A light-fronts approach to a two-center time-dependent Dirac equation

Bilha Segev[1] and J.C. Wells[2]

[1] Department of Chemistry,
Ben-Gurion University of the Negev
POB 653, Beer-Sheva 84105, ISRAEL

[2]Center for Engineering Science Advanced Research
Computer Science and Mathematics Division
Oak Ridge National Laboratory
Oak Ridge, Tennessee, 37831-6203, USA

October 2, 2001

Abstract:

The two center time dependent Dirac equation, for an electron in the external ¯eld of two colliding ultrarelativistic heavy ions is considered. In the ultrarelativistic limit, the ions are practically moving at the speed of light and the electromagnetic ¯elds of the ions are con¯ned to the light fronts by the extreme Lorentz contraction and by the choice of gauge, designed to remove the long-range Coulomb e®ects. An exact solution to the ultrarelativistic limit of the two-center Dirac equation is found by using light-front variables and a light-fronts representation. Previously unexplained experimental results obtained at CERN's SPS are explained in this way and predictions are made as to where one should look, in momentum space, and in space-time, if one wants to study and observe non-perturbative electromagnetic pair-production e®ects in extremely relativistic heavy-ion collisions.

1 Introduction

Consider the relativistic scattering problem of an electron in the external field of two point-like charges (ions), moving on parallel, straight-line trajectories in opposite directions at speeds which approach the speed of light, and at an impact parameter $2\vec{b}$. An external-field approach to the influence of the ions on the electron is appropriate for peripheral impact parameters, heavy-ions, and high energies, where, to a very good approximation, the ions travel on parallel, straight-line trajectories, and ion recoil is negligible.

We review here our recent work on this problem [1, 2, 3]. In section 2, following Ref. [2], we show that the two center time dependent Dirac equation for the electron reduces in the high energy limit to Eq. (26) with the interactions of Eq. (29). In section 3, following Ref. [1], we solve this equation off the light fronts i.e. for an electron that both initially and asymptotically is not co-moving with an ion. The main result of our work is the transition amplitude given by Eqs. (90) and (67). In section 4, we discuss the application of this solution to electromagnetic pair production in heavy ion collisions, which we have used, for example, in Ref. [3], to explain recent experimental results. We note that one should distinguish between electron-positron pairs produced so that they are co-moving with the ions and those that are not. The two cases differ experimentally. They also differ theoretically, because they are described by different asymptotic boundary conditions. We have solved the problem only for electron-positron pairs that are not co-moving with the ions. Our solution to the two center Dirac equation in the high energy limit was confirmed by different methods, including a Green function approach [4], and resummation of the perturbation series [5]. The application to pair production on the other hand, has raised some controversy, which is also discussed in section 4. Section 5 concludes.

2 Two-center Dirac equation

We are using natural units ($c = 1$, $m_e = 1$, and $\hbar = 1$). The quantity α is the fine-structure constant, α and γ^μ are Dirac matrices in the Dirac representation, as in Ref. [6]; and I_2, 0_2, I_4, and 0_4 are the 2-dimensional and 4-dimensional unit and zero matrices.

We study relativistic heavy-ion collisions with a single active electron, e.g.

we neglect electron-electron interactions in comparison to the strong electron-ion interactions. For electrons distant from both the ions at asymptotic times, the collider (i.e. center-of-velocity) inertial frame is a natural choice. In the transverse direction, the origin of the collider frame is located equidistant from the target and projectile trajectories. The position vector of the electron in the collider frame is $\vec{r} = (x; y; z)$, and the associated time is t.

The free-particle Dirac equation has the form

$$i\frac{\partial}{\partial t}|\phi(\vec{r};t)\rangle = \hat{H}_0|\phi(\vec{r};t)\rangle\,; \tag{1}$$

where $\hat{H}_0$ is the free Dirac Hamiltonian,

$$\hat{H}_0 \equiv -i\vec{\alpha}\cdot\vec{\nabla} + \gamma^0\,. \tag{2}$$

The Dirac plane waves $\{|\phi_{\mathbf{p}}(\vec{r};t)\rangle = \exp(-iE_{\mathbf{p}}t)\exp(i\vec{r}\cdot\vec{p})|u_{\mathbf{p}}\rangle\}$ which satisfy the free Dirac equation are each characterized by the quantum numbers $\mathbf{p} \equiv \{\vec{p}; \lambda_{\mathbf{p}}; s_{\mathbf{p}}\}$; the momentum $\vec{p}$, the sign of the energy $E_{\mathbf{p}} = (-1)^{\lambda_{\mathbf{p}}}\sqrt{p^2+1}$, and the spin $s_{\mathbf{p}} = \pm$. Explicit forms for the four four-spinors $|u_{\mathbf{p}}\rangle$ are given in Ref. [6, 7, 8].

The two-center, time-dependent Dirac equation in the collider frame for an electron interacting with both colliding ions is

$$i\frac{\partial}{\partial t}|\psi(\vec{r};t)\rangle = \left[\hat{H}_0 + \hat{H}_B(t) + \hat{H}_A(t)\right]|\psi(\vec{r};t)\rangle\,; \tag{3}$$

where $|\psi(\vec{r};t)\rangle$ is the Dirac spinor wave function of the electron, $\hat{H}_B(t)$ is the electron-target interaction, and $\hat{H}_A(t)$ is the electron-projectile interaction,

$$\hat{H}_B(t) \equiv \frac{-Z_B\alpha\gamma(I_4 + \beta_c\alpha_z)}{\sqrt{(\vec{r}_\perp + \vec{b}/2)^2 + \gamma^2(z+\beta_c t)^2}}\,; \tag{4}$$

$$\hat{H}_A(t) \equiv \frac{-Z_A\alpha\gamma(I_4 - \beta_c\alpha_z)}{\sqrt{(\vec{r}_\perp - \vec{b}/2)^2 + \gamma^2(z-\beta_c t)^2}}\,. \tag{5}$$

2.1 Asymptotic solution

Consider, in the collider frame, at asymptotic times, an electron distant from both the target and projectile ions. The electron-projectile and electron-target distances then have the following asymptotic limits,

$$
\begin{aligned}
\lim_{|t|\to\infty} r_A(\vec{r};t) &\approx r_A^{\infty}(\vec{r};t) = \sqrt{(b/2)^2 + \gamma^2(z-\beta_c t)^2}\,; \\
\lim_{|t|\to\infty} r_B(\vec{r};t) &\approx r_B^{\infty}(\vec{r};t) = \sqrt{(b/2)^2 + \gamma^2(z+\beta_c t)^2}\,.
\end{aligned} \tag{6}
$$

Using these distances, the asymptotic, two-center Dirac equation is

$$
i\frac{\partial}{\partial t}|\Psi^{\infty}(\vec{r};t)\rangle = \left[\hat{H}_0 + \hat{H}_B^{\infty}(t) + \hat{H}_A^{\infty}(t)\right]|\Psi^{\infty}(\vec{r};t)\rangle\,; \tag{7}
$$

where $|\Psi^{\infty}(\vec{r};t)\rangle$ is the Dirac spinor wave function of the electron asymptotic channel solution, $\hat{H}_B^{\infty}(t)$ is the asymptotic electron-target interaction, and $\hat{H}_A^{\infty}(t)$ is the asymptotic electron-projectile interaction,

$$
\hat{H}_B^{\infty}(t) \approx \frac{-Z_B\alpha\gamma(I_4+\beta_c\alpha_z)}{\sqrt{(b/2)^2+\gamma^2(z+\beta_c t)^2}}\,; \tag{8}
$$

$$
\hat{H}_A^{\infty}(t) \approx \frac{-Z_A\alpha\gamma(I_4-\beta_c\alpha_z)}{\sqrt{(b/2)^2+\gamma^2(z-\beta_c t)^2}}\,. \tag{9}
$$

For the solutions of Eq. (7), consider an ansatz of a space-time dependent phase factor times a Dirac plane-wave state.

$$
|\Psi^{\infty}(\vec{r};t)\rangle = e^{-i\chi(z;t)}|\phi^{\infty}(\vec{r};t)\rangle\,; \tag{10}
$$

where

$$
\begin{aligned}
\chi(z;t) \approx{}& \frac{Z_A\alpha}{\beta}\ln\left[\gamma(z-\beta_c t) + \sqrt{(b/2)^2+\gamma^2(z-\beta_c t)^2}\right] \\
&- \frac{Z_B\alpha}{\beta}\ln\left[\gamma(z+\beta_c t) + \sqrt{(b/2)^2+\gamma^2(z+\beta_c t)^2}\right]\,.
\end{aligned} \tag{11}
$$

Substituting Eq. (11) into Eq. (7), multiplying from the left by $e^{+i\chi(z;t)}$, and collecting like terms gives

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\phi^{\infty}(\vec{r};t)\rangle ={}& \Bigg[\hat{H}_0 + \left(\frac{1}{\gamma^2-1}\right)\frac{Z_B\alpha\gamma\beta_c\alpha_z}{\sqrt{(b/2)^2+\gamma^2(z+\beta_c t)^2}} \\
&- \left(\frac{1}{\gamma^2-1}\right)\frac{Z_A\alpha\gamma\beta_c\alpha_z}{\sqrt{(b/2)^2+\gamma^2(z-\beta_c t)^2}}\Bigg]|\phi^{\infty}(\vec{r};t)\rangle\,.
\end{aligned} \tag{12}
$$

The scalar component of the asymptotic electron-projectile and electron-target interactions cancel exactly, and the vector component vanishes in the $\beta_c \to \infty$ limit. In this limit, the remaining equation is identical to the free Dirac equation, Eq. (1), and $|\phi^{\infty}(\vec{r}, t)\rangle \to |\phi(\vec{r}, t)\rangle$, is a Dirac plane-wave eigenstate. We conclude that in the extreme, high-energy limit, the ansatz in Eq. (10) with the Dirac plane wave, is the exact solution to the asymptotic, two-center Dirac equation, Eq. (7),

$$\lim_{\beta_c \to \infty} |\Phi^{\infty}(\vec{r}, t)\rangle = e^{-i\chi(z,t)}|\phi(\vec{r}, t)\rangle \, . \tag{13}$$

2.2 Definition of transition amplitudes

Following the notation of Ref. [9], let $|\Psi_j^{(+)}(t_f)\rangle$ be the exact outgoing-wave solution evolving from an initial channel solution $|\Phi_j^{\infty}(t_i)\rangle$, i.e.

$$\lim_{t \to -\infty} |\Psi_j^{(+)}(t)\rangle = |\Phi_j^{\infty}(t)\rangle \, ; \tag{14}$$

and $|\Phi_k^{\infty}(t_f)\rangle$ be the final asymptotic channel. Then, by definition, the exact transition amplitude between these two channels is given in the post form as

$$A_{kj}^{(+)} = \lim_{t_f \to \infty} \langle \Phi_k^{\infty}(t_f)|\Psi_j^{(+)}(t_f)\rangle \, . \tag{15}$$

The prior form of the amplitude is defined at $t \to -\infty$ as the projection of the exact incoming wave solution $|\Psi_j^{(-)}(t_i)\rangle$ evolving backward in time from the final channel $|\Phi_k^{\infty}(t_f)\rangle$, i.e.

$$\lim_{t \to \infty} |\Psi_k^{(-)}(t)\rangle = |\Phi_k^{\infty}(t)\rangle \, ; \tag{16}$$

onto the initial channel solution $|\Phi_j^{\infty}(t_i)\rangle$,

$$A_{kj}^{(-)} = \lim_{t_i \to -\infty} \langle \Psi_k^{(-)}(t_i)|\Phi_j^{\infty}(t_i)\rangle \, . \tag{17}$$

The post and prior forms of the amplitude may be unified using the time-evolution operator $\hat{U}(t_f, t_i)$ to relate the full outgoing-wave (incoming-wave) solution to its initial (final) state as

$$\begin{aligned} |\Psi_j^{(+)}(t_f)\rangle &= \hat{U}(t_f, t_i)|\Phi_j^{\infty}(t_i)\rangle \\ |\Psi_k^{(-)}(t_i)\rangle &= \hat{U}^{\dagger}(t_f, t_i)|\Phi_k^{\infty}(t_f)\rangle \, . \end{aligned} \tag{18}$$

Inserting Eqs. (18) into Eq. (15) or Eq. (17), one obtains,

$$A_k^{(j)} = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \langle \Phi_k^{\infty}(t_f)|\hat{U}(t_f,t_i)|\Phi_j^{\infty}(t_i)\rangle . \tag{19}$$

2.3 Short-range representation

The factored forms of the asymptotic solutions to the two-center Dirac equation, Eq. (13), obtained in the previous section, invite the definition of a new representation for the time-dependent Dirac equation. In this section, we introduce this representation, which we call the short-range representation.

Consider the extreme, high-energy limit $\beta_c \to 1$ of the two-center Dirac equation in the collider frame, Eq. (3), so that the asymptotic channels for an electron interacting with a distant target and projectile ion has the factored form of Eq. (13). We substitute this solution into the expression for the transition amplitudes for the initial state j and final state k,

$$A_k^{(j)} = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \langle e^{-i\chi(z,t_f)}\phi^{(k)}(t_f)|\hat{U}(t_f,t_i)|e^{-i\chi(z,t_i)}\phi^{(j)}(t_i)\rangle . \tag{20}$$

Rearranging the exponential factors in the expression so that they are applied directly to the evolution operator, one obtains,

$$A_k^{(j)} = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \langle \phi^{(k)}(t_f)|e^{+i\chi(z,t_f)}\hat{U}(t_f,t_i)e^{-i\chi(z,t_i)}|\phi^{(j)}(t_i)\rangle . \tag{21}$$

Defining the short-range representation,

$$|\psi^{(S)}(\vec{r},t)\rangle \equiv e^{+i\chi(z,t)}|\psi(\vec{r},t)\rangle \tag{22}$$

$$\hat{U}^{(S)}(t_f,t_i) \equiv e^{+i\chi(z,t_f)}\hat{U}(t_f,t_i)e^{-i\chi(z,t_i)} . \tag{23}$$

gives the formal expression of a transition amplitude between plane-wave states,

$$A_k^{(j)} = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \langle \phi^{(k)}(t_f)|\hat{U}^{(S)}(t_f,t_i)|\phi^{(j)}(t_i)\rangle . \tag{24}$$

To obtain the two-center Dirac equation in the collider frame in the short-range representation, we begin with Eq. (3), and make the substitution

$$|\psi(\vec{r},t)\rangle = e^{-i\chi(z,t)}|\psi^{(S)}(\vec{r},t)\rangle . \tag{25}$$

After multiplying from the left by $e^{+i\chi(z;t)}$, the equation of motion has the form

$$i\frac{\partial}{\partial t}|\psi^{(S)}(\vec{r};t)\rangle = \left[\hat{H}_0 + \hat{W}_B(t) + \hat{W}_A(t)\right]|\psi^{(S)}(\vec{r};t)\rangle\;; \tag{26}$$

where $\hat{W}_B(t)$ and $\hat{W}_A(t)$ are the time-dependent electron-target and electron-projectile interactions in the short-range representation,

$$\begin{aligned}\hat{W}_B(t) &\equiv \hat{H}_B(t) - \frac{-Z_B\alpha\gamma(I_4 + (1/\beta_c)\alpha_z)}{\sqrt{(b/2)^2 + \gamma^2(z+\beta_c t)^2}}\;;\\ \hat{W}_A(t) &\equiv \hat{H}_A(t) - \frac{-Z_A\alpha\gamma(I_4 - (1/\beta_c)\alpha_z)}{\sqrt{(b/2)^2 + \gamma^2(z-\beta_c t)^2}}\;:\end{aligned} \tag{27}$$

In the high-energy limit, $\beta_c \to 1$, and

$$\begin{aligned}\lim_{\beta_c\to 1}\hat{W}_B(t) &\equiv \hat{H}_B(t) - \hat{H}_B^{1}(t)\;;\\ \lim_{\beta_c\to 1}\hat{W}_A(t) &\equiv \hat{H}_A(t) - \hat{H}_A^{1}(t)\;:\end{aligned} \tag{28}$$

The asymptotic dependence of the time-dependent interaction has been canceled exactly (in the $\beta \to 1$ limit). Likewise, the phase distortion in the asymptotic channel solutions is canceled by the phase transformation defining the short-range representation, and the asymptotic channels are effectively the Dirac plane waves.

Applying the sharp limit of $\beta_c \to 1;\;\; r_\perp; b \ll \gamma$ to Eqs. (28), we obtain the following factored forms for the time-dependent interaction [10, 11, 1],

$$\begin{aligned}\lim_{\substack{\beta_c\to 1\\ r_\perp;b\ll\gamma}}\hat{W}_B(t) &= (I_4+\alpha_z)Z_B\alpha\delta(t+z)\ln\left[\frac{(\vec{r}_\perp+\vec{b}/2)^2}{(b/2)^2}\right]\;;\\ \lim_{\substack{\beta_c\to 1\\ r_\perp;b\ll\gamma}}\hat{W}_A(t) &= (I_4-\alpha_z)Z_A\alpha\delta(t-z)\ln\left[\frac{(\vec{r}_\perp-\vec{b}/2)^2}{(b/2)^2}\right]\;:\end{aligned} \tag{29}$$

Consider the physical nature of this limit. A δ function over time alone would indicate a sudden interaction of the ions with the electron. In the gauge-transformed equation, as they move, the ions are continuously interacting with the electronic wave function. Naturally, this interaction is singular on the trajectories of the ions, as it was before the ultrarelativistic limit has been taken; but an additional singularity is induced in the ultrarelativistic limit by the extreme Lorentz contraction of the field. In this limit, the interaction is infinite on the two planes perpendicular to the ions trajectories, and vanishes elsewhere.

The interactions have zero range in the longitudinal direction and a logarithmic behavior in the transverse direction, similar to the potential of a line of charge. In the limit $\beta_c \rightarrow 1$, the two ions are moving at the speed of light and thus the interaction planes described above coincide with the light fronts, given by $z = \pm t$. Finally, we note that $(I_4 \pm \alpha_z)/2$ are orthonormal projection operators. The 4-Dirac spinor wave function of the electron can be decomposed into two orthogonal components,

$$|\Psi_+(\vec{r};t)\rangle \equiv \frac{1}{2}(I_4 + \alpha_z)|\Psi^{(S)}(\vec{r};t)\rangle \tag{30}$$

$$|\Psi_-(\vec{r};t)\rangle \equiv \frac{1}{2}(I_4 - \alpha_z)|\Psi^{(S)}(\vec{r};t)\rangle . \tag{31}$$

Each ion interacts directly only with one of these components; Z_A with $|\Psi_-(\vec{r};t)\rangle$ and Z_B with $|\Psi_+(\vec{r};t)\rangle$.

3 Light-fronts representation

In this section, the two-center time-dependent Dirac equation will be further simplified and solved by changing into light-front variables and by introducing a new representation for the Dirac spinors, the light-fronts representation. This is an appropriate choice of variables and representation, since, in the ultrarelativistic limit the interactions are confined to the two light fronts.

3.1 Definitions and notations

In terms of light-front variables, space-time and energy-momentum are described by the 4-vectors $(\vec{r}_\perp;\tau_+;\tau_-)$ and $(\vec{p}_\perp;p_+;p_-)$, where

$$\tau_\pm \equiv (t \pm z)/2 \tag{32}$$

$$p_\pm \equiv E_p \pm p_z \tag{33}$$

$$p_+ p_- = 1 + p_\perp^2 \tag{34}$$

The sign and absolute value of $(p_+ + p_-)/2$ are λ_p and ε_p, respectively. Equation (34) defines the energy-shell. These variables were often used previously for quantization on one of the two light fronts, $\tau_+ = 0$ or $\tau_- = 0$ [12]. For the problem considered here, it is useful to keep the symmetry between τ_+ and τ_-.

The projection operators $(I_4 \pm \alpha_z)/2$ acquire a simple form and the interaction is diagonalized by introducing the light-fronts representation for the Dirac matrices,

$$\gamma^\mu_{\text{light}-\text{fronts}} = \Lambda \gamma^\mu_{\text{Dirac}} \Lambda^\dagger; \tag{35}$$

$$\Lambda \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & \sigma_z \\ I_2 & -\sigma_z \end{pmatrix}; \tag{36}$$

$$\Lambda \alpha_z \Lambda^\dagger = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}; \tag{37}$$

$$\Lambda \left(\frac{1}{2}(I_4 + \alpha_z) \right) \Lambda^\dagger = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}; \tag{38}$$

$$\Lambda \left(\frac{1}{2}(I_4 - \alpha_z) \right) \Lambda^\dagger = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}; \tag{39}$$

$$\Lambda \vec{\alpha}_\perp \Lambda^\dagger = i \begin{pmatrix} 0_2 & -\vec{\omega} \\ \vec{\omega} & 0_2 \end{pmatrix}; \tag{40}$$

$$\vec{\omega} \equiv (-\sigma_y, \sigma_x). \tag{41}$$

With this notation, the gauge-transformed two-center Dirac equation in the sharp ultrarelativistic limit in the light-fronts representation is

$$\begin{pmatrix} i\partial_{\tau_+} |G_+\rangle \\ i\partial_{\tau_-} |G_-\rangle \end{pmatrix} = \begin{pmatrix} \delta(\tau_+) B(\vec{r}_\perp, \vec{b}) & \hat{h}_0 \\ \hat{h}_0^\dagger & \delta(\tau_-) A(\vec{r}_\perp, \vec{b}) \end{pmatrix} \begin{pmatrix} |G_+\rangle \\ |G_-\rangle \end{pmatrix}; \tag{42}$$

where $|G_+\rangle$ and $|G_-\rangle$ are the upper and lower bi-spinor components of the Dirac wave function in the light-fronts representation

$$\begin{pmatrix} |G_+\rangle \\ |G_-\rangle \end{pmatrix} \equiv \Lambda |\Psi\rangle; \tag{43}$$

and

$$\hat{h}_0 \equiv I_2 - i\vec{\omega} \cdot \hat{\vec{p}}_\perp; \tag{44}$$

$$A(\vec{r}_\perp;\vec{b}) \equiv Z_A\alpha \ln\left[\frac{(\vec{r}_\perp - \vec{b})^2}{b^2}\right]; \tag{45}$$

$$B(\vec{r}_\perp;\vec{b}) \equiv Z_B\alpha \ln\left[\frac{(\vec{r}_\perp + \vec{b})^2}{b^2}\right]. \tag{46}$$

The upper and lower bi-spinors are coupled by the free Hamiltonian. Each interacts directly with the external field of one ion and feels the field of the other ion through its coupling to the other bi-spinor.

Equation (42) has no discontinuities in the transverse direction. It is therefore useful to Fourier transform its solution with respect to $\vec{r}_\perp$. Two mixed bi-spinors wave-functions, $|g_\pm(\vec{q}_\perp;\tau_+;\tau_-)\rangle$, are then defined by

$$|G_\pm(\vec{r}_\perp;\tau_+;\tau_-)\rangle \equiv \int d\vec{q}_\perp e^{i\vec{r}_\perp\cdot\vec{q}_\perp}|g_\pm(\vec{q}_\perp;\tau_+;\tau_-)\rangle. \tag{47}$$

$|g_+\rangle$ and $|g_-\rangle$, like $|G_+\rangle$ and $|G_-\rangle$, are coupled by the free Hamiltonian.

3.2 Free Dirac equation off the light fronts

Off the light fronts, i.e. for $\tau_+ \neq 0$ and $\tau_- \neq 0$, the wave function satisfies the free Dirac equation and Eq. (42) reduces to two coupled equations for the mixed bi-spinors $|g_\pm(\vec{q}_\perp;\tau_+;\tau_-)\rangle$.

$$i\frac{\partial}{\partial\tau_+}|g_+\rangle = (I_2 - i\vec{\sigma}\cdot\vec{q}_\perp)|g_-\rangle; \tag{48}$$

$$i\frac{\partial}{\partial\tau_-}|g_-\rangle = (I_2 + i\vec{\sigma}\cdot\vec{q}_\perp)|g_+\rangle. \tag{49}$$

The second-order equations decouple

$$\frac{\partial^2}{\partial\tau_+\partial\tau_-}|g_\pm\rangle = -(1+q_\perp^2)|g_\pm\rangle; \tag{50}$$

where use was made of

$$(I_2 - i\vec{\sigma}\cdot\vec{q}_\perp)(I_2 + i\vec{\sigma}\cdot\vec{q}_\perp) = (1+q_\perp^2)I_2. \tag{51}$$

A solution to Eqs. (48, 49) is given, for example, by the plane waves which in the light-fronts representation are given by

$$\begin{pmatrix} |F^p_+\rangle \\ |F^p_-\rangle \end{pmatrix} \equiv \Lambda|\phi^{(p)}(\vec{r};t)\rangle; \tag{52}$$

$$|F^p_\pm\rangle \equiv \int d\vec{q}_\perp e^{i\vec{r}_\perp\cdot\vec{q}_\perp}|f^p_\pm(\vec{q}_\perp;\tau_+;\tau_-)\rangle; \tag{53}$$

$$|f^p_\pm(\vec{q}_\perp;\tau_+;\tau_-)\rangle = \delta(\vec{q}_\perp - \vec{p}_\perp)e^{-i(\tau_- p_+ + \tau_+ p_-)}|\Gamma^p_\pm\rangle. \tag{54}$$

The bi-spinors, $|\Gamma^p_\pm\rangle$, (the upper and lower parts of $\Lambda|u_p\rangle$),

$$|\Gamma^p_\pm\rangle = \frac{(2\pi)^{-3/2}}{2\sqrt{\varepsilon_p(1+\varepsilon_p)}} \times \left[I_2\left(1 + (-1)^{\lambda_p} p_\pm\right) \mp i\vec{\alpha}\cdot\vec{p}_\perp (\pm\sigma_z)^{\lambda_p}\right]|s_p\rangle; \tag{55}$$

satisfy the simple relation

$$|\Gamma^p_-\rangle = \frac{I_2 + i\vec{\alpha}\cdot\vec{p}_\perp}{p_+}|\Gamma^p_+\rangle. \tag{56}$$

These plane waves solve Eq. (42) off the light fronts in the limits $t \to \pm\infty$. They do not solve it for finite t, when $\vec{p}_\perp$ is no longer a good quantum number, as the singular interaction with the ions makes the wave function discontinuous at the light fronts.

3.3 The discontinuity across the light fronts

The discontinuities of the spinor wave function at the light fronts (at $\tau_+ = 0$ and at $\tau_- = 0$, excluding only $\tau_+ = \tau_- = 0$) are deduced from Eq. (42). At one light front, ($\tau_+ = 0$, $\tau_- \neq 0$), Eq. (42) for $|G_+\rangle$ reads,

$$i\partial_{\tau_+}|G_+\rangle = \hat{h}_0|G_-\rangle + B(\vec{r}_\perp)\delta(\tau_+)|G_+\rangle. \tag{57}$$

The δ-function singularity renders $|G_+\rangle$ discontinuous at $\tau_+ = 0$, as can be seen by integrating both hand sides of Eq. (57) with respect to τ_+ from $-\epsilon$ to ϵ and taking the limit $\epsilon \to 0$,

$$|G_+(\tau_+ = 0^+)\rangle \neq |G_+(\tau_+ = 0^-)\rangle. \tag{58}$$

An auxiliary bi-spinor can be defined by a piece-wise gauge transformation,

$$|\tilde{G}_+\rangle \equiv \exp[iB(\vec{r}_\perp)\theta(\tau_+)]|G_+\rangle. \tag{59}$$

Direct substitution gives,

$$i\partial_{\tau_+}|\tilde{G}_+\rangle = \exp[iB(\vec{r}_\perp)\theta(\tau_+)]\hat{h}_0|G_-\rangle \tag{60}$$

The auxiliary bi-spinor is continuous at $\tau_+ = 0$, as can be seen by operating on both sides of Eq. (60) with $\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\tau_+$, obtaining

$$|\tilde{G}_+(\tau_+ = 0^+)\rangle = |\tilde{G}_+(\tau_+ = 0^-)\rangle. \tag{61}$$

The continuity of $|\tilde{G}_+\rangle$ at $\tau_+ = 0$ ($\tau_- \neq 0$), implies a discontinuity of $|G_+\rangle$:

$$|G_+(\tau_+ = 0^+)\rangle = e^{-iB(\vec{r}_\perp;\vec{b})}|G_+(\tau_+ = 0^-)\rangle; \tag{62}$$

Likewise, the continuity of

$$|\tilde{G}_-\rangle \equiv \exp[iA(\vec{r}_\perp)\theta(\tau_-)]|G_-\rangle \tag{63}$$

at $\tau_- = 0$, ($\tau_+ \neq 0$,) implies the discontinuity of $|G_-\rangle$:

$$|G_-(\tau_- = 0^+)\rangle = e^{-iA(\vec{r}_\perp;\vec{b})}|G_-(\tau_- = 0^-)\rangle. \tag{64}$$

This Heavyside step-function, space-dependent, phase discontinuity was previously obtained in Ref. [11]. In earlier work [10], a gauge transformation was used to establish the fact that the electromagnetic field of a charge which is moving at the speed of light can be equivalently given by gauge potentials with a δ-function singularity at the light front, or by gauge potentials with only a step-function discontinuity there. The wave function of a particle interacting with this field is discontinuous or continuous, depending on the gauge choice. We choose to work with such a gauge that would give a sharp interaction and a discontinuous spinor wave function, yet we have used here other gauges to find the discontinuities in an explicit form.

3.4 Momentum-transfer distribution

Due to the space dependent phase-shift of Eqs. (62) and (64), the transverse momentum is not conserved. The Fourier components of Eq. (47) are mixed when the singularities at the light fronts are crossed,

$$|g_+(\mathbf{q}_\perp;\tau_+ = 0^+)\rangle = \int d\mathbf{p}_\perp Q^{-\mathbf{b}}_{Z_B}(\mathbf{p}_\perp - \mathbf{q}_\perp)|g_+(\mathbf{p}_\perp;\tau_+ = 0^-)\rangle; \tag{65}$$

$$|g_-(\mathbf{q}_\perp;\tau_- = 0^+)\rangle = \int d\mathbf{p}_\perp Q^{\mathbf{b}}_{Z_A}(\mathbf{p}_\perp - \mathbf{q}_\perp)|g_-(\mathbf{p}_\perp;\tau_- = 0^-)\rangle; \tag{66}$$

where the distribution for this momentum change, given by $Q^{\mathbf{b}}_Z(\boldsymbol{\kappa})$, contains all the dynamics of the ion-electron interaction,

$$Q^{\mathbf{b}}_Z(\boldsymbol{\kappa}) \equiv \frac{1}{(2\pi)^2}\int d\mathbf{r}_\perp\, e^{i\mathbf{r}_\perp\cdot\boldsymbol{\kappa}}\left[\frac{(\mathbf{r}_\perp - \mathbf{b})^2}{b^2}\right]^{-i\alpha Z} : \tag{67}$$

Note that here $\boldsymbol{\kappa}$ and $\mathbf{b}$ are two-dimensional vectors in the $(x;y)$ plane. The continuity is recovered in the limit $Z \to 0$, as $Q^{\mathbf{b}}_Z(\boldsymbol{\kappa}) \to \delta(\boldsymbol{\kappa})$. Integrating first over the angular variable, we find

$$Q^{\mathbf{b}}_Z(\boldsymbol{\kappa} \neq 0) = \frac{1}{2\pi}\frac{\exp(i\boldsymbol{\kappa}\cdot\mathbf{b})}{\kappa^2(b\kappa)^{-i2\alpha Z}}\int_{\xi>0} d\xi\, J_0(\xi)\xi^{1-i2\alpha Z}; \tag{68}$$

where $b = |\mathbf{b}|$, $\kappa = |\boldsymbol{\kappa}|$, and J_0 is the Bessel function.

The distribution $Q^{\mathbf{b}}_Z(\boldsymbol{\kappa})$ in general diverges even though the integral over this distribution is convergent and normalized $\int d\boldsymbol{\kappa}\, Q^{\mathbf{b}}_Z(\boldsymbol{\kappa}) = 1$. This is so, because Eq. (29) describes an interaction which continually increases in strength for large $r_\perp$ (or large b). In this very same regime, however, the limit that is taken in its derivation does not apply. When the integral is regularized so as to avoid contributions from large, transverse distances, i.e. from $\xi > \kappa\gamma$, by several different regularization schemes one gets (for $\boldsymbol{\kappa} \neq 0$)

$$Q^{\mathbf{b}}_Z(\boldsymbol{\kappa}) \to \frac{-i\alpha Z}{\pi}\frac{\exp(i\boldsymbol{\kappa}\cdot\mathbf{b})}{\kappa^2}\left[\frac{\Gamma(-i\alpha Z)}{\Gamma(+i\alpha Z)}\left(\frac{b\kappa}{2}\right)^{+i2\alpha Z}\right]; \tag{69}$$

where arguments presented in Ref. [13] at p. 385, p. 393, and p. 401, can be used for an exponential, a Gaussian, and a Bessel function regularization, respectively. In the perturbative limit, $\alpha Z \ll 1$, the leading order correction to the δ-function is then given by

$$\lim_{\alpha Z \ll 1} Q_Z^{\mathbf{b}}(\boldsymbol{\kappa}) = \delta(\boldsymbol{\kappa}) - \frac{i\alpha Z}{\pi}\frac{1}{\kappa^2}\exp(i\mathbf{b}\cdot\boldsymbol{\kappa}). \tag{70}$$

Note that the perturbative expression violates unitarity while the exact expression does not.

Is the regularization procedure leading to Eq. (69) correct? As we wrote in Ref. [3], implicit in its application is an assumption that contributions from large transverse distances and large impact parameters can be neglected. Clearly this is not always true. Furthermore, for Eq. (69) to apply, self consistency requires that the regulated integral of Eq. (67) must converge to the expression of Eq. (69) for ξ such that $|\mathbf{r}_\perp - \mathbf{b}| \ll \gamma$. The case of small coupling, $\alpha Z \ll 1$, was studied in Ref. [1] and does not present any special problem. The case of large αZ can be considered by the method of stationary phase. Expansion of Eq. (67) around the stationary point $\mathbf{r}_\perp - \mathbf{b} = 2\alpha Z \boldsymbol{\kappa}/\kappa^2$ confirms Eq. (69) for this case. The procedure is consistent if the stationary point is located at small distances from the ion, i.e. if and only if

$$|\boldsymbol{\kappa}| \gg \frac{2\alpha Z}{\gamma}. \tag{71}$$

It is interesting to find that Eq. (71) is trivially satisfied in two very different limits: in the perturbative limit of $\alpha Z \to 0$ and in the high-energy limit of $\gamma \to \infty$. This issue, previously discussed by us in Ref. [3] was ignored by other works on this subject. This, we believe, has caused much confusion. We discuss it below, in section 4.

3.5 A piecewise solution

The singular interaction on the planes perpendicular to the trajectories of the ions, cut space-time along the light fronts into four regions. A piecewise solution is defined off the light fronts by $|g_\pm(\mathbf{q}_\perp;\tau_+;\tau_-)\rangle = |g_\pm^{(i)}(\mathbf{q}_\perp;\tau_+;\tau_-)\rangle$, where (i)= I for $\tau_+ < 0$ and $\tau_- < 0$, (i)= II for $\tau_+ > 0$ and $\tau_- < 0$, (i)= III for $\tau_+ < 0$ and $\tau_- > 0$, and (i)= IV for $\tau_+ > 0$ and $\tau_- > 0$. In each region, the wave function is continuous and solves the local free Dirac equation. At any time, except for $t \to \pm\infty$, the wave function extends in space through three (or two, at $t = 0$) of these regions. The solution presented here is not complete in the sense that it does not include the solution on the light fronts; $\tau_+ = 0$ and $\tau_- = 0$ are excluded.

3.5.1 Initial condition and intermediate states

Consider the initial condition of a single plane wave with the quantum numbers $j = \{\vec{j}; \lambda_j; s_j\}$, or, using light-front variables, $j = \{\vec{j}_\perp; j_+; j_-; s_j\}$, with the constraint $j_+ j_- = 1 + j_\perp^2$. The continuity off the light fronts gives the solution in region I,

$$|g^{\mathrm{I}}_{\pm}(\vec{q}_\perp)\rangle = \delta(\vec{j}_\perp - \vec{q}_\perp)e^{-i(\tau_- j_+ + \tau_+ j_-)}|\Gamma^{j}_{\pm}\rangle; \tag{72}$$

where the bi-spinors $|\Gamma^{j}_{\pm}\rangle$ are defined as in Eq. (55).

The solution in regions II and III is obtained by first applying Eq. (65) for the discontinuity across $\tau_+ = 0$ and Eq. (66) for the discontinuity across $\tau_- = 0$ and then solving the coupled equations (48, 49) inside each of the intermediate space-time regions. We obtain in region II

$$\begin{aligned} |g^{\mathrm{II}}_{+}(\vec{q}_\perp)\rangle &= \exp\left[-i\tau_- j_+ - i\tau_+\left(\frac{1+q_\perp^2}{j_+}\right)\right] Q^{-\hat{b}}_{Z_B}(\vec{j}_\perp - \vec{q}_\perp)\,|\Gamma^{j}_{+}\rangle; \\ |g^{\mathrm{II}}_{-}(\vec{q}_\perp)\rangle &= \left(\frac{I_2 + i\vec{\omega}\cdot\vec{q}_\perp}{j_+}\right)|g^{\mathrm{II}}_{+}(\vec{q}_\perp)\rangle; \end{aligned} \tag{73}$$

and in region III,

$$\begin{aligned} |g^{\mathrm{III}}_{-}(\vec{p}_\perp)\rangle &= \exp\left[-i\tau_+ j_- - i\tau_-\left(\frac{1+p_\perp^2}{j_-}\right)\right] Q^{\hat{b}}_{Z_A}(\vec{j}_\perp - \vec{p}_\perp)\,|\Gamma^{j}_{-}\rangle; \\ |g^{\mathrm{III}}_{+}(\vec{p}_\perp)\rangle &= \left(\frac{I_2 - i\vec{\omega}\cdot\vec{p}_\perp}{j_-}\right)|g^{\mathrm{III}}_{-}(\vec{p}_\perp)\rangle: \end{aligned} \tag{74}$$

It is now apparent why the Fourier transform with respect to $\vec{r}_\perp$ and the definition of $|g_\pm(\vec{q}_\perp; \tau_+; \tau_-)\rangle$ in Eq. (47) were needed. The simple discontinuity condition (62) at $\tau_+ = 0$ applies only to $|G_+\rangle$. The other bi-spinor $|G_-\rangle$ is influenced indirectly by the field at $\tau_+ = 0$ through its coupling to $|G_+\rangle$. Likewise, at $\tau_- = 0$ the simple discontinuity condition (64) for $|G_-\rangle$ induces a non-trivial change in $|G_+\rangle$. The coupling between $|G_+\rangle$ and $|G_-\rangle$ in free space on either sides of the singular interaction is best described by

Eqs. (48,49) for their Fourier components with respect to $\vec{r}_\perp$. Thus, while the discontinuity conditions (65,66) for $|g_\pm\rangle$ seem more complicated than the discontinuity conditions (62,64) for $|G_\pm\rangle$, using $|g_\pm\rangle$ allows for a simple derivation of the complete spinor wave function in regions II and III.

The solution of the free Dirac equation in region IV is complicated by the non-trivial boundary conditions on the light fronts. Applying Eq. (65) again for the discontinuity across τ_+ and Eq. (66) for the discontinuity across τ_-, we cross from regions II and III into region IV to obtain on the hyper-surfaces adjacent to the light fronts,

$$
\begin{aligned}
|g_-^{IV}(\vec{k}_\perp;\tau_- = 0^+)\rangle &= \int d\vec{q}_\perp \exp\left[-i\tau_+\left(\frac{1+q_\perp^2}{j_+}\right)\right]\tilde{Q}_{Z_A}(\vec{q}_\perp - \vec{k}_\perp) \\
&\quad \times \tilde{Q}^{-}_{Z_B}(\vec{j}_\perp - \vec{q}_\perp)\left(\frac{I_2 + i\vec{\omega}\cdot\vec{q}_\perp}{j_+}\right)|\Gamma^j_+\rangle; \qquad (75)\\
|g_+^{IV}(\vec{k}_\perp;\tau_+ = 0^+)\rangle &= \int d\vec{p}_\perp \exp\left[-i\tau_-\left(\frac{1+p_\perp^2}{j_-}\right)\right]\tilde{Q}^{-}_{Z_B}(\vec{p}_\perp - \vec{k}_\perp) \\
&\quad \times \tilde{Q}_{Z_A}(\vec{j}_\perp - \vec{p}_\perp)\left(\frac{I_2 - i\vec{\omega}\cdot\vec{p}_\perp}{j_-}\right)|\Gamma^j_-\rangle. \qquad (76)
\end{aligned}
$$

Instead of solving now for $|g_\pm^{IV}\rangle$ at any $\tau_\pm > 0$, the transition amplitudes are obtained in the next section by defining the transition current and by applying Gauss' theorem for this current.

3.5.2 Transition current and Gauss' theorem

The transition amplitudes $A_k^{(j)}$ were defined in Eq. (24),

$$A_k^{(j)} \equiv \lim_{t_f \to \infty} \int d\vec{r}\, \phi^{(k)\dagger}(\vec{r};t_f)\,\Psi^{(j)}(\vec{r};t_f); \qquad (77)$$

where

$$\Psi^{(j)}(\vec{r};t_f) = \hat{U}^{(S)}(t_f;t_i)|\phi^{(j)}(t_i)\rangle. \qquad (78)$$

The integrand is a component of a 4-vector transition current density:

$$
\begin{aligned}
J_0^{(k;j)} &\equiv \phi^{(k)\dagger}\Psi^{(j)} \\
\vec{J}^{(k;j)} &\equiv \phi^{(k)\dagger}\vec{\alpha}\,\Psi^{(j)}. \qquad (79)
\end{aligned}
$$

An equivalent form for the transition current in terms of light-fronts representation wave-functions includes

$$J_{\pm}^{(k;j)} \equiv J_0^{(k;j)} \pm J_z^{(k;j)} = 2\, F_{\pm}^{k\dagger}\, G_{\pm}^{(j)}. \tag{80}$$

We prove first that the transition 4-current density defined in Eq. (79) is conserved. In fact, any two solutions of the free Dirac equation can be used to define a conserved current in a similar way. This proof is very similar to the one found in textbooks proving the probability current to be conserved [6]. Both $\phi^{(k)}$ and $\psi^{(j)}$ solve in region IV the free Dirac equation in the Dirac representation

$$i\frac{\partial}{\partial t}\psi^{(j)}(\mathbf{r};t) = \left[-i\boldsymbol{\alpha}\cdot\nabla + \beta m\right]\psi^{(j)}(\mathbf{r};t)\,; \tag{81}$$

$$i\frac{\partial}{\partial t}\phi^{(k)}(\mathbf{r};t) = \left[-i\boldsymbol{\alpha}\cdot\nabla + \beta m\right]\phi^{(k)}(\mathbf{r};t)\,. \tag{82}$$

Multiplying Eq. (81) from the left by the adjoint of $\phi^{(k)}$, multiplying the Hermitian conjugate of Eq. (82) from the right by $\psi^{(j)}$ and subtracting gives

$$\frac{\partial}{\partial t}\left(\phi^{(k)\dagger}\psi^{(j)}\right) = -\nabla\cdot\left(\phi^{(k)\dagger}\boldsymbol{\alpha}\,\psi^{(j)}\right)\,; \tag{83}$$

where the Hermiticity of the Dirac matrices has been used. Using the definition of the transition current in Eq. (79), Eq. (83) is reveled as the continuity equation

$$\frac{\partial}{\partial t}J_0^{(j;k)} + \nabla\cdot\mathbf{J}^{(j;k)} = \frac{\partial J^{\mu}}{\partial x^{\mu}} = 0\,; \tag{84}$$

proving the transition-current density to be conserved.

Integrating over any empty space-time hyper-volume, V, and applying Guass' theorem to convert the volume integral into a surface integral over the hyper-surface S enclosing V, in general gives,

$$\int_S d\sigma J^{\mu} n_{\mu} = 0\,; \tag{85}$$

where the unit 4-vector n_μ is defined as the outward pointing normal to S. For our purposes, it is useful to apply Eq. (85) to the space-time region IV, defined by $\tau_\pm > 0$. The closed hyper-surface S enclosing region IV is made of the following open hyper-surfaces: (i) $t = t_f \to +\infty$, (ii) $\tau_+ = 0^+$, $\tau_- > 0$,

(iii) $\tau_- = 0^+$, $\tau_+ > 0$, (iv) $x \to \pm\infty$, and (v) $y \to \pm\infty$. Writing Eq. (85) for this surface gives

$$
\begin{aligned}
0 &= \lim_{t_f \to \infty} \int d\mathbf{r}_\perp \int_{+\infty}^{-\infty} J_0(\mathbf{r}; t_f) \\
&\quad - 2 \int d\mathbf{r}_\perp \int_{+\infty}^{0^+} d\tau_- \, J_+(\mathbf{r}_\perp; \tau_+ = 0^+; \tau_-) \\
&\quad - 2 \int d\mathbf{r}_\perp \int_{0^+}^{+\infty} d\tau_+ \, J_-(\mathbf{r}_\perp; \tau_+; \tau_- = 0^+);
\end{aligned}
\tag{86}
$$

where use was made of the fact that in any physical situation, i.e. for a square-integrable wavepacket, the currents vanish as $\mathbf{r}_\perp \to \infty$. The hypersurfaces (iv) and (v) do not contribute to the integral. The factors of 2 arise from the Jacobian relating the original differentials to the differentials for the light-front variables, and the negative sign in the second and third terms arise because the unit normal vectors $\hat{\mathbf{n}}_\pm$ are directed outside the volume V, i.e. $J \cdot \hat{\mathbf{n}}_\pm = -J_\pm$. The transition currents $J_\pm^{(k;j)}$ are

$$
\begin{aligned}
J_\pm^{(k;j)}(\mathbf{r}_\perp; \tau_+; \tau_-) &= 2 \int d\mathbf{p}_\perp \int d\mathbf{l}_\perp \exp[i\mathbf{r}_\perp \cdot (\mathbf{l}_\perp - \mathbf{p}_\perp)] \\
&\quad \times \langle f_\pm^k(\mathbf{p}_\perp; \tau_+; \tau_-) | g_\pm^{IV}(\mathbf{l}_\perp; \tau_+; \tau_-) \rangle .
\end{aligned}
\tag{87}
$$

Integrating over $\mathbf{r}_\perp$ and using the explicit expression for the plane waves,

$$
\begin{aligned}
A_k^{(j)} &= 16\pi^2 \int_{0^+}^{\infty} d\tau_- \, e^{i\tau_- k_+} \langle -_+^k | g_+^{IV}(\mathbf{k}_\perp; \tau_+ = 0^+; \tau_-) \rangle \\
&\quad - 16\pi^2 \int_{0^+}^{\infty} d\tau_+ \, e^{i\tau_+ k_-} \langle -_-^k | g_-^{IV}(\mathbf{k}_\perp; \tau_+; \tau_- = 0^+) \rangle .
\end{aligned}
\tag{88}
$$

The amplitudes are finally obtained by substituting Eqs. (75,76) and integrating over $\tau_\pm$. The integration over $\tau_\pm$ would have given a δ-function conservation law for the light-front momenta, had it been on the complete line $-\infty < \tau_\pm < \infty$. Instead, the integrals on the half lines $0 < \tau_\pm < \infty$ are regulated in the usual way with an infinitesimal small constant, ϵ [14].

$$
\int_{0^+}^{\infty} d\tau \exp(i\tau\eta) = \frac{i}{\eta + i\epsilon} .
\tag{89}
$$

3.6 Transition amplitudes

The transition amplitudes corresponding to the exact solution of the sharp Dirac equation off the light fronts are

$$A_k^{(j)} = \frac{i}{\pi}\int d\vec{p}_\perp \left\{ \frac{\sigma_k^j(\vec{p}_\perp)\, \widehat{Q}_{Z_B}(\vec{k}_\perp - \vec{p}_\perp)\, \widehat{Q}_{Z_A}(\vec{j}_\perp - \vec{p}_\perp)}{p_\perp^2 + 1 - j_- k_+} - \frac{\sigma_j^{k\dagger}(\vec{p}_\perp)\, \widehat{Q}_{Z_A}(\vec{p}_\perp - \vec{k}_\perp)\, \widehat{Q}_{Z_B}(\vec{p}_\perp - \vec{j}_\perp)}{p_\perp^2 + 1 - j_+ k_-} \right\}; \quad (90)$$

which is the main result of our work. The spinor part is

$$\begin{aligned} \sigma_k^j(\vec{p}_\perp) &\equiv (2\pi)^3 \langle u_k| (I_4 - \alpha_z)(\vec{\alpha}\cdot\vec{p}_\perp + \gamma^0)(I_4 + \alpha_z) |u_j\rangle \\ &\equiv (2\pi)^3 \langle -_+^k | I_2 - i\vec{\sigma}\cdot\vec{p}_\perp | -_-^j \rangle; \end{aligned} \quad (91)$$

and $\widehat{Q}_Z(\vec{\ })$, defined in Eq. (67), is the Fourier transform of the phase shift at the light front.

4 Application to pair-production

Electron-positron pair production in extremely relativistic heavy-ion collisions has lately received a lot of attention. Recent and ongoing experiments at CERN's SPS [15, 16, 17], as well as upcoming experiments at RHIC and LHC, combine with the fundamental aspects of this process to make its investigation an important field of research.

The transition amplitudes, $A_k^{(j)}$, is the amplitude for electron scattering from the initial state at $t_i \to -\infty$,

$$e^{-i\chi(z;t_i)}\psi^{(j)}(t_i)\rangle \quad (92)$$

to the final state at $t_f \to \infty$,

$$e^{-i\chi(z;t_f)}\psi^{(k)}(t_f)\,. \quad (93)$$

Note that these asymptotic forms at $t_i \to -\infty$ and $t_f \to \infty$ are correct only in the limit $\beta \to 1$. When $\lambda_k = 0$ and $\lambda_j = 1$, $A_k^{(j)}$ is an amplitude for a transition from the negative continuum to the positive continuum, i.e. an amplitude for pair production. The probability for pair production off the light fronts will then be given by an integral over the transition amplitudes squared, $|A_k^{(j)}|^2$. This integral, however, does not give the total cross section for pair production. Pairs for which either the electron or the positron are moving at the velocity of an ion, are not accounted for. It is possible that these "left out" pairs dominate the total cross section for pair production. In fact, this integral can only give a prediction for an observable within a wave-packet formulation of of both initial and final states. This is also consistent with the physical nature of a scattered particle. One can avoid the light fronts, experimentally by placing the detector away from the ions' trajectories, and theoretically by forming a wave packet from the distorted plane waves of Eq. (92).

4.1 The perturbative limit

The small-charge perturbative-limit of the pair-production amplitude was calculated in Ref. [18]. To leading order in αZ (second order), the amplitude is given by a sum over two diagrams, where each diagram describes a two-photon exchange process. Despite the completely different derivation, here and in Ref. [18], the perturbative limit of our amplitude obtained by substituting Eq. (70) in Eq. (90) exactly reproduces the ultrarelativistic limit, of $\beta \to 1$ and large γ, of the perturbative result of Ref. [18]. Both give

$$\int d\vec{p}_\perp e^{-i\vec{b}\cdot(2\vec{p}_\perp - \vec{j}_\perp - \vec{k}_\perp)} \frac{i8\,(\alpha Z_A)(\alpha Z_B)}{(\vec{p}_\perp - \vec{k}_\perp)^2\,(\vec{p}_\perp - \vec{j}_\perp)^2} \frac{\langle -^k_+|I_2 - i\vec{\omega}\cdot\vec{p}_\perp|-^j_-\rangle}{j_- k_+ - (1+p_\perp^2)}$$
$$- \int d\vec{q}_\perp e^{i\vec{b}\cdot(2\vec{q}_\perp - \vec{j}_\perp - \vec{k}_\perp)} \frac{i8\,(\alpha Z_A)(\alpha Z_B)}{(\vec{q}_\perp - \vec{k}_\perp)^2\,(\vec{q}_\perp - \vec{j}_\perp)^2} \frac{\langle -^k_-|I_2 + i\vec{\omega}\cdot\vec{q}_\perp|-^j_+\rangle}{j_+ k_- - (1+q_\perp^2)}. \quad (94)$$

4.2 Nonperturbative effects

The nonperturbative effects are probably the most interesting subject of this research. It is important to give predictions as to both the magnitude and nature of nonperturbative effects in pair production.

Substituting the transverse-momentum transfer distribution induced by a single ion, $Q_Z^{\vec{b}}(\sim)$, as it is, with no regularization applied, into Eq. (90) the amplitude is given by three nested two-dimensional integrations, two of which diverge. As discussed in Ref. [3] and in section 3.4 above, this divergence is a result of applying the approximation $r_{\perp,b} \ll \gamma$ outside its range of validity. A regularization may be applied to overcome this divergence. It would be best to apply a physically motivated regularization and actually calculate the complete integral. We have not attempted to do that. Instead, motivated by recent experiments in CERN, we have limited our calculation to observed yields of electron positron pairs within a certain range of transverse momenta for which, we have shown, contributions from large transverse distances and large impact parameters can be safely neglected.

We had first integrated over $\vec{p}_\perp$ to obtain simple combinations of the Bessel functions of the third kind, K_0 and K_1. Note that the convergence of the $\vec{p}_\perp$ integration to the Bessel functions occurs only for pair-production amplitudes for which $1 - j_\pm k_\mp > 0$, and is directly related to the mass gap between the two continua and should be reconsidered for transitions within the same continuum. We have then used the condition that one of the two transverse momenta, $\vec{j}_\perp$ or $\vec{k}_\perp$, is much larger than $2\alpha Z/\gamma$ to apply a stationary phase calculation to one of the coordinate integrations. The last integral, over the other coordinate-integration variable, converges due to the Bessel functions which drop exponentially for large values of their arguments. Having thus proved that contributions for the 6-fold integral of Eq. (90) from large, transversal coordinates can be neglected, we have made the substitution of Eq. (69) and obtained

$$
\begin{aligned}
A_k^{(j)} \longrightarrow & \left[\left(\frac{b}{2}\right)^{+i2\alpha(Z_A+Z_B)} \frac{\Gamma(-i\alpha Z_A)}{\Gamma(+i\alpha Z_A)} \frac{\Gamma(-i\alpha Z_B)}{\Gamma(+i\alpha Z_B)} \right] \\
& \times \frac{i}{\pi^3} \alpha^2 Z_A Z_B \int d\vec{p}_\perp (\vec{p}_\perp - \vec{k}_\perp)^{-2} (\vec{p}_\perp - \vec{j}_\perp)^{-2} \\
& \times \Bigg\{ \frac{\sigma_k^j(\vec{p}_\perp)}{p_\perp^2 + 1 - j_- k_+} e^{i\vec{b}\cdot(\vec{j}_\perp + \vec{k}_\perp - 2\vec{p}_\perp)} \left[|\vec{p}_\perp - \vec{k}_\perp|^{i2\alpha Z_A} |\vec{p}_\perp - \vec{j}_\perp|^{i2\alpha Z_B} \right] \\
& - \frac{\sigma_j^{k\dagger}(\vec{p}_\perp)}{p_\perp^2 + 1 - j_+ k_-} e^{-i\vec{b}\cdot(\vec{j}_\perp + \vec{k}_\perp - 2\vec{p}_\perp)} \left[|\vec{p}_\perp - \vec{k}_\perp|^{i2\alpha Z_B} |\vec{p}_\perp - \vec{j}_\perp|^{i2\alpha Z_A} \right] \Bigg\} ; \qquad (95)
\end{aligned}
$$

We emphasize, that naively substituting the regularized result of Eq. (69) in Eq. (90) is generally incorrect and so is therefore Eq. (95). In particular, it induces mistakes when Eq. (71) does not apply. Note, for example, that the branch-point singularities for the intermediate momentum $\vec{p}_\perp = \vec{k}_\perp$ or $\vec{j}_\perp$ are an artifact of using Eq. (69) for ~ = 0, Eq. (90) has no such singularities, and an additional regularization at these points is then needed.

This substitution, however, is correct and useful when applied with care and within the appropriate restricted experimental conditions. In Ref. [3] we apply it to recent experiments in CERN SPS, where, to the best of our knowledge, these conditions do indeed apply. Based on this substitution, we were able to explain previously unexplained experimental results. We have shown that the nonperturbative solution and second-order perturbation theory give exactly the same results, in the high-energy limit, for production yields, integrated over the impact parameter, of electron-positron pairs that are not co-moving with the ions, as long as the transverse-momenta transferred in the collision from the ions to the electron are much larger than 2®Z=° or as long as one only counts what Baltz and MacLerran had called \centrally produced pairs". The $Z_B^2 Z_A^2$ charge dependence of the single-positron yields, even for very large charges, observed in these experiments is consistent with the charge dependence we have thus obtained for the nonperturbative, high-energy limit. It agrees with perturbation theory, but is not a perturbative e®ect.

The basis of this picture is the division of space-time to free regions and phases between them where the interaction acts. This also tells us where one should look, in momentum space, and in space-time, if one wants to study and observe non-perturbative e®ects, either:

² on the light fronts,

² at other observables than total yields,

² at transverse distances larger or momenta smaller than required by the sharp limit of Eq. (71).

An interesting example for such an observable may be multiple pair production. The two-photon exchange diagrams of second order perturbation theory were replaced in our amplitude by a two \kicks" mechanism in which a photon exchange is replaced by a space dependent phase shift. Higher orders in the coupling constant appear in these phase shifts instead of in higher order diagrams. New predictions for multiple-pair production because of these nonperturbative phases were considered in Ref. [19].

4.3 Controversy

In Refs. [20, 21], Ivanov, Schiller, and Serbo, calculated corrections to the Born cross section for the inclusive process

$$Z_1Z_2 \to Z_1Z_2e^+e^- ; \tag{96}$$

and obtained large negative corrections for the total pair-production cross sections. The authors also referred to our work which they interpreted as leading to the conclusion that no such corrections exist and thus claimed a disagreement with our work. Several suggestions were made as to the resolution of this controversy.

We have claimed that no real disagreement was shown [22]. Ref. [20] calculates the total cross section. We considered only electron-positron pairs produced so that neither the electron nor the positron is asymptotically co-moving with an ion, and we have neglected contributions from small momentum transfer. We have not calculated the complete cross section for electron-positron pair production in heavy-ion collisions, and we have certainly not claimed that the complete cross section is given by second order perturbation theory. On the contrary, we have emphasized that our amplitude is restricted to part of the total phase-space. To be more specific, we have written the transition amplitude of Eqs. (90) and (67) in Refs. [1, 2] yet we have not integrated over it to obtain a total cross section.

Our approach distinguished between electron-positron pairs produced so that they are co-moving with the ions and those that are not. This is because these two cases correspond to two essentially different asymptotic boundary conditions for the electron-positron pair. We have also noted that it is straight-forward to distinguish between these cases experimentally. In Ref. [3] we have integrated over the amplitude only under the restrictions discussed above and in the context of specific experimental conditions.

As for the total cross section, we did not calculate it within our approach nor were we able to show that the discrepancy between the total cross section of Refs. [20, 21], and the integral over our amplitude can indeed be attributed to that parts of the total cross section that our amplitude does not account for. It seems that this was recently shown by Lee and Milstein in Ref. [23]. They use our amplitude but improve it by correcting for the neglected contributions from small momentum transfer. They show that neglecting this regime can be misleading and prove a remarkable result, namely, that this very same contribution to the cross section which we have not calculated compensates exactly for the di®erence between our result and the total cross section obtained by Ivanov, Schiller, and Serbo in Ref. [20].

In Ref. [24] a di®erent resolution of the so-called puzzle was suggested, namely that the amplitude for pair production and the amplitude for electron scattering are no longer related by a simple crossing symmetry in the high energy limit. Hence, according to Ref. [24] it is incorrect to use our amplitude, formally derived for electron scattering, to the direct calculation of pair-production cross sections. They suggest an indirect calculation based on this amplitude and unitarity.

5 Conclusion

We have shown that the two-center time-dependent Dirac equation for an electron in the classical external ¯eld of two colliding ions reduces in the limit in which the ions are moving at the speed of light to an equation, which can be solved o® the light fronts exactly and in closed form. This special equation and its solution were further considered by several groups. Independent research has been published which is by-and-large in agreement with our work [4, 5]. Using di®erent methods di®erent research groups have all arrived at the same results. On the other hand, the application of this solution to pair-production has caused some confusion and controversy that is still in debate. What makes the debate so di±cult yet so fascinating, is that each group, coming from di®erent ¯elds of physics, and building on di®erent traditions and concepts, uses a di®erent framework and a di®erent language. Nevertheless, we believe that the di®erent approaches are likely to converge to a single result as is already being hinted in Ref. [23].

6 Acknowledgements

Research partially sponsored by the Engineering Research Program of the O±ce of Basic Energy Scineces, U.S. DOE, and partially by the Laboratory Directed Research and Development Program of Oak Ridge National Laboratory (ORNL), managed by UT-Battelle, LLC for the U.S. DOE under Contract No. DE-AC05-00OR22725.

References

[1] B. Segev and J.C. Wells, Phys. Rev. A 57, 1849 (1998).

[2] J.C. Wells, B. Segev, and J. Eichler, Phys. Rev. A 57, 1849 (1999).

[3] B. Segev and J.C. Wells, Phys. Rev. C 59, 2753 (1999).

[4] A.J. Baltz and L. McLerran, Phys. Rev. C 58, 1679 (1998).

[5] U. Eichmann, J. Reinhardt, S. Schramm, W. Greiner, Phys. Rev. A 59, 1223 (1999).

[6] W. Greiner, B. Müller, and J. Rafelski, Quantum Electrodynamics of strong fields, (Springer-Verlag, Berlin Heidelberg, 1985).

[7] C.A. Bertulani, G. Baur, Phys. Reps. 163, 299 (1988).

[8] J. Eichler and W.E. Meyerhof, Relativistic Atomic Collisions, (Academic Press, San Diego, 1995).

[9] Jörg Eichler, Phys. Rep. 193, 165 (1990).

[10] R. Jackiw, D. Kabat, and M. Ortiz, Phys. Lett. B 277, 148 (1992).

[11] A.J. Baltz, Phys. Rev. Lett. 78, 1231 (1997).

[12] J.D. Bjorken, J.B. Kogut, and E. Soper, Phys. Rev. D 3, 1382 (1971).

[13] G.N. Watson, A Treatise on the Theory of Bessel Functions, (Cambridge University Press, London, 1962).

[14] R. G. Newton, Scattering Theory of Waves and Particles, (McGraw-Hill, New York, 1966).

[15] C.R. Vane, S. Datz, P.F. Dittner, H.F.Krause, C. Bottcher, M. Strayer, R. Schuch, H. Gao, and R. Hutton, Phys. Rev. Lett. 69, 1911 (1992).

[16] C.R. Vane, S. Datz, P.F. Dittner, H.F. Krause, R. Schuch, H. Gao, and R. Hutton, Phys. Rev. A 50, 2313 (1994).

[17] C.R. Vane, S. Datz, E.F. Deveney, P.F. Dittner, H.F. Krause, R. Schuch, H. Gao, and R. Hutton, Phys. Rev. A **56**, 3682 (1997).

[18] C. Bottcher and M.R. Strayer, Phys. Rev. D **39**, 1330 (1989).

[19] M.C. GÄclÄ, Phys. Rev. C **61**, 054903 (2000).

[20] D.Y. Ivanov, A. Schiller, and V.G. Serbo, Phys. Lett. B **454**, 155 (1999).

[21] V.G. Serbo, Nucl. Phys. B **82**, 414 (2000).

[22] B. Segev and J.C. Wells, private communication with Prof. Serbo.

[23] R.N. Lee and A.I. Milstein, Phys. Rev. A **61**, 062103 (2000).

[24] U. Eichmann, J. Reinhardt, and W. Greiner, Phys. Rev. A **61**, 062710 (2000).