Series Classes and Primes in Graphs

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Abstract

In 1996, H. Stark and A. Terras noted that a special type of closed path in undirected graphs (ones with no immediate "backtracking") correspond to primes in arithmetic. Given a graph G, let Δ_G be the greatest common divisor of the lengths of primes in G, and δ'_G be the g.c.d. of the lengths of series classes in G. In Stark and Terras' work on Siegel poles for the graph theoretic Ihara zeta function, a key lemma is that Δ_G equals either δ'_G or $2\delta'_G$ for all connected G which are not cycles or trees. We present a new proof of this relationship, as well as a new structural characterization of graphs with $\Delta_G = k$ for all positive integers k.

1 Introduction

Zeta and L-functions for number fields are key objects in studying the distribution of primes. More specifically, number theorists are interested in understanding the locations of the zeros of these functions. For the classic definitions and results in this area, see for example Neukirch [1]. Recently, Stark and Terras have defined analogues of these functions (and the necessary related objects, such as primes) for graphs [3, 4]. Since the graph zeta functions are reciprocals of polynomials, they have poles (not zeros), but understanding the locations of these special points is again crucial in discussions of prime distribution. An important result in number theory gives an explicit zero-free region for Dedekind zeta functions, with the possible exception of a single first order real zero (known as a Siegel zero) - for details, see Davenport [2]. For the graph theoretic Ihara zeta function, Stark and Terras ([5]) find a similar pole-free region, and define Siegel poles to be the exceptional poles on a circle in this region. A key reduction in their proof relies on a relationship between the lengths of primes and series classes. Here we present a new proof of this relationship (Theorem 2.6), as well as a new result (2.5) which gives a structural characterization for graphs where the g.c.d of the lengths of its primes is a given integer k. In order to state these results more exactly, we need a number of definitions.

First, we establish some graph theory terminology and well-known results. All graphs in this paper are undirected and finite. We write $G = (V, E)$ for a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The members of $E(G)$ (which may be a multi-set) are unordered pairs of vertices $e = \{u, v\}$, with u and v the *endpoints* of e. If $u = v$, we say e is a *loop*. The graph $G = (V, E)$ is a *subgraph* of $H = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$. The *degree* of a vertex v is the number of non-loop edges incident with v plus twice the number of loops at v. A *leaf vertex* has degree one. A graph has *multiple edges* if for some pair of distinct vertices u, v, there is more than one edge with endpoints u and v in G.

For a graph G, and a set of vertices $S' \subseteq V$, $G \backslash V = (V', E')$, is the subgraph defined by $V' = V - S$ and $E' = \{\{u, v\} \in E \mid u, v \in V'\}$. Analogously, for a set of edges $T \subseteq E$, the graph $G \backslash T = (V', E')$, is defined by $V' = V$, and $E' = E - T$.

A *walk* W is a sequence $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ where the v_i are vertices and the e_i are edges of G, and if v_i is adjacent to e_j in the sequence, v_i and e_j are incident in the graph. The *length* of W is n. We say W is a walk *between* v_0 and v_n. The walk W is *closed* if $v_0 = v_n$. For a walk W and positive integer k, we write W^k to denote the sequence consisting of W repeated k times. If we require $e_i \neq e_j$ and $v_i \neq v_j$ for all $i \neq j$ in a walk W, then W is a *path*. The graph G is *connected* if for any $u, v \in V(G)$, there is a path between u and v. Finally, an edge $e = \{u, v\}$ is a *cut-edge* of a graph G if $G \backslash \{e\}$ is not connected.

A *series class* of edges in a graph G is a set $S = \{e_1, \ldots, e_\ell\}$ of distinct edges, with e_i having endpoints v_i and v_{i+1} so that $v_i \neq v_j$ for $i \neq j$, with the property that $v_1, v_{\ell+1}$ have degree at least 3 and $v_2, v_3, \ldots, v_\ell$ have degree exactly 2. The *size* of a series class is the number of edges, $|S| = \ell$.

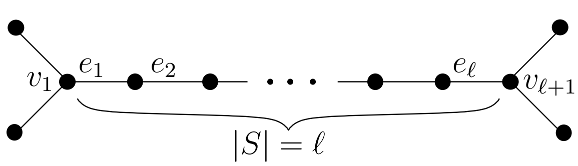

Figure 1: Example of a series class S

A graph G is a *subdivision* of the graph H if G can be obtained from H by replacing each edge of H by a non-empty series class of new edges. Note all vertices which are in G but not H are thus of degree two. We say G is a *k-subdivision* of H if each edge of H was replaced by a series class of size exactly k.

A *cycle* C is a closed walk with all vertices and edges distinct, except that the first and last vertices are the same. In other words, a set of distinct edges $E(C) = \{e_1, \ldots, e_\ell\}$ with $e_i = \{v_i, v_{i+1}\}$ so that $v_i \neq v_j$ for $i \neq j$, except $v_{\ell+1} = v_1$. We say the *length* of C is ℓ. Finally, a graph is a *tree* if it has no cycles.

Lemma 1.1. *In a graph G, if C and D are distinct cycles with $e \in E(C) \cap E(D)$ and $f \in E(C) \backslash E(D)$, then there is a cycle in $(C \cup D) \backslash \{e\}$ which contains the edge f.*

Corollary 1.2. *In a graph G, if C and D are distinct cycles with $f \in E(C) \backslash E(D)$, then there is a cycle in $(C \cup D) \backslash (C \cap D)$ which contains the edge f.*

A graph $G = (V, E)$ is *bipartite* if we can write $V = A \cup B$ with $A \cap B = \emptyset$ so that if $\{u, v\} \in E$, either $u \in A, v \in B$ or $u \in B, v \in A$. It is easily seen that G is bipartite if and only if G has no cycle of odd length.

Furthermore, we need a few definitions from Stark and Terras. Say a walk W in G is *primitive* if $W \neq U^m$ for any other walk U in G, with $m \geq 2$. A *prime* in a graph G is a primitive closed walk $v_0, e_1, \ldots, e_n, v_n$ with the property that $e_i \neq e_{i+1}$ for $i = 1, \ldots, n-1$ and $e_1 \neq e_n$. The length of a prime P is its length as a closed walk, and denoted $\nu(P)$. We can then define $\Delta_G := \gcd(\{\nu(P) \,|\, P$ a prime of $G\})$. Finally, let $\delta'_G := \gcd(\{|S|, S$ a series class in $G\})$ (as in Section 4 of [5]).

We can now state the relationship from Stark and Terras formally: for graphs G which are not cycles or trees, Δ_G is equal to either δ'_G or $2\delta'_G$. Our structural result gives a characterization of graphs with $\Delta_G = k$ as subdivisions of graphs with Δ_G at most 2, with a dependence on the parity of k.

2 Results

Lemma 2.1. *For any two graphs G, H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, any prime P in H is also a prime in G.*

Proof. This follows directly from the definition of a prime. Adding vertices and edges to the graph cannot change the properties of being closed or not using the same edge twice in a row. Similarly, if $P \neq U^m$ for any walk U in H with $m \geq 2$, then $P \neq U^m$ for all walks U in G, as any walk U' in G but not H contains a vertex or edge not in P.

Lemma 2.2. *Given a graph G with leaf vertices $V_L = \{w_1, \ldots, w_\ell\}$, then $\Delta_G = \Delta_{G \backslash V_L}$.*

Proof. Since Δ_G is defined to to be the g.c.d. of the lengths of the primes in G all we need to show is that P is a prime in G if and only if P is a prime in $G \backslash V_L$. By Lemma 2.1, it suffices to show any prime P in G is a prime in $G \backslash V_L$. Assume w_i is a leaf vertex, adjacent to edge e. Then any closed walk passing through w_i must use the edge e twice in a row, which means the walk cannot be prime, by definition. If P contains no vertices from V_L, it remains unchanged when we delete V_L, and is still prime. Thus all primes in G are also primes in $G \backslash V_L$.

Lemma 2.3. *Given a graph G and a cycle C, C is a prime of G.*

Proof. By definition, C is a closed walk. We need to check that two sequential edges in C are never the same, and that $C \neq U^m$ for any walk U with $m \geq 2$. Both conditions follow directly from the fact that all edges in C are distinct. C therefore satisfies all the conditions for being a prime of G.

Theorem 2.4. *Let G be a graph with $\Delta_G = k$. Then for any series class S in G, k divides $2|S|$.*

Proof. Throughout the proof, we will consider a generic series class $S = \{e_1, \ldots, e_\ell\}$ of distinct edges, with e_i having endpoints v_i and v_{i+1} . We also use the fact that v_1 and v_ℓ have degree at least three by choosing two arbitary neighbors of each outside of S (namely a_1, a_2, b_1, and b_2). Many diagrams depict only these two neighbors, though more could exist and are not excluded in the proof. Finally, we will repeatedly choose closed walks and show they are primes. Note that any closed walk W including some edge e exactly one time must be primitive (if $W = U^m$ with $m \geq 2$, $e \in U$ implies e appears m times in W). When we construct our closed walks, the first edge used will always appear only once in the walk, thus they are primitive, and no further proof of this will be provided. We now break our proof into several cases:

Case 0: G has a loop, multiple edges, or a leaf vertex.
If G has a loop edge e_ℓ, that loop (along with the vertex it is incident on) is a prime P_ℓ, with $\nu(P_\ell) = 1$. Thus Δ_G divides one, and must in fact equal one. Clearly, 1 divides the length of every series class S in G, thus Δ_G divides $2|S|$.

If G has more than one edge between u and v, let e and f be two such edges. Then $P = u, e, v, f, u$ is cycle, and thus a prime in G, with $\nu(P) = 2$, so $\Delta_G \in \{1, 2\}$. Again, it is clear that either value will divide $2|S|$ for any series class S, as both divide two.

If G has a leaf vertex v, it cannot be part of any series class, and thus Lemma 2.2 allows us to consider $G \setminus \{v\}$ instead.

Case 1: There are cycles C_1 and C_2 passing through v_1 and $v_{\ell+1}$, respectively, so that $C_i \cap S = \emptyset$ for $i = 1, 2$. We do not require C_1 and C_2 be disjoint, thus this includes the case when there are paths P_1 from a_1 to b_i and P_2 from a_2 to b_j (where $i \neq j$, and P_1, P_2 are edge-disjoint from S).

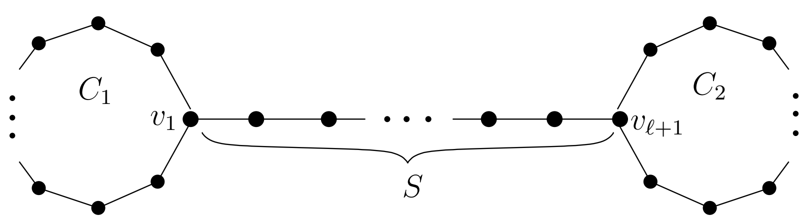

Figure 2: Case 1 - Cycles through v_1 and $v_{\ell+1}$ not using S

Consider the primitive closed walk Q starting at v_1, going around C_1, along S, around C_2, then back along S to v_1. We can now show that Q is prime by noting the only edges used twice are those in S, and all the edges in C_2 are in the sequence between two occurrences of an edge in S, so $e_i \neq e_{i+1}$ for all i, and $e_1 \neq e_{\nu(Q)}$ as $e_1 \in C_1$ and $e_{\nu(Q)} \in S$. Since Q is prime, Δ_G divides $\nu(Q) = \nu(C_1) + 2|S| + \nu(C_2)$. We now note that since every cycle is a prime by Lemma 2.3, by definition, Δ_G divides $\nu(C)$ for all cycles C in G. Thus Δ_G divides $\nu(Q)$ implies Δ_G divides $2|S|$, as desired.

We now consider two types of series classes: those of cut-edges in G, and those of non-cut-edges in G (one can easily argue that if one edge of a series class S is a cut-edge, then all edges in S must be cut-edges).

Case 2: S is a series class of non-cut-edges, i.e. there is a cycle C which includes S, and thus a path between some outneighbor of v_1 and some outneighbor of $v_{\ell+1}$, forming a cycle C as shown in Figure 3 (without loss of generality, we may assume a_2 and b_2 are connected by this path).

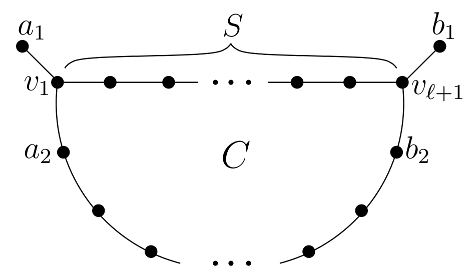

Figure 3: Case 2 - Series Class of non-cut-edges

We know that not both v_1 and $v_{\ell+1}$ are in cycles disjoint from S by Case 1, so WLOG, assume $v_{\ell+1}$ is not in such a cycle. We wish to show that the edge $f = \{v_{\ell+1}, b_1\}$ must then be a cut-edge. Suppose not (i.e. there is a cycle containing f). Every cycle through $v_{\ell+1}$ must contain S by assumption, so there is a cycle D containing the edge f and S. Now by Corollary 1.2, applied to C and D (which are distinct, since f is not in C), there is a cycle in G which contains f and not S. This is a contradiction to our assumption. Thus f must be a cut-edge of G. Follow any walk starting at $v_{\ell+1}$ with first edge f. Since G has no leaf vertices by Case 0, and there is no path from b_1 to v_1 (or we would be in case 1, as the walk had to go through a neighbor of v_1 which we could call a_1, as a_1 was arbitrary), we must eventually come back to a vertex we've already encountered. Stop the first time this occurs, say at v^*, creating a path P with a cycle C_1 at the end, as shown in Figure 4.

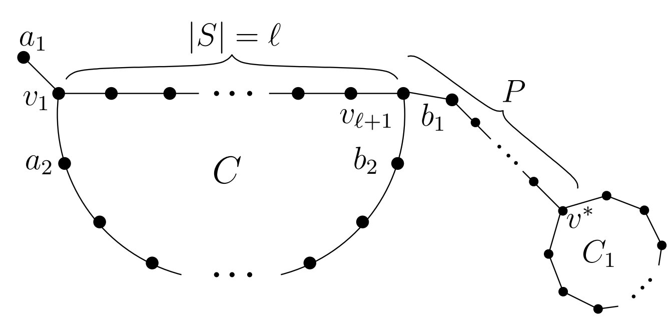

Figure 4: Case 2 - Series Class of non-cut-edges with P and C_1

Consider the primitive closed walk Q starting at v^*, going around C_1, along P to $v_{\ell+1}$, around C, then back along P to v^*. We can show Q is prime by noting the only edges used twice are those in P, and all the edges in C are in the sequence in between two occurrences of an edge in P, so $e_i \neq e_{i+1}$ for all i, and $e_1 \neq e_{\nu(Q)}$ as $e_1 \in C_1$, but $e_{\nu(Q)} \in P$. Since Q is prime, Δ_G divides $\nu(Q) = \nu(C_1) + 2|P| + \nu(C)$. As before, Δ_G divides $\nu(C)$ for all cycles C in G, thus Δ_G divides $\nu(Q)$ implies Δ_G divides $2|P|$.

We now consider what happens at v_1:

Case 2a: If there is a cycle D containing v_1 so that $D \cap S = \emptyset$, let R be the primitive closed walk starting at v^*, going around C_1, along P, then along S to v_1, around D, then back along S and P to v^*. R satisfies the properties of being a prime by construction. Now, Δ_G divides $\nu(R) = \nu(C_1) + 2|P| + 2|S| + \nu(D)$. We already know Δ_G divides $2|P|$ and $\nu(C)$ for all cycles C, thus Δ_G must divide $2|S|$, as desired.

Case 2b: Otherwise, we can again argue that we get a path P' with a cycle C_2 attaching at $w*$, analogous to the one constructed at b_1, with Δ_G dividing $2|P'|$. In this case, let R be the primitive closed walk starting at v^*, going around C_1, along P, then along S to v_1, along P' to w^*, around C_2, then back along P', S and P to v^*. R again satisfies the properties of being a prime by construction. Now, Δ_G divides $\nu(R) = \nu(C_1) + 2(|P| + |S| + |P'|) + \nu(C_2)$. We already know Δ_G divides $2|P|$, $2|P'|$, and $\nu(C)$ for all cycles C, thus Δ_G must divide $2|S|$, finishing the proof for Case 2.

Case 3: S is a series class of cut-edges, i.e., every path from the neighbors of v_1 to the neighbors of $v_{\ell+1}$ uses S.

Again, not both v_1 and $v_{\ell+1}$ are in cycles disjoint from S by Case 1, so WLOG, assume $v_{\ell+1}$ is not in such a cycle. Now, we argue that $e = \{v_{\ell+1}, b_2\}$ and $f = \{v_{\ell+1}, b_1\}$ must then be cut-edges. WLOG, suppose there is a cycle D containing f. By assumption, D contains S. But then, D contains a neighbor of v_1, and the edges in S are not cut-edges, a contradiction. As in Case 2, we can construct paths P_i from $v_{\ell+1}$ through b_i with cycles C_i attaching at w_i, as shown in Figure 5.

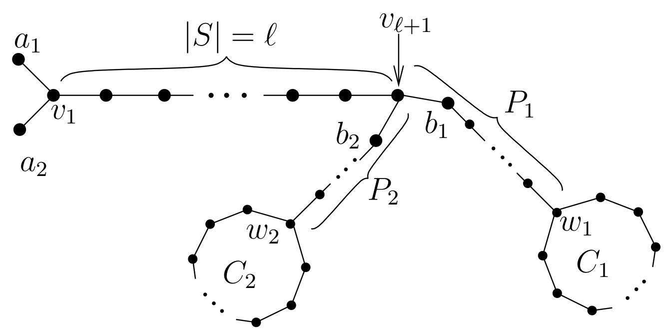

Figure 5: Case 3 - Series Class of cut-edges

There are two sub-cases to consider, based on what happens at v_1:

Case 3a: v_1 is in a cycle C, as shown in Figure 6.
In this case, we argue that Δ_G divides $2(|P_1| + |P_2|)$ using the primitive closed walk R_1 starting at w_1, going around C_1, down P_1 to $v_{\ell+1}$, along P_2 to w_2, and back along P_2 and P_1 to w_1. No

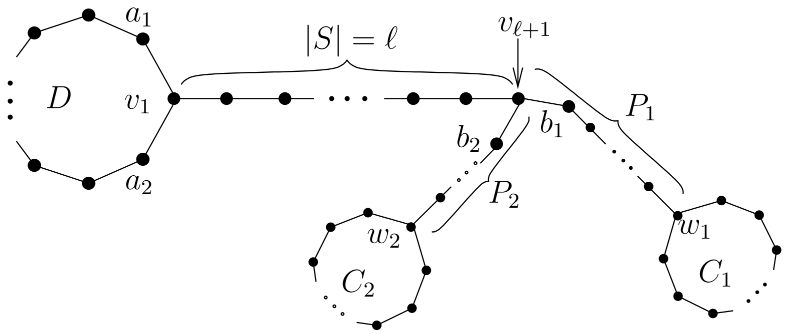

Figure 6: Case 3a - Series Class of cut-edges with a cycle D at v_1

two consecutive edges of R_1 are the same by construction, and the first edge and last edge are different, since one is in C_1 and the other in P_1, proving that R_1 is prime. By definition, Δ_G divides $\nu(R_1) = \nu(C_1) + 2(|P_1| + |P_2|) + \nu(C_2)$. Since C_1 and C_2 are primes, this gives the desired result.

We then show Δ_G divides $2|S|$ by considering the primitive closed walk R_2 starting at w_1, going around C_1, down P_1 to $v_{\ell+1}$, along P_2 to w_2, and back along P_2 to $v_{\ell+1}$. R_2 then continues along S to v_1, goes around D, then comes back along S and P_1 to w_1. No two consecutive edges of R_2 are the same by construction, and the first edge and last edge are different, since one is in C_1 and the other in P_1, proving that R_1 is prime. By definition, Δ_G divides $\nu(R_2) = \nu(C_1) + 2(|P_1| + |P_2| + |S|) + \nu(C_2) + \nu(D)$. Since C_1, C_2, and D are primes, and we already showed Δ_G divides $2(|P_1| + |P_2|)$, this completes our case.

Case 3b: v_1 is not in a cycle, so a_1 and a_2 give rise to paths and cycles Q_1, Q_2, D_1, D_2, as shown in Figure 7.

In this case, we argue that Δ_G divides $2(|P_1| + |P_2|)$ analogously to Case 3a. Similarly, Δ_G

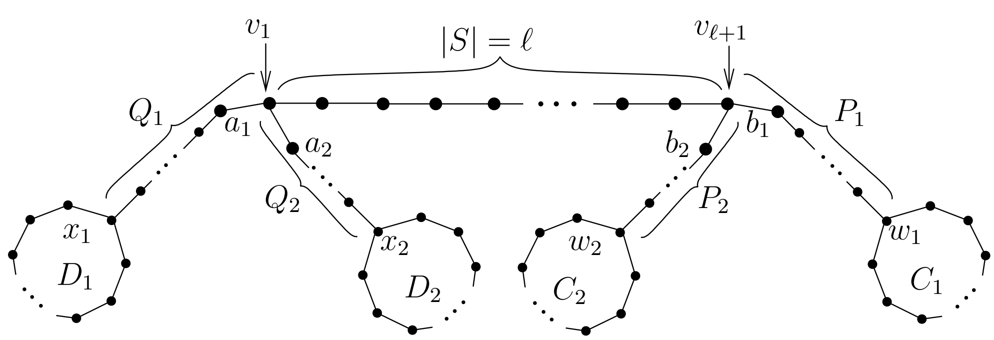

Figure 7: Case 3b - Series Class of cut-edges with no cycle at v_1

divides $2(|Q_1| + |Q_2|)$. We then show Δ_G divides $2|S|$ by considering the primitive closed walk R starting at w_1, going around C_1, down P_1 to $v_{\ell+1}$, along P_2 to w_2, and back along P_2 to $v_{\ell+1}$. R then continues along S to v_1, along Q_1 to x_1, around D_1, then back along Q_1 to x_1. R finishes by going along Q_2 to x_2, around D_2, and back along Q_2, S and P_1 to w_1. No two consecutive edges of R are the same by construction. The first edge and last edge are different, since one is in C_1 and the other in P_1, proving that R_2 is prime. By definition, Δ_G divides

$\nu(R) = \nu(C_1) + 2(|P_1| + |P_2|) + \nu(C_2) + 2|S| + 2(|Q_1| + |Q_2|) + \nu(D_1) + \nu(D_2)$. Since C_1, C_2, D_1, and D_2 are primes, and we already showed Δ_G divides $2(|P_1| + |P_2|)$, and $2(|Q_1| + 2|Q_2|)$, this completes the proof that Δ_G divides $2|S|$ for any series class S in G.

Corollary 2.5. *Let G be a graph with no multiple edges or loops, and $\Delta_G = k$. If k is even, then G is a $\frac{k}{2}$-subdivision of a bipartite graph H with $\Delta_H = 2$. If k is odd, G is a k-subdivision of a graph H with $\Delta_H = 1$.*

Proof. From Theorem 2.4, we have that k divides twice the length of each series class in G. If k is odd, k thus divides the length of each series class, and if k is even, $\frac{k}{2}$ does. Define δ_G to be the largest positive integer dividing the length of all series classes in G.

Let H be the graph obtained from G by replacing each series class $S = v_1, e_1, \ldots, e_{m\delta_G}, v_{m\delta_G+1}$ of length $m\delta_G$ ($m \geq 1$) by $S' = v_1, f_1, w_1, \ldots, , w_{m-1}, f_m, v_{m\delta_G+1}$, a series class of length m. Now G is obtained from H replacing every edge by a series class of size δ_G, that is G is a δ_G-subdivision of H.

Consider Δ_H. We can see that $r = \Delta_H \delta_G$ divides the length of every series class in G. If $\Delta_H > 2$, this is a contradiction, as r divides the length of every series class and thus the length of every prime, yet $r > \Delta_G$. Additionally, when k is odd, we get a contradiction when $\Delta_H > 1$, as $\delta_G = \Delta_G$. This shows that for odd k, G is a k-subdivision of a graph H with $\Delta_H = 1$ (since $\Delta_H \in \mathbb{Z}^+$).

Thus we may assume k is even. Let P be a prime in H. Then $P = S_1, S_2, \ldots, S_n$ for some (not neccessarily distinct) series classes in H, and $\nu(P) = \sum_{i=1}^{n} |S_i|$. Each series class S in H came from a series class T in G of size $|T| = \frac{k}{2}|S|$. Then $Q = T_1, T_2, \ldots, T_n$ was a prime in G (one can check that this will be primitive and have no repeated consecutive edges (including first and last) as long as P was prime). Then $\Delta_G = k$ divides $|Q| = \frac{k}{2}|P|$. This implies that $|P|$ must be even for all primes in H, so $\Delta_H \geq 2$. By Lemma 2.3, all cycles in H have even length, and thus H is bipartite. This proves that if k is even, G is a $\frac{k}{2}$-subdivision of a bipartite graph H with $\Delta_H = 2$, completing our corollary.

Theorem 2.6. *For all connected graphs G such that G is not a tree or a cycle, $\Delta_G = \delta'_G$ or $\Delta_G = 2\delta'_G$.*

Proof. We first note that Theorem 2.4 implies immediately that $\Delta_G | 2\delta'_G$ since δ'_G is the greatest common divisor of the sizes of series classes. So it suffices to show that $\delta'_G | \Delta_G$. If G has a loop $\delta'_G = \Delta_G = 1$, satisfying 2.6, so we may assume G is loopless. Now, as in [5], we proceed by first proving δ' is equivalent to the invariant $\delta = \gcd(\{length(P) : P \text{ is a path between two vertices of degree at least } 3\})$. It is clear from the definitions that $\delta_G | \delta'_G$, and $\delta'_G | \delta_G$ because any length considered in computing δ is a sum of lengths considered for δ'_G. We now note that $\delta_G | \Delta_G$, every cycle in G has a vertex of degree at least three (as G is not a cycle, and is connected), and this completes our proof.

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