DOES INCOMPUTABLE MEAN NOT ENGINEERABLE?

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ABSTRACT

Self-referential systems have some remarkable properties. The processes of life and mind are not only self-referential, but self-reference turns out to be a crucial property of both. However, they are difficult to understand. From a given starting point, both *endogenous systems* (self-referential natural systems) and *impredicative systems* (self-referential formal systems) have infinitely many logically consistent consequences. Both are incomputable; neither halts after a finite number of steps. Therefore, neither can produce an exact prediction of the behavior of the other in finitely many steps. Despite the fact that all engineering decisions are based on incomplete information, this inherent inability of an impredicative model to produce exact predictions of an endogenous system is troubling to some engineers. Nevertheless, self-reference leads to a more general, but no less rational, form of modeling than that provided by traditional reductionism. Although the mathematics of self-reference is unfamiliar to engineers, its power is dramatic. For example, it resolves the apparent paradox of how a brain/mind possessing freewill can operate in a deterministic Universe.

INTRODUCTION

WHY SHOULD WE CARE?

Engineers seek to develop artificial systems that exhibit behaviors similar to the cognitive processes observed in biological brains. This is a worthy goal, and reaching it will be one of the crowning achievements of 21st Century technology. By no stretch of the imagination has the goal been reached yet. Indeed, the astoundingly simple nervous system of the nematode (all 300 neurons) is completely mapped out. Nevertheless, despite years of research, nobody has been able to devise an algorithm that behaves anything like a nematode (Chomsky 1993). Is it really such a stretch to suppose that the reason that the goal has not been met is that cognitive behavior is beyond the scope of the traditional reductionistic methods used by engineers? The answer to this question is yes, and this leads to another question. What other rational methods might we invoke?

IMPREDICATIVE MATHEMATICAL OBJECTS

What is really being said by the remarkably tricky proposition, $\phi(x) = \phi(x)$? It appears to say nothing and everything simultaneously. On the one hand, it says practically nothing. As a definition of $\phi(x)$, it does not inform us how $\phi(x)$ differs from any other entity, such as $\psi(x)$. This one fact alone seems to illustrate the futility of defining an entity by circular reference. On the other hand, $\phi(x) = \phi(x)$ states a profound truth regarding every entity; it is a condensed statement of Aristotle's Law of Non-contradiction (Adler 1978). Translated into words, $\phi(x) = \phi(x)$ says roughly that a thing is what it is and does not act contrary to its nature.

This self-referential proposition is a foundation axiom of our system of rational thinking. It is more than self-referential; it is self-evident. The Law of Non-contradiction cannot be validated from any more fundamental proposition. However, no counter example has ever been produced to falsify it. Paradoxically, any effort to logically prove that it is false starts from the assumption (usually implicit and unacknowledged) that it is true. Thus, although $\phi(x) = \phi(x)$ does not directly inform us how $\phi(x)$ differs from $\psi(x)$, it does provide us with the foundation of a system of logic that may, given proper additional data, allow us to answer the question.

Mathematicians and scientists prefer to avoid self-referential structures because they often lead to paradoxes. A classic logical paradox is "The Liar." What are we to make of the claim, "This statement is false?" Out of context, it appears to be a flat contradiction. If we assume it to be true, it asserts a false proposition. If we assume it to be false, it asserts a true proposition. The simplest, and conventionally accepted, way to evade the dilemma is to exclude all self-referential propositions from the epistemological Universe of Discourse, and to ignore its distinction from the ontological Universe.

However, such a simple evasion is deeply unsatisfying for several reasons. First, our system of logic is based on the self-evident self-referential proposition, $\phi(x) = \phi(x)$. Whether we like it or not; disallowing the foundation principle from the Universe of Discourse hardly appears to be a sound way to begin a logical process of reasoning. Second, we can use language to discuss "The Liar" and be understood; a system of logic that simply disallows the admission of such propositions would be far too impoverished to allow us to use it to reason about language (Barwise and Moss 1996). Third, a taboo on self-reference produces a system of logic too impoverished to discuss mathematics. Although mathematicians are loath to admit it, self-referential, or *impredicative*, propositions are both admissible and necessary in mathematics (Kleene 1950).

An impredicative definition is one in which the object being defined participates in its own definition. For example, $x \in X$, where x is defined in terms of its relationship to X, is a legitimate definition of the set X. It is subtly but crucially different from a circular definition. As already noted, a completely circular definition provides no feature to distinguish between the object being defined and the remainder of the Universe of Discourse. An impredicative definition must include some constraint (in addition to identity) on the relationship between an object and itself, and the properties of the constraint constitute a crucial distinguishing feature of the definition.

Is this sophistry, or are impredicative structures of some practical use? Consider that a decade ago, long distance telephone service cost ten cents per minute, but now it is commonly available for half that price. How did this come to be? It happened because telephone engineers found a practical way to double the carrying capacity a telephone circuit, *with absolutely no loss of information*. They performed this seeming miracle by discovering a technique called perfect reconstruction wavelet compression.

At the foundation of wavelet compression is the wavelet scaling function. The wavelet scaling function, $\phi(x)$, is defined by the First Fundamental Wavelet Equation, $\phi(x) = 2\Sigma_n h_n \phi(2x-n)$, where n is an even finite integer. This proposition is only true for vectors h_n that satisfy certain constraints. However, for an admissible vector, h_n, $\phi(x)$ is a uniquely defined function. In addition to being unique, it has all sorts of desirable mathematical properties, including finite support, continuity, differentiability, and orthogonality to translates of itself. There is no other way to define it except by the First Fundamental Wavelet Equation (Akansu and Haddad 1992). The crucial point is that this definition of $\phi(x)$ is impredicative.

Despite its impredicative definition, the wavelet scaling function is just as logical than more conventionally defined functions. Given the definition of $\phi(x)$, logical inferences can be drawn from it, and valid engineering decisions made from those inferences. In particular, the Second Fundamental Wavelet Equation can be defined in terms of the First, namely $\psi(x) = 2\Sigma_n g_n \phi(2x\text{-}n)$, where g_n is the time reversal of the quadrature reflection of h_n. In addition to having all the desirable properties that the scaling function, $\phi(x)$, has, the wavelet function, $\psi(x)$, has an even more important property, it is orthogonal to scaled versions of itself. More importantly for engineers, h_n, g_n, and their time reversals, can be proved to be a set of coefficients for a perfect-reconstruction decimate-by-2 digital filter bank.

The reader may be tempted to think that an impredicative structure is simply a recursive algorithm. It is not. A recursive algorithmic function has a defined bottom; after a finite number of steps, it hits bottom and the bottom step returns a predefined symbol to the previous step. The First Fundamental Wavelet Equation is the definition of $\phi(x)$, but computing the *exact* value of $\phi(x)$ given h_n would require an infinite number of calls to the next finer level of scale. Since it requires infinitely many steps to complete, it is not an algorithm (Knuth 1973). Infinite recursion depth is typical of impredicative processes; they are non-algorithmic and *incomputable*. These differ fundamentally from recursive algorithms, which have a defined bottom level of recursion.

The reader may also be tempted to think that an impredicative structure is simply an analog feedback system, or perhaps the differential equation describing such a system. It is not this either. The crucial distinction is that the impredicative object is defined in terms of its relationship to itself *between levels*. To appreciate the notion of level, imagine the model of a model of a system. An impredicative model is defined in terms of its relationship with a model of the model. For example, a wavelet scaling function (itself a model of some other system) on a given scale is the weighted sum of copies of translates of the same function at the next finer scale (a model of the original model). In contrast, for an analog feedback mechanism, the self-reference is within the natural system, and if modeled by a differential equation, the feedback imposes no need to look at a model of the model. The feedback is *within a level*.

The key points of this digression into impredicativity are as follows. Logic is founded on self-reference or impredicativity. The cost of self-reference is the risk that it can lead to paradoxes. A blanket taboo against self-reference in order to avoid paradoxes is too restrictive. An epistemological Universe of Discourse that disallows self-reference is too impoverished to allow meaningful discussion of the ontological Universe, where self-referential structures (*e.g*. semantic languages) abound. In fact, an epistemological Universe of Discourse that disallows self-reference is too impoverished even to allow meaningful discussion of other useful epistemologies, such as engineering mathematics. Impredicativity has infinite recursion depth, and is not an algorithm. It requires self-reference between levels, and fundamentally differs from a feedback mechanism. Most crucially, for readers of this paper, engineers can use impredicative mathematics to guide their decisions just as reasonably as we use more traditional mathematics.

ENDOGENOUS PHYSICAL PROCESSES

In the natural world, a multi-level self-referential process is called an *endogenous system*. This is a widely used medical term that describes the property of a system that grows from within itself, or makes itself up as it goes along (Clayman 1989). The term is used for a similar concept in economics (Gavin and Kydland 1999). Some biologists consider this property to be a defining feature of living systems. For example, Margulis and Sagan (1995) flatly claim that a system is alive if and only if it is autopoietic. They identify autopoiesis as the process of life making itself.

Endogeny in the form of autopoiesis was first propounded by Maturana and Varela (1998). They begin by noting that attempting to define life by listing properties leads to the various dead ends. They offer what they call a radically different perspective, that a living process is distinguished from a non-living process by its organization. They identify organization as the relations that must be present for a thing to exist. What distinguishes living from non-living processes is autopoietic organization.

The autopoietic object is a bounded unity. It has a dynamic network of chemical transformations commonly called a metabolism. Metabolism is distinguished from other chemical networks in that metabolism produces the components that make up the network of metabolism. An integral component of the metabolic network is a membrane that serves as the boundary between the network and the rest of the world.

The membrane and the metabolic dynamics are each necessary for the existence of the other. The metabolic transformations will only occur if the system is protected from the environment by the membrane. The membrane is a product of the metabolic transformations. The emergence of the membrane and the rest of the metabolic network is not sequential. They are two different aspects of a single unity, as indicated in Fig. 1. Disrupt either, and you disrupt the whole. Maturana and Varela say that what is most striking about an autopoietic system is that it pulls itself up by its own bootstraps. It produces and repairs its own material structure. It becomes distinct from its environment through its own dynamics. It does so in a way that the metabolism and repair processes are inseparable. They identify autonomy as a system's ability to make up its own laws of behavior, and update them as it goes along.

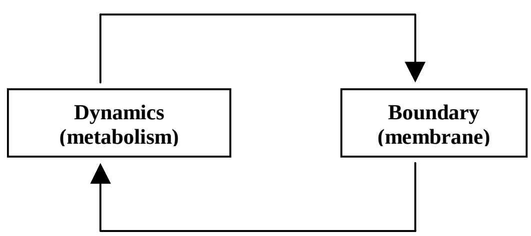


Fig. 1. Unity of Metabolism and Membrane

From direct observation, Wolpert argues that life is incomputable (Murphy and O'Neill 1995). He argues that life behaves like neither a differential equation nor a finite state machine. Differential equations exclude structure from their initial conditions, but that both cells and embryos are highly structured. However, a finite state machine, or a computable cellular automaton, is simply a discretized form of a differential equation. A formal description of a living process must be something altogether different.

CONGRUENCY: FORMAL AND NATURAL SELF-REFERENTIAL SYSTEMS

The incomputability of endogenous natural systems has some serious consequences for engineers. Attempting to project them onto an algorithm automatically discards much of the information about the process. Even worse than that, the algorithm offers no warning as to how much is lost, or where its predictions will fail.

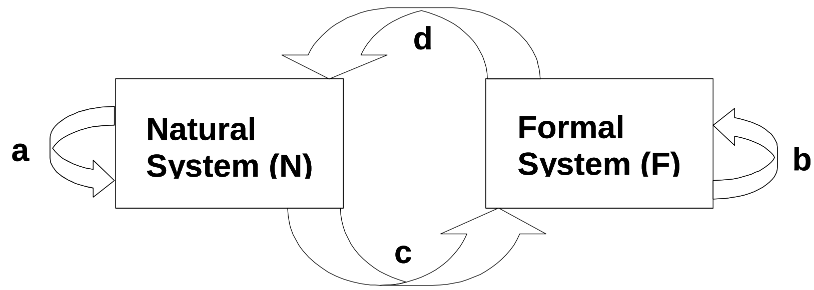
Since endogenous natural systems are incomputable, how are engineers to deal with them? The answer is to model the endogenous natural system with an impredicative formal system. What does this mean? A mathematical formalism of the Modeling Relation has existed for nearly a century, and may go back even further (Russell 1931). In mathematical biology the formalism has been popularized by Rosen (1991, p. 152), who tells us that modeling "is the art of bringing entailment structures into congruence." By itself, the statement leaves us little wiser than when we started. How does *art* enter into the discussion; are we not instead supposed to be scientific? What is an entailment, much less an entailment structure? What does it mean that two different entailment structures are congruent? In what sense are they not identical? If they are not identical, what causes us to declare them congruent? Why do these questions matter?

The first point to appreciate is that the Modeling Relation (see Fig. 2) is a formal mathematical *relation* (Rosen, 1999). Suppose that A and B are sets, and that there exists a set, R, of ordered pairs, where the first element of each pair in R is an element of A, and the second element of each pair in R is an element of B. There is also some ordering in the pairs, but there is not strict ordering. In mathematical notation: $a\in A$, $b\in B$, $(a,b)\in R <=> aRb$. In the Modeling Relation, the members a and b of each ordered pair in R are entailments from two different systems.

Entailments are the *event structures* in the organization of a system. There are two sorts of systems that might appear in the Modeling Relation, natural systems and formal systems. Natural systems are systems in physical reality that have causal linkages; if certain causative events impinge upon a natural system, then the system will behave in a certain way, or produce certain events in effect. This consequential linkage of cause and effect in a natural system is a *causal entailment*. Formal systems are conceptual systems that have inferential linkages; if certain hypothetical propositions impinge upon a formal system, then they will produce certain consequential propositions in conclusion. This linkage of hypothesis and conclusion in a formal system is an *inferential entailment*.

Entailment structures are inherent *within* a system; they are the distinguishing features that characterize the system (Rosen, 1991, p. 98). They do not cross over from one system to another. This is represented in Fig. 2, where we see a natural system, N, distinguished by its structure of causal entailments, a, and a formal system, F, distinguished by its structure of inferential entailments, b. The entailment structures of two distinct systems are distinct from one another; causes or hypotheses in one do not produce effects or conclusions in the other. In fact, this provides the answer to one of the questions posed above. Its self-contained entailment structure is what provides identity to a system and distinguishes it from other systems. This is important in living systems, since one of the distinguishing features is a living system is the unique identity of its bounded self.

The fact that distinct systems are non-identical does not preclude them from being regarded as being in some sense similar. Similar systems should have distinguishing features that closely correspond to each other. Dissimilar systems should have distinguishing features that do not closely correspond to each other. As already noted, the distinguishing feature of a system is its entailment structure. Thus, we would expect similar systems to have entailment structures in which there is some degree of correspondence between the entailments.



$$MR = \{(a,b) \mid a = c + b + d\}$$

Fig. 2.The Modeling Relation

To establish this correspondence, consider a system of encodings and decodings (Rosen 1991, p. 59). For example, we might have a system of encodings that encodes a set of events in the natural system, N, in Fig. 2, into a set of propositions in the formal system, F. We might also have a system of decodings that decodes a set of propositions in the formal system, F, into a set of phenomena in the natural system, N. Although the two systems remain independent in the sense that causes or hypotheses in one do not produce effects or conclusions in the other, the two systems can be linked by encodings and decodings.

This linkage between entailment structures provides the means of determining the similarity between two systems. Suppose that an event, e1, in N can be encoded to a proposition, p1, in F; we can think of the encoding arrow, c, in Fig. 2 as a measurement on a natural system. Suppose further that the proposition, p1, when applied as a hypothesis in the inferential structure in F entails another proposition, p2, in F as a conclusion. In other words, the propositions are entailed as an implication, b = (p1 → p2), in F. Suppose that this entailed proposition, p2, in F can be decoded into an event, e2, in N; we can think of the decoding arrow, d, in Fig. 2 as a prediction by a formal system.

Rosen defines congruency between the entailment structures as follows (Rosen, 1991, p. 61). Suppose that in the underlying reality, the event e1 in N causes event e2 in N. In other words, the two events are entailed as a causal linkage, a = (e1 → e2), in N. Suppose further that the linkages commute. Event e1 is encoded by c to proposition p1, i.e., c = (e1 → p1), which, implies (in formal system, F) proposition p2, i.e., b = (p1 → p2). Proposition p2 is decoded by d to event e2, i.e., d = (p2 → e2). Further suppose that there is an exact correspondence between the predicted event e2, and the caused event e2. The commutation is also described as a = b+c+d. (Note: + is the symbol for concatenation.) If there exists no such entailment c in F, having a commutative relationship with some entailment a in N, then the two systems do not have congruent entailment structures. Entailment structures are congruent to the extent that such correspondences between entailments exist.

CONCLUSIONS

BIZARRE BEHAVIOR IS SOMEWHAT PREDICTABLE

The idea that a behavior or effect is completely unanticipated, but fully consistent with the causal entailments of the system producing the effect is bizarre. Previously, borrowing Kirstie Bellman's idea, I had styled systems that exhibit such behavior to be bizarre systems. However, it is clear that the bizarreness is in the effect, not in the cause. Since the most compact set of distinguishing features of any system is its causal entailment structure, and not the caused behavior, it is less confusing to say that a system with a multi-level self-referential causal entailment structure is an endogenous system that produces bizarre effects, that are predicted (often, but not always) by an impredicative model with a multi-level self-referential inferential entailment structure.

In describing the systems that produce bizarre effects, we need to recall that they come in two distinct flavors, natural (ontological) and formal (epistemological). In the Modeling Relation, the two are never identical. Since much of the confusion in 20th century science arose from not noticing this distinction between ontology and epistemology, it would be wise, as we lay the foundations for a new century of thinking, to adopt terminology that never lets us forget the distinction.

Finally, how do we manipulate an impredicative model to make decisions about an endogenous system? Since the processes are incomputable, an algorithm is useless for the task. There are several strategies that have not yet been proved impossible, the super-Turing model, quantum computing, and wiring the human directly into the loop

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REFERENCES

Adler, M.J., 1978, *Aristotle For Everybody*, New York: MacMillan, pp. 160-167.

Akansu, A.N. and Haddad R.A., 1992, *Multiresolution Signal Decomposition*, San Diego: Academic Press, pp. 313-315.

Barwise, J. and Moss, L., 1996, *Vicious Circles: On The Mathematics Of Non-Wellfounded Phenomena,* Stanford, CA: Center for the Study of Language and Information.

Chomsky, N., 1993, *Language and Thought*, Wakefield, RI: Moyer Bell.

Clayman, C.B., 1989, *The American Medical Association Encyclopedia of Medicine*, New York: Random House.

Gavin, W.T. and Kydland, F.E., "Endogenous Money Supply and the Business Cycle," *Review of Economic Dynamics*, Vol. 2, No. 2, April 1999, pp. 347-369.

Kleene, S., 1950, *Introduction to Metamathematics*, Amsterdam: vanNostrand. pp. 41-43.

Knuth, D.E, 1973, *The Art Of Computer Programming*, 2d ed., Vol. 1. Reading, MA: Addison Wesley. pp. 4-6.

Margulis, L. and Sagan, D., 1995, *What is Life?* University of California Press, pp. 17-20.

Maturana, H.R. and Varela, F.J., 1998 *The Tree of Knowledge*, Shambala, pp. 41-52.

Murphy, M.P. and O'Neill, L.A.J., 1995, *What is Life? The Next Fifty Years*, Cambridge University Press, pp. 57-66.

Rosen, K., 1999, *Discrete Mathematics And Its Applications*. 4th ed., Boston: WCB/McGraw-Hill, p. 374.

Rosen, R., 1991, *Life Itself: A Comprehensive Inquiry into the Nature, Origin and Fabrication of Life,* New York: Columbia University Press.
Russell, B., 1931, *The Scientific Outlook,* First Edition, New York: W.W Norton.