Nonlinear Coupling Control Laws for a 3-DOF Overhead Crane System*

Y. Fang[1], W. E. Dixon[2], D. M. Dawson[1], E. Zergeroglu[3]
[1]Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634-0915
[2]Robotics and Process Systems Division, Oak Ridge National Laboratory, P.O. Box 2008, Oak Ridge, TN 37831-6305
[3]Lucent Technologies, Bell Lab Innovations Optical Fiber Solutions, 50 Halls Rd, Sturbridge MA, 01566

E-mail: dixonwe@ornl.gov, Telephone: (865) 574-9025

Keywords: Overhead Crane, Nonlinear Control Application, Underactuated, Setpoint Regulation

"The submitted manuscript has been authored by a contractor of the U.S. Government under contract No. DE-AC05-96OR22464. Accordingly, the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes."

Submitted to the IEEE Conference on Decision and Control, Dec. 4-7, 2001, Orlando, Florida

* This research was performed in part by a Eugene P. Wigner Fellow and staff member at the Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under contract DE-AC05-00OR22725 and is supported in part by the U.S. NSF Grants DMI-9457967, DMI-9813213, EPS-9630167, ONR Grant N00014-99-1-0589, a DOC Grant, and an ARO Automotive Center Grant.

Nonlinear Coupling Control Laws for a 3-DOF Overhead Crane System*

Y. Fang[1], W. E. Dixon[2], D. M. Dawson[1], and E. Zergeroglu[3]

[1]Department of Electrical & Computer Engineering, Clemson University, Clemson, SC 29634-0915

[2]Oak Ridge National Laboratory, P.O. Box 2008, Bldg. 7601, Mailstop 6305, Oak Ridge, TN 37831

[3]Lucent Technologies, Bell Lab Innovations Optical Fiber Solutions, 50 Halls Rd, Sturbridge MA, 01566

email: yfang, ddawson@ces.clemson.edu, dixonwe@ornl.gov, ezerger@lucent.com

Abstract

In this paper, we consider the regulation control problem for a three-degree-of-freedom (3-DOF), underactuated overhead crane system. Motivated by recent passivity-based controllers for underactuated systems, we design several controllers that asymptotically regulate the planar gantry position and the payload angle. Specifically, utilizing LaSalle's Invariant Set Theorem, we first illustrate how a simple proportional-derivative (PD) controller can be utilized to asymptotically regulate the overhead crane system. Motivated by the desire to achieve improved transient performance, we then design two nonlinear controllers that increase the coupling between the planar gantry position and the payload angle.

1 Introduction

Precise payload positioning by an overhead crane (especially when performed by a operator using only visual feedback to position the payload) is difficult due to the fact that the payload can exhibit a pendulum-like swinging motion. These payload swings can result in several performance and safety concerns including: i) damage to the payload (*e.g.*, spillage or breakage), ii) damage to the surrounding environment or personnel, and iii) large internal forces that can result in reduced payload carrying capacity or premature failure of stressed parts. Motivated by the desire to achieve fast and precise payload positioning while mitigating the above performance and safety concerns, several researchers have developed various controllers for overhead crane systems. For example, Yu *et al.* [16] utilized a time-scale separation approach to control a two degree-of-freedom (2-DOF) overhead crane system; however, an approximate linearized model of the crane was utilized to facilitate the construction of the error systems. In [15], Yashida *et al.* proposed a saturating control law based on a guaranteed cost control method for a linearized version of the 2-DOF crane system dynamics. Martindale *et al.* [9] utilized an approximate crane model to develop exact model knowledge and adaptive controllers while Butler *et al.* [2] exploited a modal decomposition

*This research was performed in part by a Eugene P. Wigner Fellow and staff member at the Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under contract DE-AC05-00OR22725 and is supported in part by the U.S. NSF Grants DMI-9457967, DMI-9813213, EPS-9630167, ONR Grant N00014-99-1-0589, a DOC Grant, and an ARO Automotive Center Grant.

technique to develop an adaptive controller. In [3], Chung and Hauser designed a nonlinear controller for regulating the swinging energy of the payload.

Several researchers have also examined the control problem for 3-DOF overhead crane systems. Specifically, Moustafa and Ebeid [10] derived the nonlinear dynamic model for a 3-DOF overhead crane and then utilized a standard linear feedback controller based on a linearized state space model. In [11], Noakes and Jansen developed a generalized input shaping approach for the linearized crane dynamics that exploited a notch filtering technique to control the motion of the bridge/trolley of a 3-DOF overhead crane system. More recently, Lee [7] developed a nonlinear model for 3-DOF overhead cranes based on a new 2-DOF swing angle definition. Based on this nonlinear model, Lee then developed an anti-swing control law for the decoupled linearized dynamics. In [12], Sakawa and Sano derived a nonlinear model for a 3-DOF crane system that was subsequently linearized to facilitate the development of a control scheme that first transferred the load to a position near the equilibrium point using an open-loop controller and then utilized a linear feedback controller to stabilize the payload about the equilibrium point.

One of the limiting factors associated with the above overhead crane control designs is that the system nonlinearities are often excluded from the closed-loop error system design and stability analysis. To overcome this drawback, several researchers have investigated control approaches that account for the nonlinear dynamics of overhead cranes and similar systems. For example in [14], Teel utilized saturation functions to develop an output feedback controller which achieves a robust, semi-global stability result for the ball-and-beam control problem. In [1], Burg *et al.* transformed the nonlinear crane dynamics into a structure that resembled the ball-and-beam problem and then adopted the research efforts of [14] to achieve asymptotic positioning from a large set of initial conditions. More recently, Fantoni *et al.* [5] and Lozano *et al.* [8] proposed passivity-based controllers for the inverted pendulum and the pendubot (*i.e.*, an inverted pendulum-like robot with an unactuated second link) based on the paradigm of driving the underactuated system to a homoclinic orbit using an energy-based nonlinear controller and then switching to a linear controller to stabilize the system around its unstable equilibrium point. Using similar stability analysis techniques, Collado *et al.* [4] proposed a proportional-derivative (PD) controller for the overhead crane problem. In [6], Kiss *et al.* developed a PD controller for a vertical crane-winch system that only requires the measurement of the winch angle and its derivative rather than a cable angle measurement.

In this paper, we utilize a similar approach as provided by [8] to develop several controllers for the 3-DOF overhead crane system. Specifically, utilizing LaSalle's Invariant Set Theorem, we illustrate how a simple PD controller can be utilized to asymptotically regulate the overhead crane dynamics. Motivated by the desire for improved transient response, we design two nonlinear energy-based coupling control laws that increase the coupling between the pendulum position and the gantry position. Simulation results are provided, which illustrate that the increased coupling of the nonlinear controllers results in improved transient response (*e.g.*, reduced overshoot and faster settling time) over the PD control law. The paper is organized as follows. In Section 2, we present the nonlinear dynamic model of the overhead crane system and in Section 3, we rewrite the open-loop system into a more convenient form. In Section 4, we develop a PD controller and two nonlinear controllers and examine the stability of the controllers through a Lyapunov-like stability analysis. The performance of the proposed controllers is illustrated through simulation results presented in Section 5. Concluding remarks are given in Section 6.

2 Dynamic Model

The dynamic model for a three degree-of-freedom (3-DOF) overhead crane system (see Figure 1) is assumed to have the following form [10]

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) = u \tag{1}$$

where $q(t) \in \mathbb{R}^4$ is defined as follows

$$q = [x \quad y \quad \theta \quad \phi]^T \tag{2}$$

where $x(t) \in \mathbb{R}^1$ denotes the gantry position along the X-Coordinate axis, $y(t) \in \mathbb{R}^1$ denotes the gantry position along the Y-Coordinate axis, $\theta(t) \in \mathbb{R}^1$ denotes the payload angle with respect to the vertical, $\phi(t) \in \mathbb{R}^1$ denotes the projection of the payload angle along the X-Coordinate axis, and $M(q) \in \mathbb{R}^{4\times 4}$, $V_m(q,\dot{q}) \in \mathbb{R}^{4\times 4}$, $G(q) \in \mathbb{R}^4$, and $u(t) \in \mathbb{R}^4$ are defined as follows

$$M = \begin{bmatrix} m_p + m_r + m_c & 0 & m_pL\cos\theta\sin\phi & m_pL\sin\theta\cos\phi \\ 0 & m_p + m_c & m_pL\cos\theta\cos\phi & -m_pL\sin\theta\sin\phi \\ m_pL\cos\theta\sin\phi & m_pL\cos\theta\cos\phi & m_pL^2 + I & 0 \\ m_pL\sin\theta\cos\phi & -m_pL\sin\theta\sin\phi & 0 & m_pL^2\sin^2\theta + I \end{bmatrix} \tag{3}$$

$$V_m = \begin{bmatrix} 0 & 0 & -m_pL\sin\theta\sin\phi\dot{\theta} + m_pL\cos\theta\cos\phi\dot{\phi} & m_pL\cos\theta\cos\phi\dot{\theta} - m_pL\sin\theta\sin\phi\dot{\phi} \\ 0 & 0 & -m_pL\sin\theta\cos\phi\dot{\theta} - m_pL\cos\theta\sin\phi\dot{\phi} & -m_pL\cos\theta\sin\phi\dot{\theta} - m_pL\sin\theta\cos\phi\dot{\phi} \\ 0 & 0 & 0 & -m_pL^2\sin\theta\cos\theta\dot{\phi} \\ 0 & 0 & m_pL^2\sin\theta\cos\theta\dot{\phi} & m_pL^2\sin\theta\cos\theta\dot{\theta} \end{bmatrix} \tag{4}$$

$$G = [0 \quad 0 \quad m_pgL\sin\theta \quad 0]^T \tag{5}$$

$$u = [F_x \quad F_y \quad 0 \quad 0]^T \tag{6}$$

where $m_p, m_r, m_c \in \mathbb{R}^1$ represent the payload mass, rail mass, and cart mass, respectively, $I \in \mathbb{R}^1$ denotes the moment of inertia of the payload, $L \in \mathbb{R}^1$ represents the length of the crane rod, $g \in \mathbb{R}^1$ represents the gravity effects, and $F_x(t), F_y(t) \in \mathbb{R}^1$ represent the control force inputs acting on the cart and rail, respectively. Based on the structure of $M(q)$ and $V_m(q,\dot{q})$ given in (3) and (4), it is straightforward to show that the following skew-symmetric relationship is satisfied

$$\xi^T\left(\frac{1}{2}\dot{M}(q) - V_m(q,\dot{q})\right)\xi = 0 \qquad \forall\,\xi \in \mathbb{R}^4 \tag{7}$$

where $\dot{M}(q)$ represents the time derivative of $M(q)$ and that the inertia matrix $M(q)$ can be upper and lower bounded by the following inequalities

$$k_1\left\|\xi\right\|^2 \le \xi^T M(q)\xi \le k_2\left\|\xi\right\|^2 \qquad \forall\xi \in \mathbb{R}^4 \tag{8}$$

where k_1, $k_2 \in \mathbb{R}^1$ are positive bounding constants. In a similar manner as in [1] and [9], we assume that the dynamic model given in (1) has the following characteristics.

Assumption 1: The payload and the gantry are connected by a massless, rigid link.

Assumption 2: The angular position and velocity of the payload and the planar position and velocity of the gantry are measurable.

Assumption 3: The gantry mass and the length of the connecting rod are exactly known.

Assumption 4: The ball joint that connects the payload link to the gantry is frictionless and that this joint does not rotate about the connecting rod (*i.e.*, the payload does not rotate about the rod axis).

Assumption 5: The angular position of the payload mass is restricted according to following inequality

$$-\frac{\pi}{2} < \theta(t) < \frac{\pi}{2} \tag{9}$$

where $\theta(t)$ is measured from the vertical position (see Figure 1).

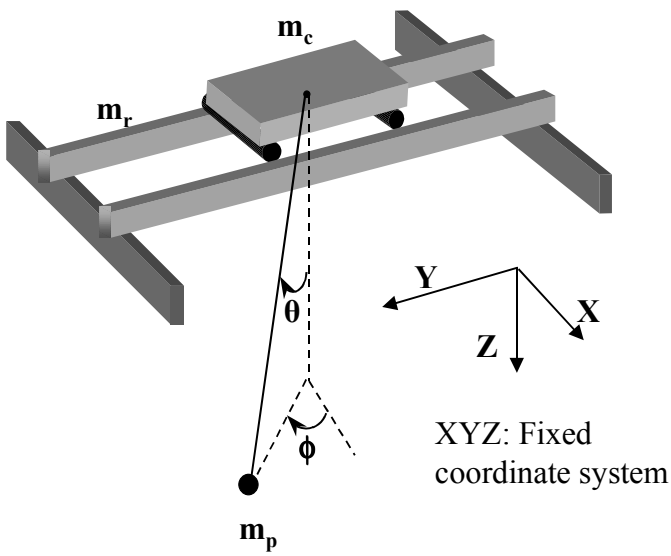

Figure 1: 3-DOF Overhead Crane System

Remark 1 *Note that the model given by (1) could be modified to include other dynamic effects associated with the gantry dynamics (e.g., gantry friction, viscous damping coefficients, mass moment of inertia of the gantry and rail motors, etc.); however, these additional dynamic effects were not included in the model since these effects can be directly cancelled by the controller.*

3 Open-Loop System Development

To express (1) in a form that facilitates the subsequent control development and stability analysis, we premultiply both sides of (1) by $M^{-1}(q)$ to obtain the following expression

$$\ddot{q} = M^{-1}\left(u - V_m\dot{q} - G\right) \tag{10}$$

where $M^{-1}(q) \in \mathbb{R}^{4\times 4}$ is guaranteed to exist due to the fact that the determinant of $M(q)$, denoted by $\det(M)$, is a positive function as shown below

$$\begin{aligned}\det(M) \;=\;& I^2\left(m_p+m_c\right)\left(m_p+m_r+m_c\right) + m_pL^2I\left[\left(m_p+m_r+m_c\right)m_c\left(1+\sin^2\theta\right)\right. \\ &\left.+\left(m_p+m_c\right)m_p\sin^2\theta + m_rm_p\left(\sin^2\phi + 2\sin^2\theta\cos^2\phi\right)\right] \\ &+m_p^2L^4\sin^2\theta\left(m_rm_p\sin^2\theta\cos^2\phi + m_c\left(m_r+m_c+m_p\sin^2\theta\right)\right).\end{aligned} \tag{11}$$

After substituting (3), (4), (5), (6) and the expression for $M^{-1}(q)$ into (10) and performing some algebraic manipulation, we can rewrite (10) as follows

$$\ddot{x} = \frac{1}{\det(M)} \left(p_{11}F_x + p_{12}F_y + w_1\right) \tag{12}$$

$$\ddot{y} = \frac{1}{\det(M)} \left(p_{12}F_x + p_{22}F_y + w_2\right) \tag{13}$$

where the measurable auxiliary terms $p_{11}(q)$, $p_{12}(q)$, $p_{22}(q)$, $w_1(q,\dot{q})$, $w_2(q,\dot{q}) \in \mathbb{R}^1$ are defined as follows

$$\begin{aligned} p_{11} &= m_p^2 L^2 I \left(\sin^2\phi + 2\sin^2\theta\cos^2\phi\right) + m_p I^2 + m_c m_p L^4 \sin^2\theta \\ &\quad + m_c m_p L^2 I \left(1 + \sin^2\theta\right) + m_c I^2 + m_p^3 L^4 \cos^2\phi \sin^4\theta \end{aligned} \tag{14}$$

$$p_{12} = -m_p^3 L^4 \sin\phi\cos\phi\sin^4\theta - m_p^2 L^2 I \sin\phi\cos\phi\left(\sin^2\theta - \cos^2\theta\right) \tag{15}$$

$$\begin{aligned} p_{22} &= m_p^3 L^4 \sin^4\theta\sin^2\phi + m_p^2 L^2 I \left[1 + \left(\sin^2\theta - \cos^2\theta\right)\sin^2\phi\right] \\ &\quad + \left(m_p + m_r + m_c\right) I^2 + \left(m_r + m_c\right) m_p^2 L^4 \sin^2\theta \\ &\quad + \left(m_r + m_c\right) m_p L^2 I \left(1 + \sin^2\theta\right) \end{aligned} \tag{16}$$

$$\begin{aligned} w_1 &= m_p L \sin\theta\sin\phi \left[\left(m_p + m_c\right) I + m_p m_c L^2 \sin^2\theta\right] \\ &\quad \left[\dot{\phi}^2 \left(m_p L^2 \sin^2\theta + I\right) + \dot{\theta}^2 \left(m_p L^2 + I\right)\right] \\ &\quad -2 I m_p L \dot{\theta}\dot{\phi}\cos\theta\cos\phi \left[\left(m_p + m_c\right) I + m_p L^2 \left(m_c + m_p \sin^2\theta\right)\right] \\ &\quad + m_p^2 g L^2 \sin\theta\cos\theta\sin\phi \left[\left(m_p + m_c\right) I + m_p m_c L^2 \sin^2\theta\right] \end{aligned} \tag{17}$$

$$\begin{aligned} w_2 &= m_p L \sin\theta\cos\phi \left[\left(m_p + m_r + m_c\right) I + \left(m_r + m_c\right) m_p L^2 \sin^2\theta\right] \\ &\quad \left[d^2 \left(m_p L^2 \sin^2\theta + I\right) + p^2 \left(m_p L^2 + I\right)\right] \\ &\quad + 2 m_p L I \dot{\theta}\dot{\phi}\cos\theta\sin\phi \left[\left(m_p + m_r + m_c\right)\left(m_p L^2 + I\right) - m_p^2 L^2 \cos^2\theta\right] \\ &\quad + m_p g L \sin\theta \left[\left(m_r + m_c\right) m_p^2 L^3 \sin^2\theta\cos\theta\cos\phi \right. \\ &\quad \left. + \left(m_r + m_c + m_p\right) m_p L I \cos\theta\cos\phi\right]. \end{aligned} \tag{18}$$

In order to write the open-loop dynamics given in (12) and (13) in a more compact form for the subsequent control development and stability analysis, we define the auxiliary signal $r(t) \in \mathbb{R}^2$ as follows

$$r = \left[x \quad y\right]^T. \tag{19}$$

After taking the second time-derivative of $r(t)$ and then utilizing the expressions given in (12)-(18), we can rewrite the open-loop dynamics given in (12) and (13) as follows

$$\ddot{r} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \frac{1}{\det(M)} \left(PF + W\right) \tag{20}$$

where $P(q) \in \mathbb{R}^{2\times 2}$ and $W(q,\dot{q}) \in \mathbb{R}^2$ are defined as follows

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \qquad W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{21}$$

and $F(t) \in \mathbb{R}^2$ is defined below

$$F = \begin{bmatrix} F_x & F_y \end{bmatrix}^T. \tag{22}$$

Given the expressions in (14)-(16), it is straightforward to prove that

$$p_{11} > 0 \qquad \text{and} \qquad p_{11}p_{22} - p_{12}^2 \geq m_p \left(m_p + m_r + m_c\right) I^4. \tag{23}$$

From the expressions given in (21) and (23), we can see that $P(q)$ is positive-definite, symmetric, and invertible, where the inverse of $P(q)$, denoted by $P^{-1}(q)$, is also positive-definite and symmetric.

To facilitate the subsequent Lyapunov-based control design, we will utilize the energy of the overhead system, denoted by $E(q, \dot{q}) \in \mathbb{R}^1$, and defined as follows

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + m_p g L(1 - \cos(\theta)) \geq 0. \tag{24}$$

After taking the time derivative of (24), substituting (1) for $M(q)\ddot{q}(t)$, and canceling common terms, we obtain the following expression for the time derivative of $E(q, \dot{q})$

$$\dot{E} = \dot{r}^T F \tag{25}$$

where (3), (4), and (7) were utilized.

4 Control Design and Analysis

Our control objective is the regulation of the planar gantry position of the overhead crane to a constant desired position, denoted by $r_d \in \mathbb{R}^2$, which is explicitly defined as

$$r_d = \begin{bmatrix} x_d & y_d \end{bmatrix}^T. \tag{26}$$

In addition, the payload angle $\theta(t)$ must also be regulated to zero. To quantify the control objective of regulating the overhead crane to a constant desired position, we define a gantry position error signal $e(t) \in \mathbb{R}^2$ as follows

$$e(t) = r - r_d. \tag{27}$$

In the subsequent control development, we will design a proportional-derivative control law and two nonlinear controllers to achieve the above control objective.

Remark 2 *The control objective is defined in terms of regulating the gantry position and the angle of the payload with the vertical. The problem of regulating the projection of the payload angle along the X-Coordinate axis, denoted by $\phi(t)$, is not required. That is, if the payload angle, denoted by $\theta(t)$, is regulated to zero, then we can see from Figure 1 that the payload has been regulated to the desired location.*

4.1 Proportional-Derivative Control Law

Based on the subsequent stability analysis, we design the following proportional-derivative (PD) control law

$$F = \frac{-k_p e - k_d \dot{r}}{k_E} \tag{28}$$

where k_d, k_E, $k_p \in \mathbb{R}^1$ are positive constant control gains.

Theorem 1 *The controller given in (28) ensures asymptotic regulation of the overhead crane system in the sense that*

$$\lim_{t\to\infty} \begin{pmatrix} x(t) & y(t) & \theta(t) \end{pmatrix} = \begin{pmatrix} x_d & y_d & 0 \end{pmatrix} \tag{29}$$

where x_d and y_d were defined in (26).

Proof: To prove (29), we define a nonnegative function $V_1(t) \in \mathbb{R}^1$ as follows

$$V_1 = k_E E + \frac{1}{2} k_p e^T e. \tag{30}$$

After taking the time derivative of (30) and then substituting (25) and the time derivative of (27) into the resulting expression, we can rewrite (30) as follows

$$\dot{V}_1 = \dot{r}^T \left(k_E F + k_p e \right). \tag{31}$$

After substituting (28) into (31) for $F(t)$ and then cancelling common terms, we obtain the following expression

$$\dot{V}_1 = -k_d \dot{r}^T \dot{r}. \tag{32}$$

Based on the expressions given in (8), (24), (27), (30) and (32), it is clear that the origin of the closed-loop system is stable in the sense of Lyapunov [13] and that $r(t), e(t), \dot{q}(t) \in \mathcal{L}_\infty$. Based on the fact that $r(t), e(t), \dot{q}(t) \in \mathcal{L}_\infty$, we can utilize (2) and (19) to prove that $x(t)$, $\dot{x}(t)$, $y(t), \dot{y}(t), \dot{r}(t), \dot{\theta}(t)$, $\dot{\phi}(t) \in \mathcal{L}_\infty$. Given that $e(t), \dot{r}(t) \in \mathcal{L}_\infty$, it is clear from (28) that $F(t) \in \mathcal{L}_\infty$. Finally, from (6) and (22), we can prove that $F_x(t), F_y(t), u(t) \in \mathcal{L}_\infty$.

Based on the fact that all of the closed-loop signals remain bounded, we can now employ LaSalle's Invariance Theorem to prove (29). To this end, we define Γ as the set of all points where

$$\dot{V}_1 = 0. \tag{33}$$

In the set Γ, it is clear from (32) and (33) that

$$\dot{r}(t) = 0 \qquad \ddot{r}(t) = 0, \tag{34}$$

and hence, we can conclude from (19), (30), (33), and (34) that $x(t)$, $y(t)$, and $V_1(t)$ are constant, and that

$$\ddot{x}(t) = 0 \qquad \ddot{y}(t) = 0. \tag{35}$$

Furthermore, from (25), (27), and (34), it is clear that

$$\dot{E}(q, \dot{q}) = \dot{e}(t) = 0. \tag{36}$$

Based on (36), it is clear that $E(q, \dot{q})$ and $e(t)$ are constant, and hence, from (28) and (34), it is clear that $F(t)$ is constant. To complete the proof, we must analyze the stability of the system for the case when $\dot{\theta} = 0$ and when $\dot{\theta} \neq 0$. In this analysis, given in Appendix A, we prove the result given in (29) under the proposition that $\dot{\theta} = 0$ and that the proposition that $\dot{\theta} \neq 0$ leads to contradictions, and hence, is an invalid proposition.

Remark 3 *In the previous stability analysis, we shown that the control objective is met and that all signals in the dynamics and the controller remain bounded for all time except for the signal $\phi(t)$ (Note that by assumption, the payload angle, denoted by $\theta(t)$, is assumed to be bounded). We note that the boundedness of $\phi(t)$ is insignificant from a theoretical point of view since $\phi(t)$ only appears in the dynamics and control as arguments of trigometric functions. We also note that the simulation results for all of the controllers indicate that the signal $\phi(t)$ remains well-behaved and is driven to constant value.*

Remark 4 *Heuristically, the only way for the energy from the payload motion to be dissipated is through the coupling between the payload dynamics and the gantry dynamics. That is, a PD feedback loop at the gantry creates an artificial spring/damper system which absorbs the payload energy through the natural gantry/payload coupling, however, from our experience with many control experiments on overhead crane testbeds, we believe that a PD feedback loop at the gantry will always provide poor performance because if the gantry friction is not compensated for perfectly (and it never will be), then the uncompensated gantry friction effects tend to retard the natural coupling between the gantry/payload dynamics, and hence, prevent payload energy from being dissipated by the PD feedback loop at the gantry. In the following sections, we develop controllers that may improve the performance of the PD feedback loop at the gantry, due to the incorporation of additional nonlinear terms in the control law that depend on the payload dynamics. Although the subsequent controllers yield the same stability result as the previous controller, we believe that the increase in gantry/payload coupling due to the additional nonlinear terms will result in improved performance when compared to the simple gantry PD controller.*

4.2 E^2 Coupling Control Law

Based on previous work presented in [8] for a 2-DOF inverted pendulum, we design the following E^2 coupling control law[1]

$$F = [\Omega]^{-1}\left(-k_d\dot{r} - k_p e - \frac{k_v}{\det(M)}W\right) \tag{37}$$

where $\Omega(t) \in \mathbb{R}^{2\times 2}$ is an auxiliary positive-definite, invertible matrix[2] defined as follows

$$\Omega = k_E E I_2 + \frac{k_v}{\det(M)}P \tag{38}$$

k_E, k_p, k_d, $k_v \in \mathbb{R}^1$ are positive constant control gains, I_2 denotes the standard 2×2 identity matrix, and $\det(M)$, $P(q)$, and $W(q,\dot{q})$ were defined in (11) and (21).

Theorem 2 *The controller given in (37) ensures asymptotic regulation of the overhead crane system in the sense that*

$$\lim_{t\to\infty}\begin{pmatrix} x(t) & y(t) & \theta(t) \end{pmatrix} = \begin{pmatrix} x_d & y_d & 0 \end{pmatrix} \tag{39}$$

where x_d and y_d were defined in (26).

Proof: See Appendix B.

4.3 Gantry Kinetic Energy Coupling Control Law

To illustrate how additional controllers can also be derived, we design the following nonlinear coupling control law[3]

$$F = \frac{-k_d\dot{r} - k_p e - k_v P^{-1}W - \frac{1}{2}k_v\left(\frac{d}{dt}\left(\det(M)P^{-1}\right)\right)\dot{r}}{k_E + k_v} \tag{40}$$

[1]The control strategy is called an E^2 coupling control law because its structure is motivated by a squared energy term in the Lyapunov function and an additional gantry squared velocity term in the Lyapunov function.

[2]Since P and I_2 are positive definite matrices, and k_E, k_v, $E(q,\dot{q})$, and $\det(M(q))$ are positive scalars, it is clear that $\Omega(t)$ is positive definite and invertible.

[3]The control strategy is called a gantry kinetic energy coupling control law because its structure is derived from an additional gantry kinetic energy-like term in the Lyapunov function.

Table 1: Control Gains

	PD Control Law	E^2 Coupling Control Law	Gantry Kinetic Energy Coupling Control Law
k_d	102	125.3	350
k_p	45	50	120
k_E	1	0.001	0.4
k_v	Not Applicable	50	0.6

where k_E, k_p, k_d, and $k_v \in \mathbb{R}^1$ are positive constant control gains, and $\det(M)$, $P(q)$, and $W(q, \dot{q})$ were defined in (11) and (21).

Theorem 3 *The controller given in (40) ensures asymptotic regulation of the overhead crane system in the sense that*

$$\lim_{t\to\infty} \begin{pmatrix} x(t) & y(t) & \theta(t) \end{pmatrix} = \begin{pmatrix} x_d & y_d & 0 \end{pmatrix} \tag{41}$$

where x_d and y_d were defined in (26).

Proof: See Appendix C.

5 Simulation Results

To illustrate the performance of the proposed controllers, we simulated the 3-DOF crane system given in (1), where the crane parameters were selected as follows

$$m_p = 160 \text{ [kg]}, \qquad m_c = 23 \text{ [kg]}, \qquad m_r = 190 \text{ [kg]}, \qquad I = 1.5 \text{ [kg.m}^2\text{]}, \qquad L = 2.5 \text{ [m]} \tag{42}$$

and the desired position of the crane gantry was selected as follows

$$\begin{bmatrix} x_d & y_d \end{bmatrix}^T = \begin{bmatrix} 10 & 3 \end{bmatrix}^T. \tag{43}$$

For each of the simulations, the initial conditions were set to zero and the control gains were tuned until the best performance was achieved. The resulting control gains from each controller is given in Table 1.

The resulting gantry position error, payload angle, and the input force are shown in Figure 2 for the PD control law, Figure 3 for the E^2 coupling control law, and Figure 4 for the gantry kinetic energy coupling control law. A summary of the performance of the controllers given in (28), (37) and (40) is provided in Table 2. In Table 2, the settling time is defined as the interval between the starting time and the time when the angle $\theta(t)$ remained within ± 0.5 degrees of the equilibrium position and the response of $x(t)$ and $y(t)$ remained within 5% of the final values. The percent overshoot is defined as

$$M_p = \frac{x_{\max} - x_d}{x_d} \times 100\%, \tag{44}$$

where $x_{\max}$ denotes the maximum gantry overshoot position.

Table 2: Performance Comparison

	PD Control Law	E^2 Coupling Control Law	Gantry Kinetic Energy Coupling Control Law
X Percent Overshoot	26%	No Overshoot	No Overshoot
Y Percent Overshoot	12%	8.9%	No Overshoot
Settling Time ($\lvert\theta\rvert \leq 0.5$ deg)	21.6[sec]	11.9[sec]	8.2[sec]

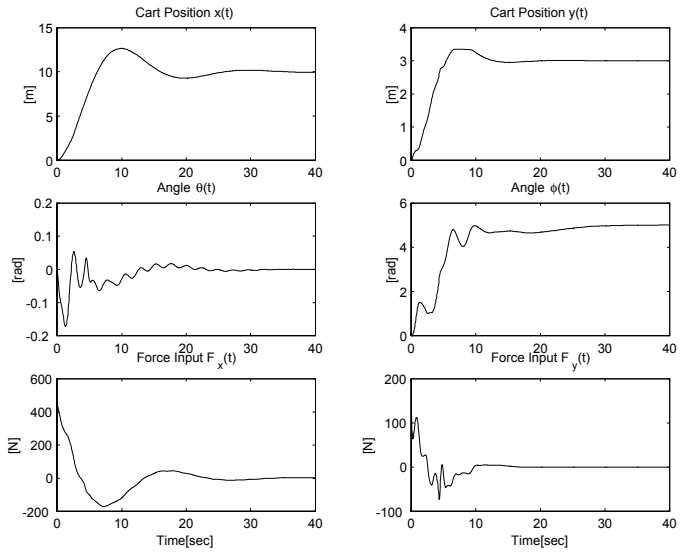

Figure 2: Results for the PD Controller

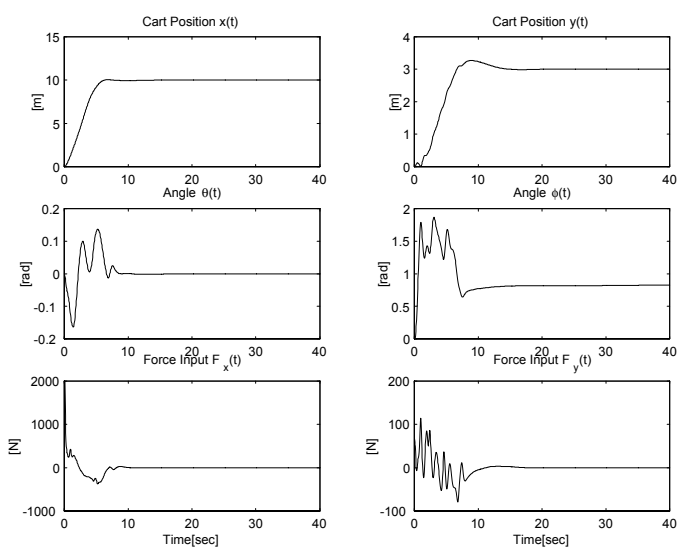

Figure 3: Results for the E^2 Coupling Control Law

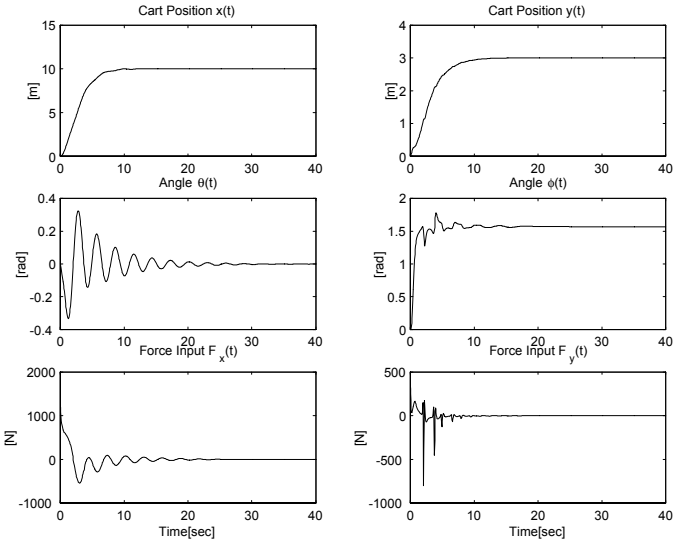

Figure 4: Results for the Gantry Kinetic Energy Coupling Control Law

6 Conclusion

In this paper, we presented three controllers for an overhead crane system. By utilizing a Lyapunov-based stability analysis along with LaSalle's Invariance Theorem, we proved asymptotic regulation of the gantry and payload position for a PD controller and two nonlinear controllers. Simulation results were utilized to demonstrate that the increased coupling between the gantry and payload that results from the additional nonlinear feedback terms in the nonlinear coupling control laws, resulted in improved transient response. Future work will focus on comparing the performance of the PD controller with the nonlinear coupling control laws through experimental results obtained from an overhead crane system with a gantry that moves in a 2-DOF Cartesian plane.

References

[1] T. Burg, D. Dawson, C. Rahn and W. Rhodes, "Nonlinear Control of an Overhead Crane via the Saturating Control Approach of Teel", *Proc. IEEE Int. Conf. Robotics and Automation,* pp. 3155-3160, 1996.

[2] H. Butler, G. Honderd, and J. Van Amerongen, "Model Reference Adaptive Control of a Gantry Crane Scale Model", *IEEE Control Systems Magazine*, pp. 57-62, January 1991.

[3] C. Chung and J. Hauser, "Nonlinear Control of a Swinging Pendulum", *Automatica*, Vol. 31, No. 6, pp. 851-862, 1995.

[4] J. Collado, R. Lozano and I. Fantoni, "Control of Convey-crane Based on Passivity", *Proc. American Control Conference*, pp.1260 -1264, 2000.

[5] I. Fantoni, R. Lozano, and M. W. Spong, "Energy Based Control of the Pendubot", IEEE Transactions on Automatic Control, Vol. 45, No. 4, pp. 725-729, 2000.

[6] B. Kiss, J. Levine, and P. Mullhaupt, "A Simple Output Feedback PD Controller for Nonlinear Cranes", *Proc. of the Conference on Decision and Control*, pp. 5097-5101, 2000.

[7] H. Lee, "Modeling and Control of a Three-Dimensional Overhead Cranes", *ASME Trans. on Dynamic Systems, Measurement, and Control,* Vol. 120, pp. 471-476, 1998.

[8] R. Lozano, I. Fantoni, D. J. Block, "Stabilization of the Inverted Pendulum Around Its Homoclinic Orbit", *Systems & Control Letters,* Vol. 40, No. 3, pp. 197-204, 2000.

[9] S. C. Martindale, D. M. Dawson, J. Zhu, and C. Rahn, "Approximate Nonlinear Control for a Two degree of Freedom Overhead Crane: Theory and Experimentation", *Proc. American Control Conference*, pp. 301-305, 1995.

[10] K. A. F. Moustafa and A. M. Ebeid, "Nonlinear Modeling and Control of Overhead Crane Load Sway", *ASME Trans. on Dynamic Systems, Measurement, and Control,* Vol. 110, pp. 266-271, 1988.

[11] M. W. Noakes and J. F. Jansen, "Generalized Inputs for Damped-Vibration Control of Suspended Payloads", *Computers and Electrical Engineering, An International Journal*, 1991.

[12] Y. Sakawa and H. Sano, "Nonlinear Model and Linear Robust Control of Overhead Traveling Cranes", *Nonlinear Analysis,* Vol. 30, Issue 4, pp.2197-2207, 1997.

[13] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*, Englewood Cliff, NJ: Prentice Hall, Inc., 1991.

[14] A. R. Teel, "Semi-global Stabilization of the 'Ball and Beam' Using 'Output' Feedback", *Proc. American Control Conference*, pp. 2577-2581, 1993.

[15] K. Yoshida and H. Kawabe, "A Design of Saturating Control with a Guaranteed Cost and Its Application to the Crane Control System", *IEEE Transactions on Automatic Control*, Vol. 37, No. 1, pp. 121-127, 1992.

[16] J. Yu, F. L. Lewis, and T. Huang, "Nonlinear Feedback Control of a Gantry Crane", *Proc. American Control Conference*, Seattle, Washington, pp.4310-4315, 1995.

A Proportional-Derivative Control Law Analysis

Case 1a: $\dot{\theta} = 0$ and $\dot{\phi} = 0$

Based on the proposition that $\dot{\theta} = 0$ and $\dot{\phi} = 0$, it is straightforward to prove that

$$\ddot{\theta} = 0 \qquad \ddot{\phi} = 0. \tag{45}$$

By rearranging the first two rows of the expression given in (1), we can obtain the following expressions

$$\frac{F_x}{m_p L} = \frac{m_p + m_r + m_c}{m_p L}\ddot{x} + \cos\theta\sin\phi\ddot{\theta} + \sin\theta\cos\phi\ddot{\phi} - \sin\theta\sin\phi\left(\dot{\theta}^2 + \dot{\phi}^2\right) + 2\cos\theta\cos\phi\dot{\theta}\dot{\phi} \tag{46}$$

$$\frac{F_y}{m_p L} = \frac{m_p + m_c}{m_p L}\ddot{y} + \cos\theta\cos\phi\ddot{\theta} - \sin\theta\sin\phi\ddot{\phi} - \sin\theta\cos\phi\left(\dot{\theta}^2 + \dot{\phi}^2\right) - 2\cos\theta\sin\phi\dot{\theta}\dot{\phi}. \tag{47}$$

Based on the expression given in (35), (45)-(47), and the proposition that $\dot{\theta} = 0$ and $\dot{\phi} = 0$, we can conclude that

$$F_x = F_y = 0. \tag{48}$$

From (22), (28), (34), and (48), it is clear that

$$e(t) = 0. \tag{49}$$

Furthermore, by rearranging the third row of the vector given in (1), we can obtain the following expression

$$\ddot{\theta} = \gamma_3 \sin\theta \cos\theta \dot{\phi}^2 - \gamma_2 \sin\theta - \frac{m_p L}{m_p L^2 + I} \left(\cos\theta \sin\phi \ddot{x} + \cos\theta \cos\phi \ddot{y}\right) \tag{50}$$

where (2)-(6) were utilized and $\gamma_2, \gamma_3 \in \mathbb{R}^1$ are positive constants defined as follows

$$\gamma_2 = \frac{m_p g L}{m_p L^2 + I} \qquad \gamma_3 = \frac{m_p L^2}{m_p L^2 + I}. \tag{51}$$

Based on (35), (45), and the proposition that $\dot{\phi} = 0$, we can utilize (50) to prove that

$$\sin\theta = 0, \tag{52}$$

and hence, from (9), it is clear that

$$\theta(t) = 0. \tag{53}$$

Given (49) and (53), we can utilize (19), (26), and (27) to prove the validity of (29) under the proposition that $\dot{\theta}(t) = 0$ and $\dot{\phi} = 0$.

Case 1b: $\dot{\theta} = 0$ and $\dot{\phi} \neq 0$

By rearranging the fourth row of the vector given in (1), we obtain the following expression

$$\gamma_1(\theta)\ddot{\phi} = -2\sin\theta\cos\theta\dot{\theta}\dot{\phi} - \left(\frac{\sin\theta\cos\phi\ddot{x} - \sin\theta\sin\phi\ddot{y}}{L}\right) \tag{54}$$

where $\gamma_1(\theta) \in \mathbb{R}^1$ is defined as follows

$$\gamma_1(\theta) = \left(\sin^2\theta + \frac{I}{m_p L^2}\right). \tag{55}$$

Based on (35), (54), and the proposition that $\dot{\theta} = 0$, it is clear that

$$\ddot{\phi} = 0 \qquad \ddot{\theta} = 0, \tag{56}$$

and hence, $\theta(t)$ and $\dot{\phi}(t)$ are constant. From (35), (46), (47), (56), the fact that $F(t)$ remains constant, and the proposition that $\dot{\phi} \neq 0$, it is straightforward to see that

$$F_x = -m_p L \sin\theta \sin\phi \dot{\phi}^2 \tag{57}$$

$$F_y = -m_p L \sin\theta \cos\phi \dot{\phi}^2. \tag{58}$$

To continue the analysis, we consider the cases of $\sin\theta = 0$ and $\sin\theta \neq 0$. Under the additional proposition that $\sin\theta \neq 0$, it is clear to see from (57) and (58) that $\sin\phi$, $\cos\phi$, and $\phi(t)$ must

be constant since $F_x(t)$ and $F_y(t)$ are constant. However, the conclusion that $\phi(t)$ is constant, contradicts with the proposition that $\dot{\phi} \neq 0$. Under the additional proposition that $\sin\theta = 0$, we can use (57) and (58) to prove that

$$F_x = F_y = \sin\theta = 0. \tag{59}$$

Given (9), (22), (28), (34), and (59), it is clear that

$$e(t) = \theta(t) = 0. \tag{60}$$

From (60), we can utilize (19), (26), and (27) to prove (29) under the propositions that $\dot{\theta}(t) = 0$, $\dot{\phi} \neq 0$, and $\sin\theta = 0$. ■

Case 2: $\dot{\theta}(t) \neq 0$

To simplify the stability analysis under the proposition that $\dot{\theta}(t) \neq 0$, we first note that if either $\sin\theta = 0$ or $\cos\theta = 0$, then $\theta(t)$ would be constant, and hence, the proposition that $\sin\theta = 0$ or $\cos\theta = 0$ would lead to a contradiction with the proposition that $\dot{\theta}(t) \neq 0$. Since $\theta(t)$ is a continuous function, it is clear that both $\sin\theta \neq 0$ and $\cos\theta \neq 0$. This fact will be utilized in the subsequent analysis.

To facilitate the stability analysis under the proposition that $\dot{\theta}(t) \neq 0$, we rewrite each row of (1) as follows

$$\sin\phi P_1 + \cos\phi P_2 = S_1 \tag{61}$$

$$\cos\phi P_1 - \sin\phi P_2 = S_2 \tag{62}$$

$$\ddot{\theta} = -\gamma_2 \sin\theta + \gamma_3 \sin\theta\cos\theta\dot{\phi}^2 + \gamma_3 S_3 \tag{63}$$

$$\gamma_1\ddot{\phi} = -2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + S_4 \tag{64}$$

where $P_1(t)$, $P_2(t)$, $S_1(t)$, $S_2(t)$, $S_3(t)$, $S_4(t) \in \mathbb{R}^1$ are defined as follows

$$P_1 = \sin\theta\left(\gamma_3\cos^2\theta\dot{\phi}^2 - \gamma_2\cos\theta - \left(\dot{\theta}^2 + \dot{\phi}^2\right)\right) + S_3\gamma_3\cos\theta \tag{65}$$

$$P_2 = \ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta \tag{66}$$

$$S_1 = \frac{1}{m_p L}\left(F_x - (m_p + m_r + m_c)\ddot{x}\right) \tag{67}$$

$$S_2 = \frac{1}{m_p L}\left(F_y - (m_p + m_c)\ddot{y}\right) \tag{68}$$

$$S_3 = -\left[\frac{\cos\theta\sin\phi\ddot{x} + \cos\theta\cos\phi\ddot{y}}{L}\right] \tag{69}$$

$$S_4 = -\left[\frac{\sin\theta\cos\phi\ddot{x} - \sin\theta\sin\phi\ddot{y}}{L}\right] \tag{70}$$

where (63) has been substituted into (65) for $\ddot{\theta}(t)$, $\gamma_1(\theta)$ was defined in (55), and γ_2, γ_3 were defined in (51). After taking the time derivative of the expressions given in (61) and (62), we obtain the following expressions

$$\cos\phi\dot{\phi}P_1 + \sin\phi\dot{P}_1 - \sin\phi\dot{\phi}P_2 + \cos\phi\dot{P}_2 = \dot{S}_1 \tag{71}$$

$$-\sin\phi\dot{\phi}P_1 + \cos\phi\dot{P}_1 - \cos\phi\dot{\phi}P_2 - \sin\phi\dot{P}_2 = \dot{S}_2 \tag{72}$$

where $\dot{P}_1(t)$, $\dot{P}_2(t)$, $\dot{S}_1(t)$, and $\dot{S}_2(t)$ can be written as follows

$$\begin{aligned}\dot{P}_1 &= \dot{\theta}\left(-\gamma_2\left(1-2\sin^2\theta\right)+\gamma_3\cos^3\theta\dot{\phi}^2-\cos\theta\left(\dot{\theta}^2+\dot{\phi}^2\right)\right)-2\gamma_3\sin\theta\dot{\phi}\left(\gamma_1\ddot{\phi}\right) \\ &\quad -2\sin\theta\dot{\theta}\ddot{\theta}-2\gamma_3\cos\theta\sin^2\theta\dot{\theta}\dot{\phi}^2-S_3\gamma_3\sin\theta\dot{\theta}+\dot{S}_3\gamma_3\cos\theta\end{aligned} \tag{73}$$

$$\dot{P}_2=\cos\theta\dot{\theta}\ddot{\phi}+\sin\theta\phi^{(3)}-2\left(\sin\theta\dot{\theta}^2\dot{\phi}-\cos\theta\ddot{\theta}\dot{\phi}-\cos\theta\dot{\theta}\ddot{\phi}\right) \tag{74}$$

$$\dot{S}_1=\frac{1}{m_pL}\left(\dot{F}_x-\left(m_p+m_r+m_c\right)x^{(3)}\right) \tag{75}$$

$$\dot{S}_2=\frac{1}{m_pL}\left(\dot{F}_y-\left(m_p+m_c\right)y^{(3)}\right) \tag{76}$$

and the expression for $\dot{S}_3(t)$ is given below

$$\begin{aligned}\dot{S}_3 &= \frac{1}{L}\left(\dot{\theta}\sin\theta\sin\phi\ddot{x}-\dot{\phi}\cos\theta\cos\phi\ddot{x}-\cos\theta\sin\phi x^{(3)}\right) \\ &\quad +\frac{1}{L}\left(\dot{\theta}\sin\theta\cos\phi\ddot{y}+\dot{\phi}\cos\theta\sin\phi\ddot{y}-\cos\theta\cos\phi\ddot{y}^{(3)}\right).\end{aligned} \tag{77}$$

After substituting (63) and (64) into (73) for $\ddot{\theta}(t)$ and $\ddot{\phi}(t)$, respectively, and then performing some algebraic manipulation, we obtain the following expression

$$\begin{aligned}\dot{P}_1 &= \dot{\theta}\left(-\gamma_2\left(1-4\sin^2\theta\right)+\gamma_3\cos^3\theta\dot{\phi}^2-\cos\theta\left(\dot{\theta}^2+\dot{\phi}^2\right)\right) \\ &\quad -\gamma_3\left(\left(2\dot{\phi}S_4+3S_3\dot{\theta}\right)\sin\theta-\dot{S}_3\cos\theta\right).\end{aligned} \tag{78}$$

After multiplying both sides of the expression given in (71) by $\sin\phi$, multiplying both sides of the expression given in (72) by $\cos\phi$, and then adding the resulting expressions, the following expression is obtained

$$\dot{P}_1-\dot{\phi}P_2=\dot{S}_1\sin\phi+\dot{S}_2\cos\phi. \tag{79}$$

By multiplying both sides of (79) by $\gamma_1(\theta)$, substituting (66) and (78) into the resulting expression for $P_2(t)$ and $\dot{P}_1(t)$, and then dividing the resulting expression by $\dot{\theta}(t)$, we obtain the following expression

$$\gamma_1P_3-2\frac{I}{m_pL^2}\cos\theta\dot{\phi}^2=\frac{S_5}{\dot{\theta}} \tag{80}$$

where (64) was utilized and the auxiliary expressions $P_3(t)$ and $S_5(t)$ are defined as follows

$$P_3=-\gamma_2\left(1-4\sin^2\theta\right)+\gamma_3\cos^3\theta\dot{\phi}^2-\cos\theta\left(\dot{\theta}^2+\dot{\phi}^2\right) \tag{81}$$

$$\begin{aligned}S_5 &= \gamma_1\left(\dot{S}_1\sin\phi+\dot{S}_2\cos\phi+\gamma_3\left(\left(3S_3\dot{\theta}+2\dot{\phi}S_4\right)\sin\theta-\dot{S}_3\cos\theta\right)\right) \\ &\quad +S_4\dot{\phi}\sin\theta.\end{aligned} \tag{82}$$

After taking the time derivative of (80), we obtain the following expression

$$\gamma_1\dot{P}_3+2\sin\theta\cos\theta\dot{\theta}P_3+2\frac{I}{m_pL^2}\left(\sin\theta\dot{\theta}\dot{\phi}^2-2\cos\theta\dot{\phi}\ddot{\phi}\right)=\frac{\dot{S}_5\dot{\theta}-S_5\ddot{\theta}}{\dot{\theta}^2} \tag{83}$$

where the expressions for $\dot{P}_3(t)$ and $\dot{S}_5(t)$ are given below

$$\dot{P}_3 = \sin\theta\dot{\theta}P_4 - 2\gamma_3\left(S_3\dot{\theta}\cos\theta + S_4\cos\theta\dot{\phi}\right) \tag{84}$$

$$\begin{aligned}\dot{S}_5 &= 2\sin\theta\cos\theta\dot{\theta}\left(\dot{S}_1\sin\phi + \dot{S}_2\cos\phi + \gamma_3\left(\left(3S_3\dot{\theta} + 2\dot{\phi}S_4\right)\sin\theta - \dot{S}_3\cos\theta\right)\right) \\ &+\gamma_1\ddot{S}_1\sin\phi + \gamma_1\dot{S}_1\dot{\phi}\cos\phi + \gamma_1\ddot{S}_2\cos\phi - \gamma_1\dot{S}_2\sin\phi\dot{\phi} \\ &+\gamma_1\gamma_3\left(\left(3\dot{S}_3\dot{\theta} + 3S_3\ddot{\theta} + 2\ddot{\phi}S_4 + 2\dot{\phi}\dot{S}_4\right)\sin\theta + \left(3S_3\dot{\theta} + 2\dot{\phi}S_4\right)\cos\theta\dot{\theta}\right) \\ &-\gamma_1\gamma_3\ddot{S}_3\cos\theta + \gamma_1\gamma_3\dot{S}_3\sin\theta\dot{\theta} + \dot{S}_4\dot{\phi}\sin\theta + S_4\ddot{\phi}\sin\theta + S_4\dot{\phi}\cos\theta\dot{\theta}\end{aligned} \tag{85}$$

where (63) and (64) were utilized, and the definitions for $P_4(t)$, $\ddot{S}_1(t)$, $\ddot{S}_2(t)$, $\ddot{S}_3(t)$ and $\dot{S}_4(t)$ are given by the following expressions

$$P_4 = 10\gamma_2\cos\theta - \gamma_3\cos^2\theta\dot{\phi}^2 + \left(\dot{\theta}^2 + \dot{\phi}^2\right) \tag{86}$$

$$\ddot{S}_1 = \frac{1}{m_pL}\left(\ddot{F}_x - (m_p + m_r + m_c)\,x^{(4)}\right) \tag{87}$$

$$\ddot{S}_2 = \frac{1}{m_pL}\left(\ddot{F}_y - (m_p + m_c)\,y^{(4)}\right) \tag{88}$$

$$\begin{aligned}\ddot{S}_3 &= \frac{1}{L}\sin\theta\sin\phi\left(\ddot{\theta}\ddot{x} + 2\dot{\theta}x^{(3)} - 2\dot{\theta}\dot{\phi}\ddot{y}\right) + \frac{1}{L}\sin\theta\cos\phi\left(2\dot{\theta}\dot{\phi}\ddot{x} + \ddot{\theta}\ddot{y} + 2\dot{\theta}y^{(3)}\right) \\ &+\frac{1}{L}\cos\theta\sin\phi\left(\dot{\phi}^2\ddot{x} + \dot{\theta}^2\ddot{x} + \ddot{\phi}\ddot{y} + 2\dot{\phi}y^{(3)} - x^{(4)}\right) \\ &+\frac{1}{L}\cos\theta\cos\phi\left(-\ddot{\phi}\ddot{x} - 2\dot{\phi}x^{(3)} + \dot{\theta}^2\ddot{y} + \dot{\phi}^2\ddot{y} - y^{(4)}\right)\end{aligned} \tag{89}$$

$$\begin{aligned}\dot{S}_4 &= -\frac{1}{L}\left(\dot{\theta}\cos\theta\cos\phi\ddot{x} - \dot{\phi}\sin\theta\sin\phi\ddot{x} + \sin\theta\cos\phi x^{(3)}\right) \\ &+\frac{1}{L}\left(\dot{\theta}\cos\theta\sin\phi\ddot{y} + \dot{\phi}\sin\theta\cos\phi\ddot{y} + \sin\theta\sin\phi\ddot{y}^{(3)}\right).\end{aligned} \tag{90}$$

By substituting (84) into (83) for $\dot{P}_3(t)$ and then multiplying the resulting expression by $\gamma_1(\theta)$, we can obtain the following expression

$$\sin\theta\dot{\theta}\left(\gamma_1^2P_4 + 2\gamma_1\cos\theta P_3 + \frac{2I}{m_pL^2}\dot{\phi}^2\left(\gamma_1 + 4\cos^2\theta\right)\right) = S_6 \tag{91}$$

where (64) was utilized, and $S_6(t)$ is defined as follows

$$S_6 = \gamma_1\left(\frac{\dot{S}_5\dot{\theta} - S_5\ddot{\theta}}{\dot{\theta}^2} + \frac{4I\cos\theta\dot{\phi}}{m_pL^2}S_4\right) + \gamma_1^2\left(2\gamma_3\left(S_3\cos\theta\dot{\theta} + S_4\cos\theta\dot{\phi}\right)\right). \tag{92}$$

After dividing (91) by $\sin\theta\dot{\theta}(t)$ and then substituting (80) into the resulting expression for $\gamma_1(\theta)P_3(t)$, the following expression is obtained

$$\gamma_1^2P_4 + \frac{2I}{m_pL^2}\dot{\phi}^2\left(\gamma_1 + 6\cos^2\theta\right) = \frac{S_6}{\sin\theta\dot{\theta}} - \frac{2S_5\cos\theta}{\dot{\theta}}. \tag{93}$$

After multiplying (80) by $\gamma_1(\theta)$, multiplying (93) by $\cos\theta$, and then adding the resulting products, we obtain the following expression

$$\gamma_1^2\left(P_3+P_4\cos\theta\right)+\frac{12I}{m_pL^2}\cos^3\theta\dot{\phi}^2=S_7 \tag{94}$$

where $S_7(t)$ is defined as follows

$$S_7=\left(\frac{S_6}{\sin\theta\dot{\theta}}-\frac{2S_5\cos\theta}{\dot{\theta}}\right)\cos\theta+\frac{S_5}{\dot{\theta}}\gamma_1. \tag{95}$$

By utilizing (81) and (86), we can rewrite (94) as follows

$$\gamma_2\gamma_1^2\left(9-6\sin^2\theta\right)+\frac{12I}{m_pL^2}\cos^3\theta\dot{\phi}^2=S_7. \tag{96}$$

To continue the analysis, we take the time derivative of (96) as follows

$$4\gamma_2\sin\theta\cos\theta\dot{\theta}\gamma_1\left(\left(9-6\sin^2\theta\right)-3\gamma_1\right)+\frac{12I}{m_pL^2}\cos^2\theta\dot{\phi}\left(-3\sin\theta\dot{\theta}\dot{\phi}+2\cos\theta\ddot{\phi}\right)=\dot{S}_7 \tag{97}$$

where the time derivative of (95) is given by the following expression

$$\begin{aligned}\dot{S}_7 &= -\left(\frac{S_6}{\sin\theta}-2S_5\cos\theta\right)\sin\theta+\left(\frac{\dot{S}_6}{\sin\theta\dot{\theta}}-\frac{S_6\cos\theta}{\sin^2\theta}-\frac{S_6\ddot{\theta}}{\sin\theta\dot{\theta}^2}\right)\cos\theta \\ &\quad -\left(\frac{2\dot{S}_5\cos\theta}{\dot{\theta}}-2S_5\sin\theta-\frac{2S_5\cos\theta\ddot{\theta}}{\dot{\theta}^2}\right)\cos\theta+\frac{\dot{S}_5}{\dot{\theta}}\gamma_1-\frac{S_5\ddot{\theta}}{\dot{\theta}^2}\gamma_1 \\ &\quad +2S_5\sin\theta\cos\theta\end{aligned} \tag{98}$$

with the expression for $\dot{S}_6(t)$ being given below

$$\begin{aligned}\dot{S}_6 &= 2\dot{\theta}\sin\theta\cos\theta\left(\frac{\dot{S}_5\dot{\theta}-S_5\ddot{\theta}}{\dot{\theta}^2}+\frac{4I\cos\theta\dot{\phi}}{m_pL^2}S_4\right) \\ &\quad +\gamma_1\left(\frac{\ddot{S}_5\dot{\theta}-S_5\theta^{(3)}}{\dot{\theta}^2}-2\ddot{\theta}\left(\frac{\dot{S}_5\dot{\theta}-S_5\ddot{\theta}}{\dot{\theta}^3}\right)\right) \\ &\quad +\gamma_1\left(-\frac{4I\sin\theta\dot{\theta}\dot{\phi}}{m_pL^2}S_4+\frac{4I\cos\theta\ddot{\phi}}{m_pL^2}S_4+\frac{4I\cos\theta\dot{\phi}}{m_pL^2}\dot{S}_4\right) \\ &\quad +4\gamma_1\dot{\theta}\sin\theta\cos\theta\left(2\gamma_3\left(S_3\cos\theta\dot{\theta}+S_4\cos\theta\dot{\phi}\right)\right) \\ &\quad +2\gamma_1^2\gamma_3\left(\dot{S}_3\cos\theta\dot{\theta}-S_3\sin\theta\dot{\theta}^2+S_3\cos\theta\ddot{\theta}\right) \\ &\quad +2\gamma_1^2\gamma_3\left(\dot{S}_4\cos\theta\dot{\phi}-S_4\sin\theta\dot{\phi}\dot{\theta}+S_4\cos\theta\ddot{\phi}\right)\end{aligned} \tag{99}$$

with the expression for $\ddot{S}_5(t)$ being given below

$$\ddot{S}_5 = \left[2\left(\cos^2\theta-\sin^2\theta\right)\dot{\theta}^2+\sin\theta\cos\theta\ddot{\theta}\right]\left(\left(\dot{S}_1\sin\phi+\dot{S}_2\cos\phi\right)\right. \tag{100}$$

$$
\begin{aligned}
&+\gamma_3\left(\left(3S_3\dot{\theta}+2\dot{\phi}S_4\right)\sin\theta-\dot{S}_3\cos\theta\right)\right)\\
&+\gamma_1\left(S_1^{(3)}\sin\phi+2\ddot{S}_1\cos\phi\dot{\phi}+\dot{S}_1\left(\cos\phi\ddot{\phi}-\sin\phi\dot{\phi}^2\right)\right)\\
&+\gamma_1\left(S_2^{(3)}\cos\phi-2\ddot{S}_2\sin\phi\dot{\phi}+\dot{S}_2\left(-\sin\phi\ddot{\phi}-\cos\phi\dot{\phi}^2\right)\right)\\
&+3\gamma_1\gamma_3\left(\ddot{S}_3\sin\theta\dot{\theta}+2\dot{S}_3\left(\cos\theta\dot{\theta}^2+\sin\theta\ddot{\theta}\right)+S_3\left(-\sin\theta\dot{\theta}^3+3\cos\theta\dot{\theta}\ddot{\theta}+\sin\theta\theta^{(3)}\right)\right)\\
&+2\gamma_1\gamma_3\left(\ddot{S}_4\sin\theta\dot{\phi}+2\dot{S}_4\left(\cos\theta\dot{\theta}\dot{\phi}+\sin\theta\ddot{\phi}\right)\right)\\
&+2\gamma_1\gamma_3\left(+S_4\left(-\sin\theta\dot{\theta}^2\dot{\phi}+\cos\theta\ddot{\theta}\dot{\phi}+2\cos\theta\dot{\theta}\ddot{\phi}+\sin\theta\phi^{(3)}\right)\right)\\
&-\gamma_1\gamma_3\left(S_3^{(3)}\cos\theta-2\ddot{S}_3\sin\theta\dot{\theta}+\dot{S}_3\left(-\cos\theta\dot{\theta}^2-\sin\theta\ddot{\theta}\right)\right)\\
&+4\sin\theta\cos\theta\dot{\theta}\left(\ddot{S}_1\sin\phi+\dot{S}_1\dot{\phi}\cos\phi+\ddot{S}_2\cos\phi-\dot{S}_2\sin\phi\dot{\phi}\right)\\
&+4\gamma_3\sin\theta\cos\theta\dot{\theta}\left(\left(3\dot{S}_3\dot{\theta}+3S_3\ddot{\theta}+2\ddot{\phi}S_4+2\dot{\phi}\dot{S}_4\right)\sin\theta+\left(3S_3\dot{\theta}+2\dot{\phi}S_4\right)\cos\theta\dot{\theta}\right)\\
&+4\gamma_3\sin\theta\cos\theta\dot{\theta}\left(-\ddot{S}_3\cos\theta+\dot{S}_3\sin\theta\dot{\theta}\right)+\ddot{S}_4\sin\theta\dot{\phi}+2\dot{S}_4\left(\sin\theta\ddot{\phi}+\cos\theta\dot{\theta}\dot{\phi}\right)\\
&+S_4\left(\sin\theta\phi^{(3)}+2\cos\theta\dot{\theta}\ddot{\phi}-\sin\theta\dot{\theta}^2\dot{\phi}+\cos\theta\ddot{\theta}\dot{\phi}\right)
\end{aligned}
$$

with the expressions for $S_1^{(3)}(t)$, $S_2^{(3)}(t)$, $S_3^{(3)}(t)$ and $\ddot{S}_4(t)$ being given below

$$
S_1^{(3)}=\frac{1}{m_pL}\left(F_x^{(3)}-\left(m_p+m_r+m_c\right)x^{(5)}\right) \tag{101}
$$

$$
S_2^{(3)}=\frac{1}{m_pL}\left(F_y^{(3)}-\left(m_p+m_c\right)y^{(5)}\right) \tag{102}
$$

$$
\begin{aligned}
S_3^{(3)} \;=\;& \frac{1}{L}\left(\dot{\theta}\cos\theta\sin\phi+\sin\theta\cos\phi\dot{\phi}\right)\left(\ddot{\theta}\ddot{x}+2\dot{\theta}x^{(3)}-2\dot{\theta}\dot{\phi}\ddot{y}\right)\\
&\frac{1}{L}\sin\theta\sin\phi\left(\theta^{(3)}\ddot{x}+\ddot{\theta}x^{(3)}+2\left(\dot{\theta}x^{(4)}+\ddot{\theta}x^{(3)}-\ddot{\theta}\dot{\phi}\ddot{y}-\dot{\theta}\ddot{\phi}\ddot{y}-\dot{\theta}\dot{\phi}y^{(3)}\right)\right)\\
&\frac{1}{L}\left(\dot{\theta}\cos\theta\cos\phi-\sin\theta\sin\phi\dot{\phi}\right)\left(2\dot{\theta}\dot{\phi}\ddot{x}+\ddot{\theta}\ddot{y}+2\dot{\theta}y^{(3)}\right)\\
&+\frac{1}{L}\sin\theta\cos\phi\left(2\ddot{\theta}\dot{\phi}\ddot{x}+2\dot{\theta}\ddot{\phi}\ddot{x}+2\dot{\theta}\dot{\phi}x^{(3)}+\theta^{(3)}\ddot{y}+3\ddot{\theta}y^{(3)}+2\dot{\theta}y^{(4)}\right)\\
&+\frac{1}{L}\left(-\dot{\theta}\sin\theta\sin\phi+\cos\theta\cos\phi\dot{\phi}\right)\left(\dot{\phi}^2\ddot{x}+\dot{\theta}^2\ddot{x}+\ddot{\phi}\ddot{y}+2\dot{\phi}y^{(3)}-x^{(4)}\right)\\
&+\frac{1}{L}\cos\theta\sin\phi\left(2\dot{\phi}\ddot{\phi}\ddot{x}+2\dot{\theta}\ddot{\theta}\ddot{x}+\dot{\phi}^2x^{(3)}+\dot{\theta}^2x^{(3)}+\phi^{(3)}\ddot{y}+3\ddot{\phi}y^{(3)}+2\dot{\phi}y^{(4)}-x^{(5)}\right)\\
&-\frac{1}{L}\left(\dot{\theta}\sin\theta\cos\phi+\cos\theta\sin\phi\dot{\phi}\right)\left(-\ddot{\phi}\ddot{x}-2\dot{\phi}x^{(3)}+\dot{\theta}^2\ddot{y}+\dot{\phi}^2\ddot{y}-y^{(4)}\right)\\
&+\frac{1}{L}\cos\theta\cos\phi\left(-\phi^{(3)}\ddot{x}-3\ddot{\phi}x^{(3)}-2\dot{\phi}x^{(4)}+2\dot{\theta}\ddot{\theta}\ddot{y}+2\dot{\phi}\ddot{\phi}\ddot{y}+\dot{\theta}^2y^{(3)}+\dot{\phi}^2y^{(3)}-y^{(5)}\right)
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
\ddot{S}_4 \;=\;& \frac{1}{L}\sin\theta\sin\phi\left(\ddot{\phi}\ddot{x}+2\dot{\phi}x^{(3)}-\dot{\theta}^2\ddot{y}-\dot{\phi}^2\ddot{y}+y^{(4)}\right)\\
&+\frac{1}{L}\cos\theta\sin\phi\left(2\dot{\theta}\dot{\phi}\ddot{x}+\ddot{\theta}\ddot{y}+2\dot{\theta}y^{(3)}\right)
\end{aligned} \tag{104}
$$

$$+\frac{1}{L}\sin\theta\cos\phi\left(\dot{\theta}^2\ddot{x}+\dot{\phi}^2\ddot{x}+\ddot{\phi}\ddot{y}++2\dot{\phi}y^{(3)}-x^{(4)}\right)$$
$$-\frac{1}{L}\cos\theta\cos\phi\left(\ddot{\theta}\ddot{x}+2\dot{\theta}x^{(3)}-2\dot{\theta}\dot{\phi}\ddot{y}\right).$$

After multiplying (97) by $\gamma_1(\theta)$, we obtain the following expression

$$\begin{aligned} & 4\gamma_2\sin\theta\cos\theta\dot{\theta}\gamma_1^2\left(\left(9-6\sin^2\theta\right)-3\gamma_1\right) \\ & -\frac{36I}{m_pL^2}\gamma_1\cos^2\theta\sin\theta\dot{\theta}\dot{\phi}^2-\frac{48I}{m_pL^2}\sin\theta\cos^4\theta\dot{\theta}\dot{\phi}^2 \\ = \; & \gamma_1\dot{S}_7-\frac{24I}{m_pL^2}\cos^3\theta\dot{\phi}S_4 \end{aligned} \tag{105}$$

where (64) was utilized. After dividing (105) by $\left(\sin\theta\cos\theta\dot{\theta}\right)$ and utilizing (96), we obtain the following expression

$$-\frac{12I\cos\theta\dot{\phi}^2}{m_pL^2}\left(8\cos^2\theta+3\gamma_1\right)-12\gamma_2\gamma_1^3=S_8-4S_7 \tag{106}$$

where $S_8(t)$ is defined as follows

$$S_8=\left(\frac{1}{\sin\theta\cos\theta\dot{\theta}}\right)\left(\dot{S}_7\gamma_1-\frac{24I}{m_pL^2}\cos^3\theta\dot{\phi}S_4\right). \tag{107}$$

By multiplying (106) by $\cos^2\theta(t)$ and then utilizing (96), we can rewrite (106) as follows

$$\left(9-6\sin^2\theta\right)\left(8\cos^2\theta+3\gamma_1\right)-12\cos^2\theta\gamma_1=\left(\frac{S_9}{\gamma_2\gamma_1^2}\right) \tag{108}$$

where $S_9(t)$ is defined as follows

$$S_9=\cos^2\theta\left(S_8-4S_7\right)+S_7\left(8\cos^2\theta+3\gamma_1\right). \tag{109}$$

Given the facts that $F_x(t)$ and $F_y(t)$ are constant and $\ddot{x}=0$ and $\ddot{y}=0$, it is clear that

$$F_x^{(k)}=0, \qquad F_y^{(k)}=0, \qquad k\geq 1 \tag{110}$$

$$x^{(k)}=0, \qquad y^{(k)}=0, \qquad k\geq 2 \tag{111}$$

and hence, the expressions given in (67)-(70), (75)-(77), (109), (85), (87)-(90), (92), (95), (76)-(104), (107), and (109) can be utilized to prove that

$$S_i=0, \qquad \dot{S}_i=0, \qquad i=1,2,...9. \tag{112}$$

After some algebraic manipulation, we can now utilize (112) to rewrite (108) as follows

$$\left(1+2\cos^2\theta\right)\left(3\left(1+\frac{I}{m_pL^2}\right)+5\cos^2\theta\right)-4\cos^2\theta\left(1+\frac{I}{m_pL^2}-\cos^2\theta\right)=0. \tag{113}$$

After some further algebraic manipulation, the expression given in (113) can be rewritten as follows

$$\cos^4\theta+\alpha_1\cos^2\theta+\alpha_2=0 \tag{114}$$

where $\alpha_1,\alpha_1\in\mathbb{R}^1$ are positive constants defined below

$$\alpha_1=\frac{1}{2}+\frac{1}{7}\frac{I}{m_pL^2} \qquad \alpha_2=\frac{3}{14}\left(1+\frac{I}{m_pL^2}\right). \tag{115}$$

Since the expression given in (114) it clearly invalid, we must conclude that the proposition that $\dot{\theta}(t)\neq 0$ must be invalid, and hence, $\dot{\theta}(t)=0$ and the analysis given in under this proposition can be utilized to prove (29). ■

B E^2 Coupling Control Law Stability Analysis

To prove (39), we define a nonnegative function $V_2(t) \in \mathbb{R}^1$ as follows

$$V_2 = \frac{1}{2}k_E E^2 + \frac{1}{2}k_p e^T e + \frac{1}{2}k_v \dot{r}^T \dot{r}. \tag{116}$$

After taking the time-derivative of (116) and then substituting (20), (25), and the time derivative of (27) into the resulting expression for $\ddot{r}(t)$, $\dot{E}(q,\dot{q})$, and $\dot{e}(t)$, respectively, we obtain the following expression

$$\dot{V}_2 = \dot{r}^T \left(\Omega F + \frac{k_v}{\det(M)} W + k_p e \right) \tag{117}$$

where (38) was utilized. After substituting (37) into (118) for $F(t)$ and then cancelling common terms, we obtain the following expression

$$\dot{V}_2 = -k_d \dot{r}^T \dot{r}. \tag{118}$$

Based on the expressions given in (24), (27), (116), and (118), it is clear that the origin of the closed-loop system is stable in the sense of Lyapunov [13] and that $r(t)$, $\dot{r}(t)$, $e(t)$, $\dot{q}(t)$, $E(q,\dot{q}) \in \mathcal{L}_\infty$, and hence, from (2) and (19) it is clear that $x(t)$, $\dot{x}(t)$, $y(t), \dot{y}(t), \dot{\theta}(t)$, $\dot{\phi}(t) \in \mathcal{L}_\infty$. Based on the expressions given in (14)-(18), (11), (21), and (23), we can prove that $\det(M(q))$, $P(q)$, $P^{-1}(q)$, $W(q,\dot{q}) \in \mathcal{L}_\infty$. Given (11), and the fact that $E(q,\dot{q})$, $P(q) \in \mathcal{L}_\infty$, we can utilize (38) to prove that $\Omega(t) \in \mathcal{L}_\infty$. Given the following expression for the determinant of $\Omega(t)$

$$\det(\Omega) = (k_E E)^2 + k_E E \frac{k_v}{\det(M)} (p_{11} + p_{22}) + \left(\frac{k_v}{\det(M)} \right)^2 \left(p_{11} p_{22} - p_{12}^2 \right), \tag{119}$$

it is easy to see from (11), (14)-(16), (23), and (24) that

$$\det(\Omega) \geq \left(\frac{k_v}{\det(M)} \right)^2 m_p \left(m_p + m_r + m_c \right) I^4. \tag{120}$$

Based on the fact that $\det(M(q))$, $\Omega(t) \in \mathcal{L}_\infty$, we can utilize (120) to prove that $\Omega^{-1}(t) \in \mathcal{L}_\infty$. Given that $e(t)$, $\dot{r}(t)$, $\det(M(q))$, $W(q,\dot{q})$, $\Omega^{-1}(t) \in \mathcal{L}_\infty$, it is clear from (37) that $F(t) \in \mathcal{L}_\infty$. From (6) and (22), we can prove that $F_x(t), F_y(t), u(t) \in \mathcal{L}_\infty$.

In a similar manner as in the proof of Theorem 1, we define Γ as the set of all points where

$$\dot{V}_2 = 0. \tag{121}$$

In the set Γ, it is clear from (118) and (121) that

$$\dot{r}(t) = 0 \qquad \ddot{r}(t) = 0 \tag{122}$$

and hence, we can utilize (19) and (121) to prove that $x(t)$, $y(t)$, and $V_2(t)$ are constant, and that

$$\ddot{x}(t) = 0 \qquad \ddot{y}(t) = 0. \tag{123}$$

Furthermore, from (25) and (122), it is clear that

$$\dot{E}(q,\dot{q}) = 0 \qquad \dot{e}(t) = 0 \tag{124}$$

and hence, $E(q,\dot{q})$ and $e(t)$ are constant.

Similar to the proof of Theorem 1, we can now divide the rest of the analysis into two cases. For the case of $\dot{\theta}=0$ and $\dot{\phi}=0$, we can easily follow Case 1a in the proof of Theorem 1 to prove the result given by (39). For the other cases, we note that (20) can be utilized to rewrite (37) in the following equivalent form[4]

$$F=\frac{-k_d\dot{r}-k_pe-k_v\ddot{r}}{k_EE}. \tag{125}$$

Based on the structure of (125), it is clear from (122), (124), and (125) that $F(t)$ is constant, and we can now utilize similar arguments as in the proof of Theorem 1 to prove the result given by (39). ■

C Gantry Kinetic Energy Coupling Control Law Stability Analysis

To prove (41), we define a nonnegative function $V_3(t)\in\mathbb{R}^1$ as follows

$$V_3=k_EE+\frac{1}{2}k_v\dot{r}^T\left(\det(M)P^{-1}\right)\dot{r}+\frac{1}{2}k_pe^Te. \tag{126}$$

After taking the time-derivative of (126) and then substituting (25) and the time derivative of (27) for $\dot{E}(q,\dot{q})$ and $\dot{e}(t)$, respectively, we obtain the following expression

$$\dot{V}_3=\dot{r}^T\left(k_EF+k_v\left(\det(M)P^{-1}\right)\ddot{r}+\frac{1}{2}k_v\left(\frac{d}{dt}\left(\det(M)P^{-1}\right)\right)\dot{r}+k_pe\right). \tag{127}$$

By substituting (20) into (127) for $\ddot{r}(t)$, we obtain the following expression

$$\dot{V}_3=\dot{r}^T\left(\left(k_E+k_v\right)F+k_vP^{-1}W+\frac{1}{2}k_v\left(\frac{d}{dt}\left(\det(M)P^{-1}\right)\right)\dot{r}+k_pe\right). \tag{128}$$

After substituting (40) for $F(t)$ and then simplifying the resulting expression, we can rewrite (128) as follows

$$\dot{V}_3=-k_d\dot{r}^T\dot{r}. \tag{129}$$

Based on the expressions given in (24), (27), (126), and (129), it is clear that the origin of the closed-loop system is stable in the sense of Lyapunov [13] and that $r(t),\dot{r}(t),e(t),\dot{q}(t)\in\mathcal{L}_\infty$, and hence, from (2) and (19) it is clear that $x(t)$, $\dot{x}(t)$, $y(t),\dot{y}(t),\dot{r}(t),\dot{\theta}(t)$, $\dot{\phi}(t)\in\mathcal{L}_\infty$. Based on the expressions in given in (11), (14)-(18), (21), and (23), we can prove that $\det(M(q)),P^{-1}(q),W(q,\dot{q})\in\mathcal{L}_\infty$. Based on the fact that

$$\frac{d}{dt}P^{-1}=-P^{-1}\left(\frac{d}{dt}P\right)P^{-1}$$

we can rewrite $\frac{d}{dt}\left(\det(M)P^{-1}\right)$ as follows

$$\frac{d}{dt}\left(\det(M)P^{-1}\right)=\left(\frac{d}{dt}\det(M)\right)P^{-1}-\det(M)P^{-1}\left(\frac{d}{dt}P\right)P^{-1} \tag{130}$$

[4]Since either $\dot{\theta}(t)\neq 0$ or $\dot{\varphi}(t)\neq 0$, we can use (24) to show that $E(q,\dot{q})>0$, and hence, the denominator of (125) does not go to zero for the remaining cases.

where the time derivative of the determinant of $M(q)$ is given by the following expression

$$\begin{aligned}\frac{d}{dt}\det(M) &= m_pL^2I\left[\left(\left(m_p+m_r+m_c\right)m_c+\left(m_p+m_c\right)m_p\right)\left(2\sin\theta\cos\theta\dot{\theta}\right)\right. \\ &\quad +2m_rm_p\cos\phi\left(\sin\phi\dot{\phi}+2\sin\theta\cos\theta\dot{\theta}\cos\phi-2\sin^2\theta\sin\phi\dot{\phi}\right)\Big] \\ &\quad +m_p^2L^4\left(2\sin\theta\cos\theta\dot{\theta}\right)\left[m_rm_p\sin^2\theta\cos^2\phi+m_c\left(m_r+m_c+m_p\sin^2\theta\right)\right] \\ &\quad +m_p^2L^4\sin^2\theta\left[m_rm_p\left(\left(2\sin\theta\cos\theta\dot{\theta}\cos^2\phi\right)-\left(2\sin^2\theta\cos\phi\sin\phi\dot{\phi}\right)\right)\right. \\ &\quad \left.+m_cm_p\left(2\sin\theta\cos\theta\dot{\theta}\right)\right]\end{aligned} \tag{131}$$

and the time derivative of each element of the auxiliary matrix $P(q)$ is given below

$$\begin{aligned}\dot{p}_{11} &= m_p^2L^2I\left(2\sin\phi\cos\phi\dot{\phi}+4\sin\theta\cos\theta\cos^2\phi\dot{\theta}-4\sin^2\theta\sin\phi\cos\phi\dot{\phi}\right) \\ &\quad +2m_cm_pL^4\sin\theta\cos\theta\dot{\theta}+2m_cm_pL^2I\sin\theta\cos\theta\dot{\theta} \\ &\quad -2m_p^3L^4\sin^4\theta\sin\phi\cos\phi\dot{\phi}+4m_p^3L^4\cos^2\phi\sin^3\theta\cos\theta\dot{\theta}\end{aligned} \tag{132}$$

$$\begin{aligned}\dot{p}_{12} &= -m_p^3L^4\left(\cos^2\phi-\sin^2\phi\right)\sin^4\theta\dot{\phi}-4m_p^3L^4\sin\phi\cos\phi\sin^3\theta\cos\theta\dot{\theta} \\ &\quad -m_p^2L^2I\left(\cos^2\phi-\sin^2\phi\right)\left(\sin^2\theta-\cos^2\theta\right)\dot{\phi}-4m_p^2L^2I\sin\phi\cos\phi\sin\theta\cos\dot{\theta}\theta\end{aligned} \tag{133}$$

$$\begin{aligned}\dot{p}_{22} &= 4m_p^2L^2\sin\theta\cos\theta\sin^2\phi\dot{\theta}\left(m_pL^2\sin^2\theta+I\right) \\ &\quad +2m_p^2L^2\sin\phi\cos\phi\dot{\phi}\left[m_pL^2\sin^4\theta+I\left(\sin^2\theta-\cos^2\theta\right)\right] \\ &\quad +2\left(m_r+m_c\right)m_pL^2\sin\theta\cos\theta\dot{\theta}\left(m_pL^2+I\right).\end{aligned} \tag{134}$$

Based on the fact that $q(t), \dot{q}(t), P^{-1}(q) \in \mathcal{L}_\infty$, it is clear that the expressions given in (130)-(134) can be utilized to prove that $\frac{d}{dt}\left(\det(M)P^{-1}\right) \in \mathcal{L}_\infty$. Given that $e(t)$, $\dot{r}(t)$, $P^{-1}(q)$, $W(q,\dot{q})$, $\frac{d}{dt}\left(\det(M)P^{-1}\right) \in \mathcal{L}_\infty$, it is straightforward from (40) that $F(t) \in \mathcal{L}_\infty$. Finally, from (6) and (22), we can prove that $F_x(t), F_y(t), u(t) \in \mathcal{L}_\infty$.

In a similar manner as in the proof of Theorem 1, we define Γ as the set of all points where

$$\dot{V}_3 = 0. \tag{135}$$

In the set Γ, it is clear from (129) and (135) that

$$\dot{r}(t) = 0 \qquad \ddot{r}(t) = 0 \tag{136}$$

and hence, we can conclude from (19) that $x(t)$, $y(t)$, and $V_3(t)$ are constant, and that

$$\ddot{x}(t) = 0 \qquad \ddot{y}(t) = 0. \tag{137}$$

Furthermore, from (25) and (136), it is clear that

$$\dot{E}(q,\dot{q}) = 0 \qquad \dot{e}(t) = 0 \tag{138}$$

and hence, $E(q,\dot{q})$ and $e(t)$ are constant.

To complete the remaining analysis, we note that (20) can be utilized to rewrite (40) in the following equivalent form

$$F = \frac{-k_d \dot{r} - k_p e - k_v \left(\det(M) P^{-1}\right) \ddot{r} - \frac{1}{2} k_v \left(\frac{d}{dt}\left(\det(M) P^{-1}\right)\right) \dot{r}}{k_E}. \tag{139}$$

Based on the structure of (139), it is clear from (136) and (138) that $F(t)$ is constant, and we can now utilize similar arguments as given in the proof of Theorem 1 to prove the result given in (41). ■