Diffusion Approximation for Transport with Multiplying Boundary Conditions

V. Protopopescu[1] and L. Thevenot[2]

[1]Computer Science & Mathematics Division, Oak Ridge National Laboratory
Oak Ridge, TN 37831-6364, USA
protopopesva@ornl.gov

[2]Université de Franche-Comté, Equipe de Mathématiques CNRS UMR 6623
16 route de Gray, 25030 Besançon Cedex, France

Recently, transport equations with *multiplying* boundary conditions (BC) has generated a lot of interest [3, 4, 7, 11, and references therein]. In general, multiplying BC lead to serious technical difficulties, since the combined effect of geometry and net production of particles at the boundaries results in the absence of overall control of the flux. In slab geometry, if the velocities are bounded, the growth effect can be controlled. Under these conditions, one can prove that the corresponding transport operator generates an exponentially bounded, positivity preserving evolution semigroup [3, 4, 7, 11].

The derivation of diffusion (hydrodynamics) from transport with either dissipative or conservative boundary conditions based on scaling arguments, asymptotic analysis, and/or functional analysis has a rather long history (see Refs. 1, 2, 5, 6, 8 (Section XXI-5), 9 (Section 13), 10, 12 – 17, and references therein). Here we report the first derivation of the diffusion limit for a model transport equation with *multiplying* boundary conditions (BC). The derivation is rigorous in the sense that all formal operations are proven to be valid in specific functional frameworks. Detailed proofs will be published elsewhere [15].

We consider monoenergetic transport in a slab of width 2a for the one-particle distribution function *f* depending on position, *x*, "velocity", μ, and time, *t*. The restitution coefficient at the slab's surfaces is strictly greater than one, accounting for *multiplying* boundary conditions.

Let γ and β be two positive constants. We consider the limit, as ε tends to 0, of the following transport problem:

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t}(x,\mu,t) = -\varepsilon^{-1}\mu\dfrac{\partial f_\varepsilon}{\partial x} - \varepsilon^{-2} f_\varepsilon + \varepsilon^{-2}\dfrac{1}{2}\displaystyle\int_{-1}^{1} f_\varepsilon d\mu' - \gamma f_\varepsilon \\ \quad \text{on } (-a,a)\times(-1,1)\times\mathbb{R}^+ \\ f_\varepsilon(-a,\mu,t) = (1+\beta\varepsilon) f_\varepsilon(-a,-\mu,t), \quad \mu>0 \\ f_\varepsilon(a,-\mu,t) = (1+\beta\varepsilon) f_\varepsilon(a,\mu,t), \quad \mu>0 \\ f_\varepsilon(x,\mu,0) = f_0(x,\mu) \in L^2((-a,a)\times(-1,1)). \end{cases} \tag{1}$$

considered as an initial-boundary value problem in the space $L^2((-a,a)\times(-1,1))$.

We note that in contradistinction with the conservative or dissipative BC, the scaling of the transport system above presents an additional difficulty introduced by the presence of multiplying BC. Indeed, in this case, the restitution coefficient at the surface has to be itself suitably scaled as indicated above. Otherwise, in the limit of infinite time, the multiplying effect of the boundaries would lead to uncontrollable growth of the solution.

The main new result is contained in the following theorem [15]:

Theorem 1

Let γ *be an absorption coefficient such that* $\gamma > \dfrac{\beta}{2a} + \dfrac{\beta^2}{2}$ *and assume that the initial data* $f_0(x,\mu)$, *is sufficiently smooth. Then the solution* f_ε *of the transport system (1) satisfies*

$$\left\| f_\varepsilon - f \right\|_{L^2((-a,a)\times(-1,1))} \le \varepsilon M e^{\delta T}, \ \textit{for all } t\in\,]0,T],$$

where T>0, M and δ *are two positive constants independent of T and* ε, *and f is the solution of the diffusion problem:*

$$\begin{cases} \frac{\partial f}{\partial t}(x,t) = \frac{1}{3}\frac{\partial^2 f}{\partial x^2} - \gamma f \text{ on } (-a,a)\times\mathbb{R}^+ \\ \frac{4}{3}\frac{\partial f}{\partial x} = -\beta f, \quad at\ x=-a \\ \frac{4}{3}\frac{\partial f}{\partial x} = \beta f, \quad at\ x=a \\ f(x,0) = \frac{1}{2}\int_{-1}^{1} f_0(x,\mu)\,d\mu \end{cases} \qquad (2)$$

considered as an initial-boundary value problem in $H^1(-a,a)$.

We note that the Robin BC in system (2) are *unstable*, i.e., they describe incoming fluxes at the boundaries and, in principle, could yield solutions that grow exponentially in time. The proof of the Theorem is divided in three main steps [15]. First, we prove that the solution f_ε is exponentially bounded in time, uniformly in ε. The time behavior is controlled by adding the additional absorption term, $-\gamma f_\varepsilon$, with γ sufficiently large. Second, we need to bound, uniformly in ε, the solution of the nonhomogeneous Milne problem with specularly reflecting boundary conditions that arises from the boundary layer. The solvability of the Milne problem yields the BC for the diffusion problem (2). Finally, the proof of the convergence of the transport solution, f_ε, to the solution of the diffusion equation, f, is based on rigorous estimates of the various terms in the asymptotic expansion.

The Theorem remains true when the boundary conditions are partially absorbing. In this case, $\beta < 0$ and $\gamma = 0$, and thus we obtain a diffusive limit with *stable* Robin boundary condition. The proofs become much simpler because the original transport operator is dissipative when $1+\beta\varepsilon<1$.

Acknowledgments

V. P.'s research is supported in part by the U.S. Department of Energy, Office of Basic Energy Sciences. The Oak Ridge National Laboratory is managed for the U.S. DOE by UT-Battelle, LLC, under contract No. DE-AC05-00OR22725.

References

1. N. Angelescu, N. Marinescu, and V. Protopopescu, "On the spectral decompositions of the linear transport operator with periodic boundary conditions," *Lett. in Math. Phys.*, **1**(4), 329-333 (1976).

2. N. Angelescu and V. Protopopescu, "Asymptotic equivalence between the hydrodynamic and kinetic stages for the Lorentz model," *Lett. in Math. Phys.*, **3**, 87-92 (1979).

3. M. Boulanouar, "L'opérateur d'advection. I. Existence d'un C_0 semi-groupe," *Transp. Theory Stat. Phys.*, **31**(2), 169-176 (2002).

4. M. Boulanouar and H. Emamirad, "A transport equation in cell population dynamics," *Diff. and Int. Eqs.*, **13**, 125-144 (2000).

5. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, "Boundary layers and homogeneization of transport processes," *J. Publ. R.I.M.S., Kyoto Univ.*, **15**, 53-157 (1979).

6. C. Bardos, R. Santos, and R. Sentis, "Diffusion approximation and computation of the critical size," *Trans. Amer. Math. Soc.*, **284**(2), 617-649 (1984).

7. G. Borgioli and S. Totaro, "3D-streaming operator with multiplying boundary conditions: semigroup generation properties," *Semigroup Forum*, **55**, 110-117 (1997).

8. R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 6. Springer-Verlag, Berlin, 1993.

9. W. Greenberg, C. van der Mee, and V. Protopopescu, *Boundary Value Problems in Abstract KineticTtheory*, Birkhäuser Verlag, Basel, 1987.

10. J. B. Keller and E. W. Larsen, "Asymptotic solution of neutron transport problems for small mean free paths," *J. Math. Phys.*, **15**(1), 75-81 (1974).

11. K. Latrach and M. Mokhtar-Kharroubi, "Spectral analysis and generation results for streaming operators with multiplying boundary conditions," *Positivity*, **3**, 273-296 (1999).

12. M. Mokhtar-Kharroubi, "On the diffusion approximation for neutron transport," To appear.

13. M. Mokhtar-Kharroubi, V. Protopopescu, and L. Thevenot, " On the singular limit of a model transport semigroup," *Math. Meth. Appl. Sciences*, **23**, 1301-1322 (2000).

14. M. Mokhtar-Kharroubi and L. Thevenot, "On the diffusion theory of neutron transport on the torus," *Asymptotic Analysis*, **30**, 273-300 (2002).

15. V. Protopopescu and L. Thevenot, "Diffusion approximation of a neutron transport equation with multiplying boundary conditions," To appear.

16. R. Sentis, "Approximation and homogenization of a transport process," *SIAM J. Appl. Math*., **39**, 134-141 (1980).

17. R. Santos, "Analyse asymptotique des équations de transport dans le cas d'évolution," *Proc. Indian Acad. Sci. (Math. Sci.)*, **90**(3), 219-227 (1981).