Post-Processing of Discontinuous Galerkin Methods for Hyperbolic Equations

Jennifer K. Ryan

Computer Science and Mathematics Division

Oak Ridge National Laboratory

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Outline

- Discontinuous Galerkin Method
- Post-processing for linear hyperbolic equations
  (C.-W. Shu, H.L. Atkins)
- One-sided post-processing
- Post-processing stencil choosing techniques
  (R. Archibald, A. Gelb, S. Gottlieb)
- Post-processing for smoothly varying mesh
- Summary & Future Work

Discontinuous Galerkin Method

Properties:

- Can handle complicated geometries.
- Simple treatment of boundary conditions.
- High order accuracy.
  - Proven: • $k + \frac{1}{2}$ order accuracy for general case.
    • $k + 1$ order accuracy in special cases.
  - Numerically: $k + 1$ order accuracy
- Flexibility for adaptivity.
- Highly parallelizable.

Discontinuous Galerkin Method

Consider $u_t + f(u)_x = 0$.

Find $u_h(x,t) \in V_h$ such that

$$\int_{I_i} (u_h)_t v dx = \int_{I_i} f(u_h) v_x dx - f((u_h)_{i+\frac{1}{2}}) v_{i+\frac{1}{2}} + f((u_h)_{i-\frac{1}{2}}) v_{i-\frac{1}{2}}$$

for all $v \in V_h$.

$V_h = \text{span}\{1, \xi_i, \xi_i^2, \cdots, \xi_i^k, i = 1, \cdots, N\}$, where $\xi_i = \frac{x - x_i}{\triangle x_i}$ on $I_i = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2})$, and $u_h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) \xi_i^l$ if $x \in I_i$.

Use upwind monotone flux

Numerical Scheme:

$$\int_{I_i} (u_h)_t v dx = \int_{I_i} f(u_h) v_x dx - \hat{f}_{i+1/2} v_{i+1/2}^{-} + \hat{f}_{i-1/2} v_{i-1/2}^{+}$$

$\forall v \in V_h$.

Post-Processor

Cockburn, Luskin, Shu, & Süli (2003)

- Discontinuous Galerkin approximation allows us to use negative order error estimates:

$$||u_h - u||_{-l} = \mathcal{O}(h^{2k+1}).$$

- Post-processor extracts this information.
- Works for a locally uniform mesh:
  $\longrightarrow$ Translation invariant
  $\longrightarrow$ Post-Processor is local

Negative Order Sobolev Norm

$$||u||_{-\ell,\Omega} = \sup_{\phi \in \mathcal{C}_0^\infty} \frac{\int_\Omega u(x)\phi(x)dx}{||\phi||_{\ell,\Omega}}, \quad \ell \geq 1$$

Example:

$$u_N = \sin(2\pi N x), \quad \Omega = (-1,1), \quad \ell \geq 1$$

$$\Rightarrow \quad ||u_N||_{-\ell,\Omega} = \frac{1}{(2\pi N)^\ell}$$

Post-Processor Kernel

- Independent of the partial differential equation.
- Applied only at the final time.
- Filters out oscillations in the error.

Kernel Properties

Bramble & Schatz (1977)
Mock & Lax (1978)

- Compact Support.
- Reproduces polynomials of degree $2k+1$ by convolution.
- Linear combination of B-splines.

Post-Processed Solution

Post-processed solution: $u^*(x) = K_h^{2(k+1),k+1} * u_h$.

$$K_h^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \psi^{(k+1)} \left(\frac{x}{h} - \gamma \right)$$

$h = \triangle x_i$ for all i, and $c_\gamma^{2(k+1),k+1} \in \mathbb{R}$.

$\psi^{(0)} = \delta_0$, $\psi^{(n)} = \psi^{(n-1)} * \chi$ for $n \geq 2$, where

$$\chi(x) = \begin{cases} 1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{else.} \end{cases}$$

Example

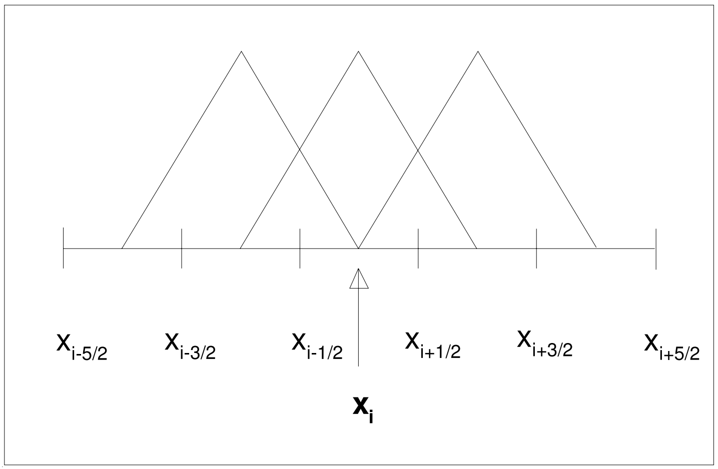
Second Order Approximation

$$u^*(y) = \frac{1}{h}\int_{-\infty}^{\infty} K^{4,2}(\frac{x-y}{h})u_h(x)dx$$

where

$$K_h^{4,2}(y) = \frac{1}{h}\left(c_{-1}^{4,2}\psi^{(2)}(\frac{y}{h}-1) + c_0^{4,2}\psi^{(2)}(\frac{y}{h}) + c_1^{4,2}\psi^{(2)}(\frac{y}{h}+1)\right)$$

and $\psi^{(2)}(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{else.} \end{cases}$



Example

Second Order Approximation

Find $c_\gamma,\ \gamma = -1, 0, 1:$

Use $K_h^{4,2} * p = p$ for $p = 1, x, x^2$

$$\begin{bmatrix} 1 & 1 & 1 \\ x+1 & x & x-1 \\ x^2+2x+\frac{7}{6} & x^2+\frac{1}{6} & x^2-2x+\frac{7}{6} \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

$$\Rightarrow K_h^{4,2}(y) = \frac{1}{h}\left(\frac{-1}{12}\psi^{(2)}(\frac{y}{h}-1) + \frac{7}{6}\psi^{(2)}(\frac{y}{h}) - \frac{1}{12}\psi^{(2)}(\frac{y}{h}+1)\right)$$

Implementation

Know form of approximation and kernel $\Rightarrow$

$$
\begin{aligned}
u^*(x) &= \frac{1}{h}\int_{-\infty}^{\infty} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right) u_h(y)\,dy \\
&= \frac{1}{h}\sum_{j=-2k}^{2k}\int_{I_{i+j}} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right)\sum_{l=0}^{k} u_{i+j}^{(l)}\left(\frac{y-x_{i+j}}{h}\right)^{l} dy \\
&= \sum_{j=-2k}^{2k}\sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x)
\end{aligned}
$$

$C(j,l,k,x) =$

$\frac{1}{h}\sum_{\gamma=-k}^{k} c_{\gamma}^{2(k+1),k+1}\int_{I_{i+j}} \psi^{(k+1)}\left(\frac{y-x}{h}-\gamma\right)\left(\frac{y-x_{i+j}}{h}\right)^{l} dy \in \mathbb{P}^{2k+1}$

$k' = \lceil (3k+1)/2 \rceil \leq 2k$

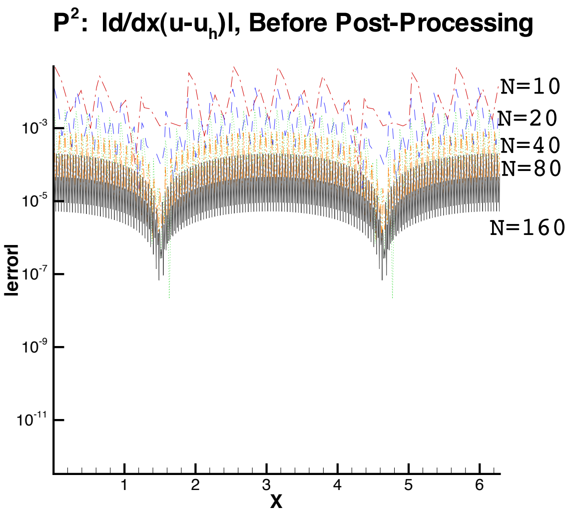
Derivatives

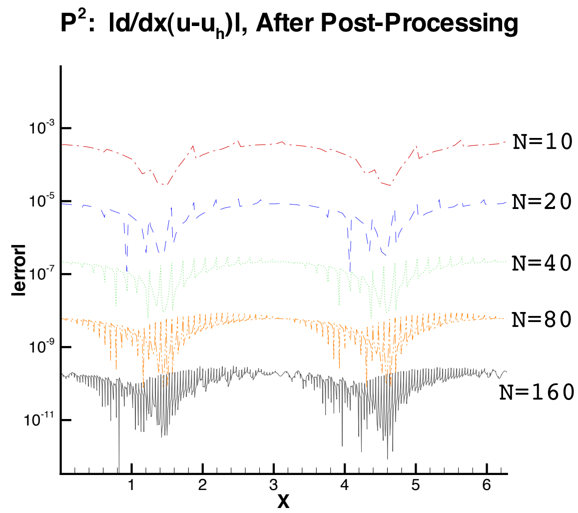
	Approximation		Post-Processed	
N	L^2 error	order	L^2 error	order
	Errors in First Derivative for $\mathbb{P}^2$			
20	3.48E-03	—	6.24E-06	—
40	8.72E-04	2.00	1.61E-07	5.28
80	2.18E-04	2.00	4.51E-09	5.16
160	5.45E-05	2.00	1.39E-10	5.02
	Errors in Second Derivative for $\mathbb{P}^2$			
20	6.78E-02	—	3.64E-05	—
40	3.39E-02	1.00	2.15E-06	4.08
80	1.70E-02	1.00	1.32E-07	4.03
160	8.48E-03	1.00	8.19E-09	4.01

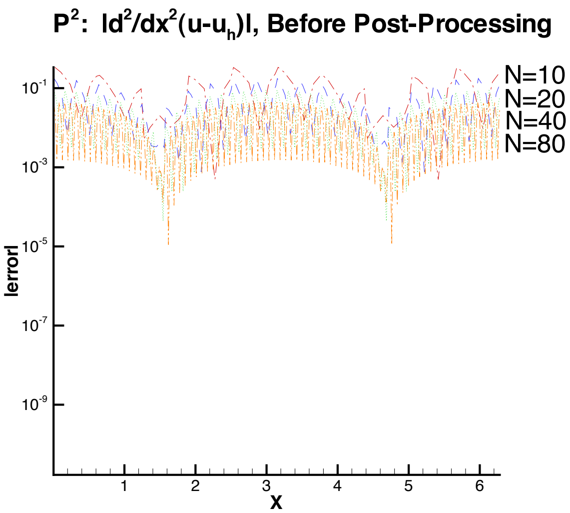
$\frac{du^*(x)}{dx} \in \mathbb{P}^{2k}$

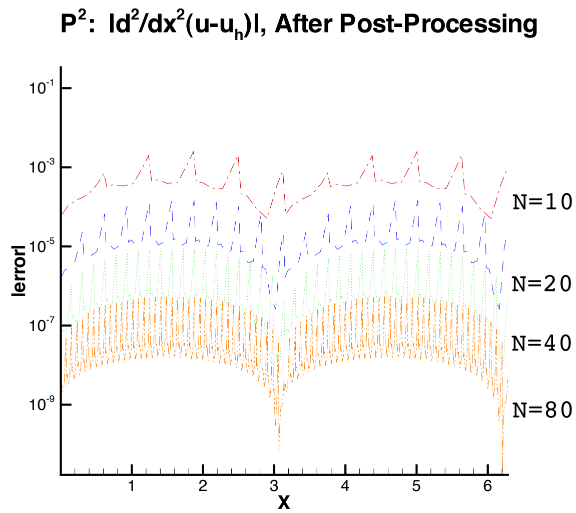
$$u_t + u_x = 0$$
$$u(x,0) = \sin(x)$$
$$x \in (0, 2\pi)$$
$$T = 12.5$$

Derivatives of Post-Processed Solution









2–D Approximation & Kernel

$K_h^{2(k+1),k+1}(x,y) =$

$$\frac{1}{h^2}\sum_{-\gamma_x}^{\gamma_x}\sum_{-\gamma_y}^{\gamma_y} c_{\gamma_x+\gamma_y}^{2(k+1),(k+1,k+1)}\psi^{(k+1)}\left(\frac{x}{h_x}-\gamma_x\right)\psi^{(k+1)}\left(\frac{y}{h_y}-\gamma_y\right)$$

Solving: $u_t + f(u)_x + g(u)_y = 0$

$$V_h = \text{span}\{1, \xi_i, \eta_j, \xi_i^2, \xi_i\eta_j, \eta_j^2, \cdots, \xi_i^k, \cdots, \eta_j^k,$$
$$i = 1, \cdots, N_x, j = 1, \cdots, N_y\}$$

on $I_{i,j} = (x_i - \frac{h_x}{2}, x_i + \frac{h_x}{2}) \times (y_j - \frac{h_y}{2}, y_j + \frac{h_y}{2})$ for $\xi_i = \frac{x-x_i}{h_x},\ \eta_j = \frac{y-y_j}{h_y}$.

Approximation form: $u_h(x,y,t) = \sum_{l=0}^{k}\sum_{m=0}^{k-l} u_{i,j}^{(l,m)}(t)\xi_i^l\eta_j^m$ for $x, y \in I_{i,j}$.

$2-D$ System

	$u_h(x,12.5)$		$u^*(x,12.5)$	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10^2	1.22E-01		1.22E-01	
20^2	1.96E-02	2.63	1.90E-02	2.68
40^2	2.85E-03	2.78	2.48E-03	2.93
80^2	4.71E-04	2.59	3.14E-04	2.98
	$\mathbb{P}^2$			
10^2	2.66E-03		1.97E-03	
20^2	2.52E-04	3.40	5.66E-05	5.12
40^2	3.10E-05	3.02	1.67E-06	5.08
80^2	3.88E-06	3.00	5.06E-08	5.05

Errors in u

$$u_t - u_x - v_y = 0$$
$$v_t + v_x - u_y = 0$$

$$u(0,t) = u(2\pi,t)$$
$$v(0,t) = v(2\pi,t)$$
$$T = 12.5$$

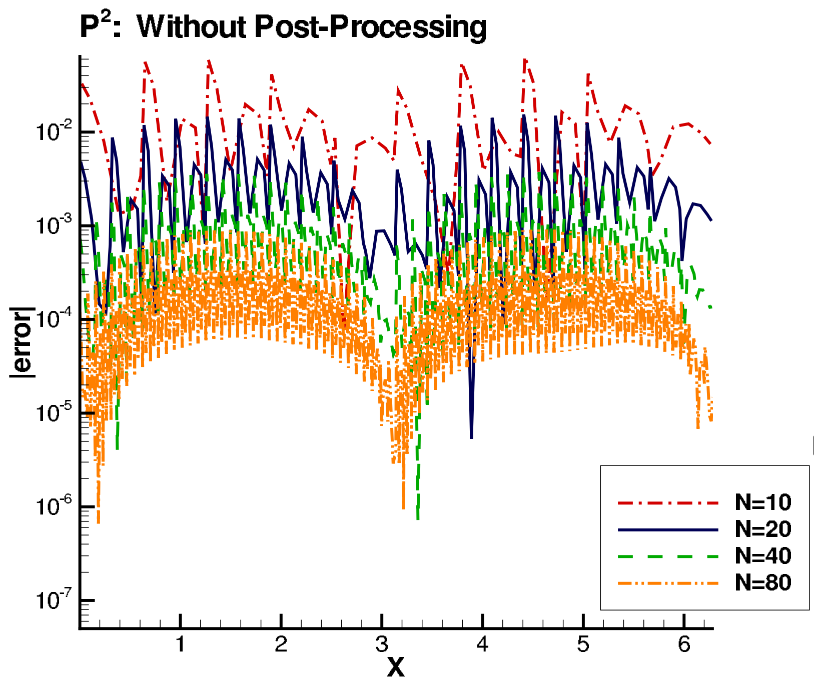
$1-D$ Variable Coefficient Equation

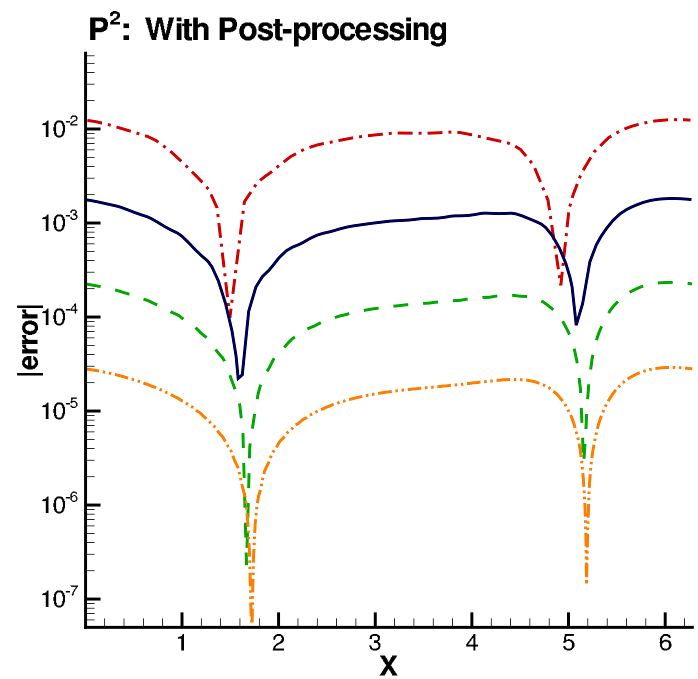
	$u_h(x,12.5)$		$u^*(x,12.5)$	
mesh	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.83E-02	—	7.82E-02	—
20	4.35E-03	2.07	1.08E-03	2.86
40	1.07E-03	2.03	1.39E-04	2.96
80	2.66E-04	2.01	1.75E-05	2.99
	$\mathbb{P}^2$			
10	8.61E-04	—	1.34E-04	—
20	1.07E-04	3.01	2.34E-06	5.84
40	1.34E-05	3.00	4.55E-08	5.69
80	1.67E-06	3.00	1.09E-09	5.38

$$u_t + (au)_x = f$$

$$a(x) = 2 + \sin(x)$$
$$u(x,0) = \sin(3x)$$
$$u(0,t) = u(2\pi,t)$$
$$T = 12.5$$

1-D Variable Coefficient

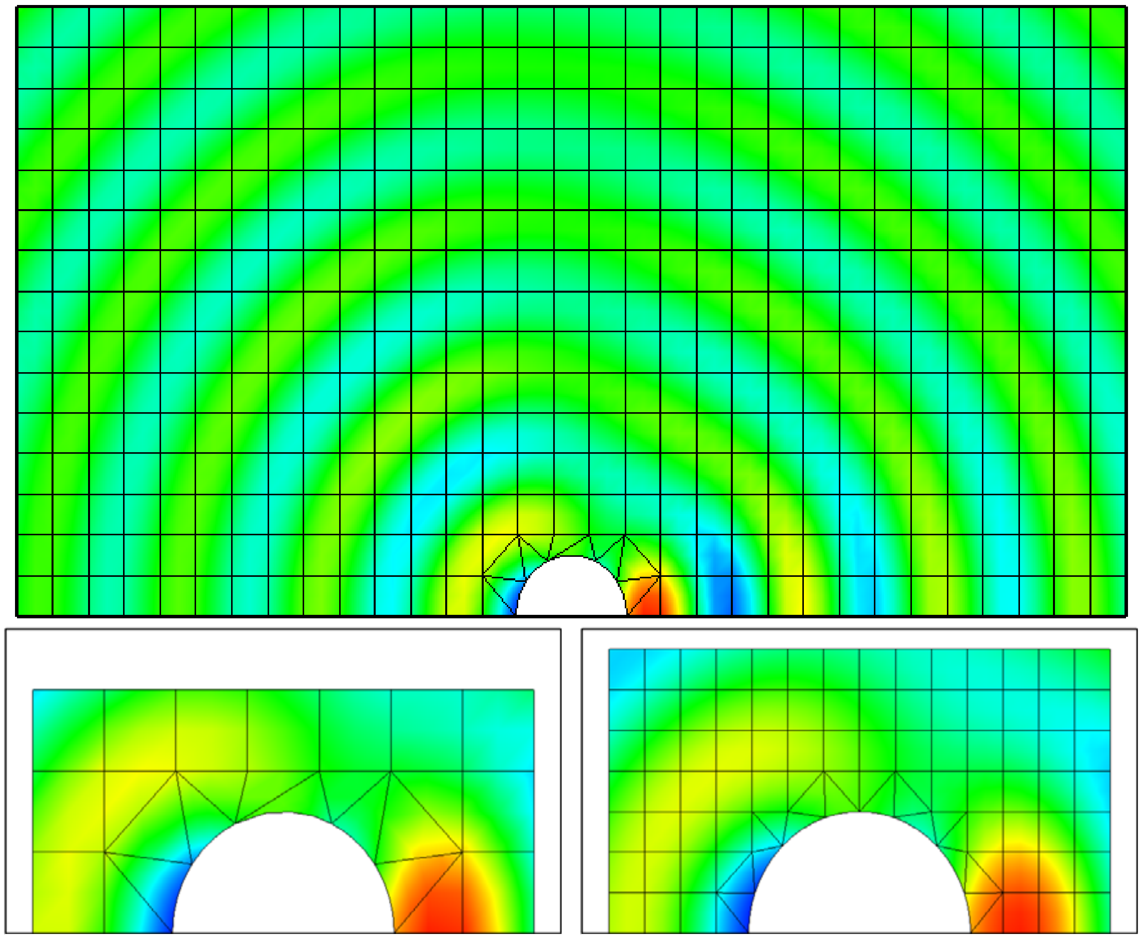




$u_t+(a(x)u)_x=f(x,t)$
$a(x)=2+\sin(x)$
$u(x,0)=\sin(3x)$
x in (0,2pi), T=12.5

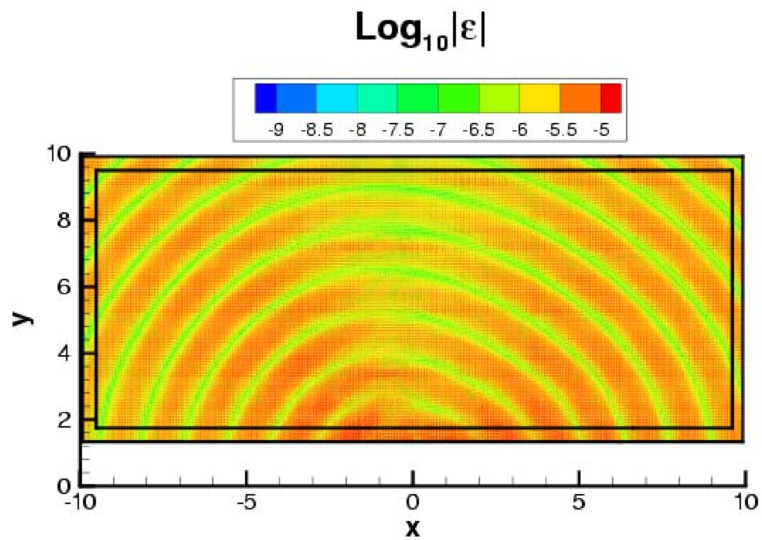
Aeroacoustic Example

Scatter of a plane wave off of a cylinder: wavelength $\lambda = 2.5r$

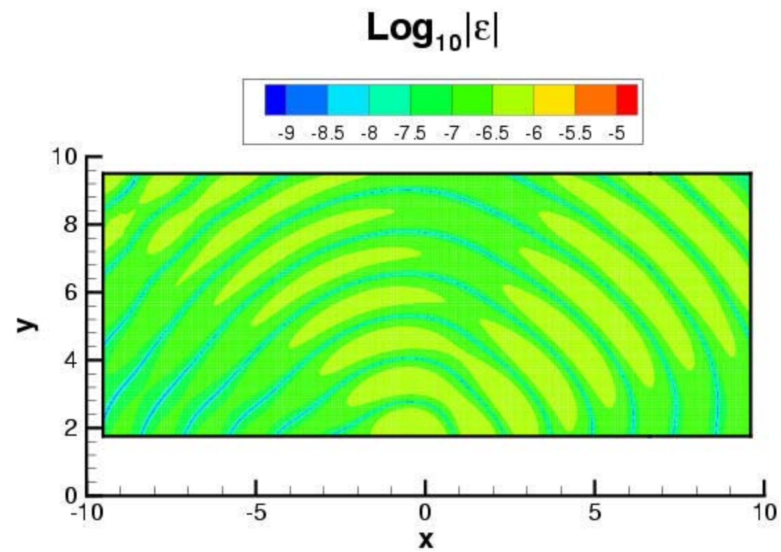


Aeroacoustic Example

Scatter of a plane wave off of a cylinder: Fine Mesh

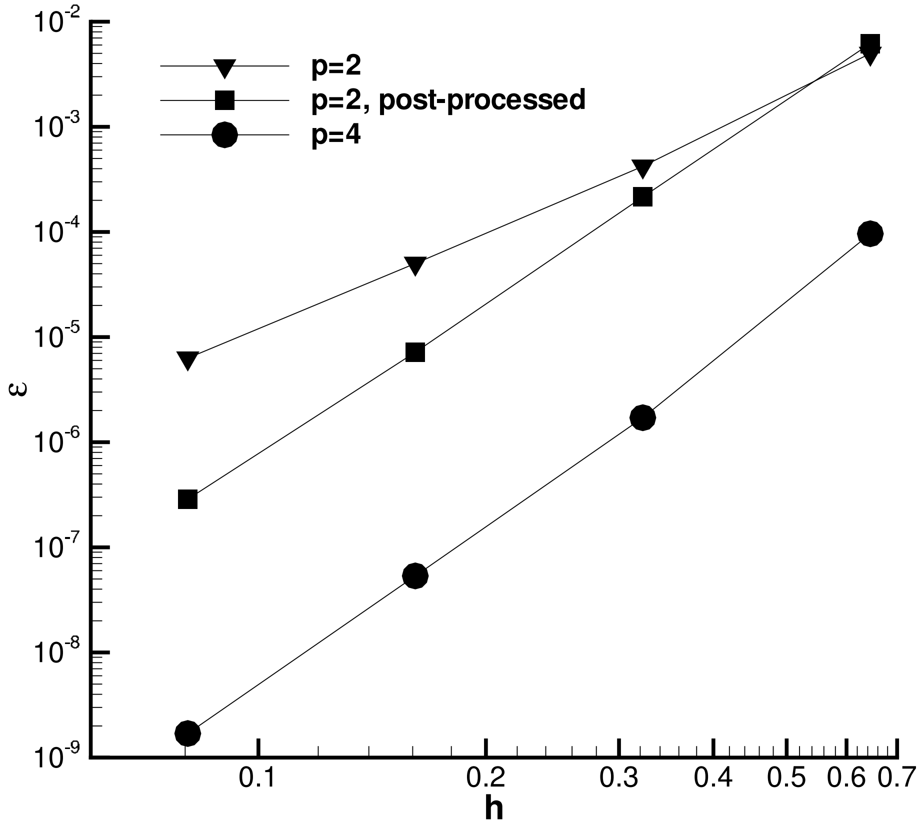


Without Post-Processing

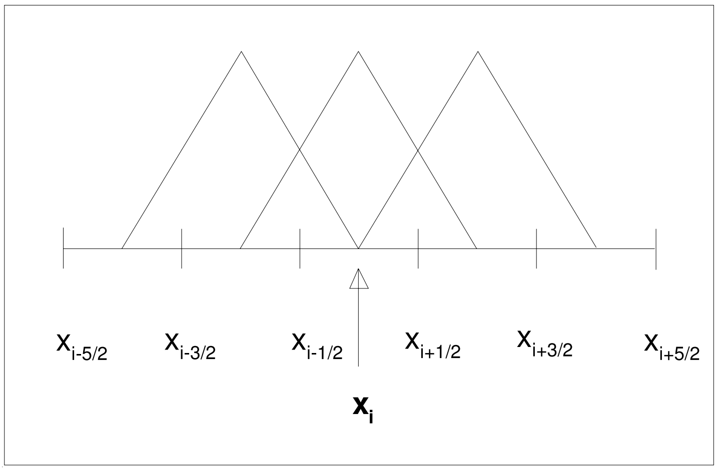


With Post-Processing

Aeroacoustic Test Problem



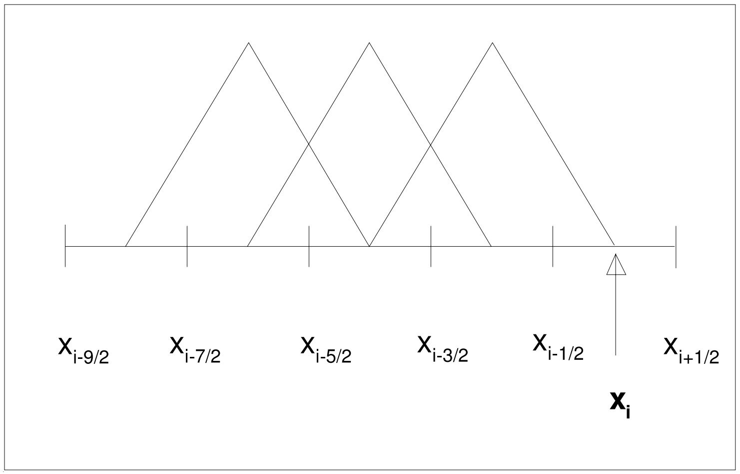
Symmetric Post-Processor



$$u^*(x) = \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x)$$

$$C(j,l,k,x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \int_{I_{i+j}} \psi^{(k+1)} \left(\frac{y-x}{h} - \gamma\right) \left(\frac{y-x_{i+j}}{h}\right)^l \, dy$$

Left Post-Processor



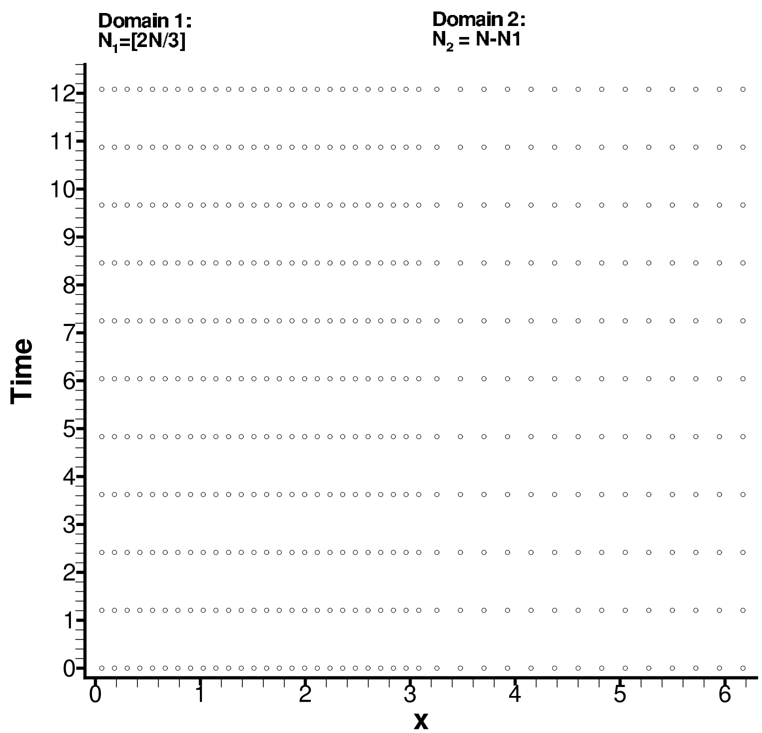
$$u^*(x) = \sum_{j=-2k'}^{0} \sum_{l=0}^{k} u_{i+j}^{(l)} \, C(j,l,k,x)$$

$$C(j,l,k,x) = \frac{1}{h} \sum_{\gamma=-2k-1}^{-1} c_\gamma^{2(k+1),k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}(\eta)\,(\xi_i+\eta+\gamma-j)^l \, dy$$

For $k = 1$:

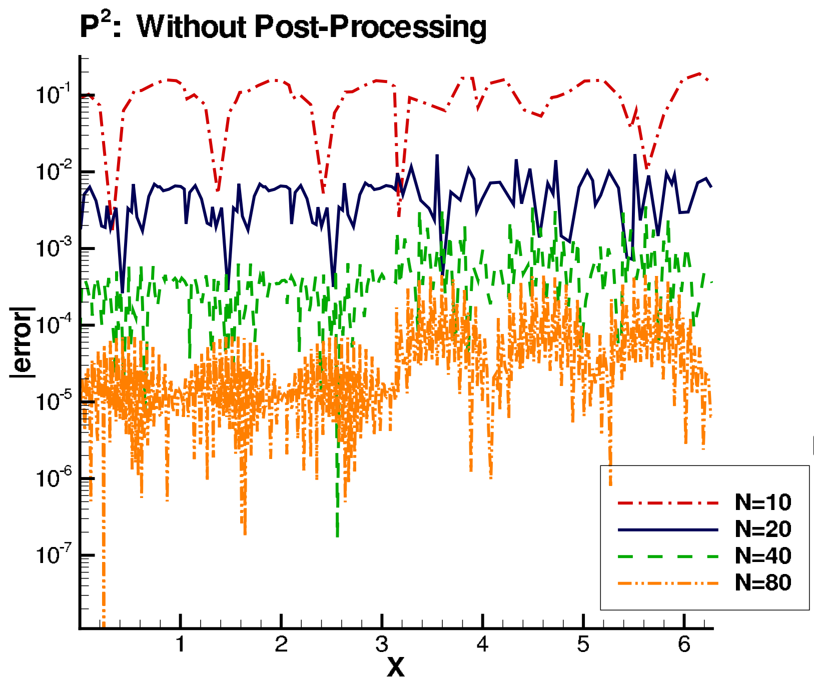
$$K(x) = \frac{11}{12}\psi^{(2)}(x+3) - \frac{17}{6}\psi^{(2)}(x+2) + \frac{35}{12}\psi^{(2)}(x+1)$$

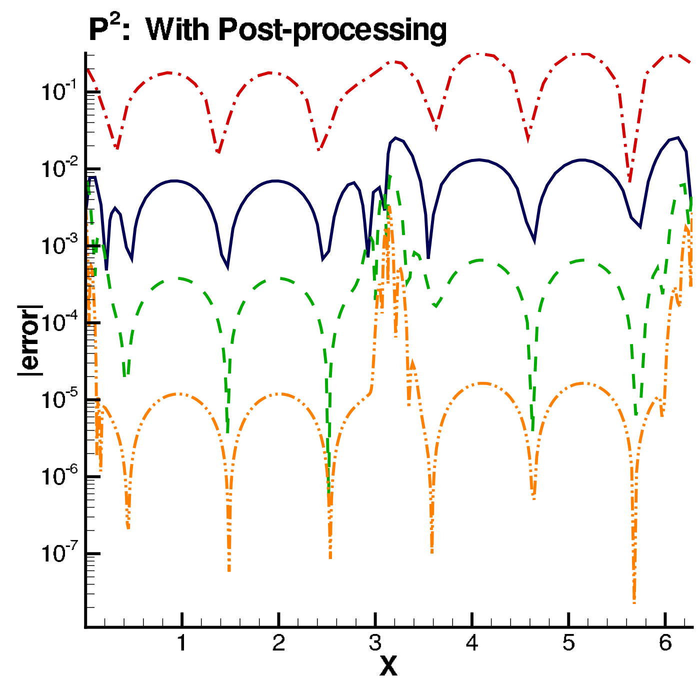
Domains with Different Mesh Sizes



- N=total number of elements for the 2 domains combined.
- $[0, \pi)$, has a more refined mesh.
- Solving a hyperbolic equation with smooth initial conditions over $[0, 2\pi]$.
- Calculating approximation for 2 periods in time.

1-D Multi-Domain



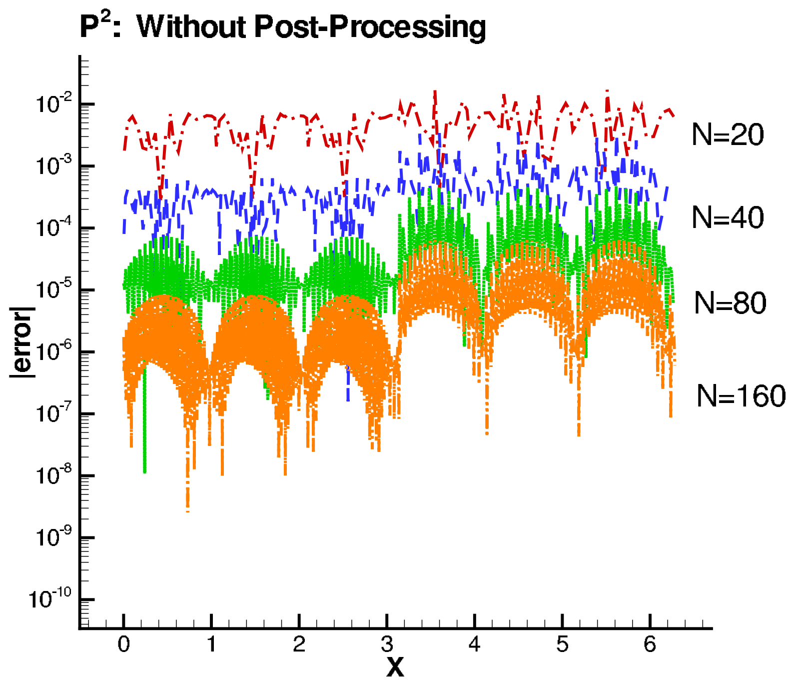


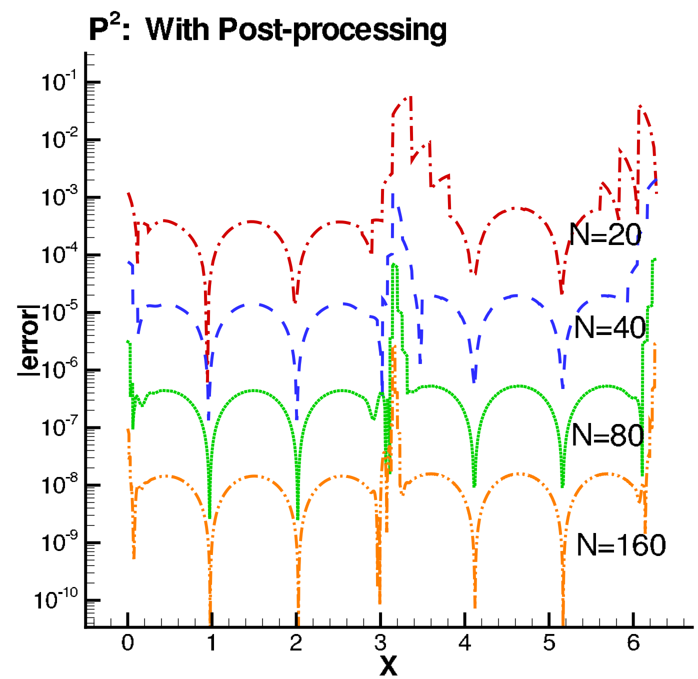
$u_t + u_x = 0$

$u(x,0) = \sin(3x)$

x in (0,2pi), T=12.5

1-D Multi-Domain Using One-Sided Post-Processor





$u_t+u_x=0$

$u(x,0)=\sin(3x)$

x in (0,2pi), T=12.5

Multi-Domain Problem

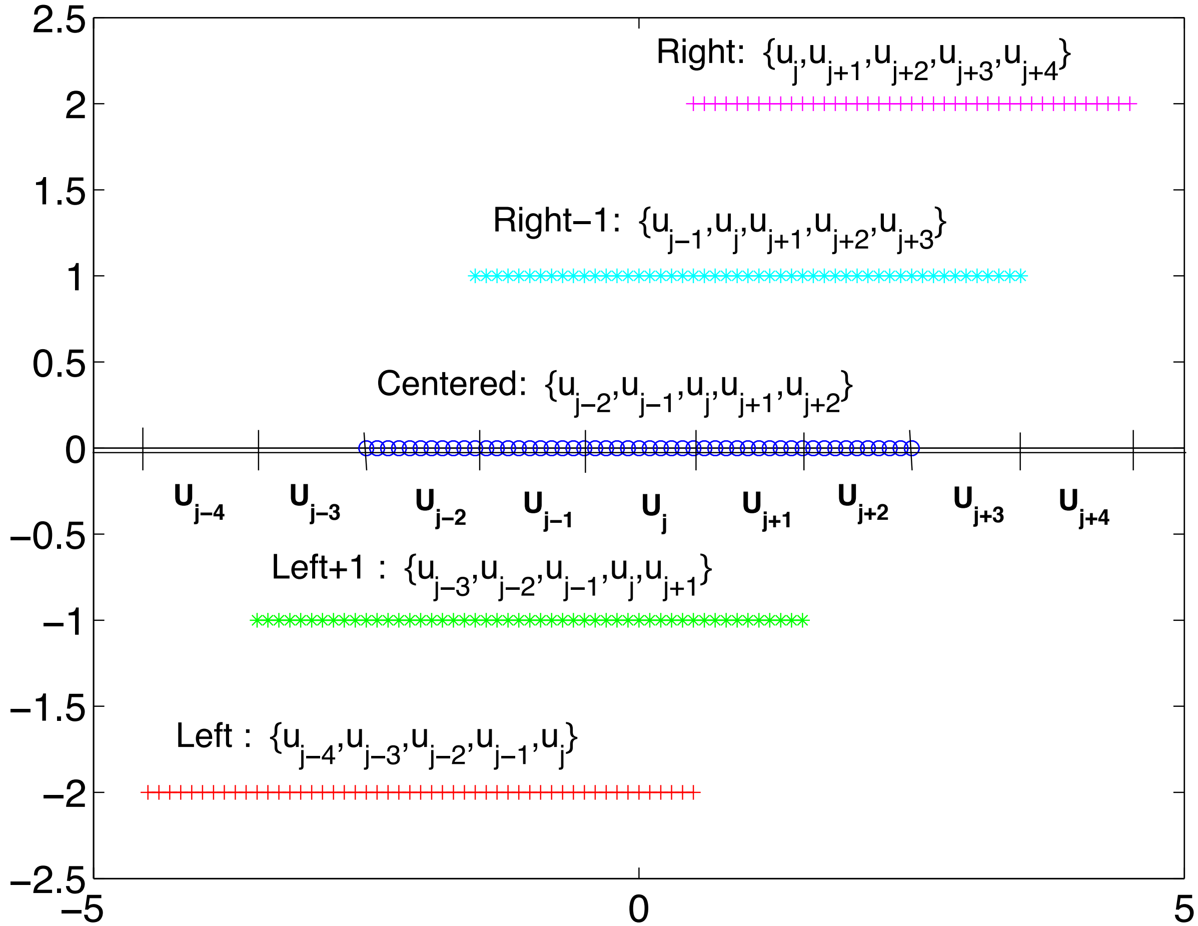
mesh	L^2 error	order	L^2 error	order
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
	$\mathbb{P}^1$			
20	3.48E-01	—	3.76E-01	—
40	6.20E-02	2.49	6.32E-02	2.57
80	9.55E-03	2.70	1.02E-02	2.63
160	1.46E-03	2.71	1.25E-03	3.03
	$\mathbb{P}^2$			
20	9.77E-03	—	3.20E-01	—
40	7.95E-04	3.62	9.96E-03	5.01
80	1.05E-04	2.92	2.67E-04	5.22
160	1.31E-05	3.01	1.03E-05	4.69

$$u_t + u_x = 0$$

$$u(x, 0) = \sin(3x)$$
$$x \in (0, 2\pi)$$
$$T = 12.5$$

Stencil Choices

For $\mathbb{P}^1$, 5 candidate stencils:



Choosing the Post-Processing Stencil

Two Approaches

- Using Essentially Non-Oscillatory (ENO) type method: Smoothness of candidate post-processing stencils. *(S. Gottlieb)*
- Edge Detection method: Finds shock location based on numerical solution. *(R. Archibald, A. Gelb)*

ENO Type Stencil Choosing

- Calculate undivided differences at downwind point:

$$D_j^1 = |u_{j+1}(x_{j-1/2}) - u_j(x_{j-1/2})|,$$
$$D_j^{r+1} = |D_{j+1}^r - D_j^r|,\ r = 1, \cdots, 3$$

- Find $\max_{j-4\le i\le j+3} D_i^1, \cdots, \max_{j-4\le i\le j} D_i^4$.

ENO Type Stencil Choosing, $\mathbb{P}^1$

$$
\begin{tikzcd}
D^4_{j-4} \arrow[d] \arrow[dr] & D^4_{j-3} \arrow[d] \arrow[dr] & D^4_{j-2} \arrow[d] \arrow[dr] & D^4_{j-1} \arrow[d] \arrow[dr] & D^4_{j} \arrow[d] \arrow[dr] & & & \\
D^3_{j-4} \arrow[d] \arrow[dr] & D^3_{j-3} \arrow[d] \arrow[dr] & D^3_{j-2} \arrow[d] \arrow[dr] & D^3_{j-1} \arrow[d] \arrow[dr] & D^3_{j} \arrow[d] \arrow[dr] & D^3_{j+1} \arrow[d] \arrow[dr] & & \\
D^2_{j-4} \arrow[d] \arrow[dr] & D^2_{j-3} \arrow[d] \arrow[dr] & D^2_{j-2} \arrow[d] \arrow[dr] & D^2_{j-1} \arrow[d] \arrow[dr] & D^2_{j} \arrow[d] \arrow[dr] & D^2_{j+1} \arrow[d] \arrow[dr] & D^2_{j+2} \arrow[d] \arrow[dr] & \\
D^1_{j-4} \arrow[d] & D^1_{j-3} \arrow[d] & D^1_{j-2} \arrow[d] & D^1_{j-1} \arrow[d] & D^1_{j} \arrow[d] & D^1_{j+1} \arrow[d] & D^1_{j+2} \arrow[d] & D^1_{j+3} \arrow[d] \\
\text{Center} & \text{Center} & \text{Left+1} & \text{Left} & \text{Right} & \text{Right-1} & \text{Center} & \text{Center}
\end{tikzcd}
$$

$1 - D$ Discontinuous Coefficient Equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0$$

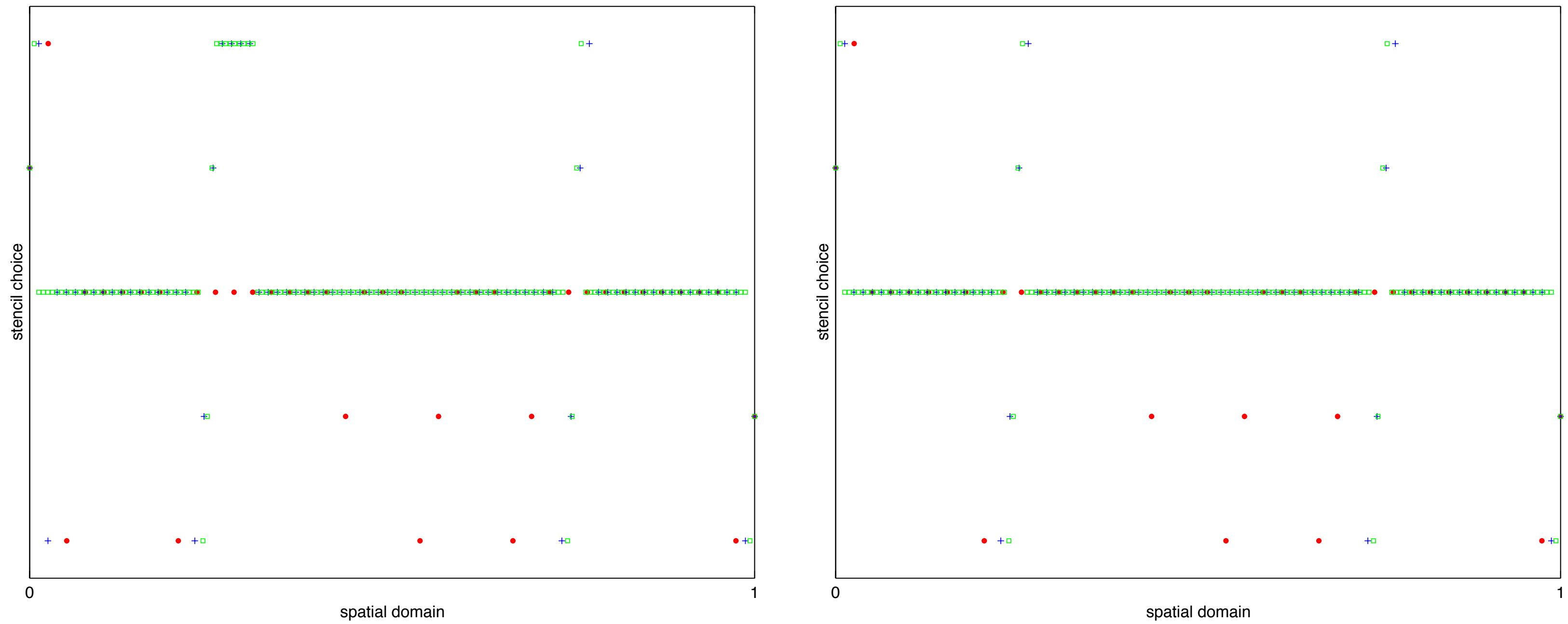
$$a(x) = \begin{cases} 1, & x \in [-1,1] \backslash (-\frac{1}{2}, \frac{1}{2}), \\ \frac{1}{2}, & x \in (-\frac{1}{2}, \frac{1}{2}) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(2\pi x), & x \in [-1,1] \backslash (-\frac{1}{2}, \frac{1}{2}), \\ -2\pi \cos(4\pi x) & x \in (-\frac{1}{2}, \frac{1}{2}). \end{cases}$$

$$u(-1,t) = u(1,t), \quad T = 12.5$$

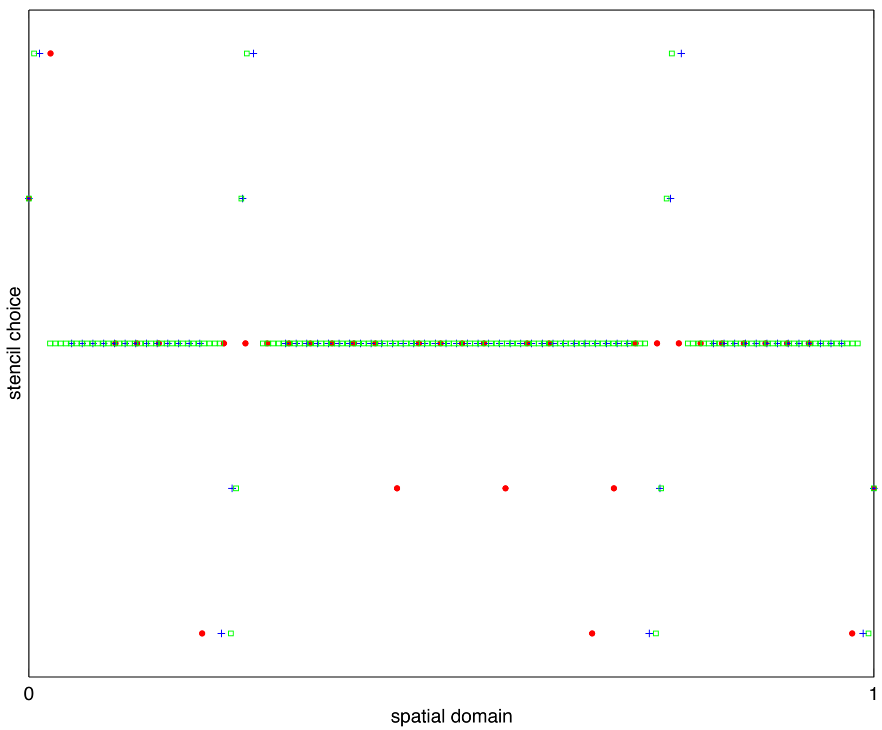
Case of two stationary shocks.

Stencil Choices



Two stationary shocks problem for $k = 1$. The S_{32} stencil is on the left and the S_{432} and S_{5432} stencils on the right. The S_{32} stencil has smeared the left shock location and biases unnecessarily, especially for 40 points. The S_{432} and S_{5432} are perfect.

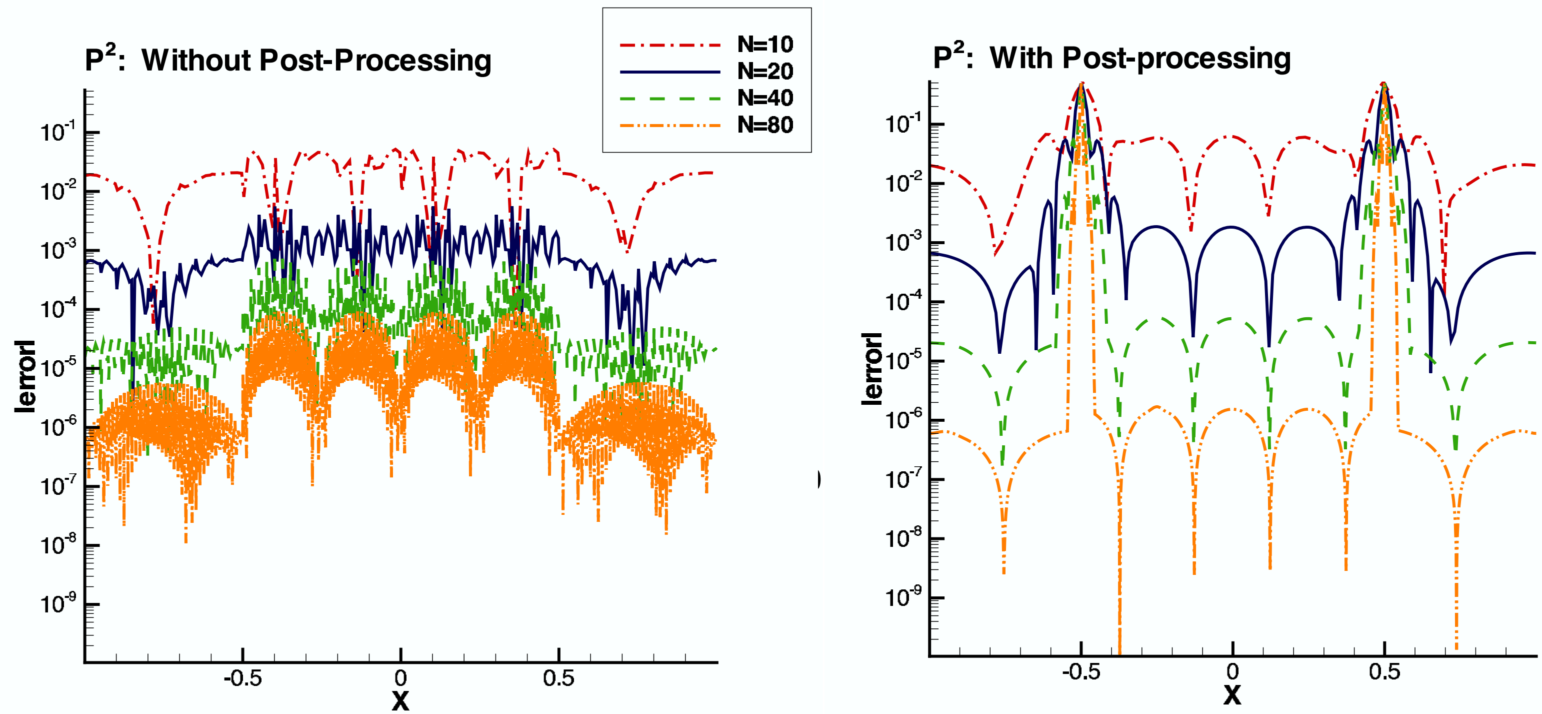
Stencil Choices



Two stationary shocks problem for $k = 2$.

The S_{32} S_{432} and S_{5432} stencils are all the same.

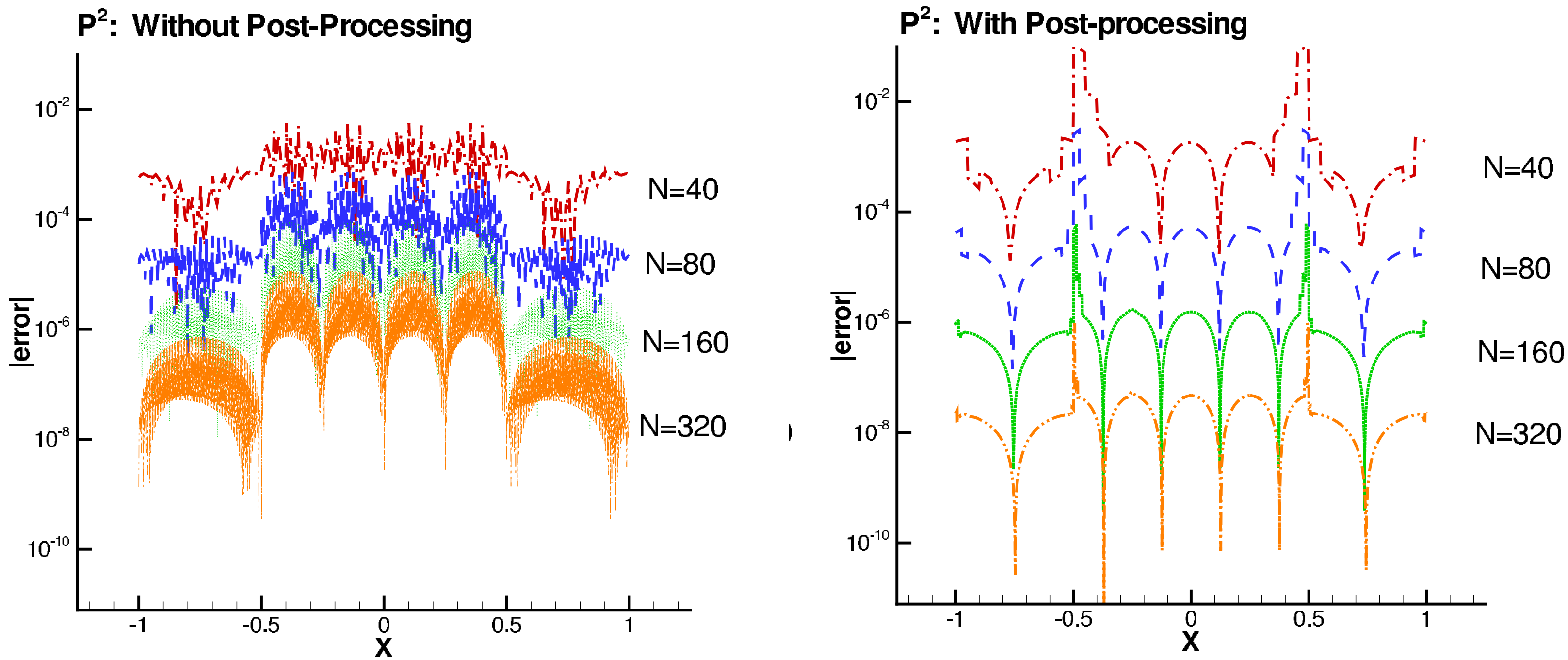
1-D Discontinuous Coefficient



$u_t + (a(x)u)_x = 0$

x in (-1,1), T=12.5

1-D Discontinuous Coefficient Using One-Sided Post-Processor



$$u_t+(au)_x=0$$

x in (-1,1), T=12.5

$1 - D$ Discontinuous Coefficient Equation

Case of two stationary shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
	$\mathbb{P}^1$			
20	8.55E-01	—	8.17E-01	—
40	1.93E-01	2.15	1.80E-01	2.18
80	2.72E-02	2.83	2.58E-02	2.80
160	3.69E-03	2.88	3.34E-03	2.95
	$\mathbb{P}^2$			
40	1.45E-03	—	2.04E-02	—
80	1.54E-04	3.24	4.48E-04	5.51
160	1.90E-05	3.02	5.87E-06	6.25

$1 - D$ Discontinuous Coefficient Equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0,$$

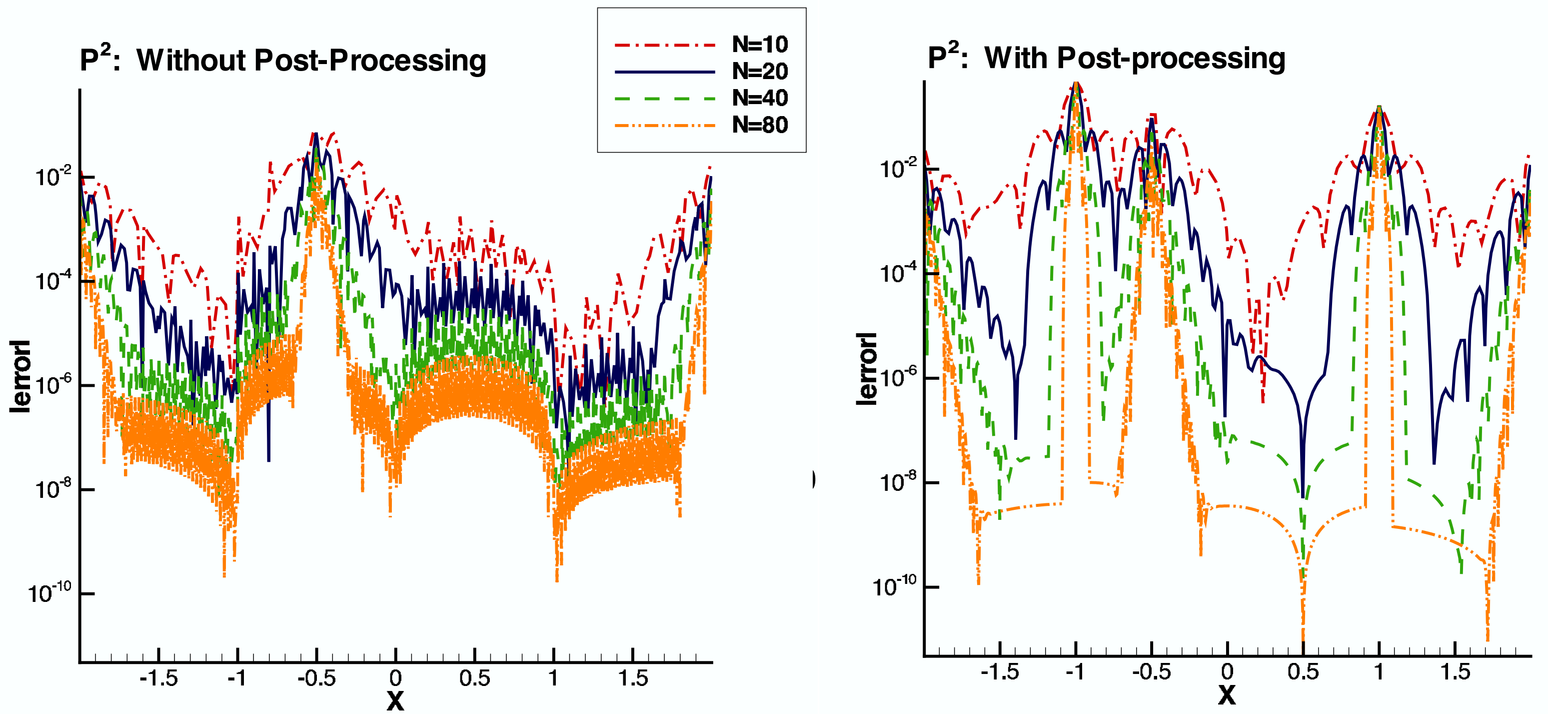
$$a(x) = \begin{cases} 1, & x \in [-2,2]\backslash(-1,1), \\ \frac{1}{2}, & x \in (-1,1) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(\frac{\pi}{2}x), & x \in [-2,2]\backslash(-1,1), \\ \frac{2}{3}\sin(\pi x) & x \in (-1,1). \end{cases}$$

$$u(-2,t) = u(2,t), \quad T = 1.$$

Case of two stationary and two moving shocks.

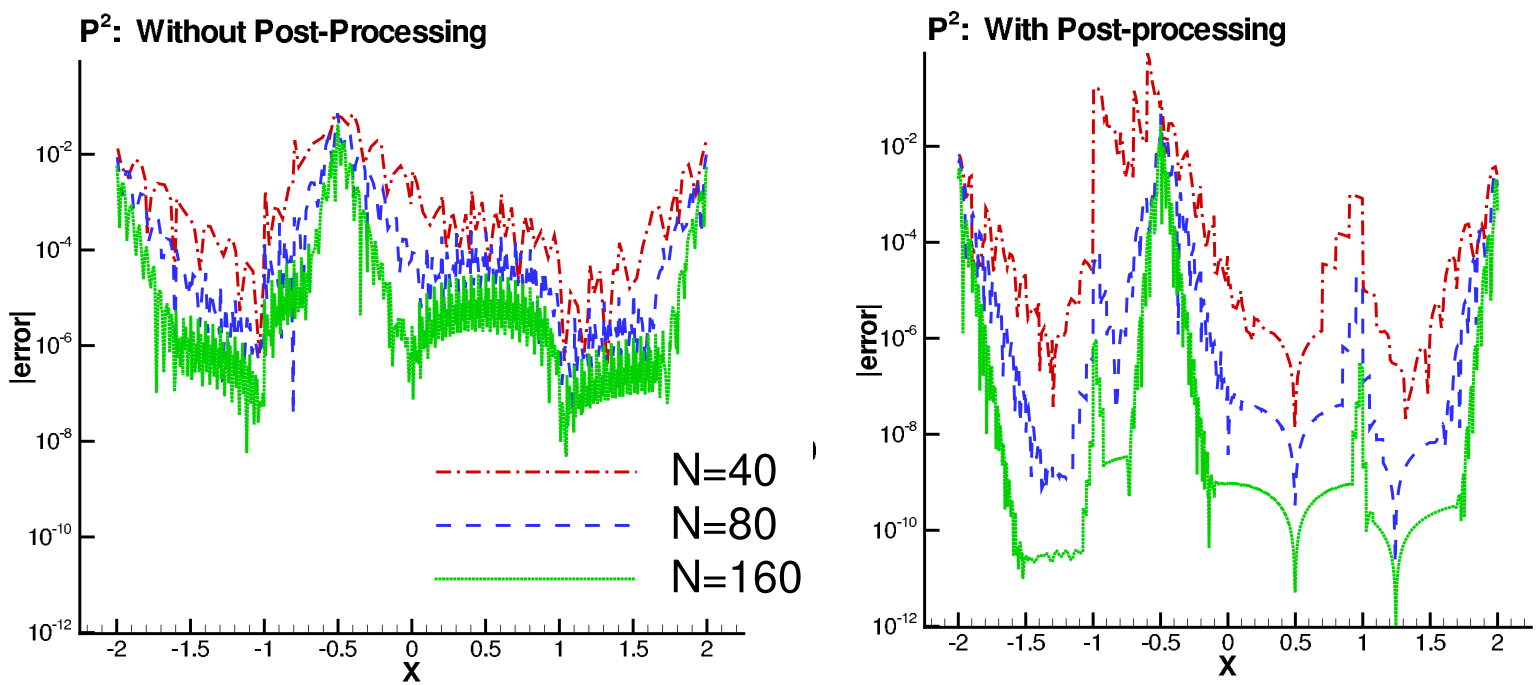
1-D Discontinuous Coefficient



$$u_t+(a(x)u)_x=0$$

x in (-2,2), T=1

1-D Discontinuous Coefficient Using One-Sided Post-Processor



$$u_t+(au)_x=0$$

x in $(-2,2)$, $T=1$

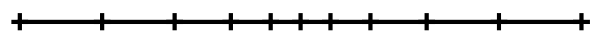
$1 - D$ Discontinuous Coefficient Equation

Case of two stationary and two moving shocks

mesh	L^2 error	order	L^2 error	order
	$u_h(x,2)$		$u^*(x,2)$	
	$\mathbb{P}^1$			
40	1.36E-03	—	2.05E-03	—
80	3.35E-04	2.02	9.57E-05	4.42
160	8.30E-05	2.01	4.61E-06	4.38
320	2.07E-05	2.01	3.40E-07	3.76
	$\mathbb{P}^2$			
40	3.79E-05	—	1.90E-04	—
80	4.77E-06	2.99	2.44E-06	6.28
160	5.98E-07	3.00	2.80E-08	6.45

Smoothly Varying Mesh

Mesh defined by $x = \xi + \frac{1}{2}\sin(\xi)$ where $\xi = ih,\ i = 1, \cdots, N$ is the uniform mesh variable:

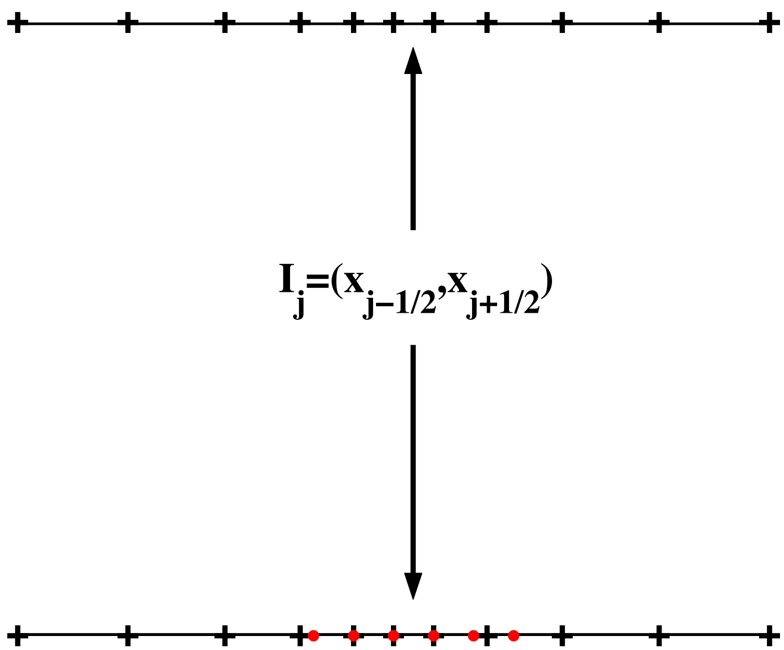


$$u^*(x) = \frac{1}{\triangle x_i} \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1}$$

$$\int_{I_{i+j}} \psi^{(k+1)} \left(\frac{y-x}{\triangle x_i} - \gamma \right) \left(\frac{y - x_{i+j}}{\triangle x_{i+j}} \right)^l dy$$

$k' = \lceil (3k+1)/2 \rceil$

Smoothly Varying Mesh

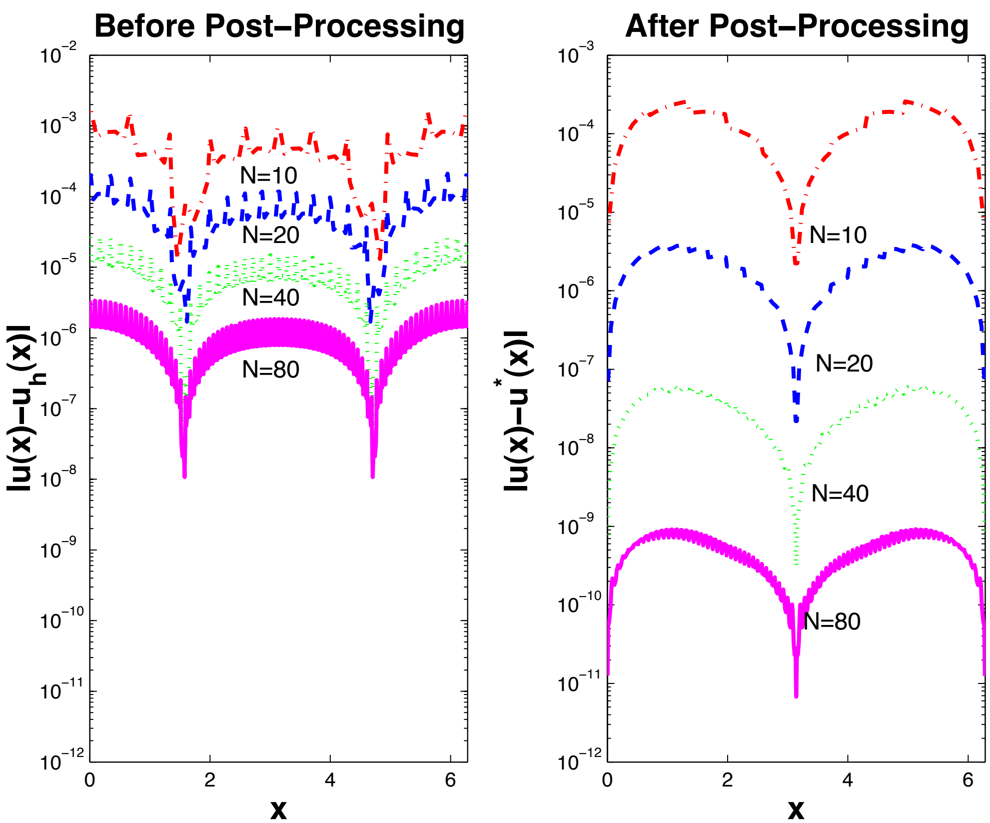


Post-processing solution on cell I_j.

- Create locally uniform mesh of mesh size: $h = x_j$
- Project $u_h(x,T)$ to locally uniform mesh for all x_i in the post-processing region.
- Use $u_n(x,T)$ to find post-processed solution on I_j : $u^*(x) = \sum_{j=-2k}^{2k} \sum_{l=0}^{k} u_n{}_{(i+j)}^{(l)} C(j,l,k,x)$

Smoothly Varying Mesh

Approximation level errors for mesh type $x = \xi + \frac{1}{2}\sin(\xi)$.



Smoothly Varying Mesh

	Approximation		Post-Processed	
N	L^2 error	order	L^2 error	order
	$\mathbb{P}^1$			
10	1.5358E-02	—	3.2149E-03	—
20	3.9029E-03	1.98	2.2069E-04	3.86
40	9.7975E-04	1.99	1.5756E-05	3.81
80	2.4519E-04	2.00	1.2682E-06	3.64
	$\mathbb{P}^2$			
10	1.2175E-03	—	6.1679E-04	—
20	1.5490E-04	2.97	1.0484E-05	5.88
40	1.9448E-05	2.99	1.6048E-07	6.03
80	2.4337E-06	3.00	2.3281E-09	6.11

Approximation level errors.

$$u(x) = \sin(x)$$

$$x = \xi + \tfrac{1}{2}\sin(\xi)$$

Summary

- $(2k+2-d)$-th order accuracy in for the d-th derivative.
- $(2k+1)$-th order accuracy for $2-$D linear hyperbolic systems.
- One-Sided post-processing is able to handle computational boundaries, mesh interfaces or discontinuous coefficients.
- ENO stencil choosing is able to find discontinuities in numerical solution.
- Post-processor is able to improve accuracy for smoothly varying meshes.

Future Work

- Comparing different accuracy enhancement methods.
  - Adjerid, Devine, Flaherty, Krivodonova
  - Cockburn, Luskin, Shu, Süli
  - Zienkiewicz, Zhu
- Nonlinear hyperbolic equations.
- Applications ...

References

ryanjk@ornl.gov

www.csm.ornl.gov/~ryq/home.html

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