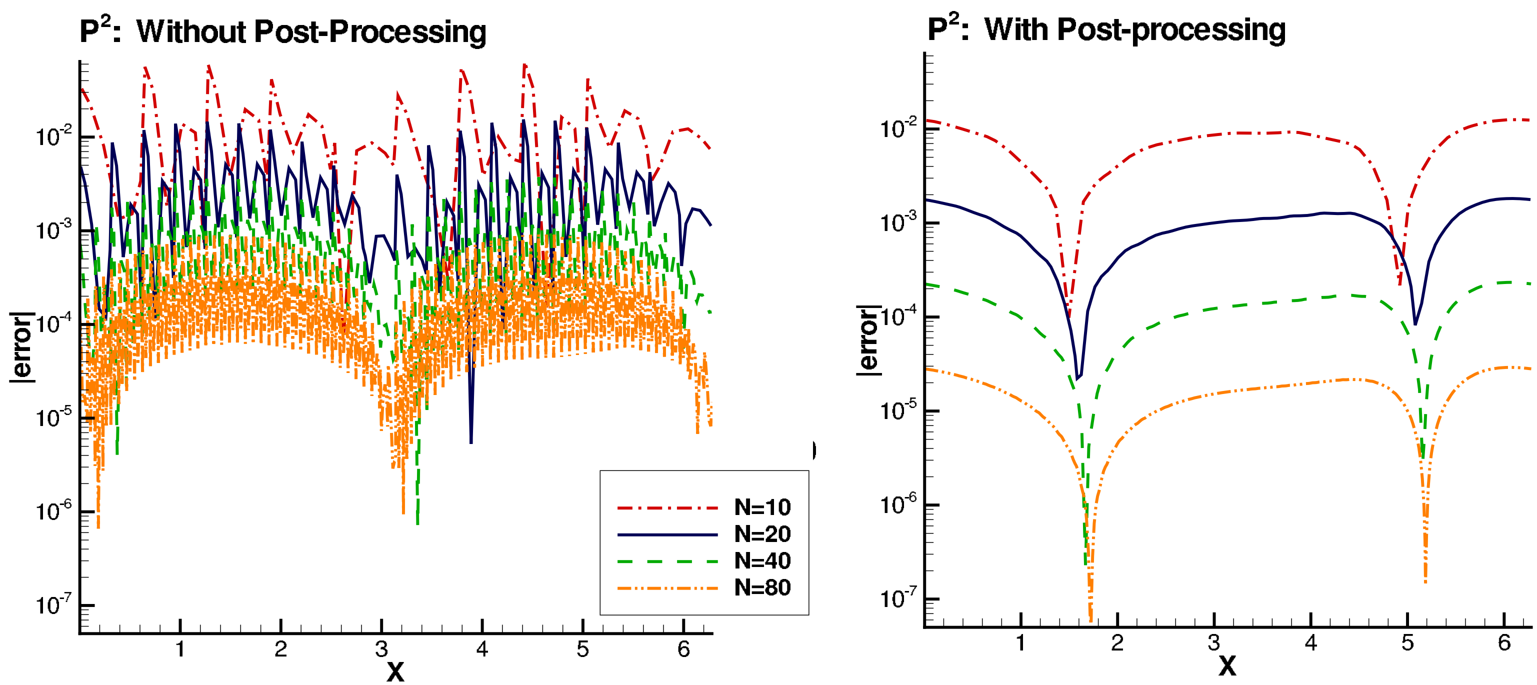
$1-D$ Variable Coefficient Equation

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.83E-02	—	7.82E-02	—
20	4.35E-03	2.07	1.08E-03	2.86
40	1.07E-03	2.03	1.39E-04	2.96
80	2.66E-04	2.01	1.75E-05	2.99
	$\mathbb{P}^2$			
10	8.61E-04	—	1.34E-04	—
20	1.07E-04	3.01	2.34E-06	5.84
40	1.34E-05	3.00	4.55E-08	5.69
80	1.67E-06	3.00	1.09E-09	5.38

$$u_t + (au)_x = f$$

$$a(x) = 2 + \sin(x)$$
$$u(x,0) = \sin(3x)$$
$$u(0,t) = u(2\pi, t)$$
$$T = 12.5$$

1-D Variable Coefficient



$u_t + (a(x)u)_x = f(x,t)$

$a(x) = 2 + \sin(x)$

$u(x,0) = \sin(3x)$

x in (0,2pi), T=12.5

Derivatives

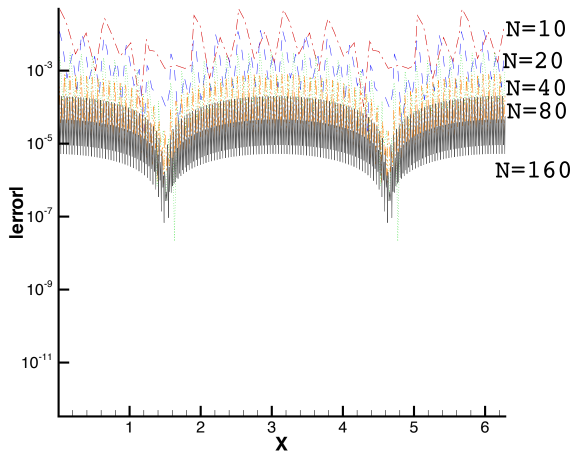
	Approximation		Post-Processed	
N	L^2 error	order	L^2 error	order
	Errors in First Derivative for $\mathbb{P}^2$			
20	3.48E-03	—	6.24E-06	—
40	8.72E-04	2.00	1.61E-07	5.28
80	2.18E-04	2.00	4.51E-09	5.16
160	5.45E-05	2.00	1.39E-10	5.02
	Errors in Second Derivative for $\mathbb{P}^2$			
20	6.78E-02	—	3.64E-05	—
40	3.39E-02	1.00	2.15E-06	4.08
80	1.70E-02	1.00	1.32E-07	4.03
160	8.48E-03	1.00	8.19E-09	4.01

$\frac{du^*(x)}{dx} \in \mathbb{P}^{2k}$

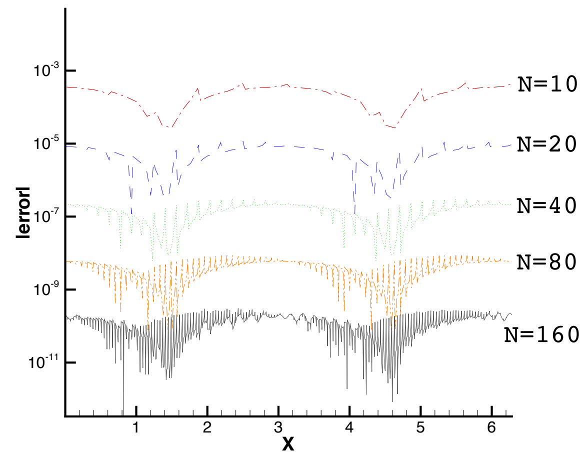
$$u_t + u_x = 0$$
$$u(x,0) = \sin(x)$$
$$x \in (0, 2\pi)$$
$$T = 12.5$$

Derivatives of Post-Processed Solution

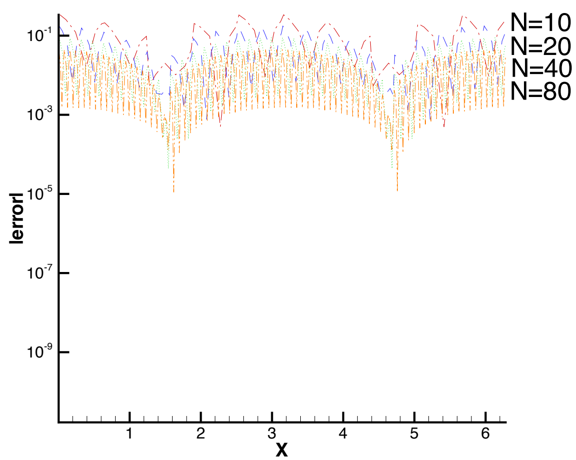
P^2: $|d/dx(u-u_h)|$, Before Post-Processing



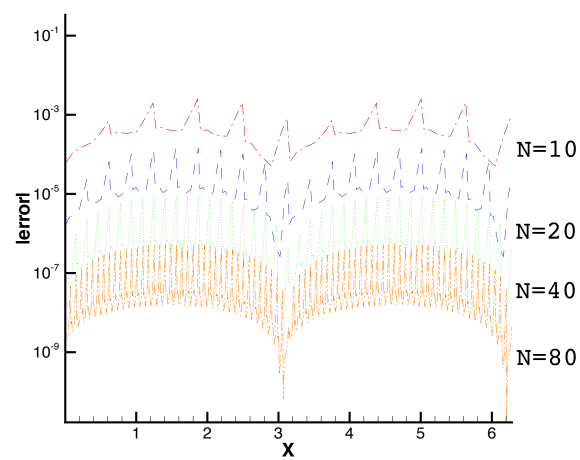
P^2: $|d/dx(u-u_h)|$, After Post-Processing



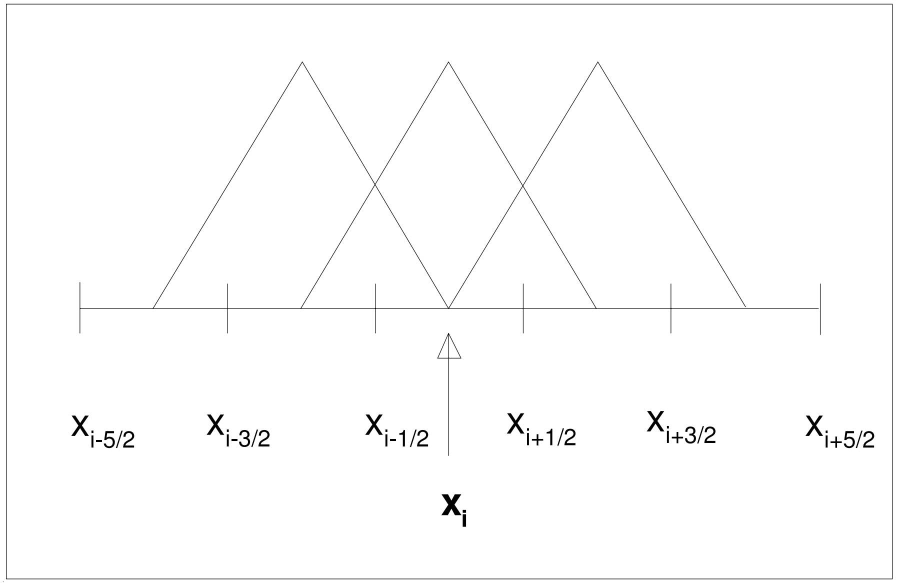
P^2: $|d^2/dx^2(u-u_h)|$, Before Post-Processing



P^2: $|d^2/dx^2(u-u_h)|$, After Post-Processing



Symmetric Post-Processor



$$u^*(x) = \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x), \quad k' = \lceil \frac{3k+1}{2} \rceil$$

$$C(j,l,k,x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}\,(\eta)\,(\xi_i+\eta+\gamma-j)^l\,dy$$

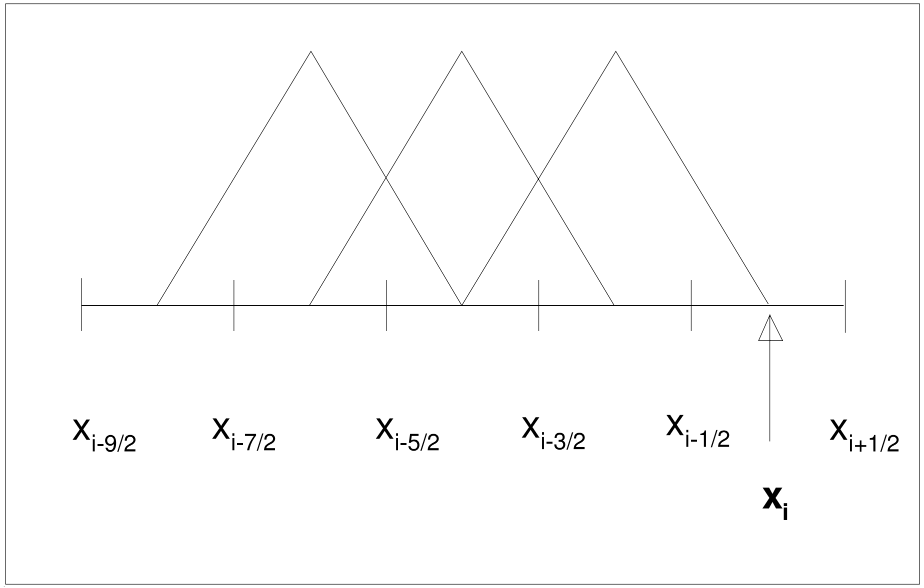
One-Sided Post-Processing

Implemented in the following regions:

- Change in mesh size
- Computational boundary
- Discontinuity

* Ryan, Shu, **One-sided post-processing for the discontinuous Galerkin method,** *Methods and Applications of Analysis,* 10(2003), 295-307.

Left Post-Processor

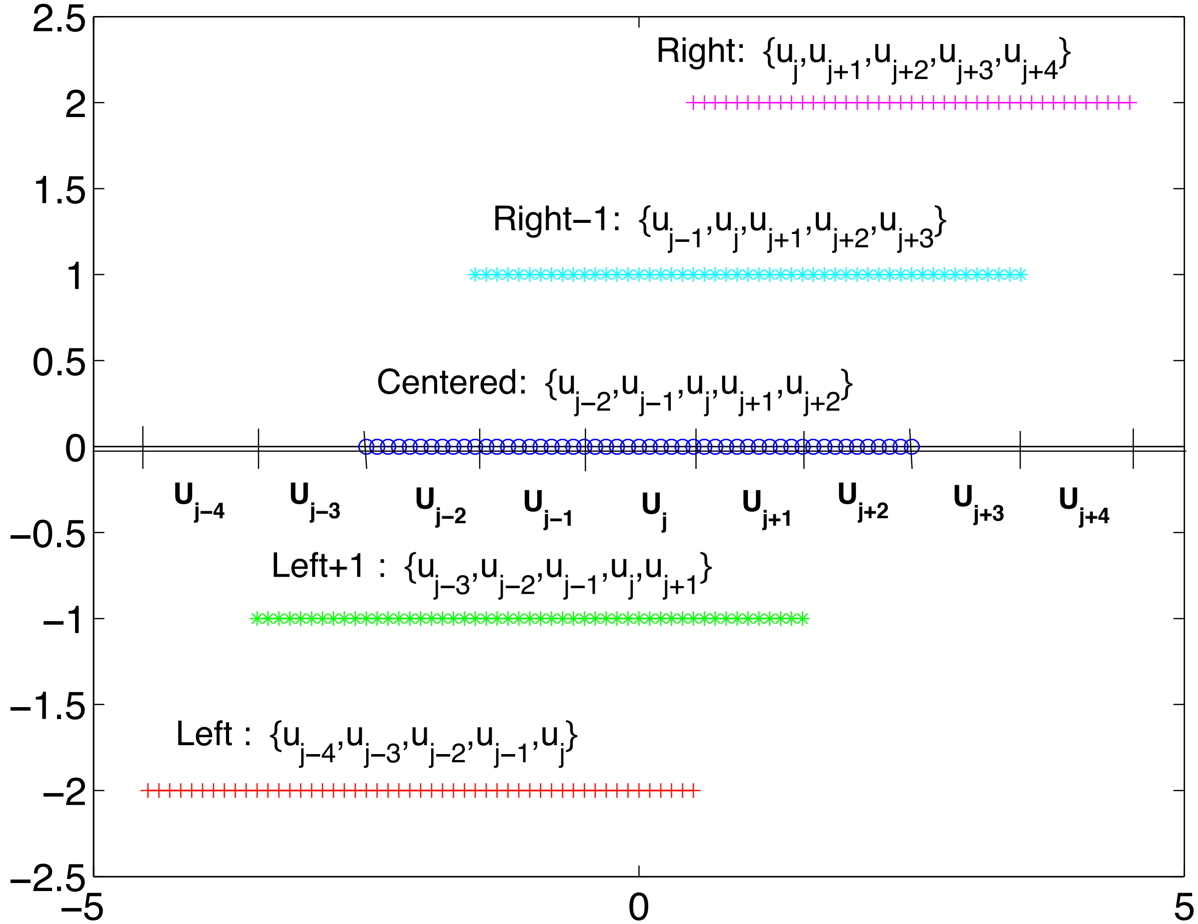


$$u^*(x) = \sum_{j=-2k'}^{0} \sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x)$$

$$C(j,l,k,x) = \frac{1}{h} \sum_{\gamma=-2k-1}^{-k} c_{\gamma}^{2(k+1),k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}\,(\eta)\,(\xi_i+\eta+\gamma-j)^l\,dy$$

Stencil Choices

For $\mathbb{P}^1$, 5 candidate stencils:



Choosing the Post-Processing Stencil

Two Approaches

- Using Essentially Non-Oscillatory (ENO) type method: Smoothness of candidate post-processing stencils.

- Edge Detection method: Finds shock location based on numerical solution.

ENO Type Stencil Choosing

- Calculate undivided differences at downwind point:

$$D_j^1 = |u_{j+1}(x_{j-1/2}) - u_j(x_{j-1/2})|,$$
$$D_j^{r+1} = |D_{j+1}^r - D_j^r|, \ r = 1, \cdots, 3$$

- Find $\max_{j-4\le i\le j+3} D_i^1, \cdots, \max_{j-4\le i\le j} D_i^4$.

ENO Type Stencil Choosing, $\mathbb{P}^1$

$$
\begin{tikzcd}
D^4_{j-4} \arrow[d] \arrow[dr] & D^4_{j-3} \arrow[d] \arrow[dr] & D^4_{j-2} \arrow[d] \arrow[dr] & D^4_{j-1} \arrow[d] \arrow[dr] & D^4_{j} \arrow[d] \arrow[dr] & & & \\
D^3_{j-4} \arrow[d] \arrow[dr] & D^3_{j-3} \arrow[d] \arrow[dr] & D^3_{j-2} \arrow[d] \arrow[dr] & D^3_{j-1} \arrow[d] \arrow[dr] & D^3_{j} \arrow[d] \arrow[dr] & D^3_{j+1} \arrow[d] \arrow[dr] & & \\
D^2_{j-4} \arrow[d] \arrow[dr] & D^2_{j-3} \arrow[d] \arrow[dr] & D^2_{j-2} \arrow[d] \arrow[dr] & D^2_{j-1} \arrow[d] \arrow[dr] & D^2_{j} \arrow[d] \arrow[dr] & D^2_{j+1} \arrow[d] \arrow[dr] & D^2_{j+2} \arrow[d] \arrow[dr] & \\
D^1_{j-4} \arrow[d] & D^1_{j-3} \arrow[d] & D^1_{j-2} \arrow[d] & D^1_{j-1} \arrow[d] & D^1_{j} \arrow[d] & D^1_{j+1} \arrow[d] & D^1_{j+2} \arrow[d] & D^1_{j+3} \arrow[d] \\
\text{Center} & \text{Center} & \text{Left+1} & \text{Left} & \text{Right} & \text{Right-1} & \text{Center} & \text{Center}
\end{tikzcd}
$$

$1-D$ Discontinuous Coefficient Equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0$$

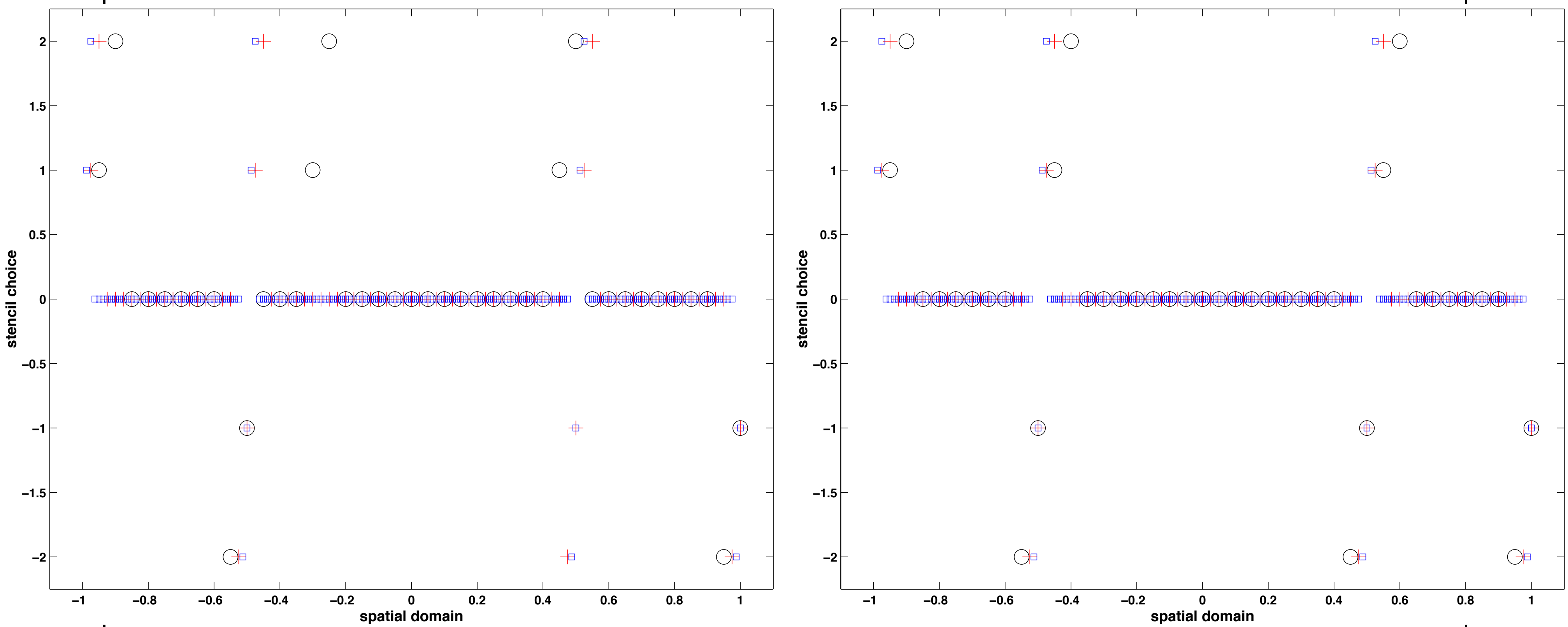
$$a(x) = \begin{cases} 1, & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ \frac{1}{2}, & x \in (-\frac{1}{2},\frac{1}{2}) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(2\pi x), & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ -2\pi\cos(4\pi x) & x \in (-\frac{1}{2},\frac{1}{2}). \end{cases}$$

$$u(-1,t) = u(1,t), \quad T = 12.5$$

Case of two stationary shocks.

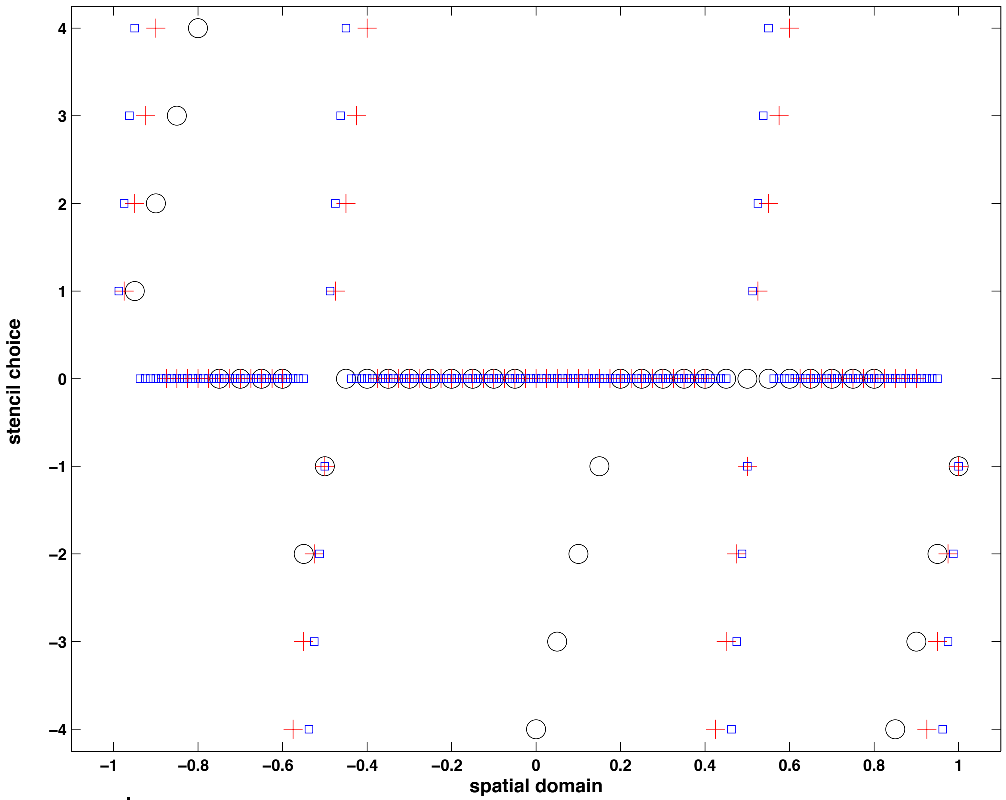
$\mathbb{P}^1$ Stencil Choices: Two stationary shocks

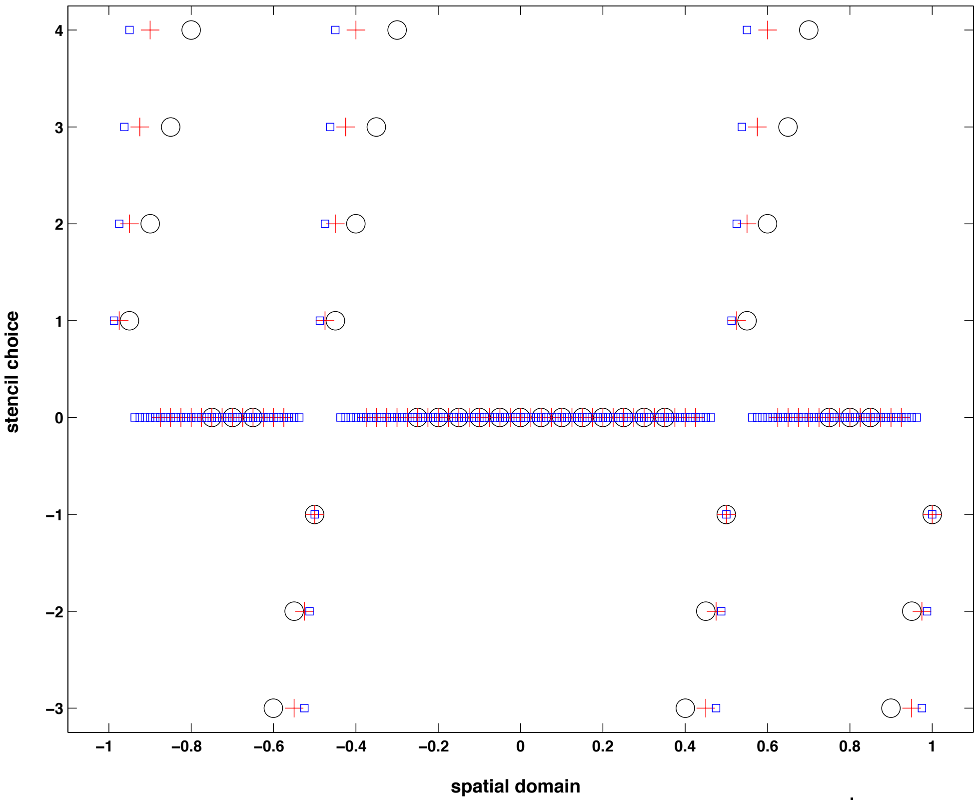


Two stationary shocks problem for $k = 1$. Stencil choices for the ENO method (left) and LED method (right).

$$N = 40 : +; \quad N = 80 : \bullet; \quad N = 160 : \square$$

$\mathbb{P}^2$ Stencil Choices: Two stationary shocks

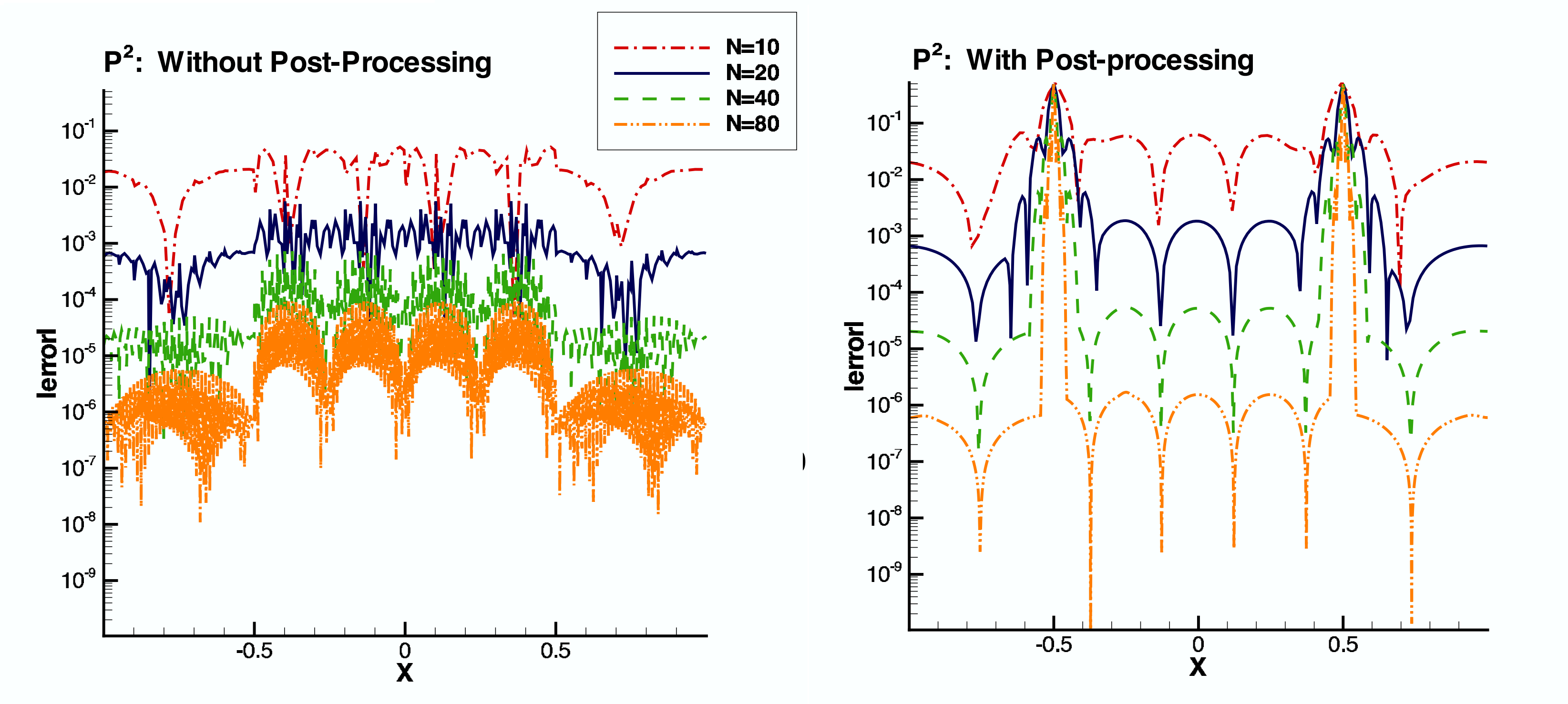




Two stationary shocks problem for $k = 2$. Stencil choices for the ENO method (left) and LED method (right).

$$N = 40 : +; \quad N = 80 : \bullet; \quad N = 160 : \square$$

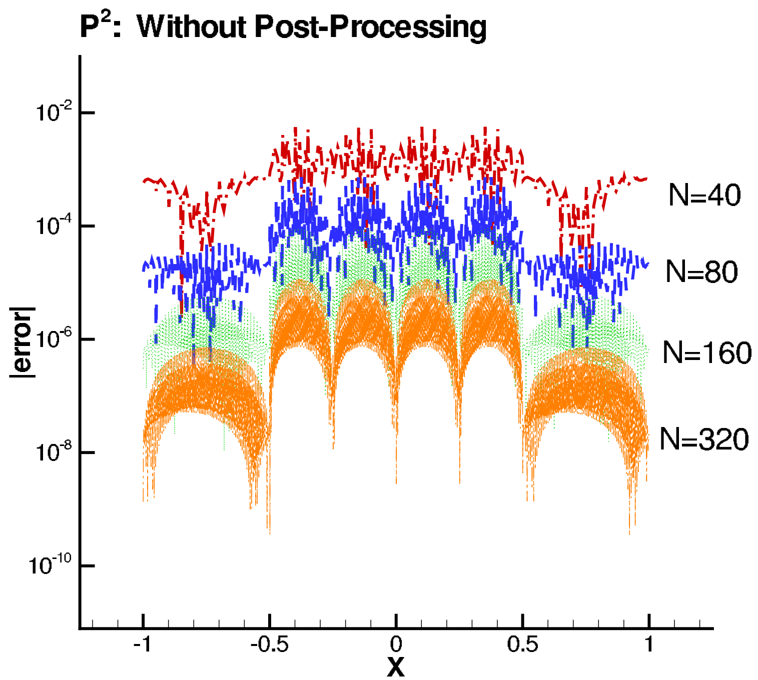
1-D Discontinuous Coefficient

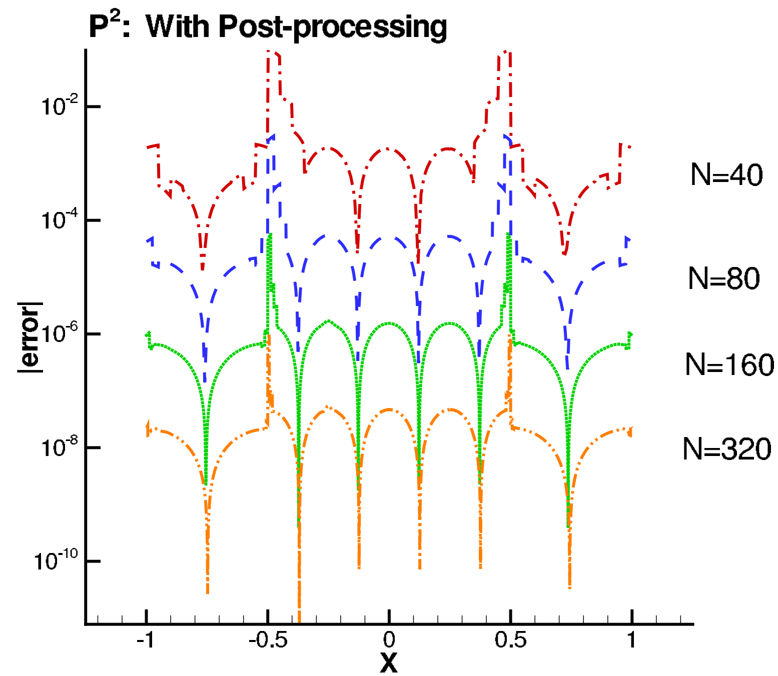


$u_t+(a(x)u)_x=0$

x in (-1,1), T=12.5

1-D Discontinuous Coefficient Using One-Sided Post-Processor





$$u_t+(au)_x=0$$

x in $(-1,1)$, $T=12.5$

$1 - D$ Discontinuous Coefficient Equation

Case of two stationary shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
	$\mathbb{P}^1$			
20	8.55E-01	—	8.17E-01	—
40	1.93E-01	2.15	1.80E-01	2.18
80	2.72E-02	2.83	2.58E-02	2.80
160	3.69E-03	2.88	3.34E-03	2.95
	$\mathbb{P}^2$			
40	1.45E-03	—	2.04E-02	—
80	1.54E-04	3.24	4.48E-04	5.51
160	1.90E-05	3.02	5.87E-06	6.25

$1 - D$ Discontinuous Coefficient Equation:

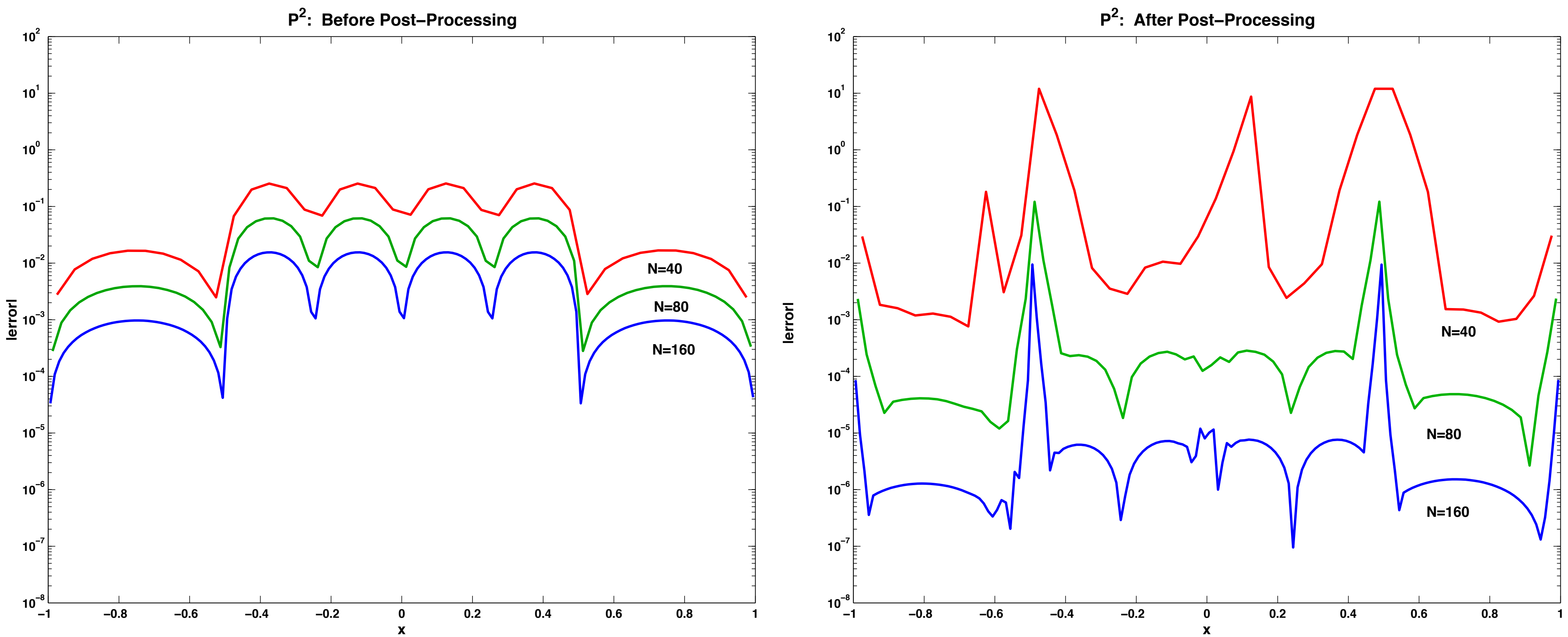
Errors in first derivative

Case of two stationary shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
	$\mathbb{P}^1$			
40	1.51E+00	—	3.04E+00	—
80	6.69E-01	1.17	1.31E-01	4.54
160	3.32E-01	1.01	1.56E-02	3.07
	$\mathbb{P}^2$			
40	1.27E-01	—	3.59E+00	—
80	3.12E-02	2.03	1.92E-02	7.55
160	7.77E-03	2.01	1.06E-03	4.18

Discontinuous Coefficient Equation

Errors in First Derivative



$1-D$ Discontinuous Coefficient Equation

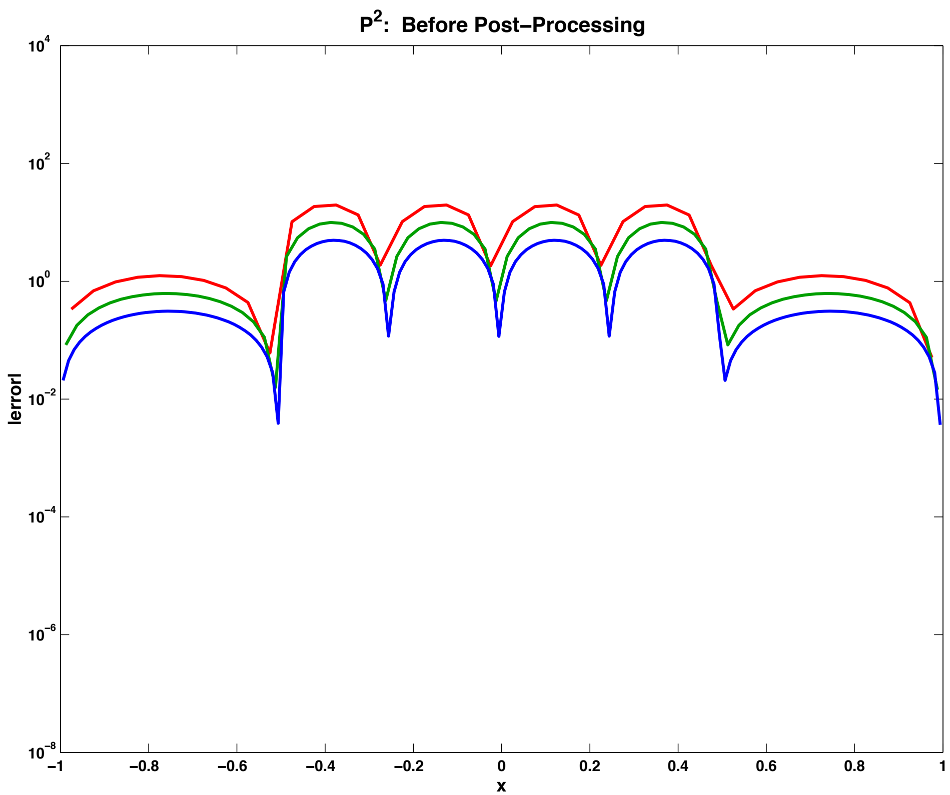
Errors in second derivative

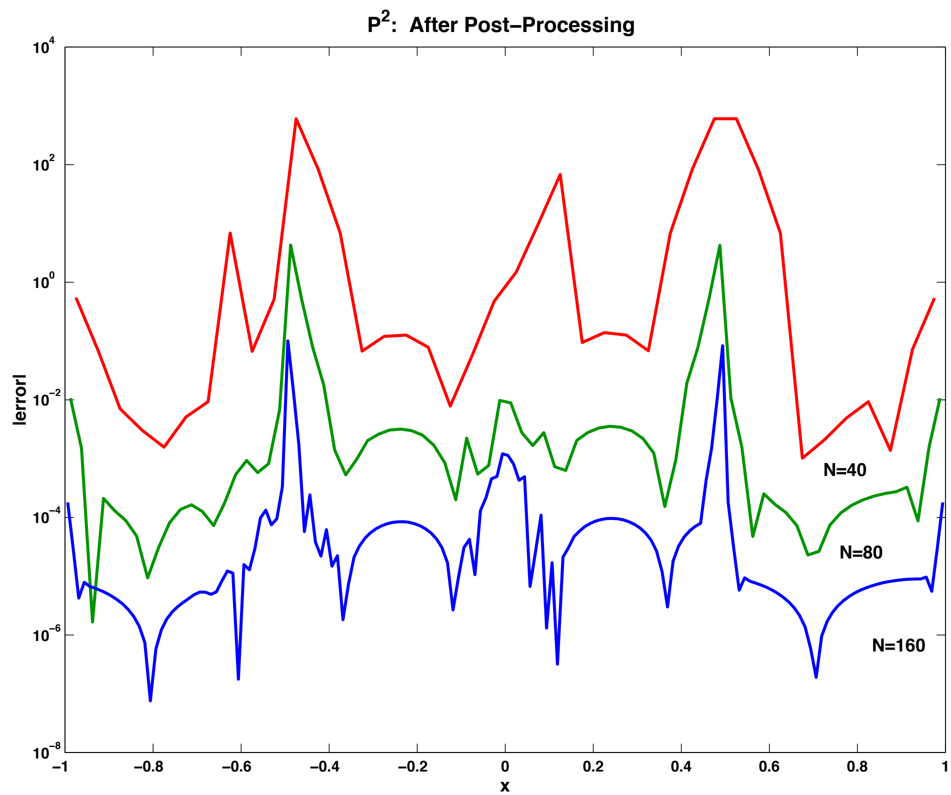
Case of two stationary shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
	$\mathbb{P}^1$			
40	1.59E+02	—	3.98E+02	—
80	1.59E+02	—	1.91E+00	4.38
160	1.59E+02	—	3.27E-01	2.55
	$\mathbb{P}^2$			
40	1.01E+01	—	1.67E+02	—
80	4.99E+00	1.02	6.78E-01	4.62
160	2.49E+00	1.00	1.04E-02	6.03

Discontinuous Coefficient Equation

Errors in Second Derivative





$1-D$ Discontinuous Coefficient Equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0,$$

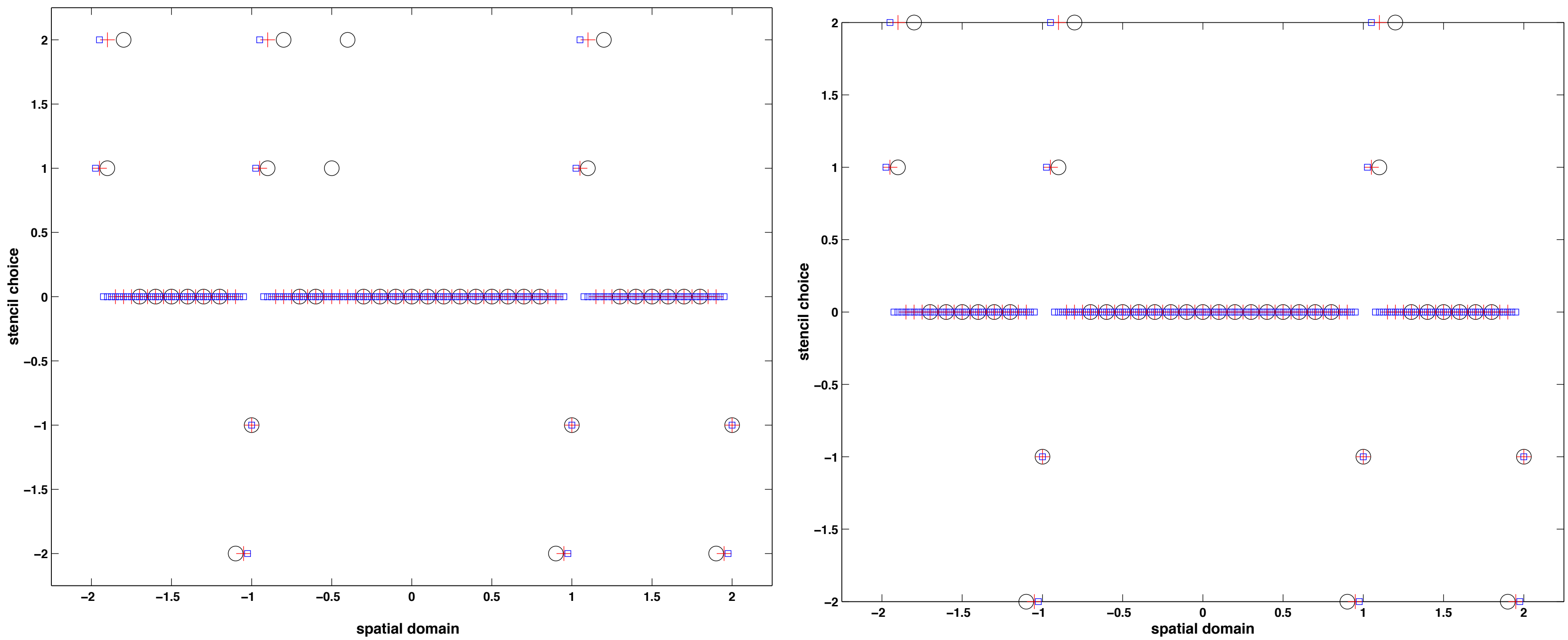
$$a(x) = \begin{cases} 1, & x \in [-2,2]\backslash(-1,1), \\ \frac{1}{2}, & x \in (-1,1) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(\frac{\pi}{2}x), & x \in [-2,2]\backslash(-1,1), \\ \frac{2}{3}\sin(\pi x) & x \in (-1,1). \end{cases}$$

$$u(-2,t) = u(2,t), \quad T = 1.$$

Case of two stationary and two moving shocks.

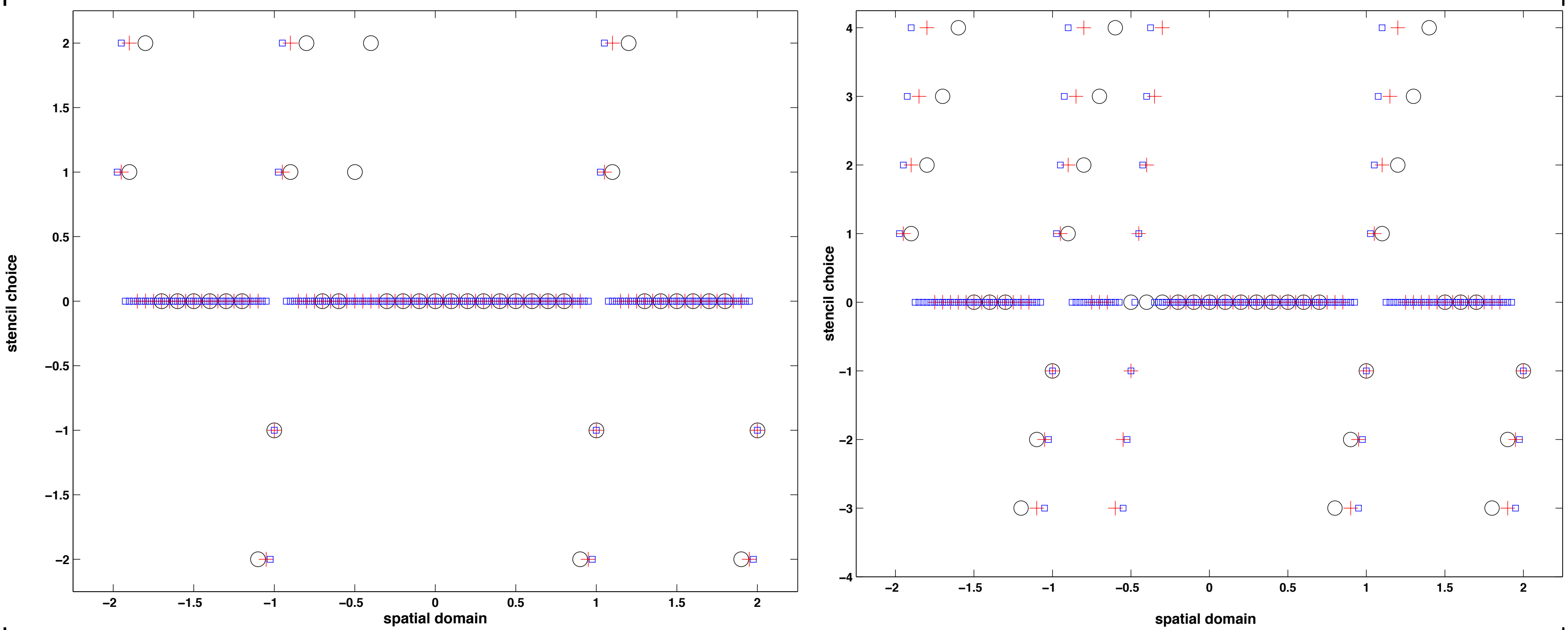
$\mathbb{P}^1$ Stencil Choices: Two stationary + two moving shocks



Two stationary + two moving shock problem for $k = 1$. Stencil choices for the ENO method (left) and LED method (right).

$$N = 40 : +; \quad N = 80 : \bullet; \quad N = 160 : \Box$$

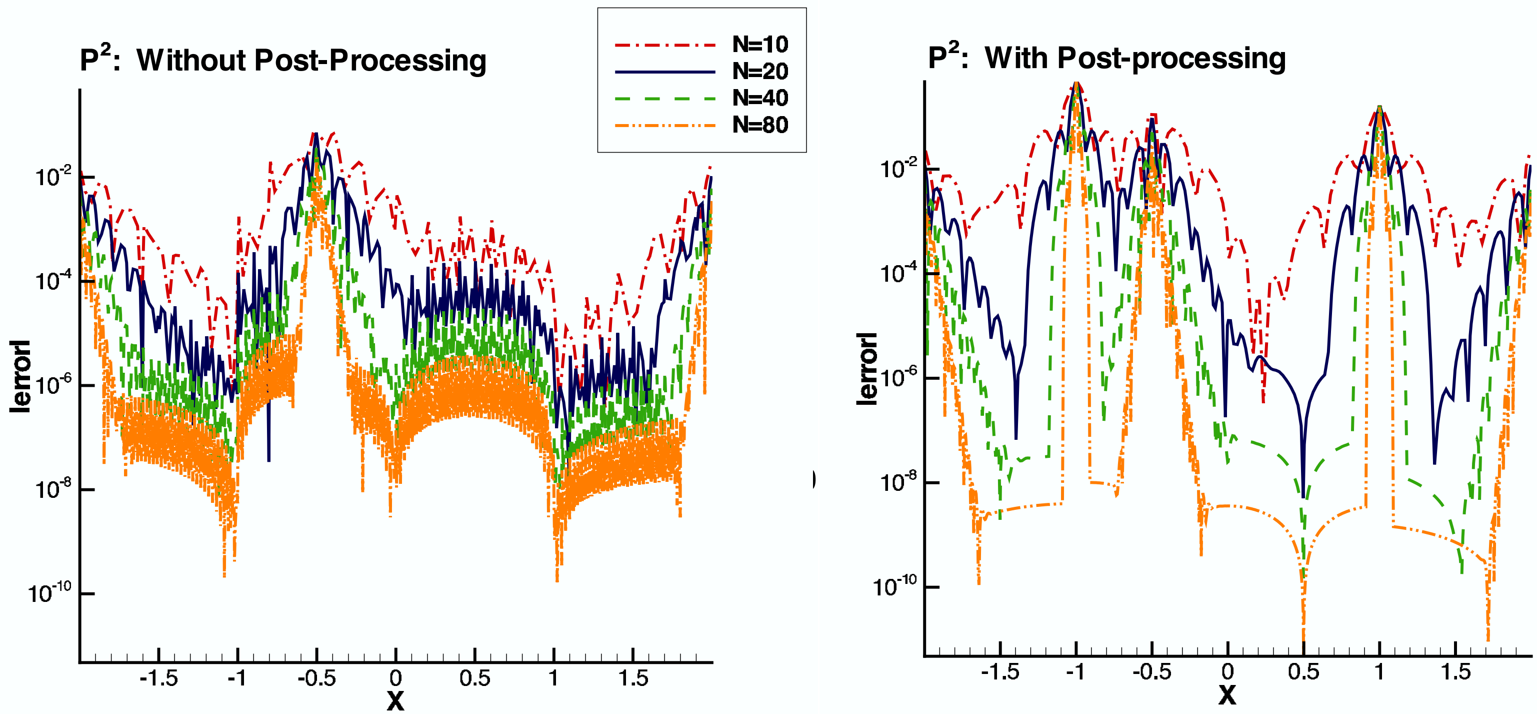
$\mathbb{P}^2$ Stencil Choices: Two stationary + two moving shocks



Two stationary + two moving shock problem for $k = 2$. Stencil choices for the ENO method (left) and LED method (right).

$$N = 40 : +; \quad N = 80 : \bullet; \quad N = 160 : \square$$

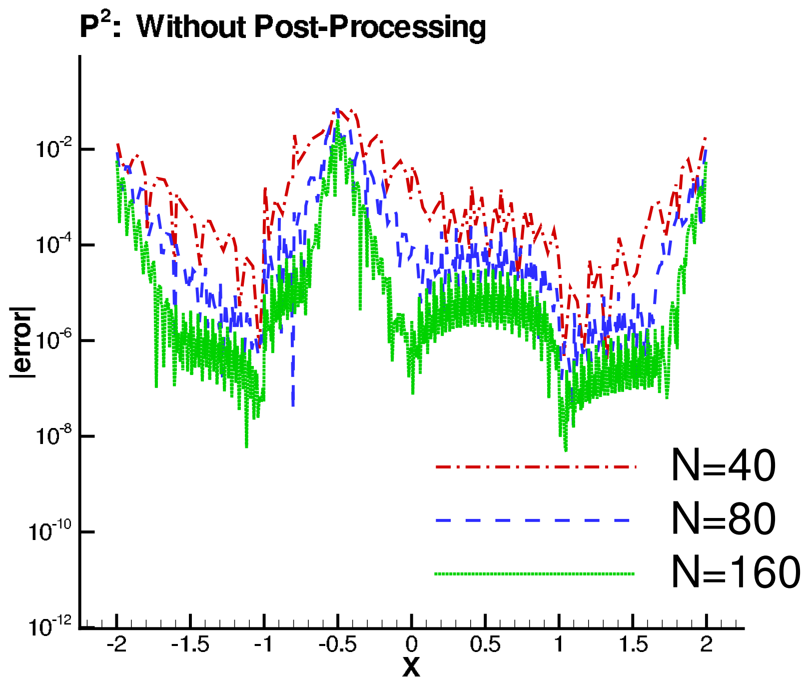
1-D Discontinuous Coefficient

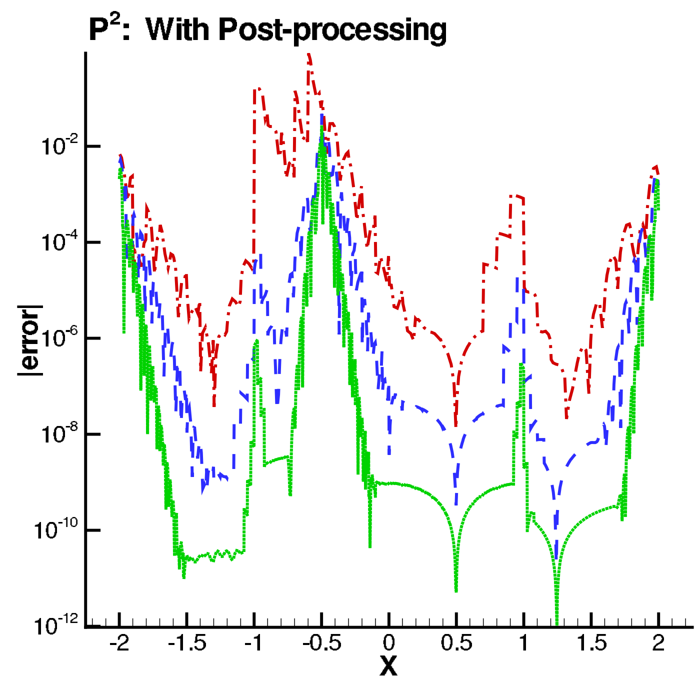


$$u_t + (a(x)u)_x = 0$$

x in $(-2,2)$, $T=1$

1-D Discontinuous Coefficient Using One-Sided Post-Processor





$$u_t + (au)_x = 0$$

x in $(-2,2)$, $T=1$

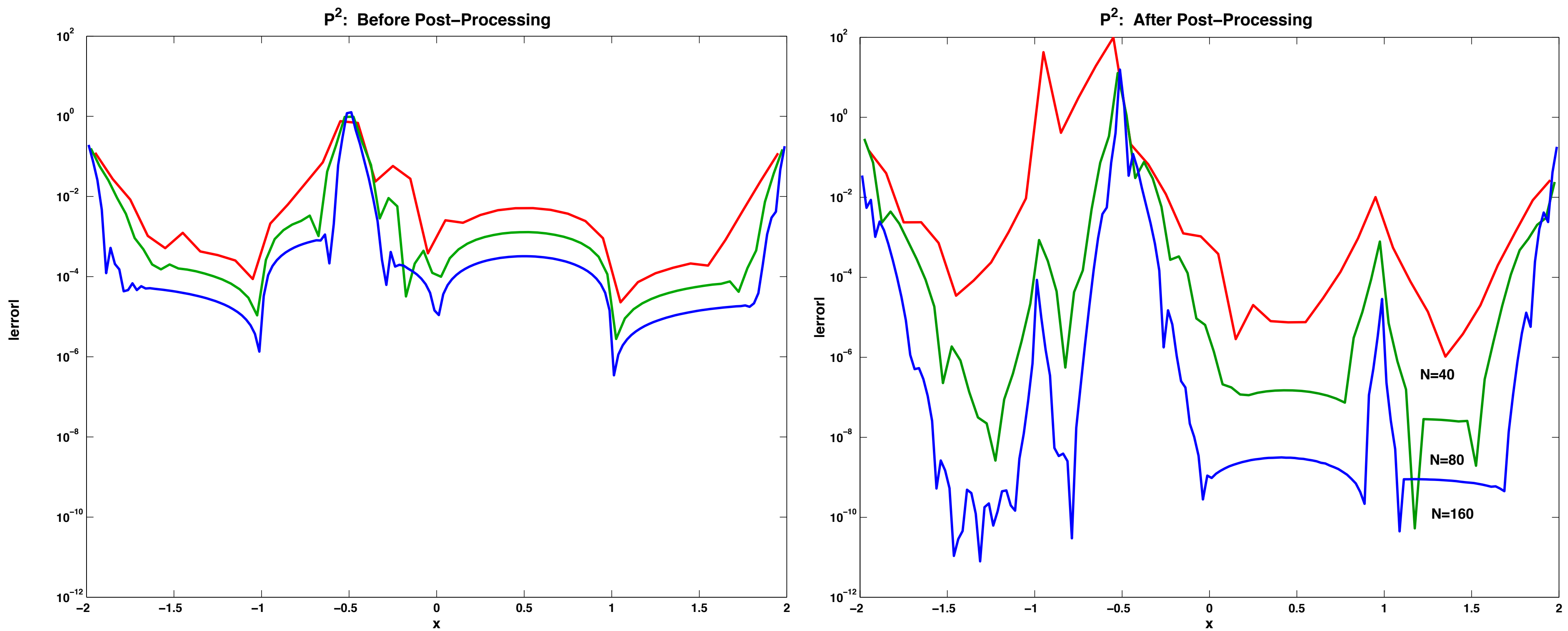
$1 - D$ Discontinuous Coefficient Equation

Case of two stationary and two moving shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x,2)$		$u^*(x,2)$	
	$\mathbb{P}^1$			
40	1.36E-03	—	2.05E-03	—
80	3.35E-04	2.02	9.57E-05	4.42
160	8.30E-05	2.01	4.61E-06	4.38
320	2.07E-05	2.01	3.40E-07	3.76
	$\mathbb{P}^2$			
40	3.79E-05	—	1.90E-04	—
80	4.77E-06	2.99	2.44E-06	6.28
160	5.98E-07	3.00	2.80E-08	6.45

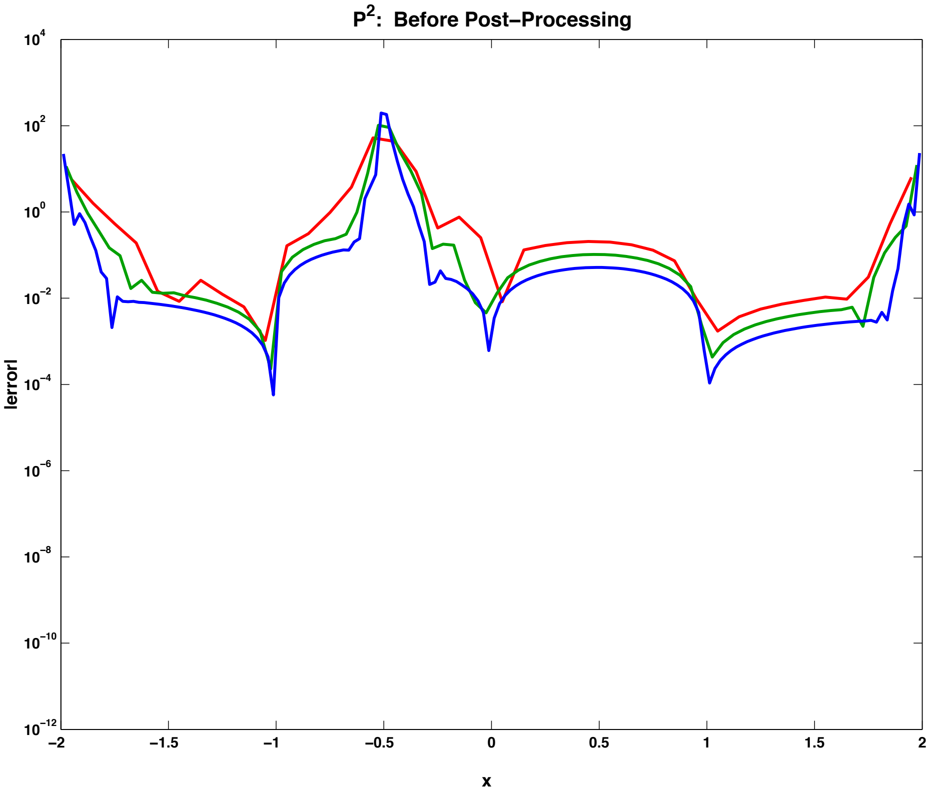
Discontinuous Coefficient Equation

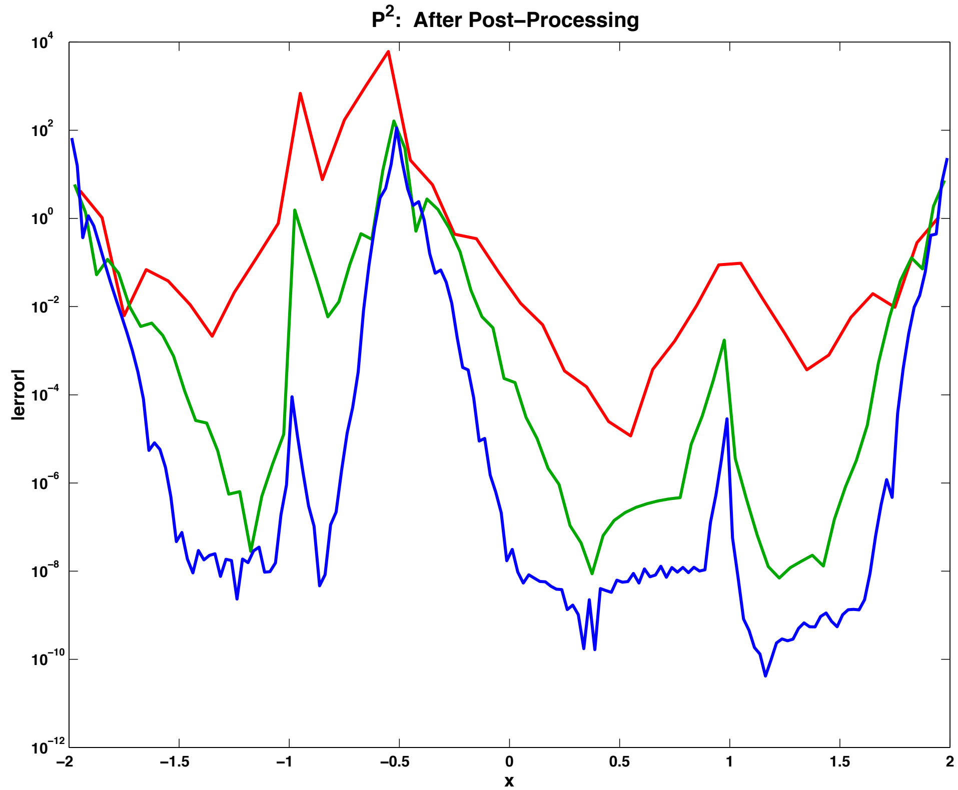
Errors in First Derivative



Discontinuous Coefficient Equation

Errors in Second Derivative





Summary

- $2k+1$ accuracy throughout entire domain when using one-sided post-processing in regions of computational domain boundaries and discontinuities.
- $(2k+2-d)$-th order accuracy in for the d-th derivative.
- ENO stencil choosing is able to find discontinuities in numerical solution.

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