Extension of a Post-Processing Technique for the Discontinuous Galerkin Finite Element Methods for Hyperbolic Equations

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Outline

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Discontinuous Galerkin Method

Discontinuous Galerkin Method is a finite-element method.

- Properties of DGM
  - Can handle complicated geometries.
  - Simple treatment of boundary conditions.
  - high order accuracy.
  - flexibility for adaptivity.
  - highly parallelizable.

Discontinuous Galerkin Method

Consider $u_t + f(u)_x = 0$.

Find $u_h(x,t) \in V_h$ such that

$$\int_{I_i} (u_h)_t v dx = \int_{I_i} f(u_h) v_x dx - f((u_h)_{i+\frac{1}{2}}) v_{i+\frac{1}{2}} + f((u_h)_{i-\frac{1}{2}}) v_{i-\frac{1}{2}}$$

for all $v \in V_h$. $V_h = span\{1, \xi_i, \xi_i^2, \cdots, \xi_i^k, i = 1, \cdots, N\}$, where $\xi_i = \frac{x - x_i}{\triangle x_i}$ on $I_i = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2})$.

$u_h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t) \xi_i^l$ if $x \in I_i$.

Use upwind monotone flux

Numerical Scheme:

$$\int_{I_i} (u_h)_t v dx = \int_{I_i} f(u_h) v_x dx - \hat{f}_{i+1/2} v_{i+1/2}^- + \hat{f}_{i-1/2} v_{i-1/2}^+$$

$\forall v \in V_h$.

Example

$$u_t + u_x = 0$$

Choose $v = \xi_i^m = (\frac{x-x_i}{\triangle x_i})^m, \; m = 0, \cdots, k$

$$\int_{I_i} (u_h)_t \xi_i^m dx = \frac{m}{\triangle x_i} \int_{I_i} u_h \xi_i^{m-1} dx - (u_h)^-_{i+\frac{1}{2}} \left(\frac{1}{2}\right)^m + (u_h)^-_{i-\frac{1}{2}} \left(\frac{-1}{2}\right)^m$$

Approximation: $u_h(x,t) = \sum_{l=0}^k u_i^{(l)}(t) \xi_i^l$ for $x \in I_i$:

$$\Rightarrow M \frac{du_h^{(l)}(t)}{dt} = RHS$$

Time Discretization

3^{rd} Order Total Variation Diminishing Runge-Kutta

$$u_t = L(u)$$

$$\begin{aligned} u^{(1)} &= u^n + \triangle t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\triangle t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\triangle t L(u^{(2)}) \end{aligned}$$

where $TV(u^{n+1}) \leq TV(u^n)$

$$TV(u) = \sum_j |u_{j+1} - u_j|.$$

Time step: $dt = \left(\frac{N_{old}}{N}\right)^{\frac{Order_x}{Order_t}} cfl$

Accuracy

Proven: • $k + \frac{1}{2}$ order accuracy for general case.
• $k + 1$ order accuracy in special cases.

Numerically: $k + 1$ order accuracy

- 1992: Zienkiewicz, J.Z. Zhu: Patch Recovery Technique (PRT) for general FEM's (elliptic, $k + 2$).
- 1995: Adjerid, Aiffa, Flaherty: $2k + 1$ super-convergence at Radau points (numerical).
- 1994: Biswas, Devine, Flaherty: error estimate for hyperbolic equations.
- 1999: Cockburn, Luskin, Shu, Süli: post-processing ($2k + 1$).
- 2000: Celani Duarte, Dutra do Carmo: PRT for DGM.
- 2001: Adjerid, Devine, Flaherty, Krivodonova: Radau points superconvergent points of DGM (proven).

Post-Processor

B. Cockburn, M. Luskin, C.-W. Shu, A. Süli, *Enhanced accuracy by post-processing for finite element methods for hyperbolic equations,* Mathematics of Computation, 72 (2003), pp 577-606.

- Discontinuous Galerkin approximation allows us to use negative order error estimates:

$$||u_h - u||_{-l} = \mathcal{O}(h^{2k+1}).$$

- Post-processor extracts this information.
- Works for a locally uniform mesh:
  $\longrightarrow$ Translation invariant
  $\longrightarrow$ Post-Processor is local

Negative Order Sobolev Norm

$$||u||_{-\ell,\Omega} = \sup_{\phi \in \mathcal{C}_0^\infty} \frac{\int_\Omega u(x)\phi(x)dx}{||\phi||_{\ell,\Omega}}, \quad \ell \geq 1$$

Example:

$$u_N = \sin(2\pi N x), \quad \Omega = (-1,1), \ \ \ell \geq 1$$

$$\Rightarrow \quad ||u_N||_{-\ell,\Omega} = \frac{1}{(2\pi N)^\ell}$$

Post-Processor Kernel

- Independent of the partial differential equation.
- Applied only at the final time.
- Filters out oscillations in the error.

Kernel Properties

Bramble & Schatz (1977)
Mock & Lax (1978)

- Compact Support.
- Reproduces polynomials of degree $2k+1$ by convolution.
- Linear combination of B-splines.

Post-processed solution

Post-processed solution: $u^*(x) = K_h^{2(k+1),k+1} * u_h$.

$$K_h^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \psi^{(k+1)}\left(\frac{x}{h} - \gamma\right)$$

$h = \triangle x_i$ for all i, and $c_\gamma^{2(k+1),k+1} \in \mathbb{R}$.

$\psi^{(0)} = \delta_0$, $\psi^{(n)} = \psi^{(n-1)} * \chi$ for $n \geq 2$, where

$$\chi(x) = \begin{cases} 1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{else.} \end{cases}$$

Example

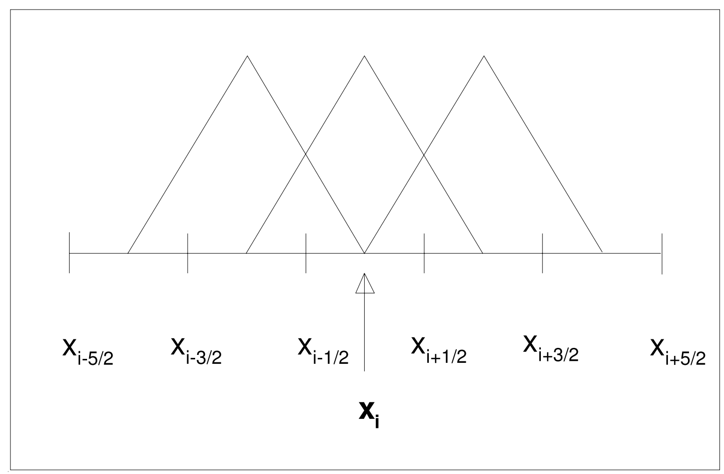
Second Order Approximation

$$u^*(y) = \frac{1}{h}\int_{-\infty}^{\infty} K^{4,2}(\frac{x-y}{h})u_h(x)dx$$

where

$$K_h^{4,2}(y) = \frac{1}{h}(c_{-1}^{4,2}\psi^{(2)}(\frac{y}{h}-1) + c_0^{4,2}\psi^{(2)}(\frac{y}{h}) + c_1^{4,2}\psi^{(2)}(\frac{y}{h}+1))$$

and $\psi^{(2)}(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{else.} \end{cases}$



Example

Second Order Approximation

Find $c_\gamma, \ \gamma = -1, 0, 1:$

Use $K_h^{4,2} * p = p$ for $p = 1, x, x^2$

$$\begin{bmatrix} 1 & 1 & 1 \\ x+1 & x & x-1 \\ x^2+2x+\frac{7}{6} & x^2+\frac{1}{6} & x^2-2x+\frac{7}{6} \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

$$\Rightarrow K_h^{4,2}(y) = \frac{1}{h}\left(\frac{-1}{12}\psi^{(2)}\left(\frac{y}{h}-1\right) + \frac{7}{6}\psi^{(2)}\left(\frac{y}{h}\right) - \frac{1}{12}\psi^{(2)}\left(\frac{y}{h}+1\right)\right)$$

Implementation

Know form of approximation and kernel $\Rightarrow$

$$\begin{aligned} u^*(x) &= \frac{1}{h}\int_{-\infty}^{\infty} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right) u_h(y)\,dy \\ &= \frac{1}{h}\sum_{j=-2k}^{2k}\int_{I_{i+j}} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right) u_h(y)\,dy \\ &= \frac{1}{h}\sum_{j=-2k}^{2k}\int_{I_{i+j}} K^{2(k+1),k+1}\left(\frac{y-x}{h}\right) \sum_{l=0}^{k} u_{i+j}^{(l)} \left(\frac{y-x_{i+j}}{h}\right)^l dy \\ &= \sum_{j=-2k}^{2k}\sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x) \end{aligned}$$

where

$$C(j,l,k,x) \in \mathbb{P}^{2k+1}$$

Derivatives

$$u^*(x) = \sum_{j=-2k}^{2k} \sum_{l=0}^{k} u_{i+j}^{(l)} \frac{d}{dx} C(j,l,k,x) \in \mathbb{P}^{2k}$$

$$u_t + u_x = 0 \quad u(x,0) = \sin(x), \; x \in (0, 2\pi), T = 12.5$$

$\mathbb{P}^2$								
	Before Post-Processing				**After Post-Processing**			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	Errors in First Derivative							
10	1.38E-02	—	4.93E-02	—	2.63E-04	—	4.63E-04	—
20	3.48E-03	1.99	1.28E-02	1.95	6.24E-06	5.40	1.14E-05	5.34
40	8.72E-04	2.00	3.21E-03	1.99	1.61E-07	5.28	3.02E-07	5.24
80	2.18E-04	2.00	8.03E-04	2.00	4.51E-09	5.16	8.65E-09	5.13
160	5.45E-05	2.00	2.00E-04	2.00	1.39E-10	5.02	3.07E-10	4.82
	Errors in Second Derivative							
10	1.35E-01	—	3.37E-01	—	6.77E-04	—	2.48E-03	—
20	6.78E-02	0.99	1.76E-01	0.94	3.64E-05	4.22	1.44E-04	4.11
40	3.39E-02	1.00	8.88E-02	0.99	2.15E-06	4.08	8.81E-06	4.03
80	1.70E-02	1.00	4.45E-02	1.00	1.32E-07	4.03	5.45E-07	4.02
160	8.48E-03	1.00	2.22E-02	1.00	8.19E-09	4.01	3.41E-08	4.00

2–D Approximation & Kernel

$K_h^{2(k+1),k+1}(x,y) =$

$$\frac{1}{h^2}\sum_{-\gamma_x}^{\gamma_x}\sum_{-\gamma_y}^{\gamma_y} c_{\gamma_x+\gamma_y}^{2(k+1),(k+1,k+1)}\psi^{(k+1)}(\frac{x}{h}-\gamma_x)\psi^{(k+1)}(\frac{y}{h}-\gamma_y)$$

Solving: $u_t + f(u)_x + g(u)_y = 0$

$$V_h = span\{1, \xi_i, \eta_j, \xi_i^2, \xi_i\eta_j, \eta_j^2, \cdots, \xi_i^k, \cdots, \eta_j^k,$$
$$i = 1, \cdots, N_x, j = 1, \cdots, N_y\}$$

on $I_{i,j} = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2}) \times (y_j - \frac{\triangle y_j}{2}, y_j + \frac{\triangle y_j}{2})$ for $\xi_i = \frac{x - x_i}{\triangle x_i}$, $\eta_j = \frac{y - y_j}{\triangle y_j}$.

Approximation form: $u_h(x,y,t) = \sum_{l=0}^{k}\sum_{m=0}^{k-l} u_{i,j}^{(l,m)}(t)\xi_i^l\eta_j^m$ for $x, y \in I_{i,j}$.

$2-D$ system

$$u_t - u_x - v_y = 0, \qquad v_t + v_x - u_y = 0.$$

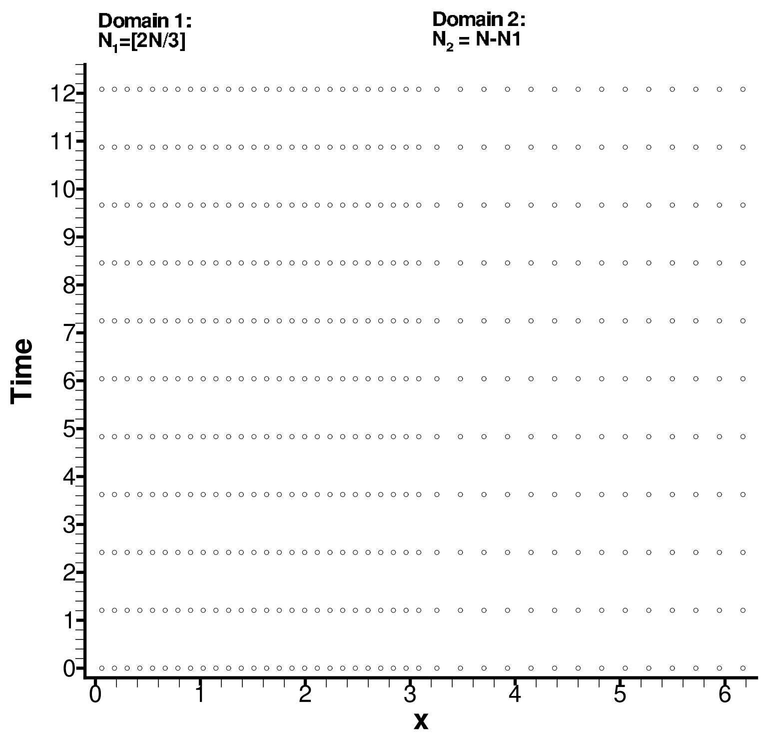
$$u(x,y,0) = \frac{1}{2\sqrt{2}}(\sin(x+y) - \cos(x+y)),$$

$$v(x,y,0) = \frac{1}{2\sqrt{2}}((\sqrt{2}-1)\sin(x+y) + (1+\sqrt{2})\cos(x+y)).$$

Periodic boundary conditions, $T = 12.5$.

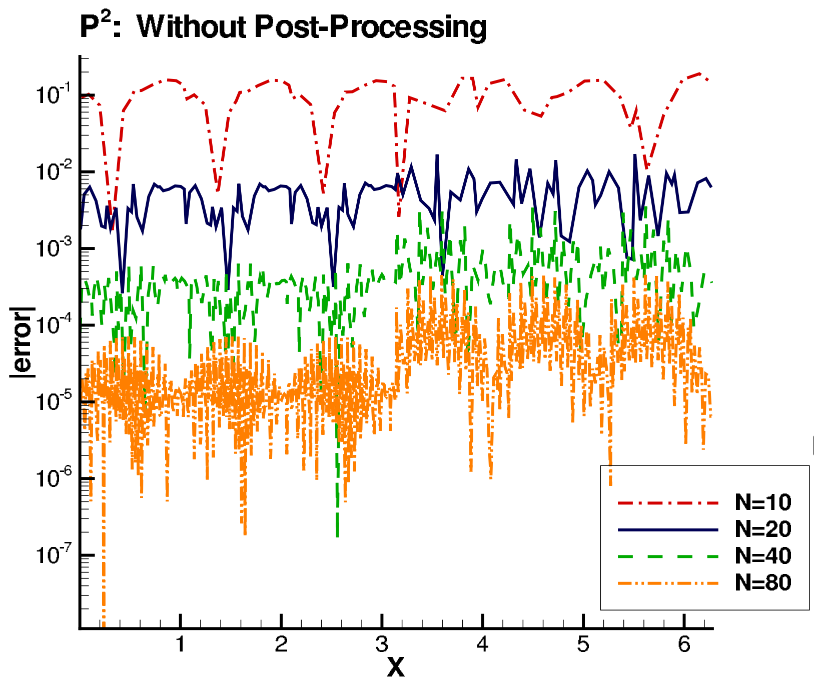
	Before post-processing				**After post-processing**			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	$\mathbb{P}^1$							
10^2	1.22E-01		1.99E-01		1.22E-01		1.72E-01	
20^2	1.96E-02	2.63	3.99E-02	2.31	1.90E-02	2.68	2.68E-02	2.68
40^2	2.85E-03	2.78	7.23E-03	2.47	2.48E-03	2.93	3.51E-03	2.93
80^2	4.71E-04	2.59	1.41E-03	2.36	3.14E-04	2.98	4.44E-04	2.98
	$\mathbb{P}^2$							
10^2	2.66E-03		9.47E-03		1.97E-03		2.79E-03	
20^2	2.52E-04	3.40	1.43E-03	2.73	5.66E-05	5.12	7.99E-05	5.12
40^2	3.10E-05	3.02	1.85E-04	2.94	1.67E-06	5.08	2.36E-06	5.08
80^2	3.88E-06	3.00	2.34E-05	2.98	5.06E-08	5.05	7.15E-08	5.05

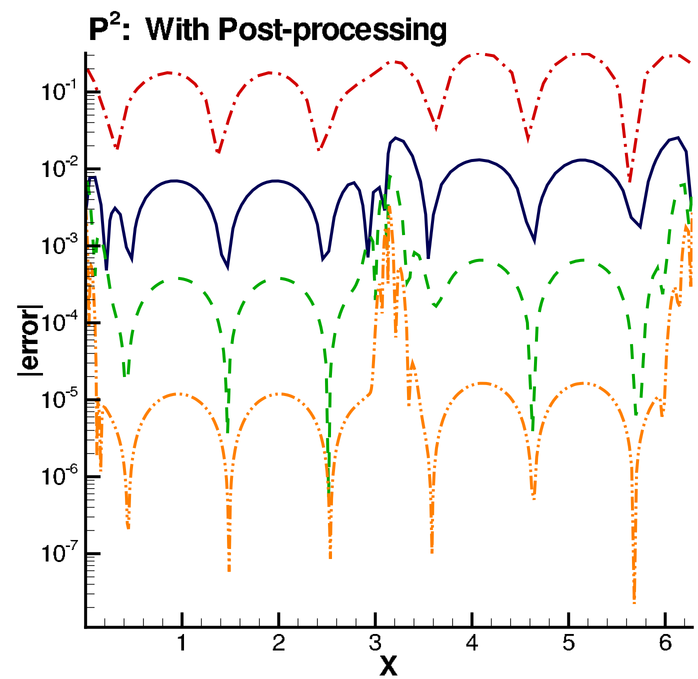
Domains with different mesh sizes



- N=total number of elements for the 2 domains combined.
- $[0, \pi)$, has a more refined mesh.
- Solving a hyperbolic equation with smooth initial conditions over $[0, 2\pi]$.
- Calculating approximation for 2 periods in time.

1-D Multi-Domain





$u_t+u_x=0$

$u(x,0)=\sin(3x)$

x in (0,2pi), T=12.5

$1 - D$ Multi-Domain Problem

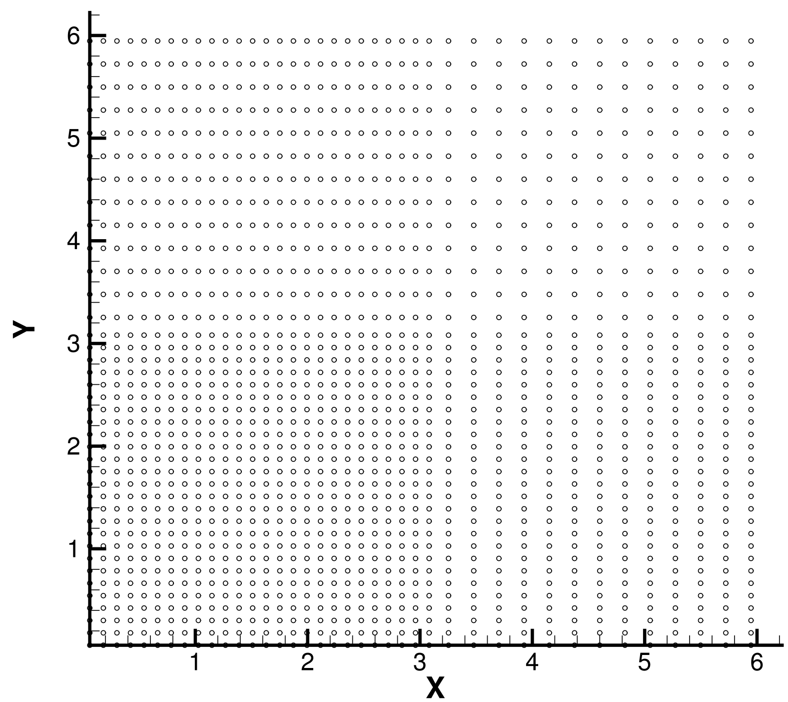
$$u_t + u_x = 0,$$

$$u(x,0) = \sin(3x),$$

$$x \in (0, 2\pi),\ T = 12.5$$

	Before Post-Processing				*After Post-Processing*			
mesh	L^2 Error	Order	L^∞	Order	L^2 Error	Order	L^∞	Order
Finer Mesh, k=2								
6	9.10E-04		1.73E-03		1.14E-03		2.81E-02	
12	7.02E-05	3.70	2.11E-04	3.04	1.73E-05	6.04	5.18E-05	5.00
26	7.19E-06	3.29	2.10E-05	3.33	4.12E-07	5.39	5.39E-07	6.61
52	8.97E-07	3.00	2.63E-06	3.00	1.15E-08	5.16	1.58E-08	5.08
Coarser Mesh, k=2								
4	1.51-03		5.16E-03		1.22E-03		3.74E-03	
8	1.76E-04	3.10	7.16E-04	2.85	6.56E-05	4.22	2.22E-04	4.07
14	2.87E-05	2.62	1.35E-04	2.41	3.93E-07	7.38	1.14E-06	7.60
28	3.58E-06	3.00	1.69E-05	3.00	9.32E-09	5.40	2.11E-08	5.76

Domains with different mesh sizes (2-D)



- $N = N_x = N_y$.
- $(\pi, 2\pi) \times (\pi, 2\pi)$, has the coarsest mesh.
- Solving a hyperbolic equation with smooth initial conditions over $[0, 2\pi] \times [0, 2\pi]$.
- Calculating approximation for 2 periods in time.

$2-D$ Multi-Domain Problem

$$u_t + u_x + u_y = 0, \quad u(x,y,0) = \sin(3(x+y)),$$

$$x, y \in (0, 2\pi), \quad T = 12.5$$

	Before Post-Processing				*After Post-Processing*			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
Finest Mesh, k=2								
24^2	1.29E-02	—	2.11E-02	—	1.29E-02	—	1.94E-02	—
48^2	5.20E-04	4.64	2.90E-03	2.86	4.36E-04	4.89	6.78E-04	4.84
96^2	3.90E-05	3.74	3.73E-04	2.95	1.39E-05	4.97	2.26E-05	4.91
Fine Mesh in x, Coarse in y, k=2								
24×12	9.01E-03	—	2.15E-02	—	9.29E-03	—	1.59E-02	—
48×24	4.06E-04	4.47	2.93E-03	2.87	3.02E-04	4.94	5.23E-04	4.93
96×48	3.63E-05	3.48	3.73E-04	2.97	9.46E-06	5.00	1.85E-05	4.82
Coarsest Mesh, k=2								
12^2	1.56E-02	—	4.89E-02	—	1.37E-02	—	2.03E-02	—
24^2	1.07E-03	3.87	7.41E-03	2.72	4.50E-04	4.92	7.00E-04	4.86
48^2	1.22E-04	3.13	9.44E-04	2.97	1.40E-05	5.01	2.30E-05	4.93

$1 - D$ Variable coefficient equation

$$u(x,t)_t + (a(x)u(x,t))_x = f(x,t),$$

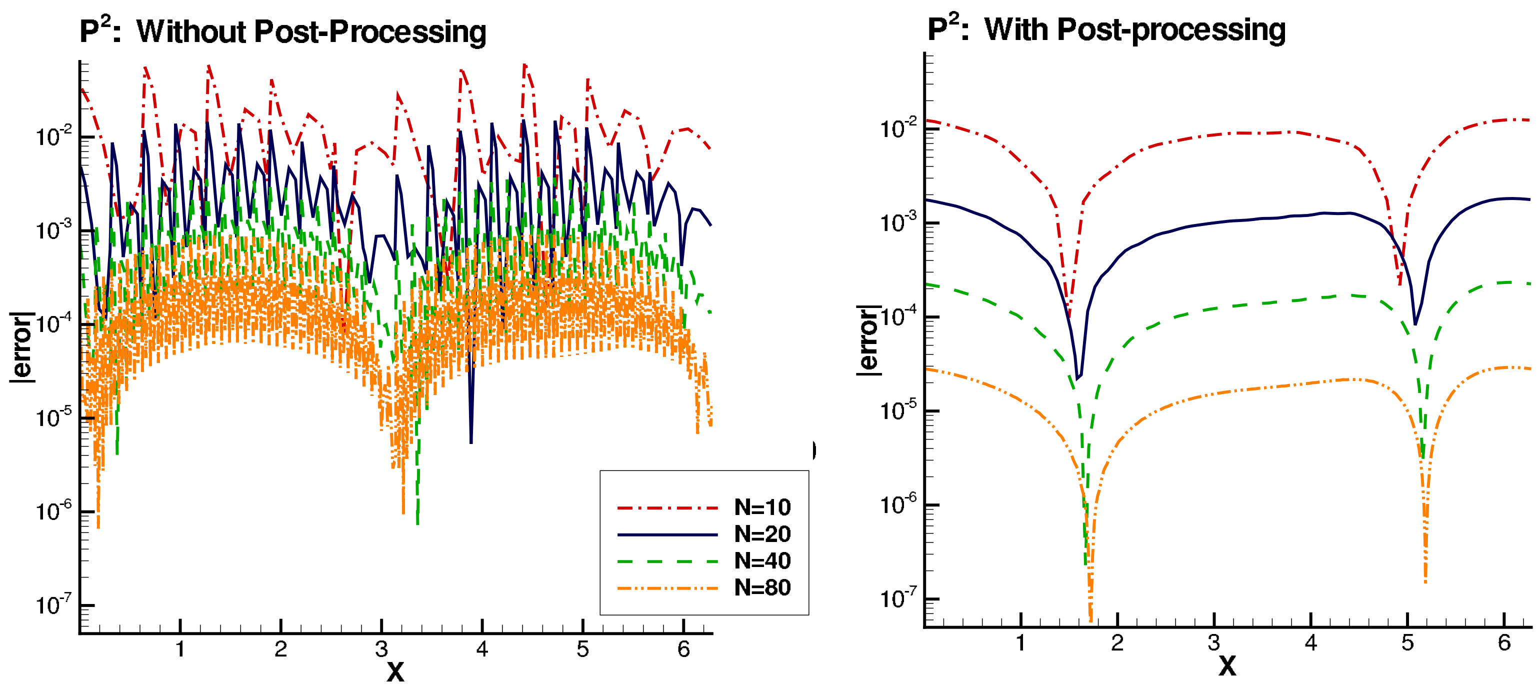
$$a(x) = 2 + \sin(x),$$

$$u(x,0) = \sin(3x)$$

Periodic boundary conditions, $T = 12.5$

	Before post-processing				After post-processing			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$\mathbb{P}^1$								
10	1.83E-02	—	6.40E-02	–	7.82E-02	—	1.26E-02	—
20	4.35E-03	2.07	1.54E-02	2.05	1.08E-03	2.86	1.82E-03	2.79
40	1.07E-03	2.03	3.73E-03	2.05	1.39E-04	2.96	2.34E-04	2.96
80	2.66E-04	2.01	9.12E-04	2.03	1.75E-05	2.99	2.91E-05	3.01
$\mathbb{P}^2$								
10	8.61E-04	—	3.05E-03	—	1.34E-04	—	2.07E-04	—
20	1.07E-04	3.01	3.82E-04	3.00	2.34E-06	5.84	4.02E-05	5.68
40	1.34E-05	3.00	4.71E-05	3.02	4.55E-08	5.69	8.69E-08	5.53
80	1.67E-06	3.00	5.84E-06	3.01	1.09E-09	5.38	2.16E-09	5.33

1-D Variable Coefficient



$u_t+(a(x)u)_x=f(x,t)$

$a(x)=2+\sin(x)$

$u(x,0)=\sin(3x)$

x in $(0,2\pi)$, $T=12.5$

$1 - D$ Variable coefficient equation

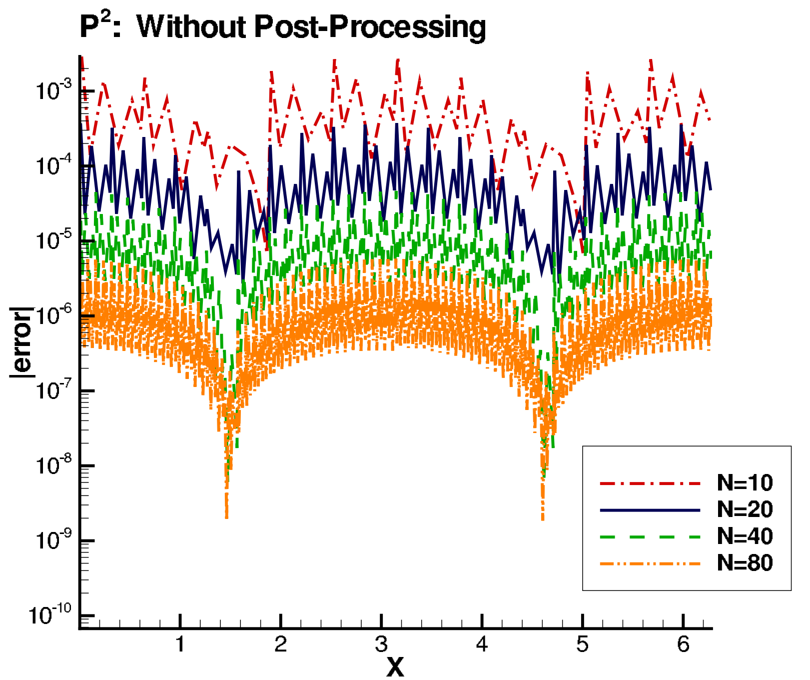
$$u(x,t)_t + (a(x)u(x,t))_x = f(x,t)$$

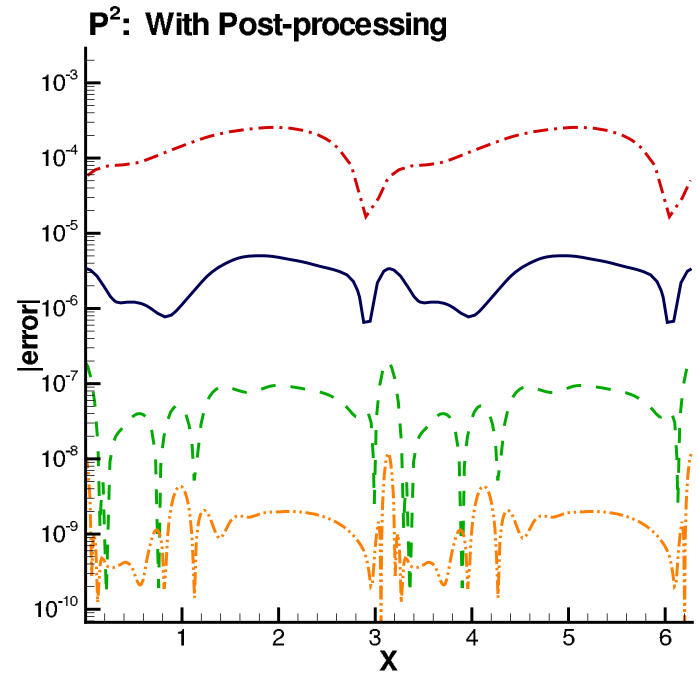
$$a(x) = 2 + |\sin(x)|, \qquad u(x,0) = \sin(x)$$

$$u(0,t) = u(2\pi,t), \quad T = 12.5$$

	Before post-processing				**After post-processing**			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$\mathbb{P}^1$								
10	1.70E-02		5.20E-02		5.07E-03		7.51E-03	
20	4.26E-03	1.99	1.37E-02	1.92	6.02E-04	3.07	9.41E-04	2.99
40	1.06E-03	2.00	3.50E-03	1.97	7.36E-05	3.03	1.22E-04	2.95
80	2.66E-04	2.00	8.84E-04	1.99	9.04E-06	3.02	1.51E-05	3.01
$\mathbb{P}^2$								
10	8.51E-04		2.89E-03		1.50E-04		2.25E-04	
20	1.07E-04	2.99	3.72E-04	2.96	2.83E-06	5.73	4.53E-06	5.64
40	1.34E-05	2.99	4.66E-05	2.99	6.70E-08	5.40	1.93E-07	4.55
80	1.67E-06	2.99	5.80E-06	3.00	2.24E-09	4.90	1.18E-08	4.03

1-D Variable Coefficient





$u_t+(a(x)u)_x=f(x,t)$
$a(x)=2+|\sin(x)|$
$u(x,0)=\sin(x)$
x in (0,2pi), T=12.5

$1-D$ Variable coefficient equation

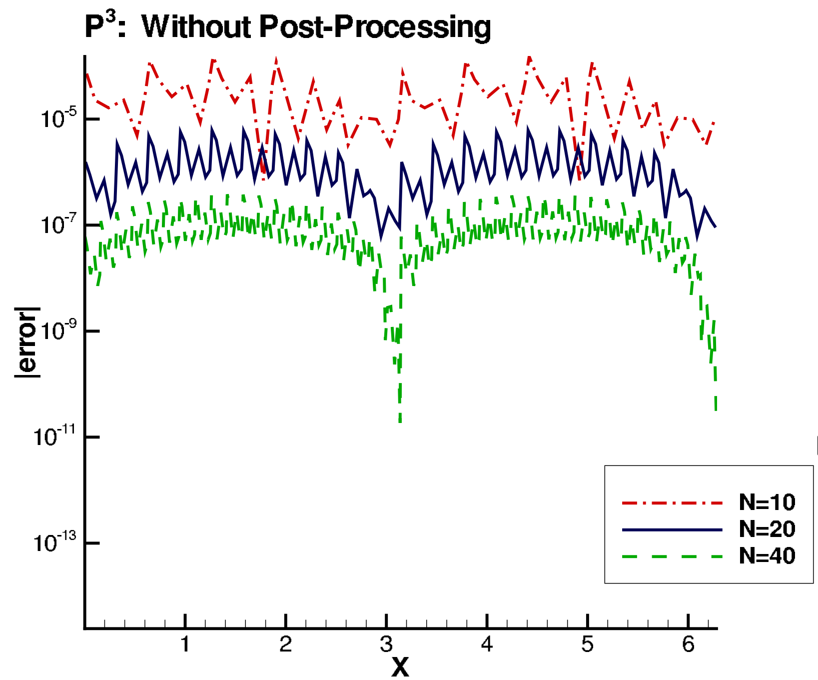
$$u(x,t)_t + (a(x)u(x,t))_x = f(x,t)$$

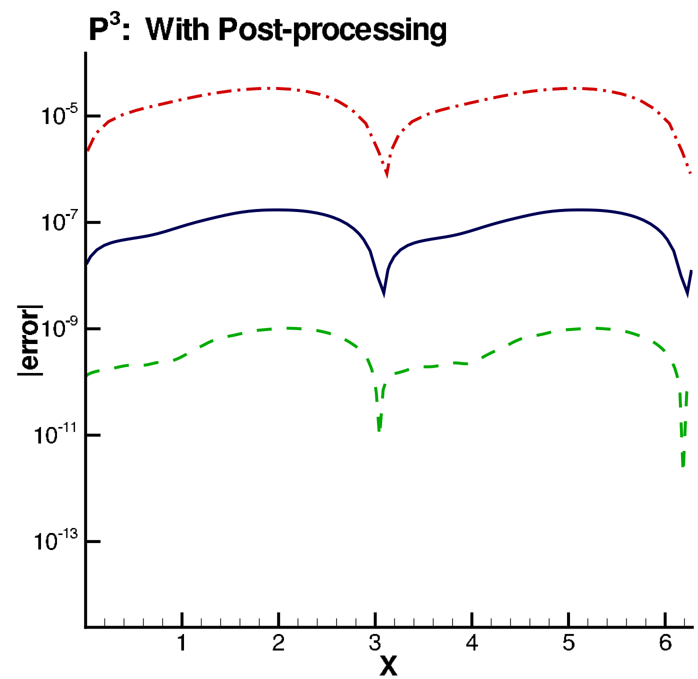
$$a(x) = 2 + |\sin(x)|, \qquad u(x,0) = \sin(x)$$

$$u(0,t) = u(2\pi,t), \quad T = 12.5$$

	$\mathbb{P}^3$							
	Before post-processing				**After post-processing**			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
10	4.38E-05		1.56E-04		2.22E-05		3.31E-05	
20	2.07E-06	4.40	6.10E-06	4.63	7.44E-08	8.22	1.07E-07	8.27
40	1.29E-07	4.00	3.81E-07	4.00	2.93E-10	7.98	4.27E-10	7.97

1-D Variable Coefficient





$u_t+(a(x)u)_x=f(x,t)$

$a(x)=2+|\sin(x)|$

$u(x,0)=\sin(x)$

x in $(0,2pi)$, $T=12.5$

$2-D$ Variable coefficient equation

$$u_t + (a(x,y)u)_x + (a(x,y)u)_y = f(x,y,t)$$

$$a(x,y) = 2 + \sin(x+y), \qquad u(x,y,0) = \sin(x+y)$$

Periodic boundary conditions, $T = 12.5$.

	Before post-processing				**After post-processing**			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	$\mathbb{P}^1$							
10^2	3.54E-02	—	2.01E-01	—	1.44E-02	—	2.24E-02	—
20^2	8.43E-03	2.07	5.10E-02	2.02	1.47E-03	3.30	2.69E-03	3.06
40^2	2.09E-03	2.01	1.28E-02	2.00	1.97E-04	2.89	3.61E-04	2.90
80^2	5.23E-04	2.00	3.17E-03	2.01	2.72E-05	2.86	5.12E-05	2.82
	$\mathbb{P}^2$							
10^2	3.87E-03	—	3.39E-02	—	2.61E-04	—	4.62E-04	—
20^2	4.79E-04	3.02	4.06E-03	3.06	6.57E-06	5.31	1.06E-05	5.45
40^2	5.97E-05	3.01	4.94E-04	3.04	2.41E-07	4.77	4.19E-07	4.66
80^2	7.45E-06	3.00	6.05E-05	3.03	8.11E-09	4.89	1.52E-08	4.79

$1-D$ Discontinuous coefficient equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0$$

$$a(x) = \begin{cases} 1, & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ \frac{1}{2}, & x \in (-\frac{1}{2},\frac{1}{2}) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(2\pi x), & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ -2\pi\cos(4\pi x) & x \in (-\frac{1}{2},\frac{1}{2}). \end{cases}$$

$$u(-1,t) = u(1,t), \quad T = 12.5$$

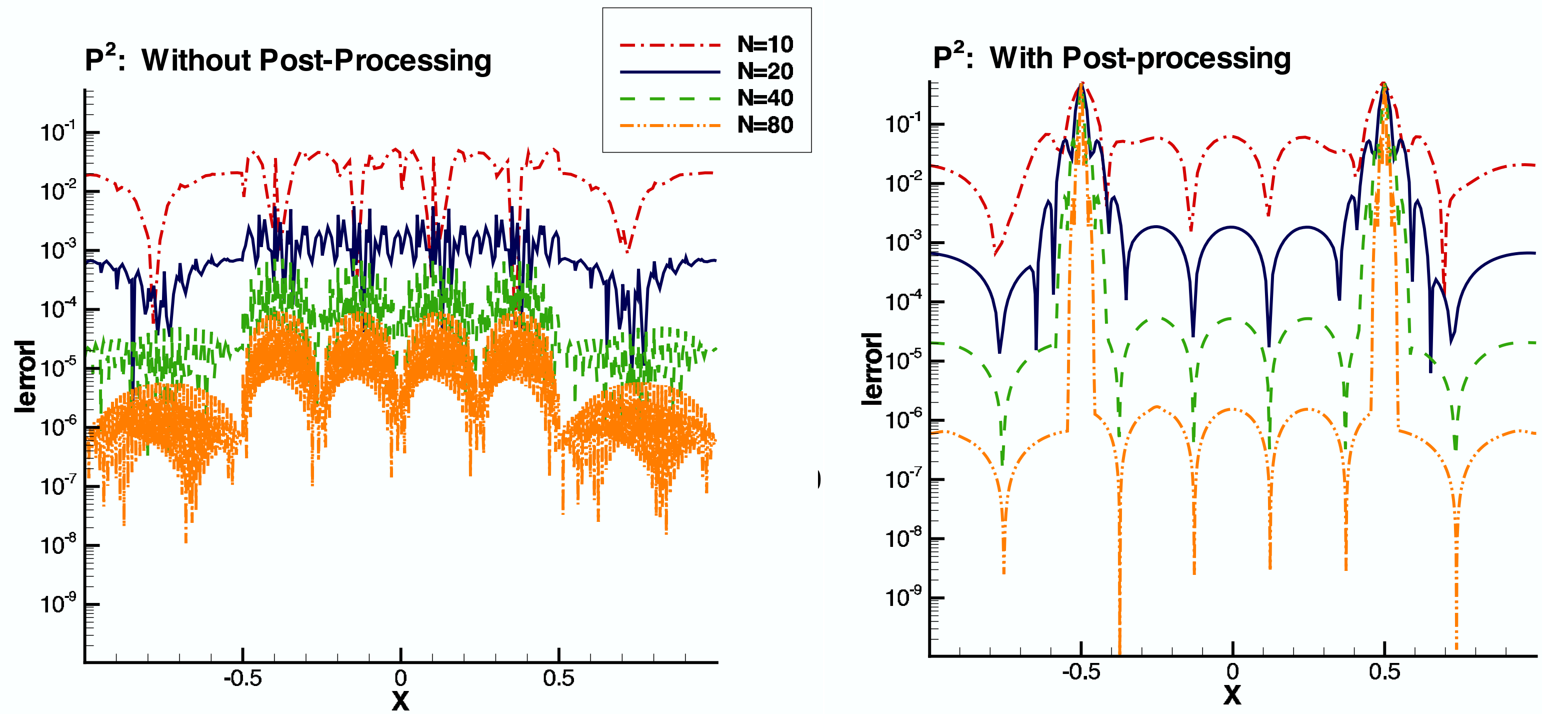
Case of two stationary shocks.

$1 - D$ Discontinuous coefficient equation

Case of two stationary shocks

	Before post-processing				After post-processing			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	$\mathbb{P}^1$							
20	4.41E-01	—	1.56E+00	—	4.44E-01	—	1.54E+00	—
40	9.89E-02	2.16	3.78E-01	2.04	9.92E-02	2.16	3.49E-01	2.14
80	1.39E-02	2.83	5.81E-01	2.70	1.37E-02	2.86	4.80E-02	2.86
160	1.88E-03	2.89	8.74E-03	2.73	1.73E-03	2.98	6.08E-03	2.98
	$\mathbb{P}^2$							
20	1.28E-02	—	5.19E-02	—	1.66E-02	—	6.23E-02	—
40	6.23E-04	4.36	5.10E-03	3.35	4.95E-04	5.07	1.83E-03	5.09
80	7.42E-06	3.03	8.56E-05	2.95	1.44E-05	5.10	5.22E-05	5.13
160	7.42E-06	3.03	8.56E-05	2.95	4.30E-07	5.06	1.54E-06	5.08

1-D Discontinuous Coefficient



$u_t+(a(x)u)_x=0$

x in (-1,1), T=12.5

$1-D$ Discontinuous coefficient equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0,$$

$$a(x) = \begin{cases} 1, & x \in [-2,2]\backslash(-1,1), \\ \frac{1}{2}, & x \in (-1,1) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(\frac{\pi}{2}x), & x \in [-2,2]\backslash(-1,1), \\ \frac{2}{3}\sin(\pi x) & x \in (-1,1). \end{cases}$$

$$u(-2,t) = u(2,t), \quad T = 1.$$

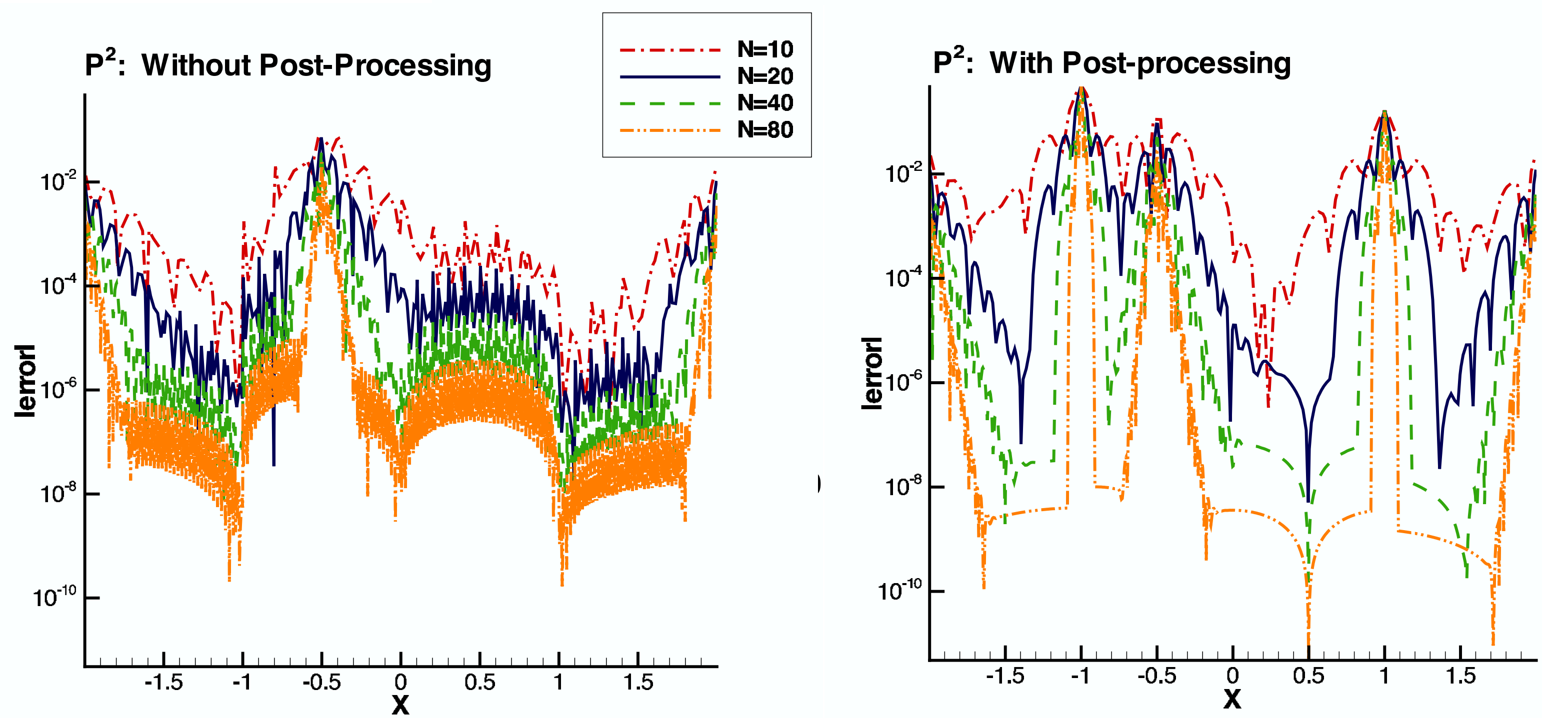
Case of two stationary and two moving shocks.

$1 - D$ Discontinuous coefficient equation

Case of two stationary and two moving shocks

	Before post-processing				After post-processing			
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	$\mathbb{P}^1$							
80	8.50E-04		6.46E-03		6.23E-05		3.79E-04	
160	2.15E-04	1.98	1.66E-03	1.96	7.64E-06	3.03	3.26E-05	3.54
320	5.39E-05	2.00	4.20E-04	1.98	9.29E-07	3.04	3.43E-06	3.25
640	1.35E-05	2.00	1.06E-04	1.99	1.15E-07	3.02	4.26E-07	3.01
	$\mathbb{P}^2$							
40	6.61E-05		2.45E-04		3.43E-05		3.25E-04	
80	7.90E-06	3.07	3.08E-05	2.99	2.55E-08	10.39	6.14E-08	12.37
160	9.86E-07	3.00	3.85E-06	3.00	5.05E-10	5.66	9.99E-10	5.94
320	1.23E-07	3.00	4.82E-07	3.00	1.35E-11	5.23	3.67E-11	4.77

1-D Discontinuous Coefficient

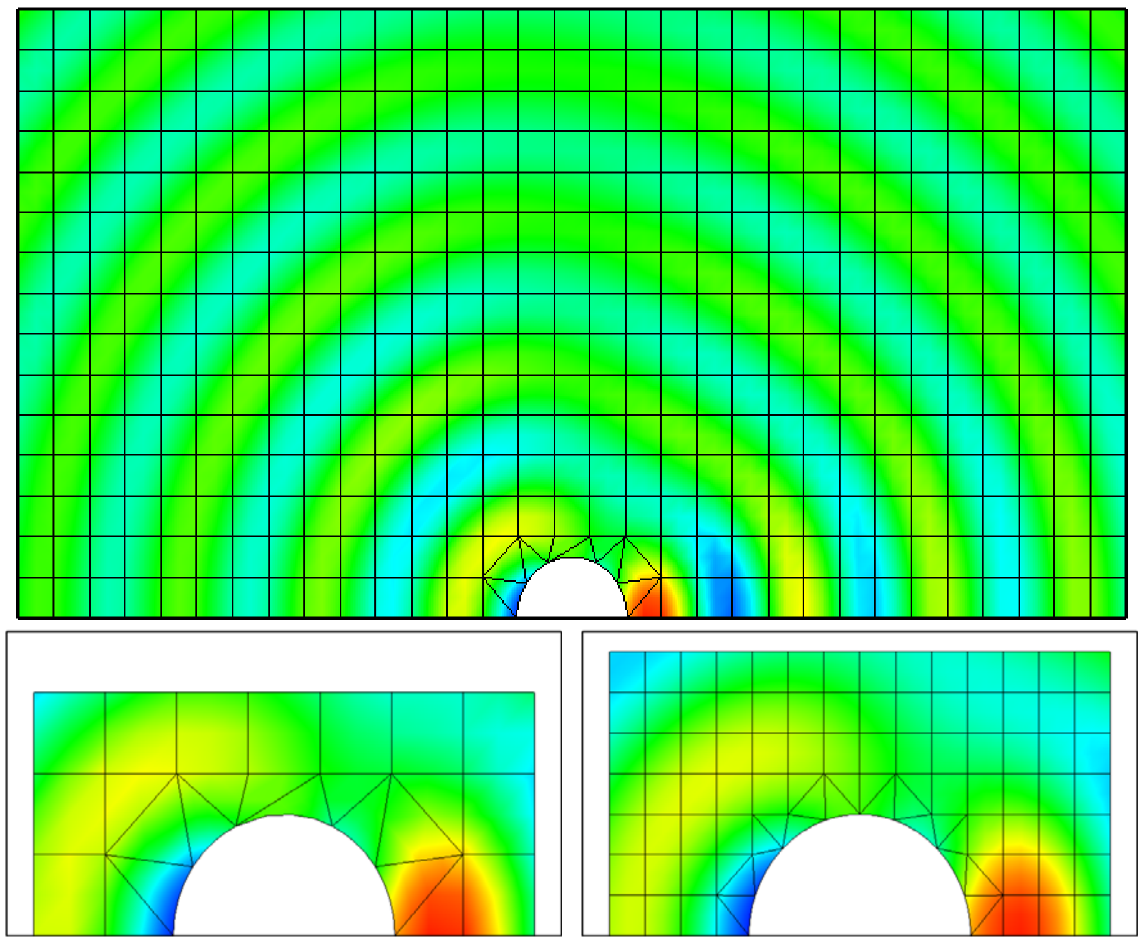


$$u_t+(a(x)u)_x=0$$

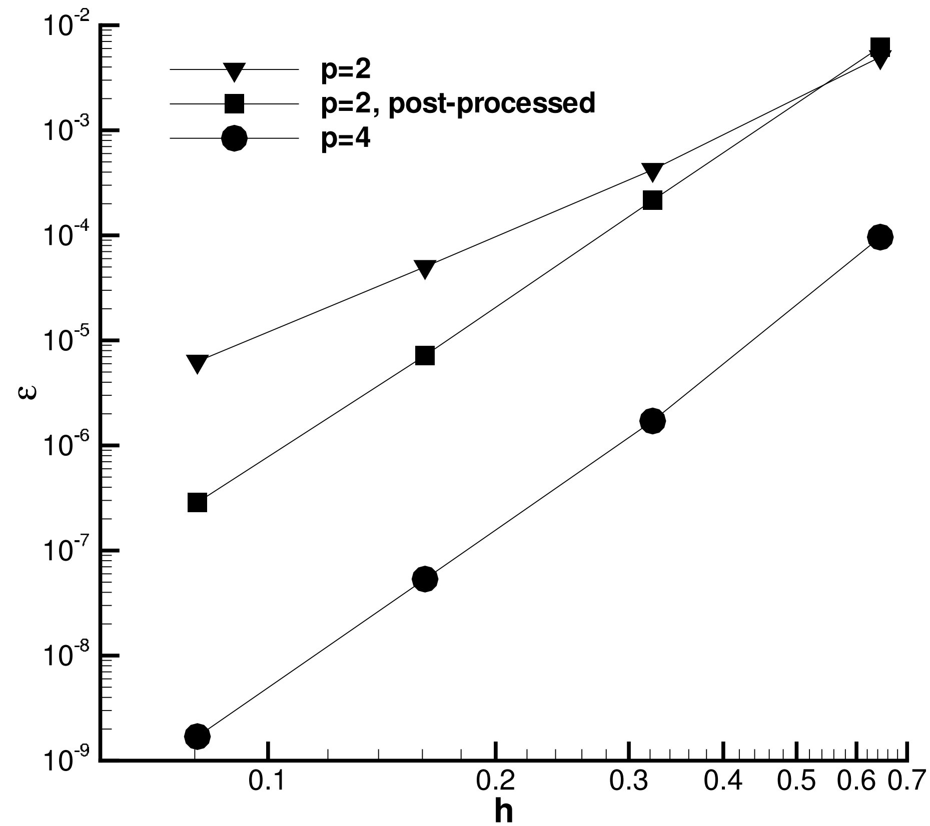
x in $(-2,2)$, $T=1$

Linearized Euler Equations

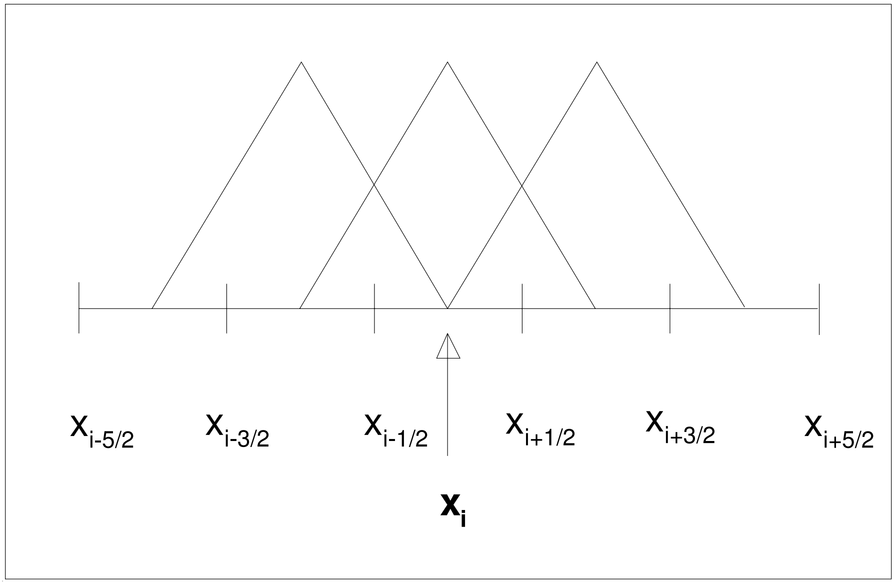
Scatter of a plane wave off of a cylinder: wavelength $\lambda = 2.5r$



Aeroacoustic Test Problem



Symmetric Post-Processor

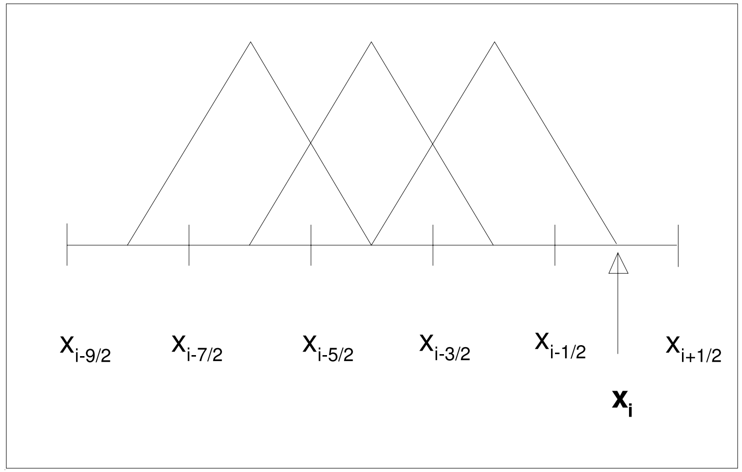


$$u^*(x) = \sum_{j=-2k}^{2k} \sum_{l=0}^{k} u_{i+j}^{(l)} \, C(j, l, k, x)$$

where

$$C(j, l, k, x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_{\gamma}^{2(k+1),k+1} \int_{I_{i+j}} \psi^{(k+1)} \left(\frac{y - x}{h} - \gamma \right) \left(\frac{y - x_{i+j}}{h} \right)^{l} dy$$

Left Post-Processor



$$u^*(x) = \sum_{j=-4k}^{0} \sum_{l=0}^{k} u_{i+j}^{(l)} \, C(j, l, k, x)$$

where

$$C(j, l, k, x) = \frac{1}{h} \sum_{\gamma=-2k-1}^{-1} c_\gamma^{2(k+1),k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}(\eta)\,(\xi_i + \eta + \gamma - j)^l \, dy$$

For $k = 1$:

$$K(x) = \frac{11}{12}\psi^{(2)}(x+3) - \frac{17}{6}\psi^{(2)}(x+2) + \frac{35}{12}\psi^{(2)}(x+1)$$

Multi-Domain Problem

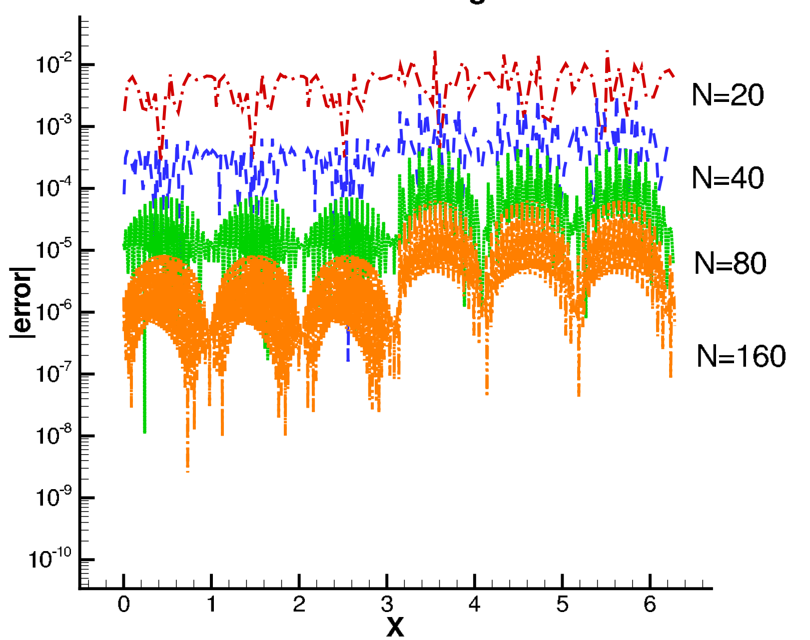
$$u_t + u_x = 0, \qquad u(x,0) = \sin(3x)$$

$$x \in (0, 2\pi),\ T = 12.5$$

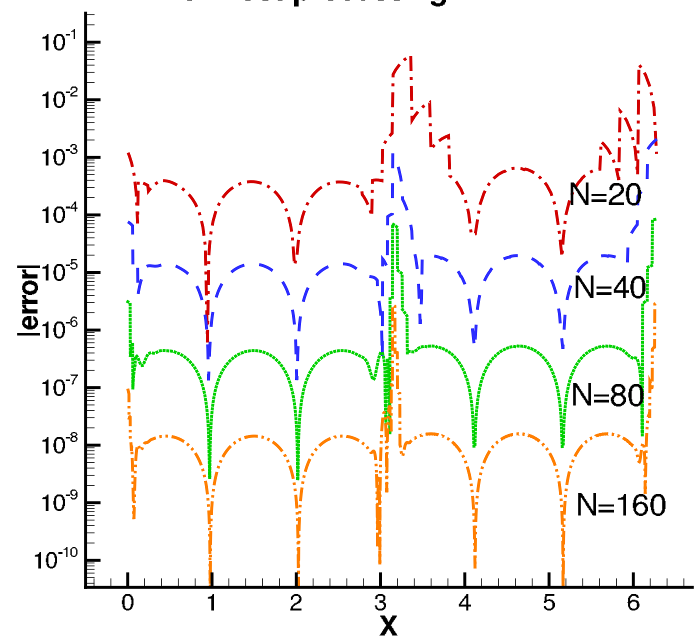
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	Before post-processing				**After post-processing**			
	$\mathbb{P}^1$							
20	3.48E-01	—	5.40E-01	—	3.76E-01	—	5.81E-01	—
40	6.20E-02	2.49	1.07E-01	2.34	6.32E-02	2.57	1.09E-01	2.41
80	9.55E-03	2.70	1.89E-02	2.50	1.02E-02	2.63	2.93E-02	1.89
160	1.46E-03	2.71	3.22E-03	2.55	1.25E-03	3.03	4.26E-03	2.78
320	2.82E-04	2.38	9.21E-04	1.80	1.53E-04	3.03	5.80E-04	2.88
	$\mathbb{P}^2$							
20	9.77E-03	—	2.44E-02	—	3.20E-01	—	1.40E-00	—
40	7.95E-04	3.62	3.55E-03	2.78	9.96E-03	5.01	5.88E-02	4.58
80	1.05E-04	2.92	5.08E-04	2.81	2.67E-04	5.22	2.08E-03	4.82
160	1.31E-05	3.01	6.37E-05	2.99	1.03E-05	4.69	8.52E-05	4.61
320	1.68E-06	2.96	8.19E-06	2.96	2.74E-07	5.24	3.00E-06	4.83

1-D Multi-Domain
Using One-Sided Post-Processor

P²: Without Post-Processing



P²: With Post-processing



$u_t + u_x = 0$

$u(x,0) = \sin(3x)$

x in (0,2pi), T=12.5

$1 - D$ Discontinuous coefficient equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0$$

$$a(x) = \begin{cases} 1, & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ \frac{1}{2}, & x \in (-\frac{1}{2},\frac{1}{2}) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(2\pi x), & x \in [-1,1]\backslash(-\frac{1}{2},\frac{1}{2}), \\ -2\pi\cos(4\pi x) & x \in (-\frac{1}{2},\frac{1}{2}). \end{cases}$$

$$u(-1,t) = u(1,t), \quad T = 12.5$$

Case of two stationary shocks.

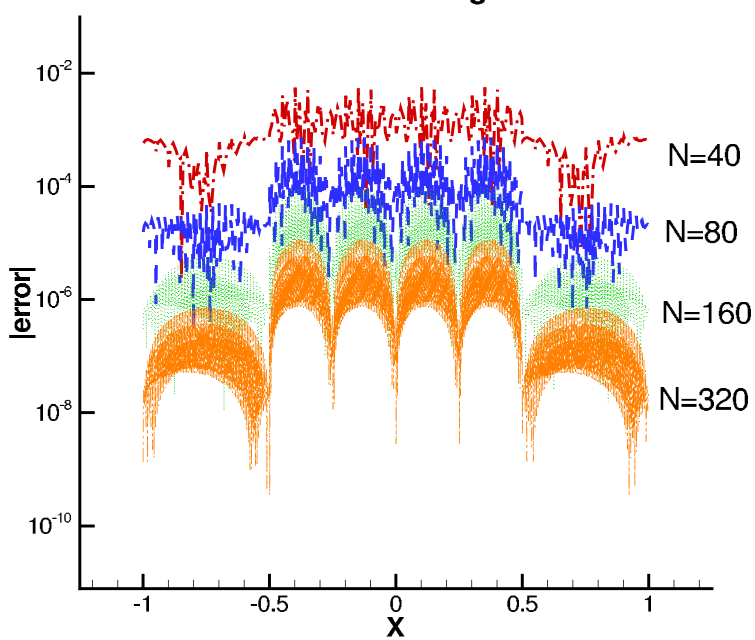
$1 - D$ Discontinuous coefficient equation

Case of two stationary shocks

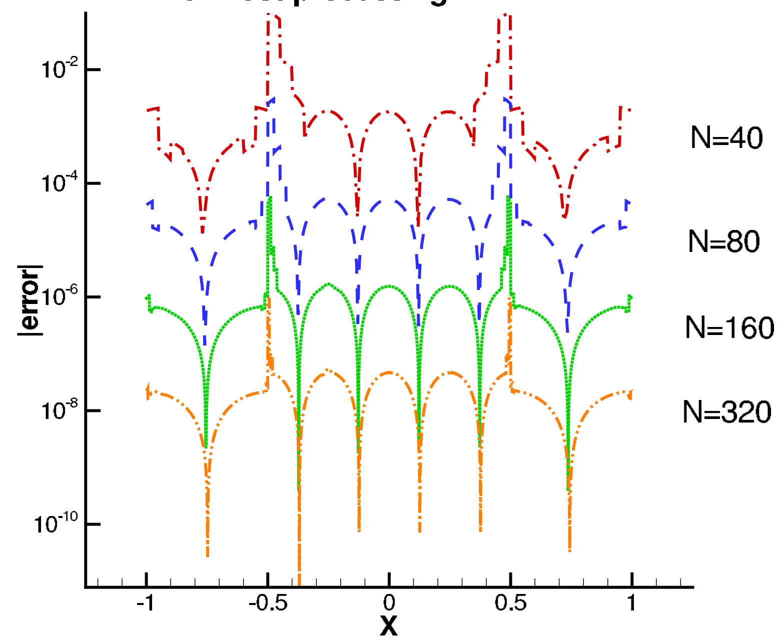
mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	Before post-processing				**After post-processing**			
	$\mathbb{P}^1$							
20	8.55E-01	—	1.56E+01	—	8.17E-01	—	1.55E+01	—
40	1.93E-01	2.15	3.80E-01	2.04	1.80E-01	2.18	3.50E-01	2.15
80	2.72E-02	2.83	5.84E-02	2.70	2.58E-02	2.80	4.90E-02	2.84
160	3.69E-03	2.88	8.84E-03	2.72	3.34E-03	2.95	6.21E-03	2.98
320	5.67E-04	2.70	1.46E-03	2.60	4.21E-04	2.99	7.87E-04	2.98
	$\mathbb{P}^2$							
40	1.45E-03	—	5.65E-03	—	2.04E-02	—	9.85E-02	—
80	1.54E-04	3.24	7.29E-04	2.95	4.48E-04	5.51	3.04E-03	5.02
160	1.90E-05	3.02	9.22E-05	2.98	5.87E-06	6.25	5.81E-05	5.71
320	2.37E-06	3.00	1.16E-05	3.00	7.26E-08	6.34	9.72E-07	5.90

1-D Discontinuous Coefficient Using One-Sided Post-Processor

P^2: Without Post-Processing



P^2: With Post-processing



$u_t+(au)_x=0$

x in (-1,1), T=12.5

$1-D$ Discontinuous coefficient equation

$$u(x,t)_t + (a(x)u(x,t))_x = 0,$$

$$a(x) = \begin{cases} 1, & x \in [-2,2]\backslash(-1,1), \\ \frac{1}{2}, & x \in (-1,1) \end{cases}$$

$$u(x,0) = \begin{cases} \cos(\frac{\pi}{2}x), & x \in [-2,2]\backslash(-1,1), \\ \frac{2}{3}\sin(\pi x) & x \in (-1,1). \end{cases}$$

$$u(-2,t) = u(2,t), \quad T = 1.$$

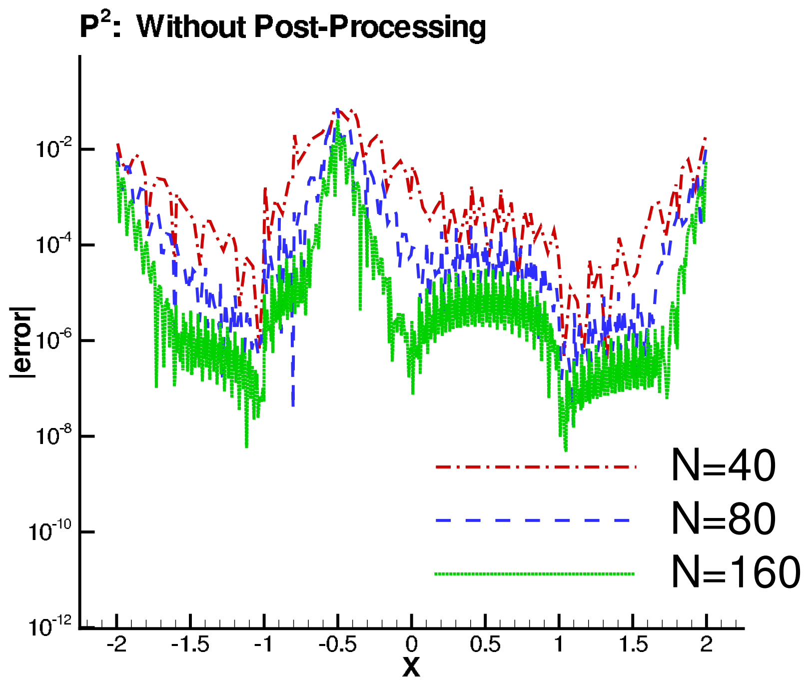
Case of two stationary and two moving shocks.

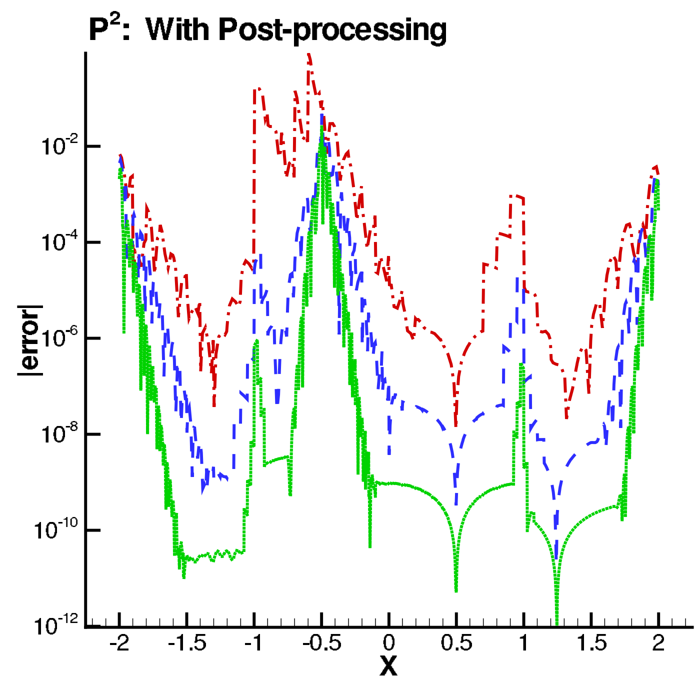
$1 - D$ Discontinuous coefficient equation

Case of two stationary and two moving shocks

mesh	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
	Before post-processing				**After post-processing**			
	$\mathbb{P}^1$							
40	1.36E-03	—	8.88E-03	—	2.05E-03	—	1.17E-02	—
80	3.35E-04	2.02	2.31E-03	1.94	9.57E-05	4.42	7.82E-04	3.91
160	8.30E-05	2.01	5.87E-04	1.98	4.61E-06	4.38	4.66E-05	4.07
320	2.07E-05	2.01	1.48E-04	1.99	3.40E-07	3.76	2.49E-06	4.23
640	5.16E-06	2.00	3.71E-05	2.00	3.87E-08	3.14	1.32E-07	4.24
	$\mathbb{P}^2$							
40	3.79E-05	—	2.45E-04	—	1.90E-04	—	1.00E-03	—
80	4.77E-06	2.99	3.08E-05	2.99	2.44E-06	6.28	1.89E-05	5.73
160	5.98E-07	3.00	3.85E-06	3.00	2.80E-08	6.45	3.09E-07	5.93

1-D Discontinuous Coefficient
Using One-Sided Post-Processor





$$u_t+(au)_x=0$$

x in (-2,2), T=1

Conclusions

- $(2k+2-d)$-th order accuracy in for the d-th derivative.
- $(2k+1)$-th order accuracy for 2−D linear hyperbolic systems.
- For domains with different mesh sizes the post-processor gives $2k+1$ order accuracy away from domain boundaries.
- Works well for Variable & Discontinuous coefficient Hyperbolic equations, even in the case of moving shocks.
- One-Sided post-processing is able to handle computational boundaries, mesh interfaces or discontinuous coefficients.

Future Work

- Applying post-processor to smoothly varying mesh.
- Non-linear Equations.
- Comparing different accuracy enhancement methods.

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