Model Derivation

Development of Three-Dimensional Model

Dynamic three-dimensional finite difference computer models were developed for each considered wall assembly. A heat-conduction, finite-difference computer code called HEATING 7.2 [Childs, 1993] was used for this analysis. HEATING 7.2 generates overall resistances, capacitances, response factors and structure factors. To generate a 3-D model of a wall assembly means to describe its geometry and material properties in the frames of HEATING 7.2. The node density was varied to provide an accurate result in a reasonable run-time. A description of HEATING 7.2 is included in Appendix D.

Response factors for the three-dimensional models were calculated with the help of the HEATING 7.2. The model assumes boundary conditions of the first kind. All envelope details were analyzed in steady-state and dynamic conditions represented by a unit triangular temperature excitation. The complex assemblies were thus simulated by the dynamic conditions represented by:

Time: 0 hours	Temperature: 0.0
Time: 1.0 hours	Temperature: 1.0
Time: 2.0 hours	Temperature: 0.0

Linear course between time 0 and 1, and also between 1 and 2 is assumed.

Given the overall resistance value and the normal response factors generated from HEATING 7.2, one may calculate dimensionless response factors. Dimensionless response factors represent ratios of the wall responses to unit triangular temperature excitations, to the steady state heat flux, due to the unit boundary temperature difference, equal to 1/R.

For plane walls, response factors with sufficiently high indices, above some M, satisfy the condition:

$$\frac{X_{m+1}}{X_m} = \frac{Y_{m+1}}{Y_m} = const = \alpha, \qquad m > M \tag{1}$$

$$\alpha = e^{-\frac{\delta}{\tau_1}} \tag{2}$$

where τ_1 is the first, of largest value, time constant of the wall. Response factors for three-dimensional assemblies, in general, have similar properties; however their ratios show small variations even for large indices, where they drop several orders of magnitude, as compared with the first ones. Transfer functions of the first order, defined as:

$$X_0^{'} = X_0 \,, \qquad Y_0^{'} = Y_0 \tag{3}$$

$$X_n^{'} = X_n - \alpha X_{n-1} \,, \qquad Y_n^{'} = Y_n - \alpha Y_{n-1} \,; \qquad n \geq 1 \tag{4}$$

were calculated with α assumed as the average value of ratios of the response factors X_n and Y_n with highest indices. The quantity τ_1 was calculated as –1/ln(α), to compare with the one-dimensional, equivalent wall model.

The response factors are used as input data to determine the z-transfer function coefficients from the primarily infinite set of linear equations. The conduction z-transfer function coefficients are determined by solving the set of linear equations, which constitute relationships with the response factors:

$$b_n = R \sum_{k=0}^{n} Y_{n-k} d_k, \qquad c_n = R \sum_{k=0}^{n} X_{n-k} d_k \tag{5}$$

and compatibility equations:

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} d_n \tag{6}$$

see Equations A13, A14, A15.

Conditions imposed by the structure factors on the z-transfer function coefficients b_n, c_n, d_n Equations A35, A36 [Kossecka, 1998], were also included as subsidiary equations:

$$\sum_{n=1}^{\infty} n b_n - \sum_{n=1}^{\infty} n d_n = \frac{RC}{\delta} \varphi_{ie} \sum_{n=0}^{\infty} d_n \tag{7}$$

$$\sum_{n=1}^{\infty} nc_n - \sum_{n=1}^{\infty} nd_n = -\frac{RC}{\delta}\varphi_{ii}\sum_{n=0}^{\infty} d_n \tag{8}$$

For each case, different kinds of cut-off were considered and minimization procedures were applied to satisfy, as best as possible, compatibility conditions. In general, minimization methods deliver many solutions; thus a good mathematical solver and knowledge of the heat transfer theory is necessary to select the appropriate solution. The z-transfer coefficients were calculated with the help of a mathematical computer tool (MathCAD).

Development of Equivalent Wall Model

An alternative way to perform whole building energy simulations is to employ a one-dimensional model. In the equivalent wall method, developed by Kossecka and Kosny [1996, 1997], a multi-layer structure is created with the same resistance, capacity and structure factors, as in the three-dimensional wall assembly.

The equivalent wall method is incorporated into a specialized version of HEATING 7.2, called EQV_WALL. This computer tool aids in the modeling of dynamic thermal performance of complex wall systems with significant thermal mass, and is utilized for these simulations.

The equivalent wall method uses, as its mathematical basis, conditions imposed on the response factors and z-transfer function coefficients by the thermal structure factors. Thermal structure factors are dimensionless quantities representing the fractions of heat stored in the wall volume, in transition between two different states of steady heat flow, which are transferred across each wall surface, see (A19) and (A20). Relationships of the structure factors with other dynamic thermal characteristics of walls are presented below.

Structure factors, φ_{ii} and φ_{ie}, for a composite wall element, adiabatically cut off from the surroundings, are given by:

$$\varphi_{ie} = \frac{1}{C}\int_V \rho\, c_p \theta\left(1-\theta\right) dv \tag{9}$$

$$\varphi_{ii} = \frac{1}{C}\int_V \rho\, c_p \left(1-\theta\right)^2 dv \tag{10}$$

where *C* is the total thermal capacity of the wall element of volume *V*:

$$C = \int_V \rho\, c_p \, dv \tag{11}$$

and θ is the dimensionless temperature for the problem of steady-state heat transfer, for ambient or surface temperatures T_i = 0 and T_e = 1.

Structure factors φ_{ii} and φ_{ie} for a wall of thickness L, composed of n plane homogeneous layers, numbered from 1 to n with layer 1 at the interior surface, are given as follows:

$$\varphi_{ii} = \frac{1}{R^2 C} \sum_{m=1}^{n} C_m \left[\frac{R_m^2}{3} + R_m R_{m-e} + R_{m-e}^2 \right] \tag{12}$$

$$\varphi_{ie} = \frac{1}{R^2 C} \sum_{m=1}^{n} C_m \left[-\frac{R_m^2}{3} + \frac{R_m R}{2} + R_{i-m} R_{m-e} \right] \tag{13}$$

where R is the total wall's thermal resistance per unit cross section area, R_m and C_m denote the thermal resistance and capacity of the m-th layer, whereas $R_{o\text{-}m}$ and $R_{m\text{-}L}$ denote the resistances for heat transfer from surfaces of the m-th layer to the interior and exterior surface of the wall, respectively. Layers with C_m=0 represent contact resistances. If the surface film resistances, R_i and R_e are included, then $R_{o\text{-}m}$ and $R_{m\text{-}L}$ are replaced by $R_{i\text{-}m}$ and $R_{m\text{-}e}$; resistances for heat transfer from surfaces of the m-th layer to inner and outer surroundings, respectively.

Conditions imposed by the structure factors on the response factors are as follows [Kossecka and Kosny 1996, 1997]:

$$\delta \sum_{n=1}^{\infty} n\, X_n = -C\, \varphi_{ii} \tag{14}$$

$$\delta \sum_{n=1}^{\infty} n Y_n = C\, \varphi_{ie} \tag{15}$$

Thermal structure factors are calculated using (9) and (10), or (14) and (15), using the response factors generated by the three-dimensional model as ‘inputs’.

There are several ways the equivalent wall technique may generate a simple one-dimensional multi-layer structure with the same thermal properties and dynamic behavior as the actual wall. The first step is to assume some number of ‘equivalent’ layers for the wall structure. In this study, three-layer equivalent wall models were developed for all wall assemblies considered. It was found that a three-layer structure produced the best results for all the wall assemblies considered. A simple way to solve for equivalent layer properties is to first generate, randomly or with some logic, a set of resistances R_n (or capacitances C_n) for each layer, then seek the capacitances C_n (or resistances R_n) to satisfy Equations (12) and (13). The thermal structure factors and overall R-value must match those for the 3-D wall assembly. Thermo-physical properties of the layers may then be established, if necessary, to match R_n and C_n values and total thickness of the wall. A detailed description of the equivalent wall procedure is included in Appendix B.

Response factors are calculated for the equivalent wall to provide a clear physical interpretation and a comparison with the three-dimensional model response factors. Response factors illustrate better the similarities and differences in dynamic thermal response than does comparison of z-transfer function coefficients.

Given that the equivalent wall is a one-dimensional model, response factors and z-transfer function coefficients are calculated using the standard, Laplace transform method, [Kusuda, Stephenson and Mitalas, Clarke]. Note that the method of obtaining z-transfer function coefficients described in the development of the three-dimensional model may also be employed for the one-dimensional equivalent wall. However, this would be disadvantageous since its mathematical basis constitutes a series of linear equations, which must be solved using minimization procedures and assumes some level of heat transfer knowledge on behalf of the user.

The development of the equivalent wall model is an iterative procedure. By adjusting the number and capacitances of equivalent wall layers, the equivalent wall model can generate results that more closely resemble those of the 3-D model.

Therefore, the relationship between the thermal structure factors and response factors play a pivotal role in the development of the equivalent wall model. The thermal structure factors, together with total thermal resistance R and capacity C, determine the dynamic thermal properties of a wall element – through the conditions they impose on the response factors. Those conditions however, do not determine the response factors in a unique way, but rather play the role of constraints. The equivalent wall model is not unique; however different equivalent walls have, in general, very similar dynamic thermal properties. One may then examine generated models to choose the best one.