APPENDIX A.

RELATIONSHIPS BETWEEN DYNAMIC THERMAL CHARACTERISTICS OF COMPOSITE WALL ASSEMBLIES USED TO DETERMINE CONDUCTION TRANSFER FUNCTION COEFFICIENTS

Three-dimensional Model

Relationship between response factors and z-transfer function coefficients

Three-dimensional z-transfer function coefficients may be used effectively for the dynamic simulations of the heating and cooling loads if the whole building simulation program enables this type of wall data input. Method of derivation of the conduction z-transfer function coefficients from the response factors for three-dimensional wall assemblies is presented.

It is assumed that the response factors, which represent surface heat flow due to triangular temperature excitations, at discrete time instants, may be calculated with the help of a computer code to simulate three-dimensional heat conduction. They are used as the “input data”, to determine z-transfer function coefficients from the set of linear equations, which includes relationships with the response factors and compatibility conditions. This infinite set of equations is to be solved applying cut-off and using minimization procedures.

In terms of the response factors, heat flux across the interior surface of a wall element at time instant $n\delta$, $Q_{i,n\delta}$, can be represented as follows [Kusuda 1969, Clarke 1985]:

$$Q_{i,n\delta} = \sum_{k=0}^{n} \left[X_n T_{i,(n-k)\delta} - Y_n T_{e,(n-k)\delta} \right] \qquad \text{(A1)}$$

where R is the resistance of the wall, $\{T_{i,n\delta}\}$ and $\{T_{e,n\delta}\}$ are sequences of the ambient temperatures values, and $\{X_n\}$ and $\{Y_n\}$ are sequences of the response factors. As far as three-dimensional problems are considered, all quantities are to be understood as average values over the surface of a wall element adiabatically cut from the surroundings.

The z-transform of the interior heat flux, $Z\{Q_i\}$ is related to the z-transforms of the interior and exterior temperature, $Z\{T_i\}$ and $Z\{T_e\}$, by the following equation [Jury 1964]:

$$Z\{Q_i\} = Z\{X\} Z\{T_i\} - Z\{Y\} Z\{T_e\} \qquad \text{(A2)}$$

where $Z\{X\}$ and $Z\{Y\}$ are the z-transforms of the sequences of the response factors X_n and Y_n.:

$$Z\{Q_i\} = \sum_{n=0}^{\infty} Q_{i,n\delta} z^{-n}, \quad Z\{X\} = \sum_{n=0}^{\infty} X_n z^{-n}, \quad Z\{Y\} = \sum_{n=0}^{\infty} Y_n z^{-n} \tag{A3}$$

The condition, response factors X_n and Y_n should satisfy:

$$\sum_{n=0}^{\infty} X_n = \sum_{n=0}^{\infty} Y_n = \frac{1}{R} \tag{A4}$$

is equivalent to the following condition for the z-transforms $Z\{X\}$ and $Z\{Y\}$:

$$\lim_{z \to 1} Z\{X\} = \lim_{z \to 1} Z\{Y\} = \frac{1}{R} \tag{A5}$$

Let now $Z\{X\}$ and $Z\{Y\}$ be given as the quotients:

$$Z\{Y\} = \frac{1}{R} \frac{N_b(z)}{D(z)}, \qquad Z\{X\} = \frac{1}{R} \frac{N_c(z)}{D(z)} \tag{A6}$$

where

$$N_b(z) = \sum_{n=0}^{M_b} b_n z^{-n}, \quad N_c(z) = \sum_{n=0}^{M_c} c_n z^{-n}, \quad D(z) = \sum_{n=0}^{M_d} d_n z^{-n} \tag{A7}$$

b_n and c_n are the normalized conduction z-transfer function coefficients, which correspond to the coefficients b_n, c_n from the ASHRAE Handbook Fundamentals [1989, 1997] multiplied by R. M_b, M_c and M_d represent numbers of coefficients significantly different from zero.

Equation (A2) can be rewritten in the form:

$$D(z) Z\{Q_i\} = \frac{1}{R} \left[N_c(z) Z\{T_i\} - N_b(z) Z\{T_e\} \right] \tag{A8}$$

Equation (A1) for $Q_{i,n\delta}$, for n exceeding M_b, M_c and M_d, assuming $d_0 = 1$, is now replaced by [Stephenson and Mitalas, 1971]:

$$Q_{i,n\delta} = \frac{1}{R} \left[\sum_{m=0}^{M_c} c_m T_{i,(n-m)\delta} - \sum_{m=0}^{M_b} b_m T_{e,(n-m)\delta} \right] - \sum_{m=1}^{M_d} d_m Q_{i,(n-m)\delta} \tag{A9}$$

Relationships (A6) for the z-transforms, rewritten in the form:

$$N_b(z) = R \cdot Z\{Y\} D(z), \quad N_c(z) = R \cdot Z\{X\} D(z) \tag{A10}$$

are equivalent to the convolution type relationships between the response factors Y_n, X_n and conduction transfer function coefficients b_n, c_n, d_n:

$$b_n = R \sum_{k=0}^{n} Y_{n-k} d_k, \qquad c_n = R \sum_{k=0}^{n} X_{n-k} d_k \tag{A11}$$

The conditions (A5), for $Z\{Y\}$ and $Z\{X\}$ now have the form:

$$\left.\frac{N_b(z)}{D(z)}\right|_{z=1} = \left.\frac{N_c(z)}{D(z)}\right|_{z=1} = 1 \tag{A12}$$

which gives the following conditions for the z-transfer function coefficients:

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} d_n \tag{A13}$$

Determining the z-transfer function coefficients from the response factors

On the basis of Equations (A11) and (A13) one may try to determine z-transfer function coefficients from the response factors Y_n, X_n. This is the most straightforward method; z-transfer function obtained in this way are expected to “exactly” reproduce the output for any input function composed of straight-line segments, joining the coordinates that correspond to its z-transform coefficients.

Assuming that values of the coefficients with indices above some n are negligibly small and $d_0 = 1$, we obtain the following set of linear equations:

$$b_0 + b_1 + b_2 + b_3 + \ldots.. + b_n = 1 + d_1 + d_2 + d_3 + \ldots.. + d_n \tag{A14.1}$$

$$b_0 = R Y_0 \tag{A14.2}$$

$$b_1 = R(Y_1 + Y_0 d_1) \tag{A14.3}$$

$$b_2 = R(Y_2 + Y_1 d_1 + Y_0 d_2) \tag{A14.4}$$

$$b_3 = R\left(Y_3 + Y_2 d_1 + Y_1 d_2 + Y_0 d_3\right) \quad \text{(A14.5)}$$

.....

$$b_n = R\left(Y_n + Y_{n-1} d_1 + Y_{n-2} d_2 + Y_{n-3} d_3 + \ldots\ldots + Y_0 d_n\right) \quad \text{(A14.n)}$$

$$0 = R\left(Y_{n+1} + Y_n d_1 + Y_{n-1} d_2 + Y_{n-2} d_3 + \ldots\ldots + Y_1 d_n\right) \quad \text{A14.n+1)}$$

$$0 = R\left(Y_{n+2} + Y_{n+1} d_1 + Y_n d_2 + Y_{n-1} d_3 + \ldots\ldots + Y_2 d_n\right) \quad \text{(A14.n+2)}$$

.....

$$c_0 + c_1 + c_2 + c_3 + \ldots\ldots + c_n = 1 + d_1 + d_2 + d_3 + \ldots\ldots + d_n \quad \text{(A15.1)}$$

$$c_0 = R\, X_0 \quad \text{(A15.2)}$$

$$c_1 = R\left(X_1 + X_0 d_1\right) \quad \text{(A15.3)}$$

$$c_2 = R\left(X_2 + X_1 d_1 + X_0 d_2\right) \quad \text{(A15.4)}$$

$$c_3 = R\left(X_3 + X_2 d_1 + X_1 d_2 + X_0 d_3\right) \quad \text{(A15.5)}$$

.....

$$c_n = R\left(X_n + X_{n-1} d_1 + X_{n-2} d_2 + X_{n-3} d_3 + \ldots\ldots + X_0 d_n\right) \quad \text{(A15.n)}$$

$$0 = R\left(X_{n+1} + X_n d_1 + X_{n-1} d_2 + X_{n-2} d_3 + \ldots\ldots + X_1 d_n\right) \quad \text{(A15.n+1)}$$

$$0 = R\left(X_{n+2} + X_{n+1} d_1 + X_n d_2 + X_{n-1} d_3 + \ldots\ldots + X_2 d_n\right) \quad \text{(A15.n+2)}$$

.....

When the structure factors are calculated together with the resistance and the response factors, one may use conditions imposed by the structure factors on the z-transfer function coefficients, see Equations (A31) and (A32) below, as subsidiary equations.

One may use more equations than the number of unknowns and apply minimizing procedures to get the solution. Maximum indices M_b, M_c, M_d, of the coefficients b_n, c_n, d_n, depend on the specific dynamic thermal properties of given wall assembly. In general, total number of the z-transfer function coefficients increases with the resistance and

mass of the wall; however it is not the rule. Trying different kinds of the cut-off one should control the following quantity:

$$E_r = \left[\sum_{n=0}^{M_b} b_n - \sum_{n=0}^{M_c} c_n \right] \times \left[\sum_{n=0}^{M_d} d_n \right]^{-1} \qquad \text{(A16)}$$

E_r is the relative error of the heat flux in steady state conditions, simulated using the z-transfer function method.

Solving back equations (A11) for Y_n, X_n , with $d_0 = 1$, gives the recurrence formulae, which may be used to additionally verify the solution obtained for the z-transfer function coefficients:

$$Y_0 = \frac{b_0}{R} \quad , \qquad X_0 = \frac{c_0}{R} \qquad \text{(A17)}$$

$$Y_n = \frac{b_n}{R} - \sum_{k=1}^{n} Y_{n-k} d_k, \qquad X_n = \frac{c_n}{R} - \sum_{k=1}^{n} X_{n-k} d_k; \qquad n \geq 1 \qquad \text{(A18)}$$

Equivalent wall model

This discussion presents the idea of the “thermally equivalent wall”; a plane multi-layer structure with dynamic characteristics similar to those of a complex structure in which three-dimensional heat flow occurs. The effect of internal thermal structure on the dynamic characteristics of walls is analyzed below. It introduces the idea of structure factors and shows the conditions they impose on response factors.

Relationships between response factors and thermal structure factors

Relationships between the structural and dynamic thermal characteristics of building walls are analyzed below to answer the question how should the response factors (or transfer function coefficients) be modified to account for the effects of thermal bridges in computerized whole building energy calculations

The simplest method is to calculate the overall resistance of a wall with imperfections - solving the steady state heat transfer problem - and multiply response factors by the resulting correction factor. This approach is suitable for light walls for which storage effects are insignificant.

However, imperfections in plane walls do not only change the resistance of the walls but also modify their dynamic properties, which may be represented by response factors in computer whole building energy simulations. To account for this effect, the general conditions between structural and dynamic characteristics for walls must be noted. Such conditions follow from the asymptotic formulae for the heat flow across the surfaces of the separated wall element due to temperature difference on its two sides.

Consider the heat flow through an element of a building envelope of complex material and geometrical structure. The element is embedded in a plane wall, homogeneous in every cross section and parallel to the wall surfaces. It is assumed that the thermophysical properties of the structure—thermal conductivity k, density ρ and specific heat c—are constant in time. The element, together with its nearest neighborhood, is represented by the region D. Region D is bounded by the inner surface facing room temperature T_i; the outer surface facing environmental temperature T_e; and the adiabatic surface of the cut which separates it from those parts of the wall where the heat flow can be considered as one-dimensional.

Asymptotic expressions for total heat flow have been developed, based on the following set of conditions:

- across the internal and external surface of a wall element
- time interval [0, t]
- sufficiently large t
- ambient temperatures held constant at T_i and T_e
- initial conditions zero

The equations have the following simple form [Kossecka 1992, 1996, 1998, Kossecka and Kosny 1996, 1997]:

$$\int_0^t Q_i(t')dt' = \frac{t}{R}[T_i - T_e] + C\varphi_{ii}T_i + C\varphi_{ie}T_e \tag{A19}$$

$$\int_0^t Q_e(t')dt' = \frac{t}{R}[T_i - T_e] - C\varphi_{ie}T_i - C\varphi_{ee}T_e \tag{A20}$$

where t' is the integration variable, t denotes time and R and C are the total thermal resistance and capacity, respectively, of the wall element of volume V, adiabatically cut from the surroundings:

$$C = \int_V \rho c_p \, dv \tag{A21}$$

whereas φ_{ii}, φ_{ie} and φ_{ee} are given by:

$$\varphi_{ii} = \frac{1}{C} \int_V \rho c_p \left(1-\theta\right)^2 dv \tag{A22}$$

$$\varphi_{ie} = \frac{1}{C} \int_V \rho c_p \, \theta \left(1-\theta\right) dv \tag{A23}$$

$$\varphi_{ee} = \frac{1}{C} \int_V \rho c_p \, \theta^2 \, dv \tag{A24}$$

where θ is the dimensionless temperature for the problem of steady-state heat transfer through a wall, for ambient temperatures $T_i = 0$ and $T_e = 1$.

In other words, the asymptotic expression for steady-state heat flow across the wall surface in time interval [0, t] due to a step change in excitation temperature is the sum of the term proportional to t, and the capacity term, proportional to the wall's capacity. Capacity terms corresponding to four possible heat transfer modes (combinations of the surface of excitation and the surface of response) are represented here by the products $C\varphi_{ii}T_i$, $C\varphi_{ie}T_e$, $C\varphi_{ie}T_i$ and $C\varphi_{ee}T_e$.

Dimensionless quantities φ_{ii}, φ_{ie} and φ_{ee} are called the "thermal structure factors" for a wall or a wall detail. For a plane wall they are determined directly by its structure, represented by the thermal capacity and resistance distribution across its thickness. In three-dimensional case, to calculate them effectively, one has to solve the steady state heat transfer problem.

The following identity is satisfied:

$$\varphi_{ii} + 2\varphi_{ie} + \varphi_{ee} = 1 \tag{A25}$$

Structure factors φ_{ii}, φ_{ie}, and φ_{ee} for a wall composed of n plane homogeneous layers, numbered from 1 to n with layer 1 at the interior surface, are given as follows:

$$\varphi_{ii} = \frac{1}{R^2 C} \sum_{m=1}^{n} C_m \left[\frac{R_m^2}{3} + R_m R_{m-e} + R_{m-e}^2 \right] \tag{A26}$$

$$\varphi_{ie} = \frac{1}{R^2 C} \sum_{m=1}^{n} C_m \left[-\frac{R_m^2}{3} + \frac{R_m R}{2} + R_{i-m} R_{m-e} \right] \tag{A27}$$

$$\varphi_{ee} = \frac{1}{R^2 C} \sum_{m=1}^{n} C_m \left[\frac{R_m^2}{3} + R_m R_{i-m} + R_{i-m}^2 \right] \tag{A28}$$

where: R_m and C_m denote the thermal resistance and capacity of the *m*-th layer respectively, whereas R_{i-m} and R_{m-e} denote the resistance for heat transfer from surfaces of the *m*-th layer to inner and outer surroundings, respectively. Structure factors φ_{ii} and φ_{ee} take values from the range (0, 1), whereas φ_{ie} from the range (0, 1/4); for 2-layer wall from the range (0, 3/16). For a homogeneous wall, contact resistances being neglected, $\varphi_{ii} = \varphi_{ee} = 1/3$, $\varphi_{ie} = 1/6$.

Products $C\varphi_{ii}$, $C\varphi_{ie}$, $C\varphi_{ee}$, are equivalent to the thermal mass factors, introduced by Anderson [1989]; see also ISO/DIS 9869.2. 1991.

Correspondence with the dynamic thermal characteristics of walls one deals with using Laplace transform method is as follows [Kossecka 1998]:

$$C\varphi_{ii} = \left[\frac{d}{ds} \frac{A(s)}{B(s)} \right]_{s=0}, \qquad C\varphi_{ie} = -\left[\frac{d}{ds} \frac{1}{B(s)} \right]_{s=0}, \qquad C\varphi_{ee} = \left[\frac{d}{ds} \frac{D(s)}{B(s)} \right]_{s=0} \tag{A29}$$

where $A(s)$, $B(s)$ and $D(s)$ are elements of the s-transfer matrix for a multi-layer wall.

The s-transfer function 1/B may be represented as [Brown and Stephenson 1993]:

$$\frac{1}{B(s)} = \frac{1}{R} \frac{1}{\prod_{n=1}^{\infty} (1 + \tau_n s)} \tag{A30}$$

where τ_n are the time constants of the wall, i.e. the poles of $1/B$ are at $s = -1/\tau_n$. Differentiating (A30) gives:

$$\frac{d}{ds} \left[\frac{1}{B(s)} \right]_{s=0} = -\frac{1}{R} \sum_{n=1}^{\infty} \tau_n \tag{A31}$$

Compatibility of Equations (A29) and (A31) gives the condition:

$$\sum_{n=1}^{\infty} \tau_n = RC\varphi_{ie} \tag{A32}$$

which means that walls of the same value of the product $RC\varphi_{ie}$ have the same sum of time constants. For multi-layer walls, in general, sum of the first five time constants is about 0.9 of the product $R{\cdot}C{\cdot}\varphi_{ie}$ whereas first time constant τ_1 is between 0.5 to 0.9 of this quantity.

Compatibility of the asymptotic formula (A19) with the expressions for the heat flow, given in terms of the response factors, yields the following constraints conditions [see Kossecka 1992, 1996, 1998, Kossecka and Kosny 1996, 1997]:

$$\delta \sum_{n=1}^{\infty} n X_n = -C\varphi_{ii} \tag{A33}$$

$$\delta \sum_{n=1}^{\infty} n Y_n = C\varphi_{ie} \tag{A34}$$

Conditions analogous to (A33), (A34), for the dimensionless z-transfer function coefficients b_n, c_n, d_n, have the form [Kossecka 1998]:

$$\sum_{n=1}^{\infty} nb_n - \sum_{n=1}^{\infty} nd_n = \frac{RC}{\delta}\varphi_{ie} \sum_{n=0}^{\infty} d_n \tag{A35}$$

$$\sum_{n=1}^{\infty} nc_n - \sum_{n=1}^{\infty} nd_n = -\frac{RC}{\delta}\varphi_{ii} \sum_{n=0}^{\infty} d_n \tag{A36}$$

Response factors Y_n, with $n \geq 1$, describe the storage effects - heat fluxes after the time of duration of the triangular temperature impulse. They are all positive. High value of the structure factor φ_{ie} indicates that response factors with number $n \geq 1$, are comparatively large; on the contrary, small value of this structure factor indicates that they are comparatively small. However equations (A33), (A34) must be satisfied simultaneously with (A4), which states that the sum of all response factors must be equal to 1. Therefore the larger are the values of Y_n for $n \geq 1$, the smaller is the value Y_0 and vice versa. Taking into account that Y_n should be relatively smooth functions of the number n, one can expect that, for given C, response factors corresponding to small values of structure factors decay relatively quickly whereas those corresponding to large values of structure factors decay relatively slowly. Constraint condition (A32), for the time constants τ_n, indicates the same.

Structure factors for multilayer walls depend on the arrangement of wall materials. This effect may be demonstrated using the simple example of walls of the same resistance and capacity, composed of different configurations of concrete and insulation (Kossecka and Kosny 1998, 2001). Walls 1 through 6 are depicted in Figure A1. Their structure factors are collected in Table A1. The main part of each wall is a composition of heavyweight concrete layers, of 6 in. total thickness, and insulation layers, of 4 in. total thickness. The interior layer is ½ in. thick gypsum plaster; the exterior layer is ¾ in. thick stucco.

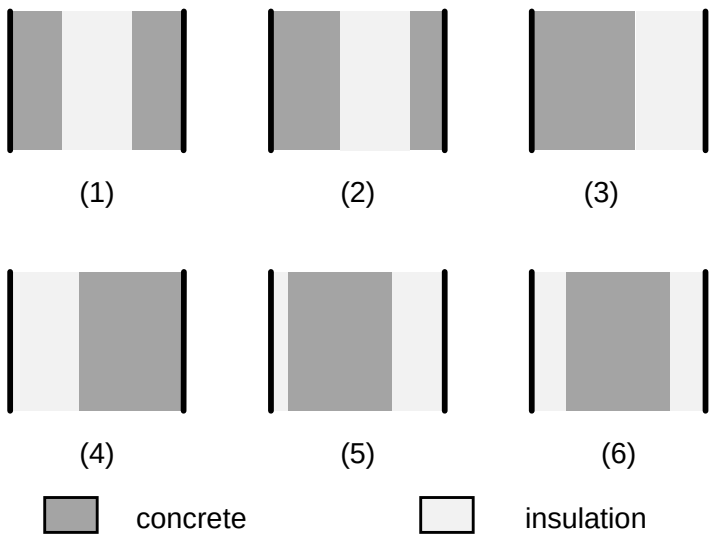

Figure A1 *Walls of different structure composed of concrete and insulation*

Thermophysical properties of the wall materials are as follows:

- *Heavyweight concrete*: k = 10 Btu-in/h ft^2 ^{0}F, ρ = 140 lb/ft^3, c_p = 0.2 Btu/lb ^{0}F;
- *Insulation*: k = 0.25 Btu-in/h ft^2 ^{0}F, ρ = 1 lb/ft^3, c_p = 0.29 Btu/lb ^{0}F;
- *Gypsum board*: k = 1.11Btu-in/h ft^2 ^{0}F, ρ = 50 lb/ft^3, c_p = 0.26 Btu/lb ^{0}F; and
- *Stucco*: k = 5 Btu-in/h ft^2 ^{0}F, ρ = 116 lb/ft^3, c_p = 0.2 Btu/lb ^{0}F.

Other properties of the walls are as follows:

- surface film resistances R_i = 0.69 ft^2 ^{0}F h/Btu, and R_e = 0.33 ft^2 ^{0}F h/Btu;
- total thermal resistance R_T = 18.22 h ft^2 ^{0}F/Btu;
- overall heat transfer coefficient U = 0.055 Btu/h ft^2 ^{0}F;
- wall thermal capacity C = 16.15 Btu/ft^2 ^{0}F; and
- time constant $R_T \cdot C$ = 293.14 h.

Structure factors φ_{ii}, φ_{ie} and φ_{ee} for walls 1 through 6 are collected in Table A1. The maximum φ_{ii} is attained for wall 3 (all concrete inside) and minimum for wall 4 (all insulation inside); φ_{ie} attains it's maximum for wall 6 (symmetric insulation) and minimum for wall 4. Last columns in Table A1 present values of the product $R_T C \varphi_{ie}$ and of the first time constant τ_1 for each wall. Dimensionless response factors $R_T \cdot Y_n$ for walls 1 through 6 are represented in Figure A2. Influence of the structure factor φ_{ie} on the character of variability of Y_n with n, is observed.

TABLE A1

Structure Factors and First Time Constant for Walls with Cores Composed of Heavyweight Concrete and Insulation, Shown in Figure A1

Wall No.	Layers Thickness (in.)	φ_{ii}	φ_{ie}	φ_{ee}	$R_T C \varphi_{ie}$ (h)	τ_1 (h)
Gypsum - Heavyweight Concrete - Insulation - Heavyweight Concrete – Stucco						
1	**½ - 3 - 4 - 3 – ¾**	0.408	0.048	0.496	14.02	8.47
2	**½- 4 - 4 - 2 - ¾**	0.530	0.053	0.363	15.57	11.39
3	**½- 6 - 4 - 0 – ¾**	0.770	0.068	0.094	19.92	17.61
Gypsum - Insulation - Heavyweight Concrete - Insulation – Stucco						
4	**½- 4 - 6 - 0 – ¾**	0.034	0.040	0.885	11.80	9.58
5	**½- 1 - 6 - 3 – ¾**	0.460	0.187	0.167	54.76	48.66
6	**½- 2 - 6 - 2 – ¾**	0.234	0.222	0.322	65.02	62.48

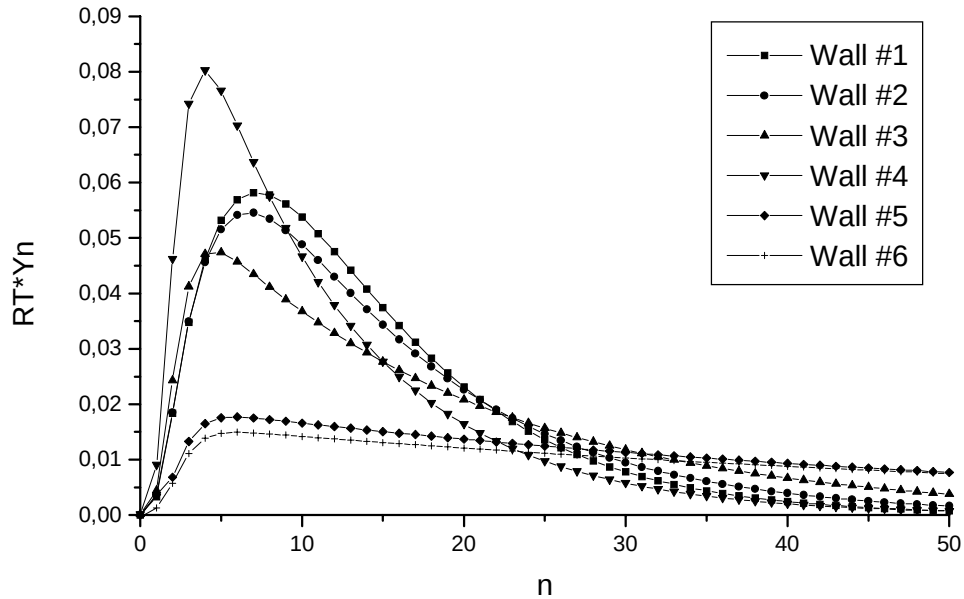

Figure A2 *Dimensionless response factors* $R_T \cdot Y_n$ *for walls 1 through 6*

One can say thus, that thermal structure factors, together with total thermal resistance *R* and capacity *C*, determine the dynamic thermal properties of a wall element - through the conditions they impose on response factors. Those conditions however do not determine the response factors in a unique way, but rather play the role of constraints. Nevertheless one may expect that walls with the same total thermal resistance, capacity and structure factors, have also similar dynamic characteristics - response factors - even if they are quite different in details.

Relationships between structure factors and response factors, and also z-transfer function coefficients, have the same form for plane and composite walls. This analogy constitutes the basis of the notion of the “thermally equivalent wall”: the plane wall of the same dynamic properties as a complex structure, which may be used as its substitute in the whole building thermal modeling.