Scaling Laws for Damage Evolution in Disordered Media[1]

Phani Kumar V.V. Nukala[2] and Srdan Simunovic

Computer Science and Mathematics Division

Oak Ridge National Laboratory

Oak Ridge, TN 37831-6359

Abstract

Scaling laws derived from mesoscopic discrete lattice models are typically used for coupling the mesoscopic damage evolution with the continuum damage response and in determining the size effects on the constitutive response of materials. This study develops the scaling laws based on the Renormalization Group (RG) methodology for the number of bonds broken in a discrete lattice before the macroscopic fracture occurs. The developed scaling laws imply the existence of a finite critical fracture threshold, defined as a fraction of number of broken bonds to the total number of bonds, below which macroscopic fracture of an infinite system does not occur. This result is in contrast with earlier results based on power law curve fit expressions, wherein the critical threshold approaches zero in the limit of an infinite system. However, the existence of a finite critical fracture threshold can be associated with a critical crack size, below which macroscopic fracture of a specimen does not occur. Further, the finite size scaling law based on the RG methodology remains accurate *with increasing system size* and also avoids certain inconsistencies associated with conventional power law type expressions reported in the literature. Numerical simulations based on two-dimensional triangular and diamond lattice networks substantiate the proposed scaling laws and are used to estimate the critical thresholds and the scaling exponents.

1 Introduction

Damage evolution leading to breakdown of disordered quasi-brittle materials has received considerable interest from material scientists, physicists, and engineers for many years [1, 2, 3, 4, 5, 6, 7]. Despite significant advances, many fundamental aspects of damage evolution leading to breakdown of materials still remain unanswered. Indeed, while definitions of damage based on average broken bond density or stiffness degradation are suitable to represent the extent of damage within the early stages of hardening regimes, where the damage levels are moderate and spatially diffused, they are inadequate around the peak load and especially in the softening regime of the stress-strain

[1]The submitted manuscript has been authored by a contractor of the U.S. Government under Contract No. DE-AC05-00OR22725. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

[2]Corresponding author: nukalapk@ornl.gov

curve. Most importantly, these definitions of damage are size-dependent; however, defining damage as an intensive variable independent of size is essential for comparing the amount of damage across different specimen sizes and for obtaining the scaling laws of fracture that are valid even in the softening regime of the material stress-strain curve.

2 Renormalization Group (RG) Approach

The Renormalization Group (RG) approach presents an attractive theoretical framework for obtaining the scaling laws of damage in disordered media. Damage evolution leading to material breakdown in disordered media under quasi-static loading can be described as a model system heading towards its critical point (phase transition) [8, 9, 10, 11, 12]. In the above description of material breakdown, the critical point is defined as that equilibrium state at which the material breaks apart, i.e., the secant stiffness becomes zero. The main idea of the RG transformation is to perform coarse graining followed by rescaling of lengths such that in the new length units the coarse grained description of the system is statistically similar to the original system at the finer scale, but with a different set of parameter values. Repeating this coarse graining procedure leads to a sequence of parameter values at coarser length scales and the system moves away from the critical fixed point of the RG transformation in the parameter space. Unless the system is at the fracture threshold (critical fixed point), successive coarse graining transformations lead the system to a trivial fixed point represented by the bulk-phase of the damaged material. At the critical fixed point, a singular behavior is obtained due to infinite number of repetitions of the RG transformation. This singular behavior is the origin of universality of the critical exponents observed at fracture.

Based on the RG approach, we propose a finite size scaling law of the form

$$p_{f_L} - p^\star \quad = \quad cL^{-\alpha_\rho} \tag{1}$$

for the fraction of broken bonds at lattice system failure. In the Eq. (1), p_{f_L} and $p^\star$ represent the fracture thresholds in a lattice system size of L and infinity, respectively, and c_p is a constant. This finite size scaling law presents the influence of system size, L, on the critical fracture threshold, $p^\star$. As the system size $L \to \infty$, the average fraction of broken bonds at failure $p_{f_L} \to p^\star$. Numerical simulation results on two-dimensional triangular and diamond (square lattice with 45 degress inclined bonds) lattice networks support this scaling relation and suggest the existence of a non-zero critical fracture threshold, $p^\star$, below which macroscopic fracture does not occur in an infinite system. This critical fracture threshold, $p^\star$, may be associated with the critical crack size, a_c, below which macroscopic fracture does not occur in the continuum fracture mechanics description. This result is in contrast with the power law-type expressions given in the literature for the number of broken bonds, which suggest that the probability of bond failure, p_{f_L} tends to zero as L approaches infinity [5, 6, 13]. Moreover, the finite size scaling form developed in this study eliminates some of the inconsistencies associated with the conventional power-law type scaling expressions used for the number of broken bonds at the peak load and at fracture.

3 Numerical Simulations

Progressive damage evolution leading to failure of disordered quasi-brittle materials has been studied numerically using various types of discrete lattice models [5, 6, 7, 2]. The essential features of discrete lattice models are disorder, elastic response characteristics and a breaking rule for each of the bonds in the lattice. We shall use an electrical equivalence to the mechanical problem, thereby assuming that the successive bond failures leading ultimately to the lattice system failure is similar to the breakdown of quasi-brittle materials [13]. We consider a *random threshold* model problem, where a lattice consists of fuses having the same conductance, but the bond breaking thresholds, i_c, are based on a broad (uniform) probability distribution, which is constant between 0 and 1. A broad thresholds distribution represents large disorder and exhibits diffusive damage leading to progressive localization, whereas a very narrow thresholds distribution exhibits brittle failure in which a single crack propagation causes material failure.

Periodic boundary conditions are imposed in the horizontal direction to simulate an infinite system and a constant voltage difference (displacement or strain) is applied between the top and the bottom of lattice system. The simulation is initiated with a lattice of intact fuses in which disorder is introduced through random breaking thresholds. The voltage V across the lattice system is increased until a fuse (bond breaking) burns out. The burning of a fuse occurs whenever the electrical current (stress) in the fuse (bond) exceeds the breaking threshold current (stress) value of the fuse. The current is redistributed instantaneously after a fuse is burnt. The voltage is then gradually increased until a second fuse is burnt, and the process is repeated.

4 Results and Discussion

The main focus of this paper is to estimate the critical fracture threshold, defined as the fraction of the number of broken bonds to the total number of bonds, below which the macroscopic fracture of an infinite system does not occur. In this study, we applied the RG methodology to derive the functional form of the scaling law for the number of broken bonds at failure and substantiate this result with numerical simulations. Table 1 presents the number of broken bonds at peak load, n_p, and at fracture, n_f, for each of the lattice sizes considered.

Using Eq. (1), a simple curve fit of the simulation results obtained for the mean fraction of broken bonds at failure and at peak load yields (Figs. 1(a) and 1(b))

$$p_{f_L} - 0.0963 = 0.3995\, L^{-0.477} \tag{2}$$

$$p_{p_L} - 0.0834 = 0.1962\, L^{-0.334} \tag{3}$$

For a triangular lattice topology, the number of broken bonds at fracture, n_f, and at peak load, n_p, may be obtained from the above relations using $p_{f_L} = \frac{n_f}{N_{el}}$, and $p_{p_L} = \frac{n_p}{N_{el}}$, where $N_{el} = (3L+1)(L+1)$. It should be noted that Eqs. (2) and (3) eliminate the inconsistency associated with the number of broken bonds at the peak load and at fracture for large system sizes. That is, using Eqs. (2) and (3), the number of broken bonds at the peak load is always less than the number of broken bonds at fracture even for large system sizes.

For diamond lattice topology, similar results have been obtained. The scaling law for the mean fraction of broken bonds at failure and at peak load are given by

$$p_{f_L} - 0.0908 = 0.677\, L^{-0.666} \quad (4)$$

$$p_{p_L} - 0.0875 = 0.367\, L^{-0.654} \quad (5)$$

The total number of bonds, N_{el}, in a diamond topological lattice with periodic boundary conditions in the horizontal direction is given by $N_{el} = 2L(L+1)$.

The cumulative probability distribution for the fraction of broken bonds at failure, defined as the probability $\Pi_{pf}(p_f, L)$ that a system of size L fails when the fraction of broken bonds equals $p_f = \frac{nf}{N_{el}}$, may be plotted using $\bar{p}_f$ defined as $\bar{p}_f = \frac{(nf-\mu_{nf})}{\sigma_{nf}} = \frac{(p_f - p_{f_L})}{\Delta_{pf_L}}$, for varying system sizes, as shown in Fig. 2(a). The subscript f denotes that the cumulative distribution is considered for the fraction of broken bonds at failure. Figure 2(a) shows that $\Pi_{pf}(p_f, L)$ may be expressed in a universal scaling form such that $\Pi_{pf}(p_f, L) = \Pi_{pf}(\bar{p}_f)$ for different values of p_f and system sizes L. Figure 2(b) presents the cumulative probability distribution for the fraction of broken bonds at peak load $\Pi_{pp}(p_p, L)$, where the subscript p denotes the distribution for the fraction of broken bonds at peak load. Once again, the results in Fig. 2(b) indicate that $\Pi_{pp}(p_p, L)$ may be expressed in an universal scaling form using the standard variate $\bar{p}_p = \frac{(np-\mu_{np})}{\sigma_{np}}$ such that $\Pi_{pp}(p_p, L) = \Pi_{pp}(\bar{p}_p)$. Figure 3 provides a comparison between the cumulative probability distributions at failure, $\Pi_{pf}(\bar{p}_f)$, and at peak load, $\Pi_{pp}(\bar{p}_p)$. Based on the numerical simulation results presented in Fig. 3, it is clear that the cumulative probability distributions for the fraction of broken bonds at failure, Π_{pf}, and at the peak load, Π_{pp}, are identical. That is, $\Pi_p = \Pi_{pf} = \Pi_{pp}$. In addition, it can also be concluded that a Gaussian distribution adequately describes Π_p.

5 Conclusions

Scaling laws derived from numerical simulations based on mesoscopic discrete lattice models are typically used for coupling the mesoscopic damage evolution with the continuum damage response and in determining the size effects on the constitutive response of materials. This study develops the scaling laws based on the RG methodology for the number of bonds broken in a discrete lattice before the macroscopic fracture occurs. The developed scaling laws incorporate the finite size scaling effects and are consistent with increasing system size. Numerical simulation results on two-dimensional triangular and diamond (square lattice with 45 degress inclined bonds) lattice networks support this scaling relation and suggest the existence of a non-zero critical fracture threshold, $p^\star$, below which macroscopic fracture does not occur in an infinite system. This critical fracture threshold, $p^\star$, may be associated with the critical crack size, a_c, below which macroscopic fracture does not occur in the continuum fracture mechanics description. When the scaling relations derived in this paper are used to make predictions for the number of broken bonds at the peak load and at failure with increasing lattice system sizes, an excellent agreement is obtained between the predicted scaling law estimates and the numerical simulations.

Acknowledgment
This research is sponsored by the Mathematical, Information and Computational Sciences Division, Office of Advanced Scientific Computing Research, U.S. Department of Energy under contract number DE-AC05-00OR22725 with UT-Battelle, LLC.

References

[1] H. Liebowitz (editor). *Fracture*. Vols. I-VII, Academic, New York, 1984.

[2] Bikas K. Chakrabarti and L. Gilles Benguigui. *Statistical Physics of Fracture and Breakdown in Disordered Systems*. Oxford Science Publications, Oxford, 1997.

[3] Dietrich Stauffer and Amnon Aharony. *Introduction to Percolation Theory*. Revised second edition, Taylor and Francis, 1994.

[4] D. Krajcinovic. *Damage Mechanics*. Elsevier, Amsterdam, 1996.

[5] A. Hansen and S. Roux. *Statistical toolbox for damage and fracture*, pages 17–101. Springer, New York, 2000.

[6] H. J. Herrmann and S. Roux (eds). *Statistical Models for the Fracture of Disordered Media*. North-Holland, Amsterdam, 1990.

[7] M. Sahimi. Non-linear and non-local transport processes in heterogeneous media from long-range correlation percolation to fracture and materials breakdown. *Physics Reports*, 306:213–395, 1998.

[8] L. L. Mishnaevsky. Methods of the theory of complex systems in modelling of fracture: A brief review. *Engineering Fracture Mechanics*, 56(1):47–56, 1997.

[9] E. A. Saibal. Thermodynamic criterion for the fracture of metals. *Phys. Rev.*, 69(11/12):667, 1946.

[10] R. Furth. Relation between breaking and melting. *Nature*, 145:741, 1940.

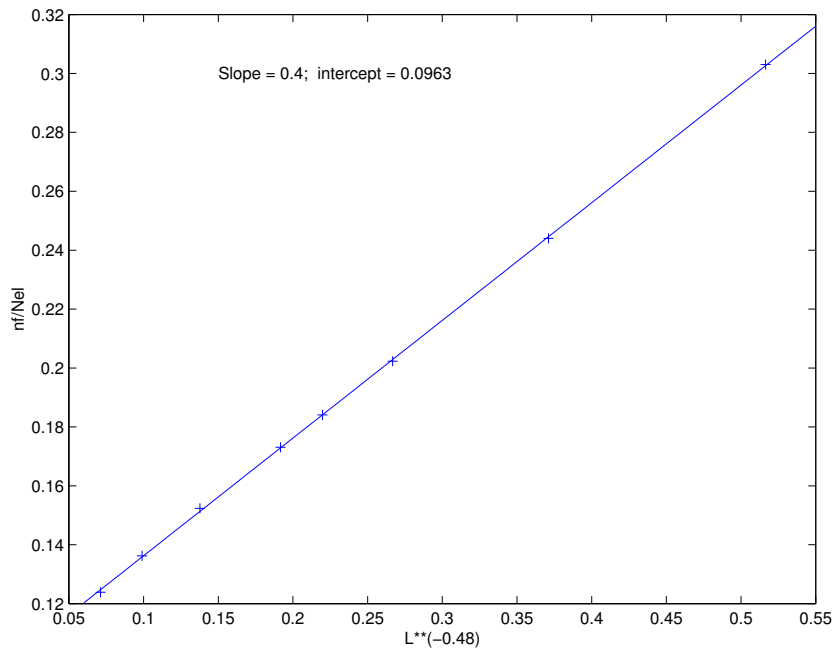
[11] C. J. Allegre, J. L. Le Mouel, and A. Prevost. Scaling rules in rock fracture and possible implications for earthquake prediction. *Nature*, 297:47–49, 1982.

[12] R. F. Smalley, D. L. Turcotte, and S. A. Solla. A renormalization group approach to the stick-slip behavior of faults. *Journal of Geophysical Research*, 90:1894–1900, 1985.

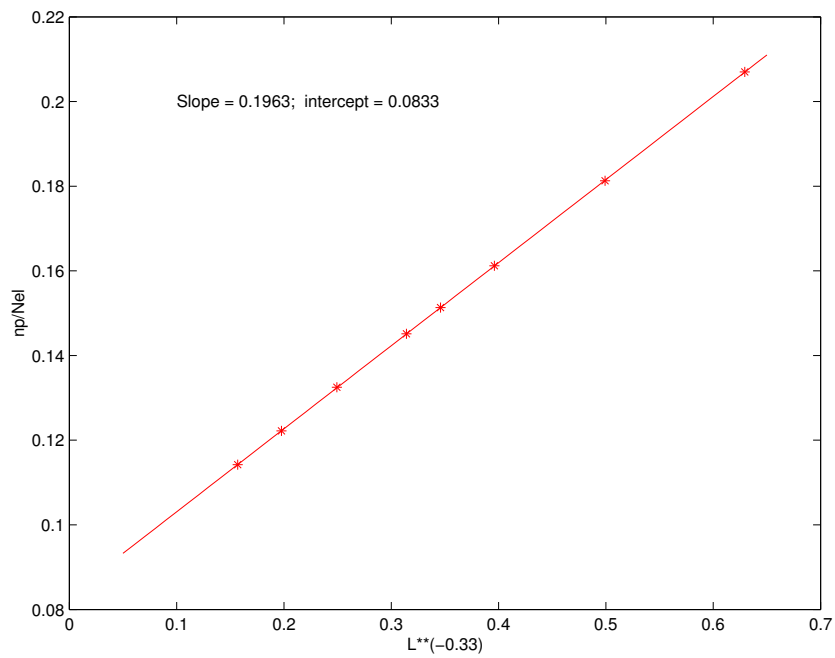
[13] Arnaud Delaplace, Gilles Pijaudier-Cabot, and Stephane Roux. Progressive damage evolution in diecrete lattice models and consequences on continuum modelling. *Journal of Mechanics and Physics of Solids*, 44:99–136, 1996.

Table 1: Number of broken bonds at peak and at failure

L	N_{config}	time (seconds)	Triangular		Diamond	
			n_p	n_f	n_p	n_f
4	50000	0.002	13	19	9	14
8	50000	0.006	41	54	26	37
16	50000	0.042	134	168	80	107
24	50000	0.186	276	335	161	208
32	50000	0.592	465	554	268	337
64	50000	10.72	1662	1911	942	1126
128	12000	212.2	6068	6766	3406	3901
256	1200	5647	22572	24474	12571	13846
512	200	93779	84487	89595		

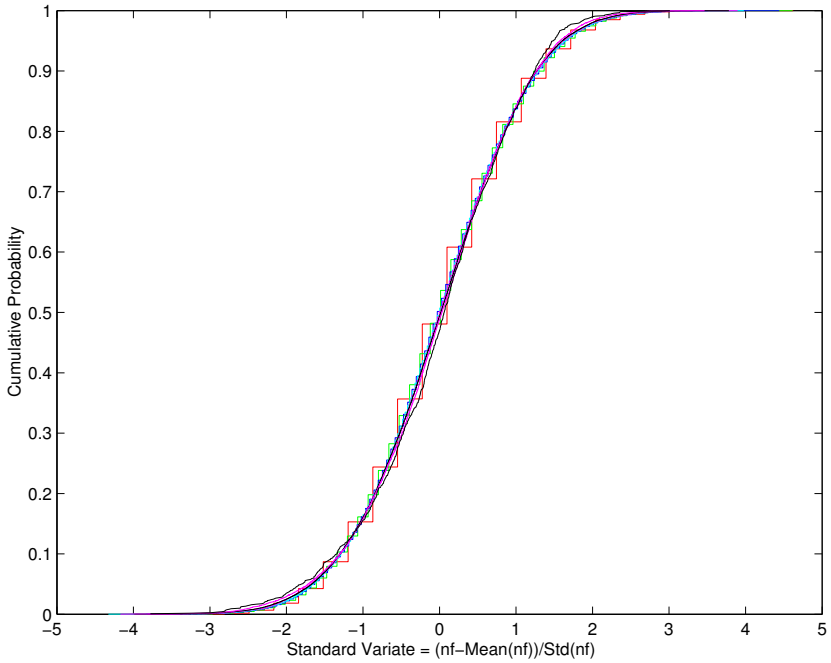


(a)

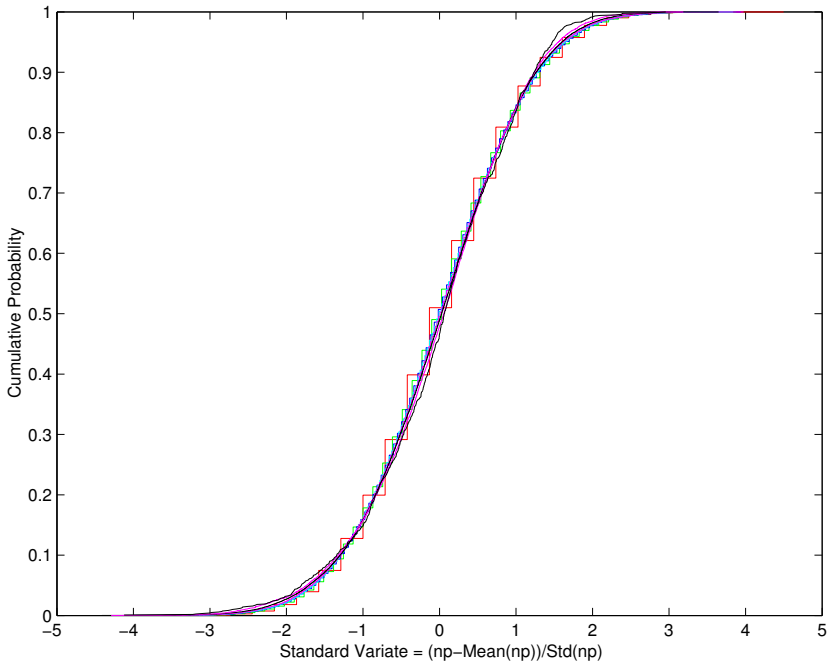


(b)

Figure 1: Scaling of critical fracture threshold, (a) p_{f_L} (b) p_{p_L}, with system size, L, in a triangular lattice



(a)



(b)

Figure 2: Cumulative probability distribution of broken bonds at (a) failure (b) peak load with varying system sizes, L, in a triangular lattice

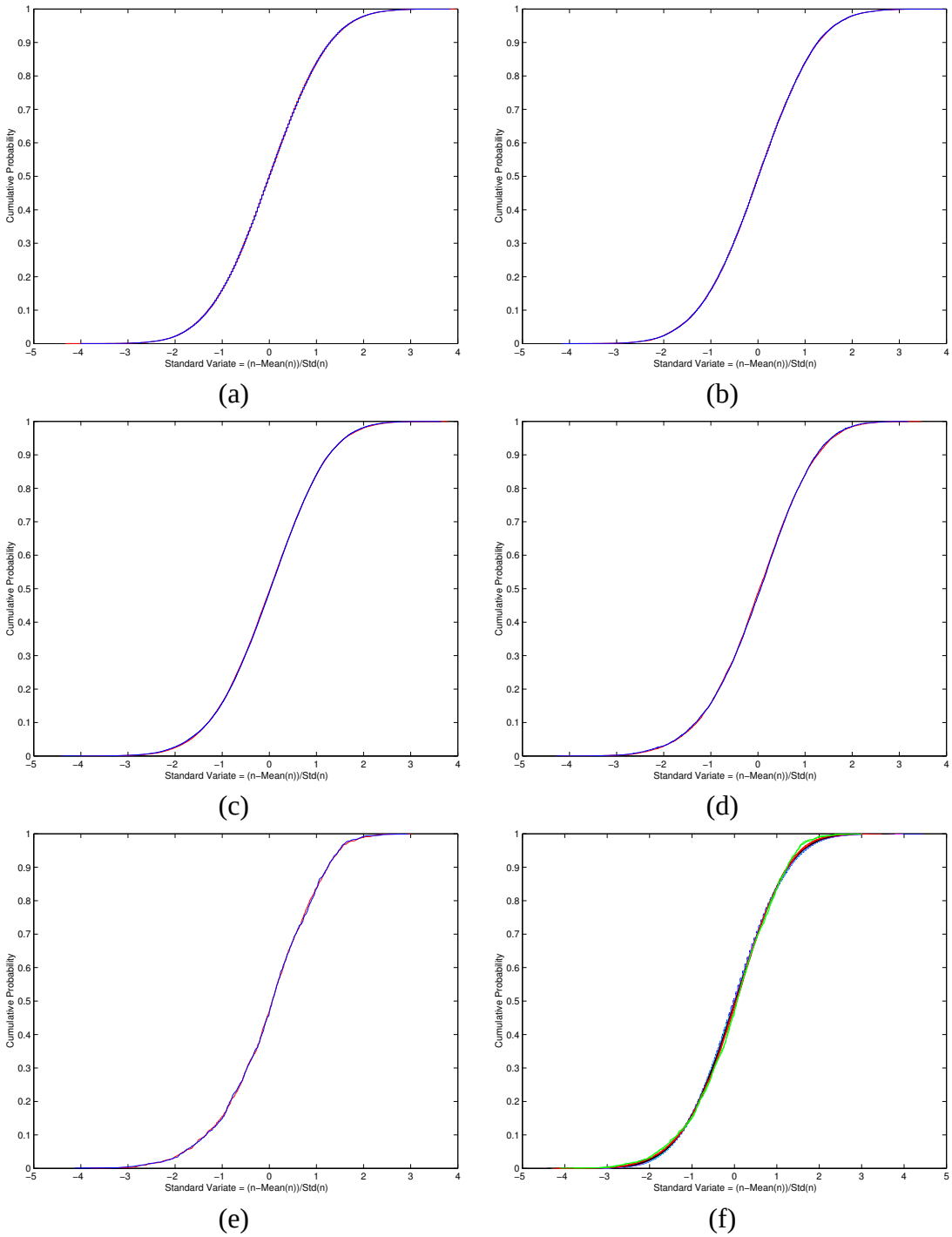


Figure 3: Comparison of cumulative probability distribution of broken bonds at the peak load and at failure with varying system sizes, L, in a triangular lattice: (a) $L = 24$ (b) $L = 32$ (c) $L = 64$ (d) $L = 128$ (e) $L = 256$ (f) $L = 16, 24, 32, 64, 128, 256$