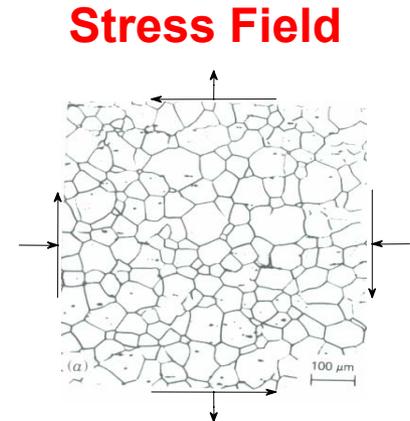

Scaling of Fracture Strength in Disordered Quasi-brittle Materials

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Motivation: Stress Induced Microcracking Evolution

Macroscopic properties and behavior of quasi-brittle materials are significantly affected by the internal microstructure and damage/microcracking evolution



Microcracking evolution

Phenomenological Material Models:

- microstructure-insensitive
- valid only for moderate damage levels
- local stress field fluctuations and interactions are not considered

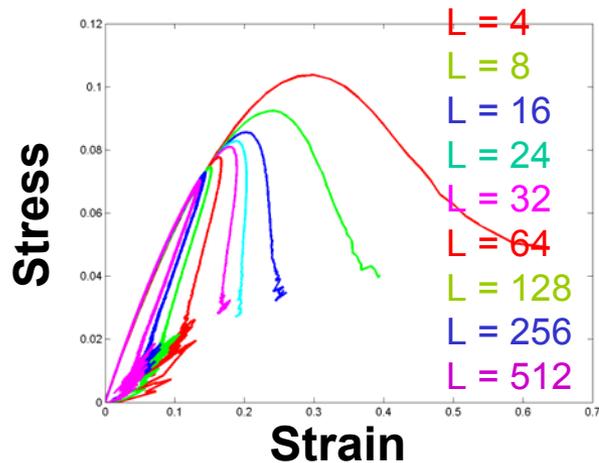
Modeling at the mesoscale will lead to a fundamental understanding of the effect of microstructural features on the microcracking evolution in brittle materials

Objective: Scaling Laws in Fracture

Mesoscale Damage Evolution: Relevant Questions:

- depends on the system size L
- computationally intractable $\approx O(L^4)$

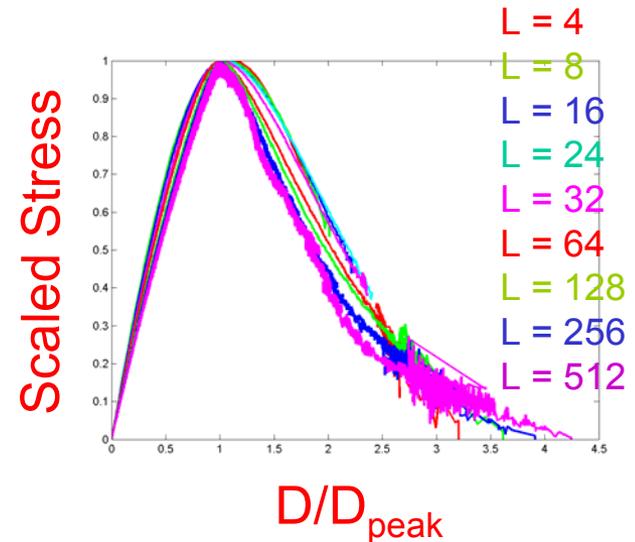
- size effect on failure
- intensive measure of damage comparing the extent of damage between two specimens?



Material response is size dependent



Scaling laws are required to obtain a "normalized" response that couple mesoscopic and continuum length scales



Explicit modeling of material **microstructure** combined with the **scaling theory** accounts for **size effects** and local stress field interactions during damage/microcrack evolution

Outline

- Numerical Methodology
- Typical Lattice Response
- Scaling Theory (Renormalization Group)
- Numerical Results
- Summary

Numerical Methodology

Mesososcopic Simulation: Discrete Lattice Models

Focus of the study is not on any particular material

But, in capturing the generic features of damage evolution

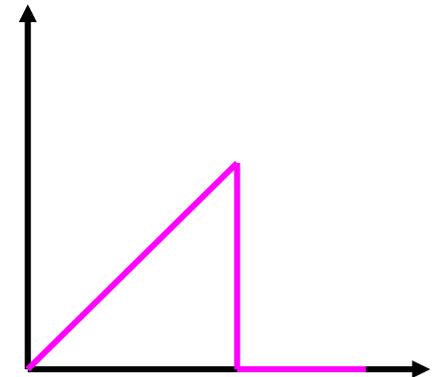
Essential ingredients of breakage process:

- Initial material disorder (inhomogeneities)
- redistribution of stresses due to damage evolution

Discrete Lattice Models:

- disorder in bond strength and stiffness
- elastic response characteristics of the bonds
- bond breaking rule (failure criteria)

Perfectly brittle bond

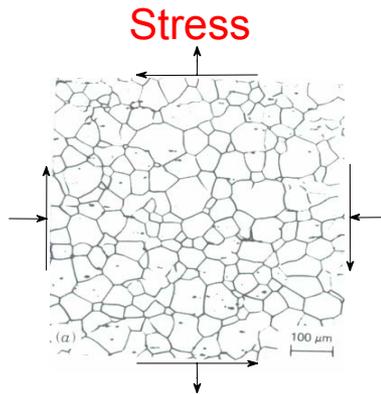


Any realistic damage evolution description must be capable of reproducing the behavior of these idealized discrete lattice models

Mesososcopic Modeling Approach

Failure of a bond is governed by

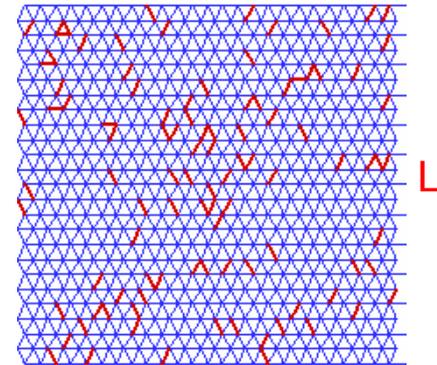
- weakest bond of the disordered medium
- stress concentration around material inhomogeneities



Discretization with random disorder distributions



Lattice system with disorder



Disorder Type

+

Lattice Topology

+

Lattice Bond Model

Applied Stress



Failure Criteria

Mesososcopic Damage Evolution

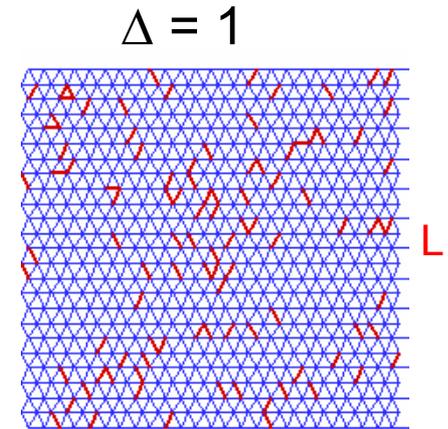
Analysis Procedure

Procedure:

Step 0: For each bond in the lattice system, assign unit stiffness and random force threshold f_i^{th}

Step 1: Impose a unit macroscopic displacement

Step 2: Calculate the force f_i in each bond through lattice equilibrium



Lattice system with disorder

$$\mathbf{K} = \sum_i \mathbf{f}_i^2 \quad \mathbf{K} \text{ Global stiffness}$$

Step 3: Determine the bond i_c for which

$$\frac{1}{\lambda} = \max_i \left(\frac{f_i}{f_i^{th}} \right)$$

Step 4: Record the lattice displacement and force $(\lambda, \mathbf{K}\lambda)$

Step 5: Remove the bond i_c and repeat steps 1-4, until the entire lattice system breaks apart

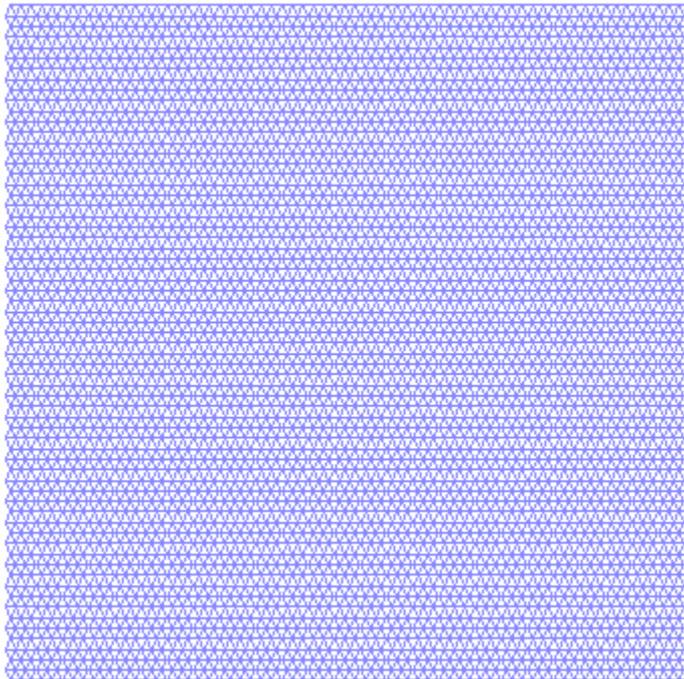
Typical Lattice Response

Typical Loading Response

512 x 512



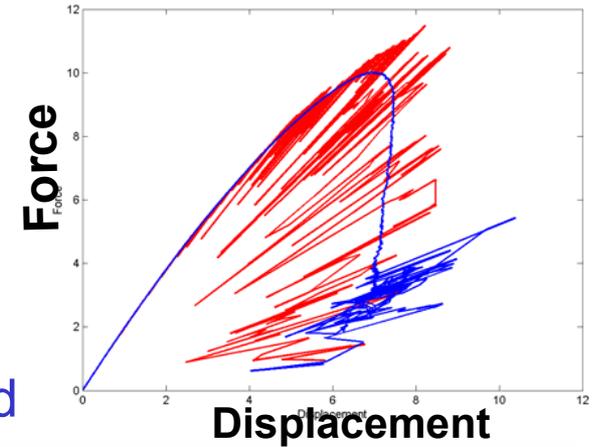
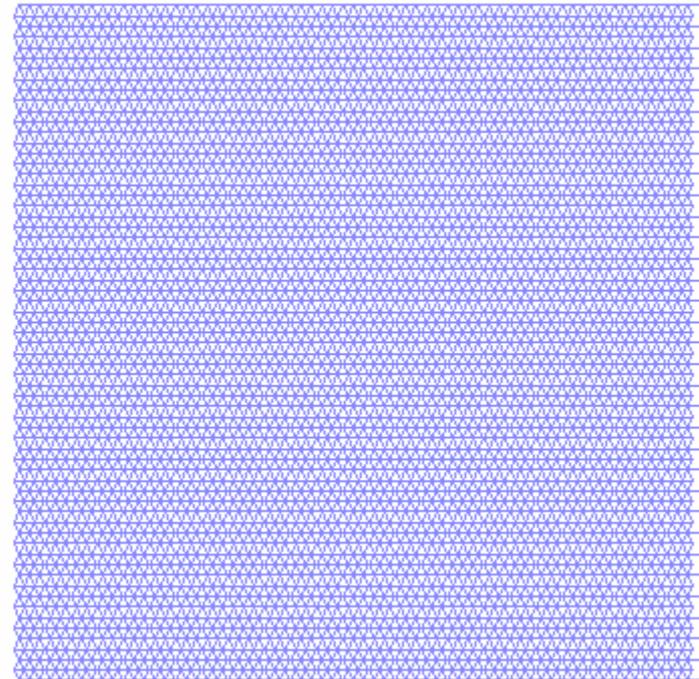
At Failure



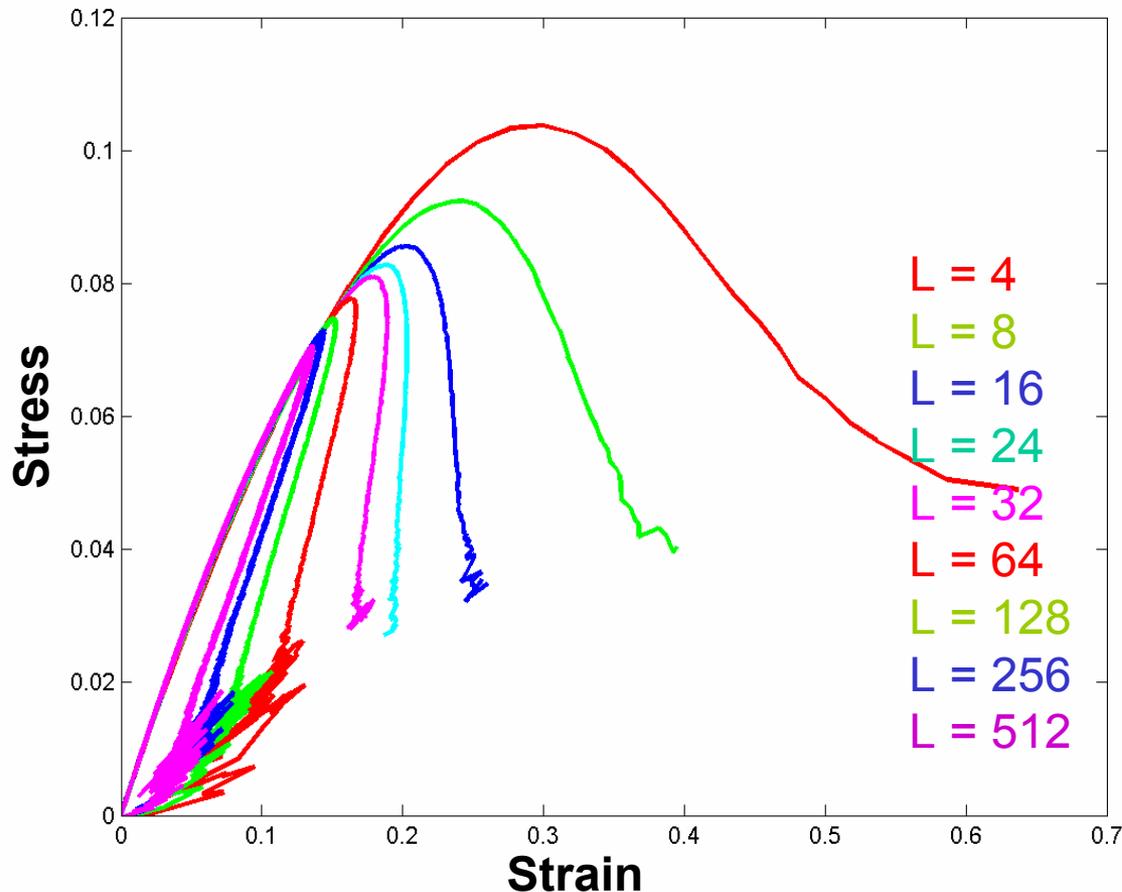
Single lattice response (64 x 64)
Lattice response averaged
over 5000 samples



At Peak Load

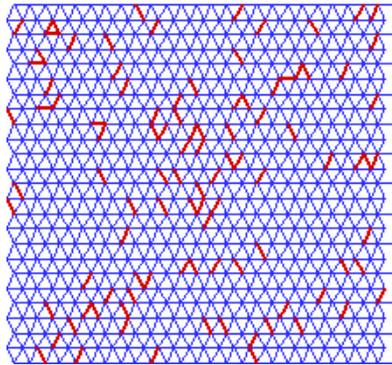


Lattice Response versus System Size

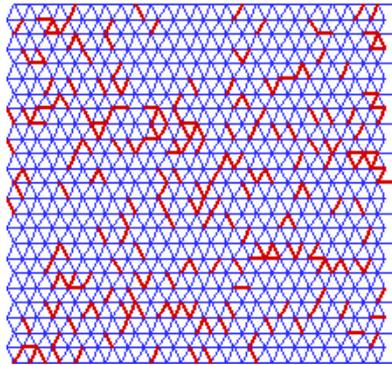
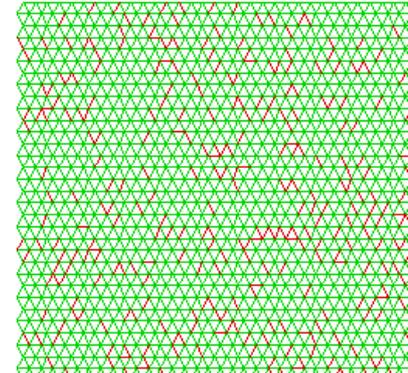


- Lattice response depends on the **system size**
- **Scaling laws** are required to obtain a “normalized” response that couples the mesoscopic scale response to the continuum scale response

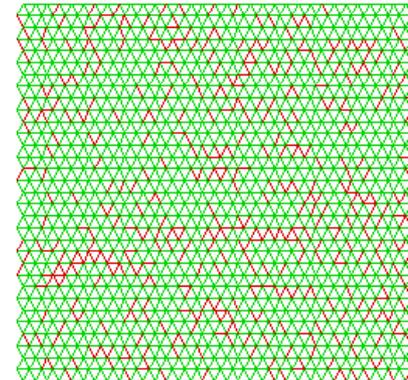
What is the Intensive Measure of Damage?



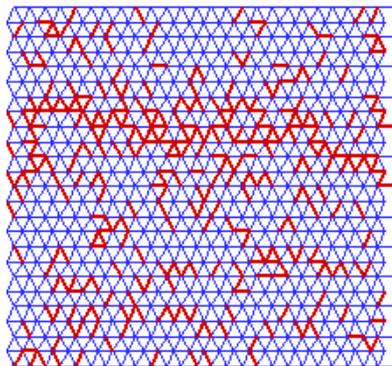
**Nucleation Phase:
Diffusive Damage**



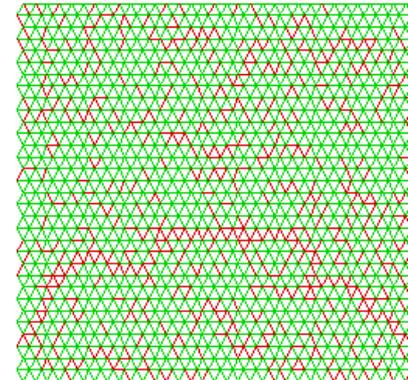
**Growth Phase:
Stress Concentration
effects are dominant**



$L = 24$



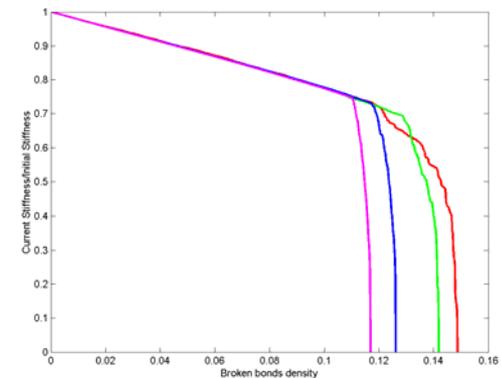
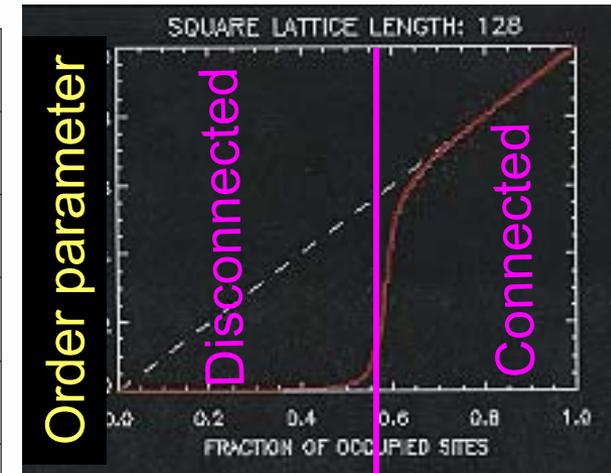
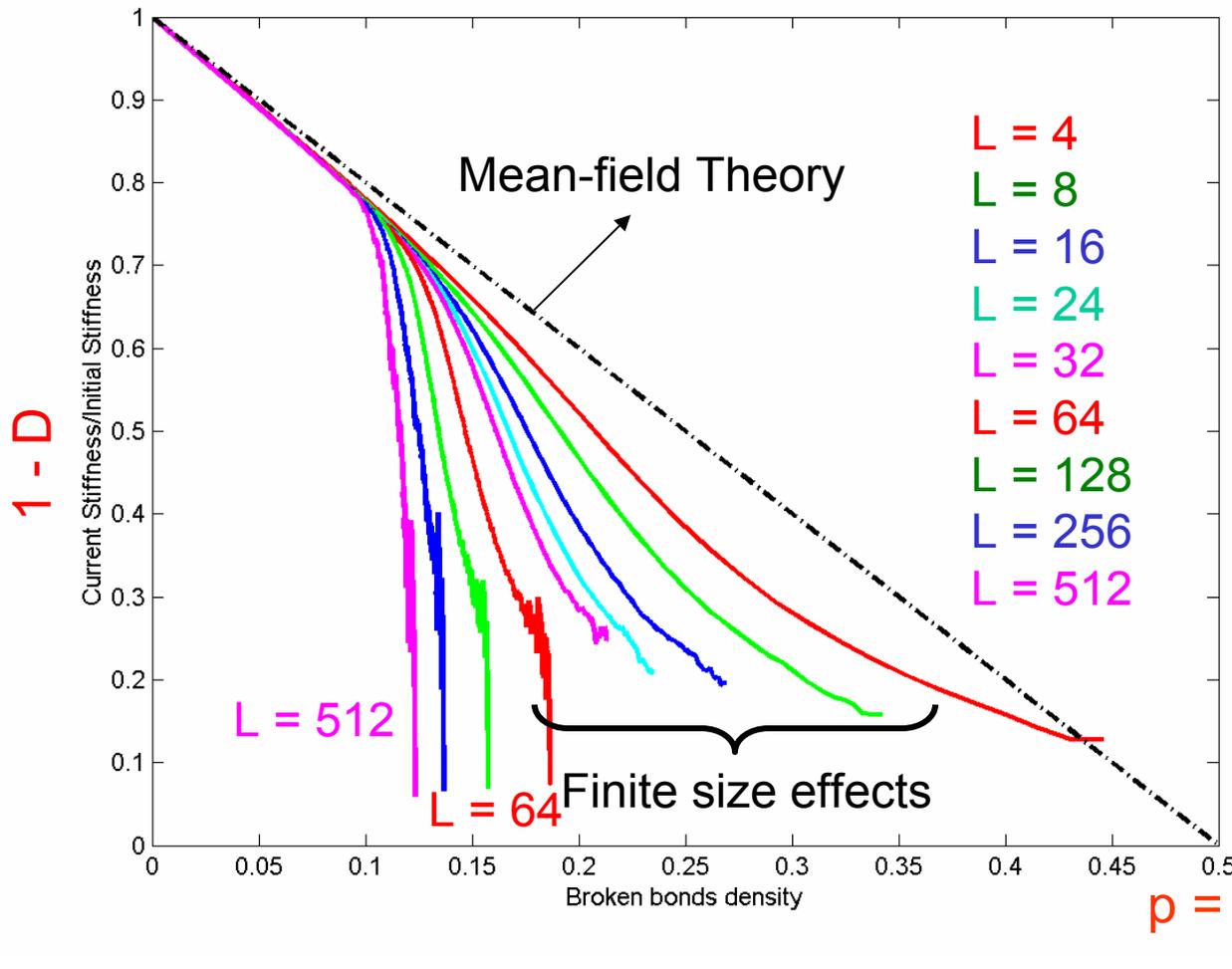
**Coalescence:
Localization of damage to
form a percolating crack**



$L = 32$

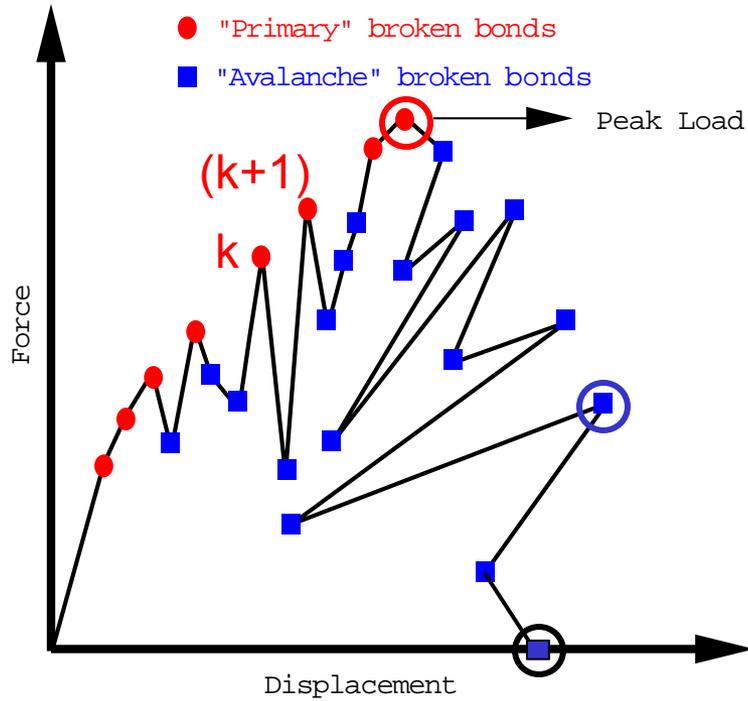
Damage Definition Based on Stiffness Degradation

$$\text{Damage Variable} = \left\{ 1 - \frac{\text{Current Stiffness}}{\text{Initial Stiffness}} \right\} \rightarrow \text{Close to being intensive!}$$



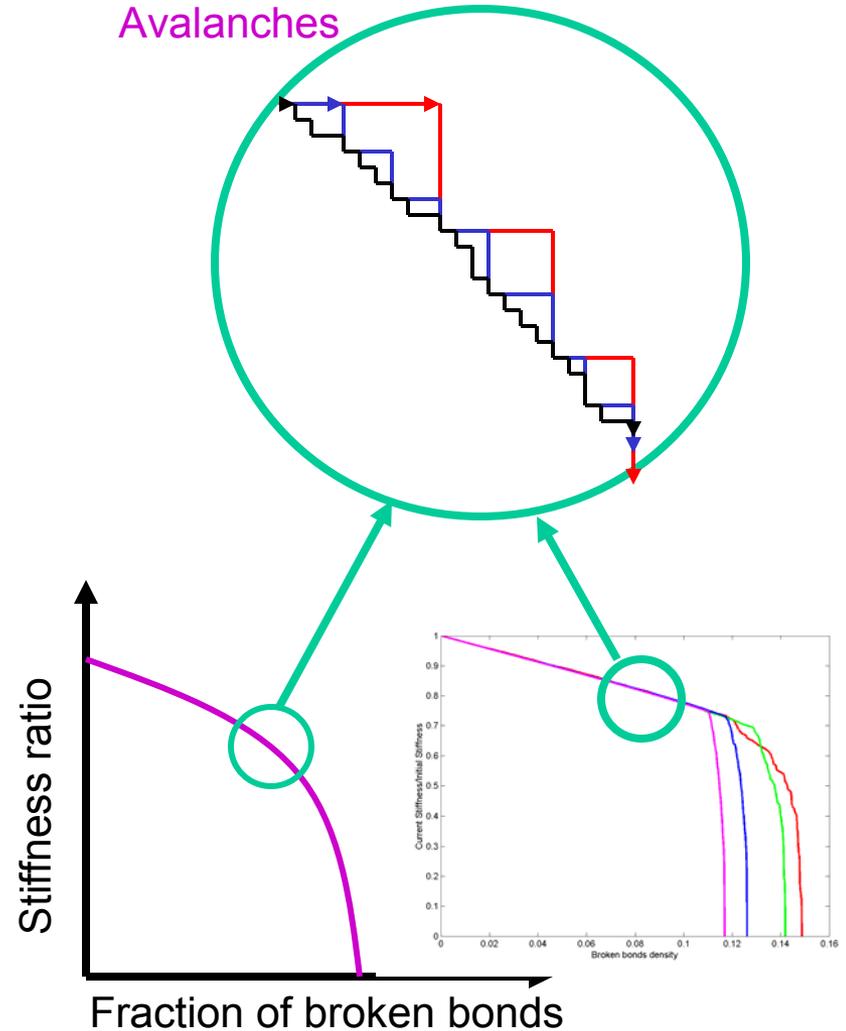
Typical Stiffness Degradation

Material Breakdown: Avalanche Process



- Load Controlled
- Displacement Controlled
- Bond Failure Controlled

Avalanches



Scaling Theory

Scaling Laws: (RG Theory)

RG Theory:

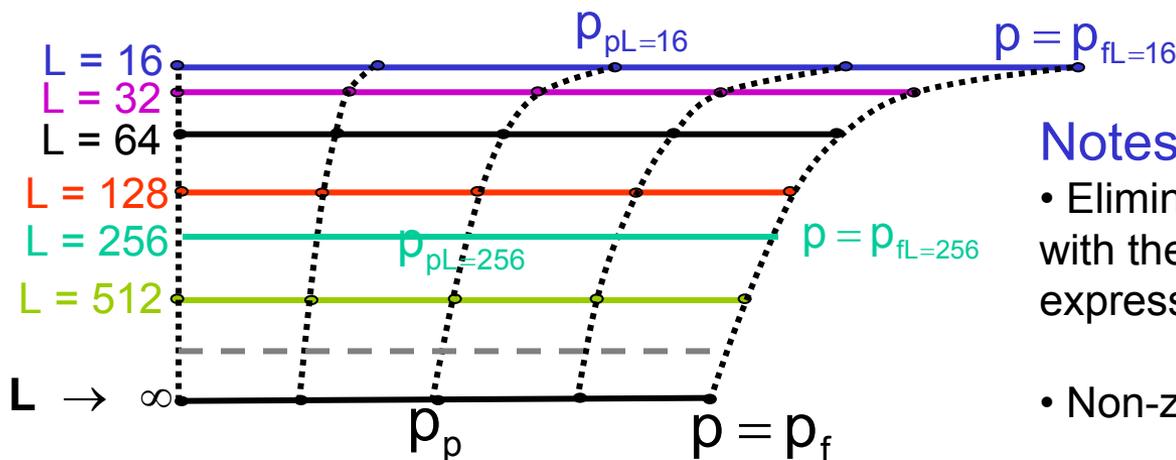
Coarse graining followed by **rescaling** of lengths such that the probability of failure under the influence of external load remains the same at both scales

Scaling Law:

$$p_L - p_\infty = c_\infty L^{-\alpha} \quad \longrightarrow \quad n_L = N_{el} \left(p_\infty + c_\infty L^{-\alpha} \right)$$

non-zero p_∞ indicates **critical crack size** needed for macroscopic fracture

$$c_\infty = c(p_\infty)$$



Notes:

- Eliminates inconsistencies associated with the conventional power-law expressions

- Non-zero critical threshold

$$p_f \rightarrow p_\infty \quad \text{as} \quad L \rightarrow \infty$$

Failure Load Distribution

Let $B_k = \{\text{set of } k \text{ "primary" broken bonds}\}$

Probability $f_{(k+1)}$ that $(k+1)^{\text{th}}$ "primary" bond fails

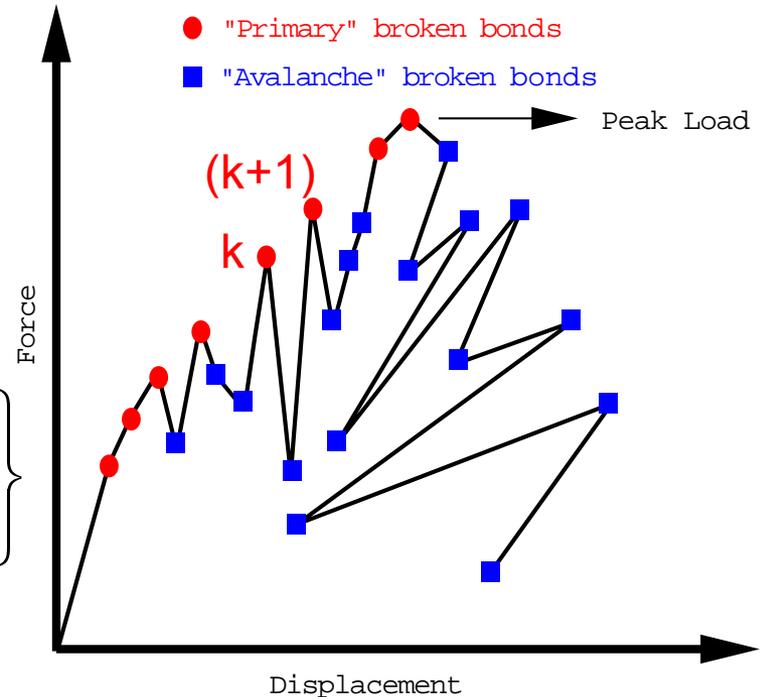
$$f_{(k+1)} = \prod_{j \in B_k} f_j$$

$$\text{Define } \mathcal{G} = \left\{ g_j = \frac{\sigma_j}{\sigma_{(j-1)}} \quad \forall j \in B_k \text{ and } g_1 = 1 \right\}$$

- g_j are independently distributed random variables.

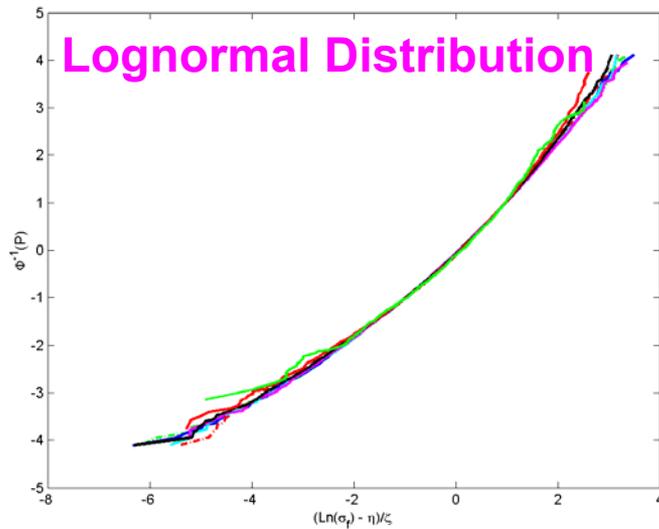
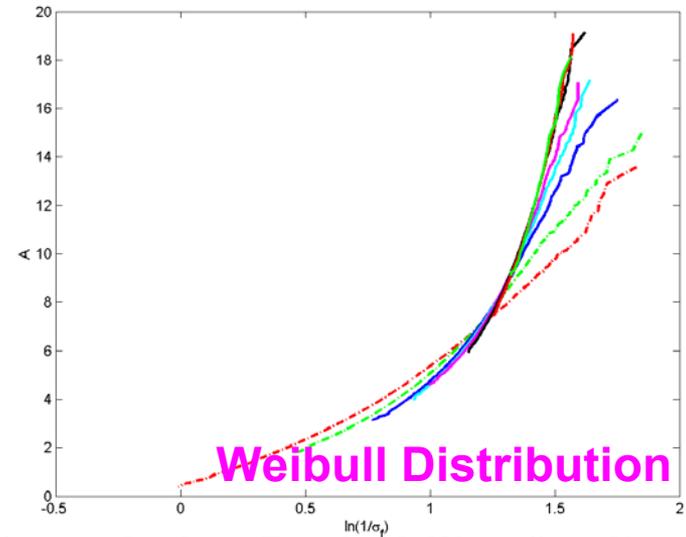
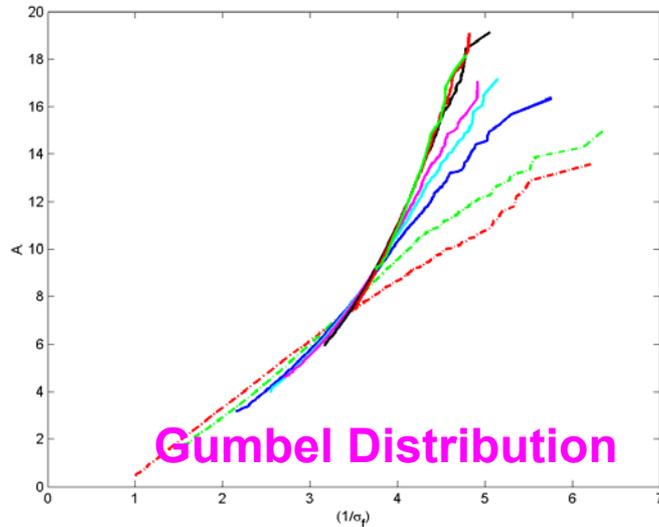
Stress $\sigma_{(k+1)}$ required to break $(k+1)^{\text{th}}$ "primary" bond $\sigma_{(k+1)} = \left(\prod_{j \in B_k} g_j \right) \sigma_1$

$$\longrightarrow \text{Prob}[\sigma_{(k+1)} \leq \sigma] \approx \text{LN}$$

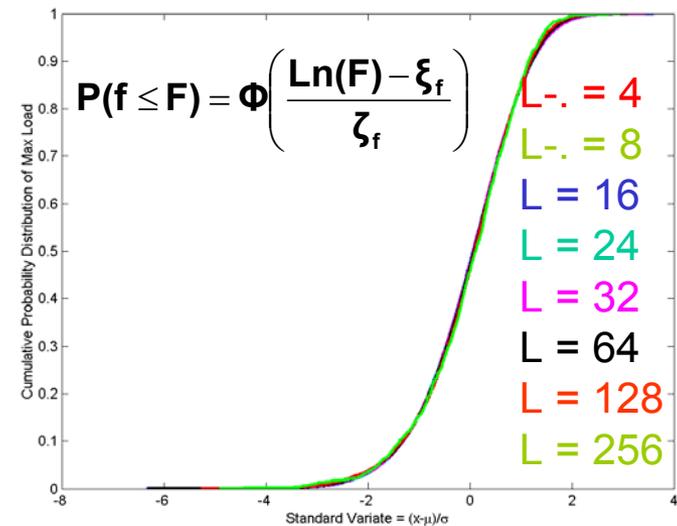


Numerical Results

Scaling of Failure Load Distribution

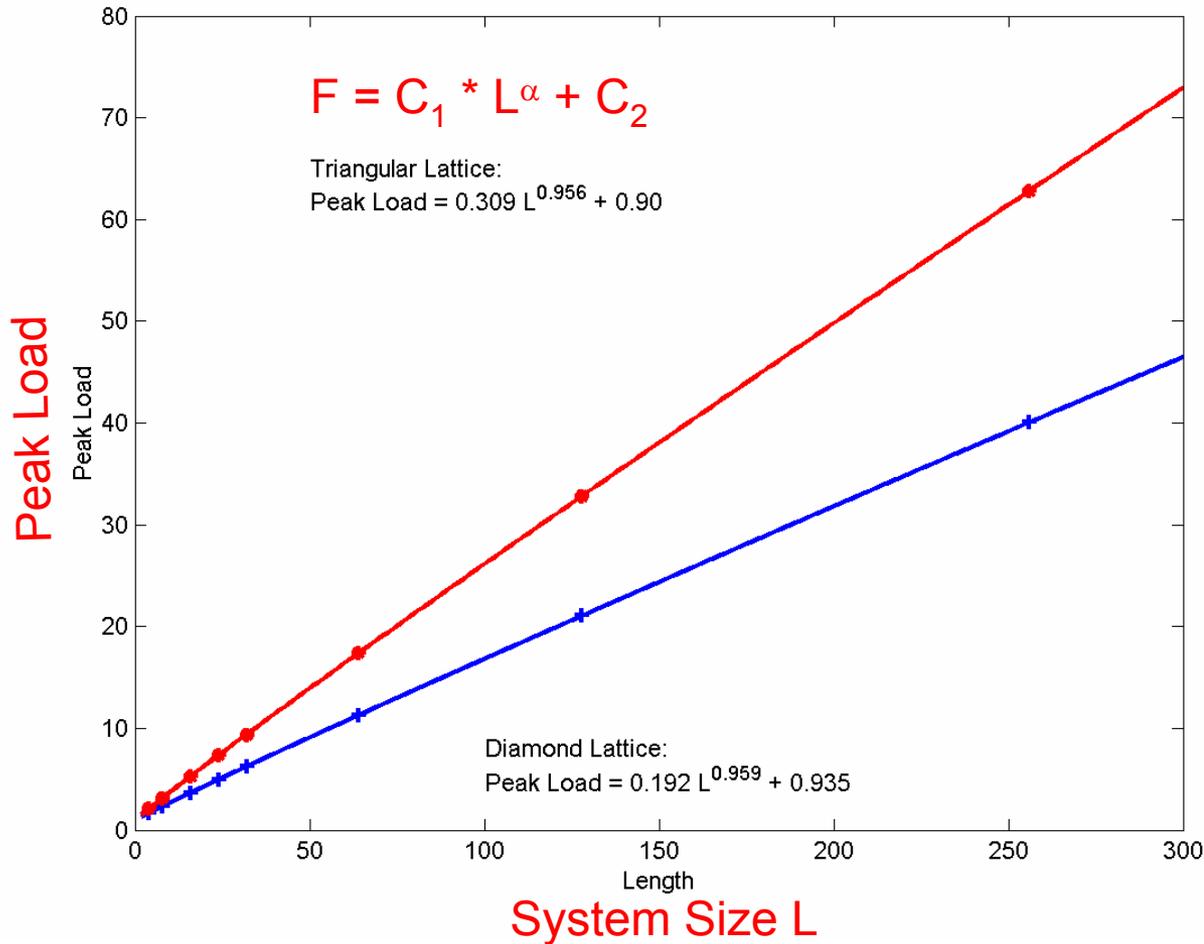


• Cumulative Probability Scaling Law



Standard Variate = $(\text{Ln}(F) - \xi) / \zeta$

Size Effect on the Mean Failure Load



$$\sigma_{\text{peak}} = C_1 L^{(\alpha-1)} + \frac{C_2}{L}$$

Since

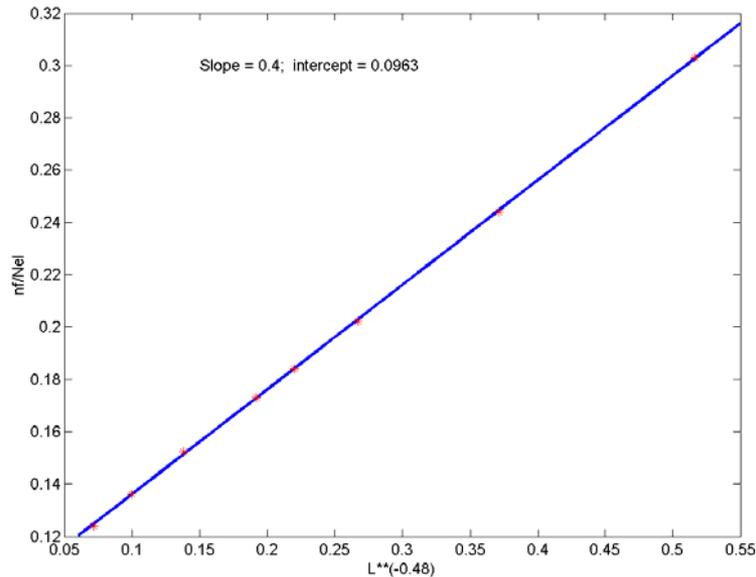
$$(1-\alpha) \ll 1 \Rightarrow L^{(1-\alpha)} \approx (\log(L))^{-\psi}$$

$$\sigma_{\text{peak}} = \frac{C_1}{(\log(L))^\psi} + \frac{C_2}{L}$$

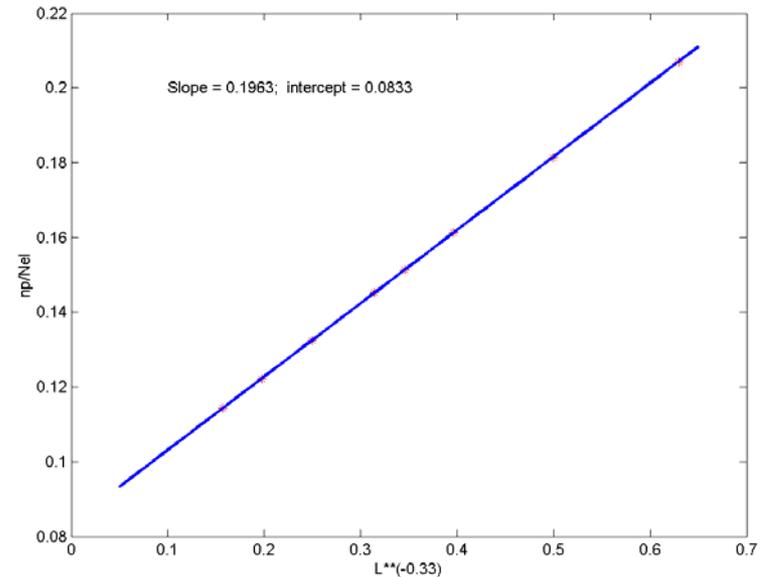
Mean fracture strength decreases very slowly with increasing system size L , and scales as $\sigma_{\text{peak}} \approx \frac{C_1}{(\log(L))^\psi}$ for very large L

Scaling of Number of Broken Bonds

At Failure



At Peak Load



$$L = \{4, 8, 16, 24, 32, 64, 128, 256, 512\}$$

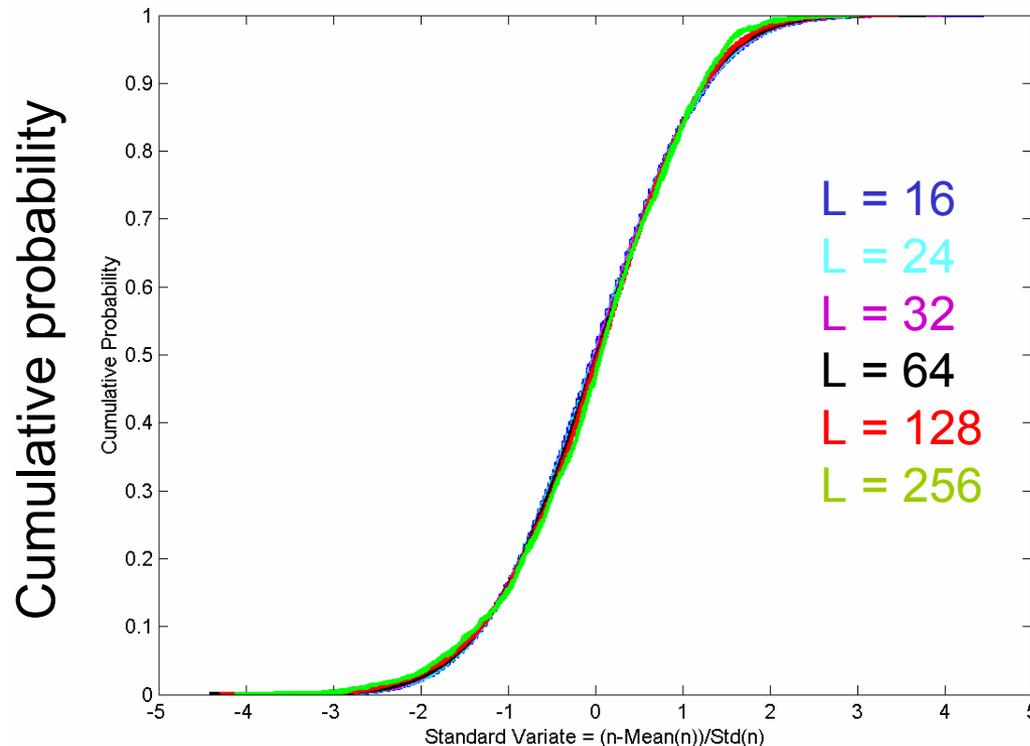
$$p_f - 0.096 = 0.4 L^{-0.48}$$

$$p_p - 0.083 = 0.196 L^{-0.33}$$

Probability Distributions for Fraction of Broken Bonds at Failure and at Peak Load

p_f = Mean fraction of broken bonds at failure

p_p = Mean fraction of broken bonds at peak load



Standard variate = $(p - \text{Mean}(p)) / \text{Std}(p)$

Probability distributions for fraction of broken bonds at failure as well as at peak load are identical and are normal

Computing Requirements

- Need to **resolve** after breaking a bond (intact lattice to total failure)
- **Ensemble averaging** of the response

Lattice Size (L)	CPU Time (sec)	CPU Time (sec)
	PCG Algorithm	New Algorithm
32	11.66	0.592
64	173.6	10.72
128	7473	212.2
256		5647
512		93779

$$\text{CPU Time (sec)} = 1.53 * 10^{-7} L^{4.36}$$

For L = 1000  Time ~ 21 days!

New Algorithm:

Direct Solver

- Breaking a bond is equivalent to rank-p update
- Use multiple-rank down-date algorithm of Cholesky factorization

Iterative Solver

- Laplacian on 2D and 3D lattice topology exhibits block structure
- Use block-circulant and optimal circulant preconditioners

Summary

For materials with **broadly** distributed heterogeneities,

- Scaling Law: $p_f - p_\infty = c_\infty L^{-\alpha}$
- Fraction of broken bonds p_f is **finite**:
 $p_f \rightarrow p_\infty$ as $L \rightarrow \infty$
- Fracture Strength: \sim **Lognormal (LN)**
- Mean fracture strength

$$\sigma_{\text{peak}} = \frac{C_1}{(\log(L))^\psi} + \frac{C_2}{L}$$

 $\sigma_{\text{peak}} \approx \frac{C_1}{(\log(L))^\psi}$ as $L \rightarrow \infty$

For materials with **narrowly** distributed heterogeneities,

- Weakest-link hypothesis
- p_f tends to **zero** threshold
 $p_f \rightarrow 0$ as $L \rightarrow \infty$
- Fracture Strength: \sim **Weibull** or **modified Gumbel**
- Mean fracture strength

Weibull  $\sigma_{\text{peak}} = \frac{C_2}{L^\psi}$

Gumbel  $\sigma_{\text{peak}} \approx \frac{C_1}{(\log(L))^\psi}$

Scaling Theory

Renormalization Approach for Scaling

Let $L_1 < L_2 < L_3$

p fraction of broken bonds at scale L_1

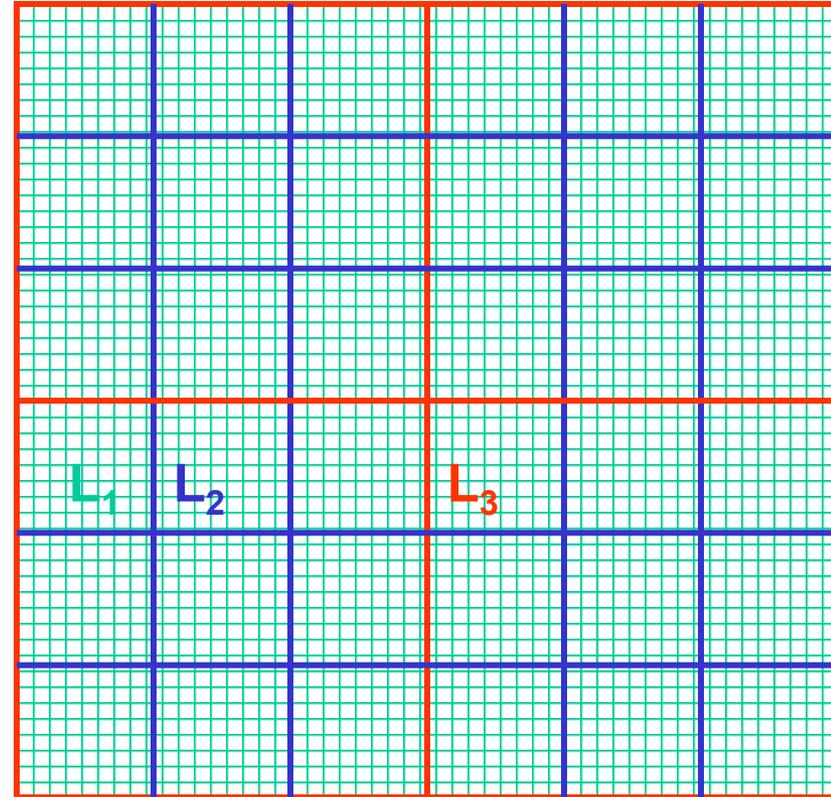
p' fraction of broken bonds at scale L_2

Coarse Graining*

$$p' = R_l(p) \quad \text{where} \quad l = \frac{L_2}{L_1}$$

$$\begin{aligned} p' - p_\infty &= R_l(p) - R_l(p_\infty) \\ &= \left. \frac{\partial R_l}{\partial p} \right|_{p=p_\infty} (p - p_\infty) \\ &= \Lambda_l (p - p_\infty) \end{aligned}$$

where $\Lambda_l = l^{-\alpha} \longrightarrow p_L - p_\infty = c_\infty L^{-\alpha}$



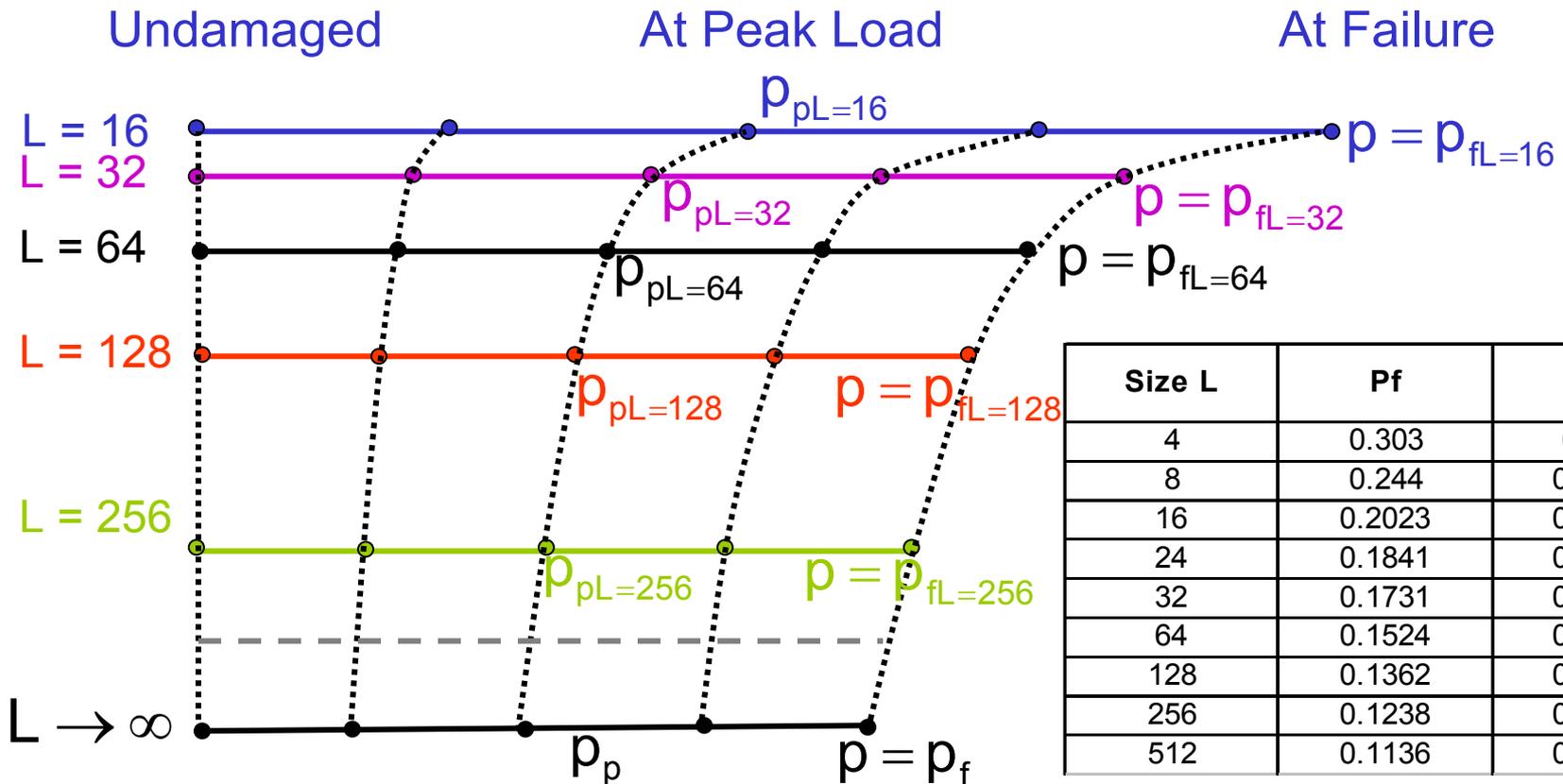
(* such that the probability of failure under the influence of external load remains the same at both scales)

non-zero p_∞ indicates critical crack size needed for macroscopic fracture

Intensive Definition of Damage

$$p_L - p_\infty = c_\infty L^{-\alpha} \quad \longrightarrow \quad n_L = N_{el} (p_\infty + c_\infty L^{-\alpha})$$

$$c_\infty = c(p_\infty)$$



Scaling of Number of Broken Bonds

$$p_f - p_\infty = c_\infty L^{-\alpha} \quad \longrightarrow \quad n_f = N_{\text{el}} (p_\infty + c_\infty L^{-\alpha})$$

In the literature, power laws have been fitted for n_f versus L

$$n_f \propto L^{1.71 \pm 0.1} \quad \longrightarrow \quad p_f = \frac{n_f}{N_{\text{el}}} \propto \frac{L^{1.71 \pm 0.1}}{L^2} \xrightarrow{\text{Limit } L \rightarrow \infty} 0$$

Maximum crack size:

$$a_c \propto \varphi(p_f) \quad \text{and} \quad a_c \rightarrow 0 \quad \text{as} \quad p_f \rightarrow 0$$

Notes:

- Number of broken bonds versus lattice size L is **NOT** a power law!
- Instead, there is fixed threshold to which p_f converges as L increases
- **Non-zero failure threshold** indicates to **critical crack size** required for fracture

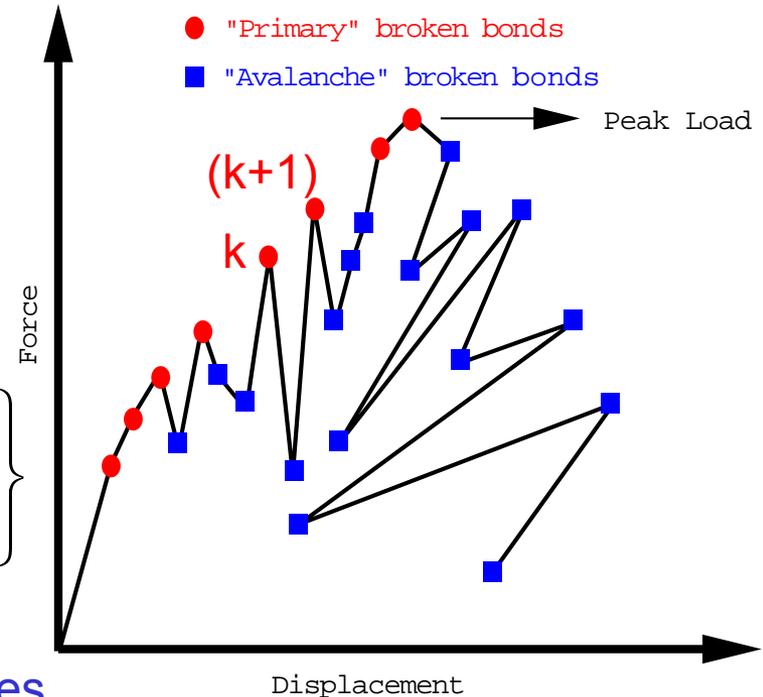
Failure Load Distribution

Let $B_k = \{\text{set of } k \text{ "primary" broken bonds}\}$

Probability $f_{(k+1)}$ that $(k+1)^{\text{th}}$ "primary" bond fails

$$f_{(k+1)} = \prod_{j \in B_k} f_j$$

Define $G = \left\{ g_j = \frac{\sigma_j}{\sigma_{(j-1)}} \quad \forall j \in B_k \text{ and } g_1 = 1 \right\}$



In the case of broadly distributed heterogeneities, the scale factors g_j become independent distributed random variables. (Since they depend not only on the stress concentration factors but also on the initial randomly distributed bond threshold values)

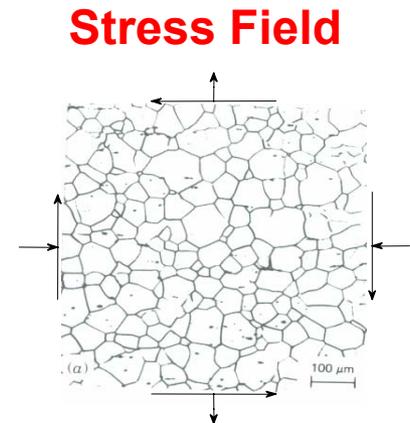
Stress $\sigma_{(k+1)}$ required to break $(k+1)^{\text{th}}$ "primary" bond $\sigma_{(k+1)} = \left(\prod_{j \in B_k} g_j \right) \sigma_1$

$\rightarrow \text{Prob}[\sigma_{(k+1)} \leq \sigma] \approx \text{LN}$

Motivation: Stress Induced Microcracking Evolution

Macroscopic properties and behavior of quasi-brittle materials are significantly affected by the internal microstructure and damage/microcracking evolution

Controlling of microstructure state and damage evolution leads to improved macroscopic behavior



Microcracking evolution

Phenomenological Material Models:

- microstructure-insensitive
- valid only for moderate damage levels
- local stress field fluctuations and interactions are not considered

Modeling at the mesoscale will lead to a fundamental understanding of the effect of microstructural features on the microcracking evolution in brittle materials

Mesoscopic Simulation: Discrete versus Continuum Models

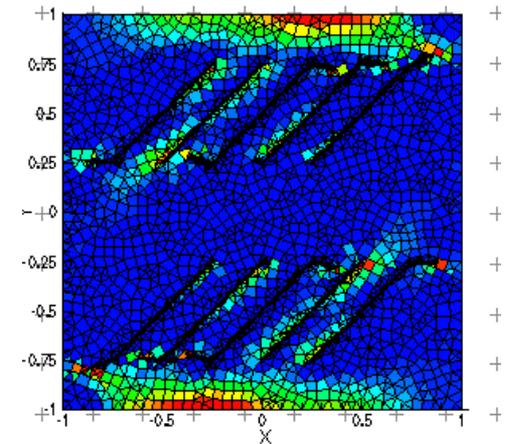
Discrete Lattice Models

- Suitable for studying the behavior of complex microstructures with heterogeneities
- Captures crack propagation and microstructure evolution with relative ease
- Ideal for studying statistical behavior including scaling and size effects
- Not readily applicable for capturing plasticity dominated phenomena

Continuum Models

- Suitable for studying the behavior of homogeneous solids
- Mesh size should be much smaller than typical inhomogeneity (crack, grain) size
- Captures inter-granular cracks using cohesive laws
- Not readily applicable for large number of heterogeneities
- Recent investigations on extended FE methods show promise in capturing inter- and trans-granular cracks and their interaction

Extended FE Models:
Multiple cracks, growth, interaction
and coalescence



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