The $m = 1$ diocotron instability in single species plasmas

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Abstract.
According to conventional theory based on the drift-Poisson equations, the $m = 1$ diocotron mode is stable, even for hollow density profiles. However, experiments[C. F. Driscoll, Phys. Rev. Lett. **64**, 645 (1990)] show instability for this mode. These results have remained unexplained since 1990. We have found two effects which lead to instability with growth rates and other properties in good agreement with the experiments. The first effect is due to curvature of the sheaths at the ends of the trap and involves compression parallel to the magnetic field. The second effect is the free boundary effect due to the linearized perturbation of the plasma length. The effects are described in terms of the *modified drift-Poisson model*, which states the conservation of the *line integrated* density. The modified drift-Poisson equations derived are analogous to the shallow water equations of geophysical fluid dynamics, and the line integrated density corresponds to the potential vorticity in the shallow water equations. This is explained in more detail in [del-Castillo-Negrete et al., this volume]. More recent experimental results[A. A. Kabantsev and C. F. Driscoll, this volume] show agreement over a wider range of parameters than the original experiments. Curvature and free boundary effects can increase the growth rate of the $m = 2$ mode, but the growth rate of this mode is still very small compared to that of the $m = 1$ mode for realistic parameters, and the critical hollowness for stability is much greater. Results are also shown for $m = 1$ modes in the analogous geophysical system. Specifically, topography variation in cylindrical geometry (in the γ-plane approximation) and free boundary effects both lead to instability with properties very similar to those in the plasma models.

According to the classical theory of diocotron modes in nonneutral plasmas [1,2], the $m = 1$ mode (m is the azimuthal mode number, in a geometry with the magnetic field in the $\hat{\mathbf{z}}$ direction) is stable even in the presence of a hollow density profile. However, experiments with such profiles in a Penning-Malmberg trap [3] have shown an instability [4], with an exponential growth rate a few percent of the $\mathbf{E} \times \mathbf{B}$ rotation frequency. There have been several theoretical attempts to explain this discrepancy, but to date none have found growth rates comparable to those found in the experiment [5–8].

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The theory of low frequency behavior such as diocotron modes is based on the drift-Poisson model. This system consists of the continuity equation with velocity equal to the $\mathbf{E}\times\mathbf{B}$ drift $\mathbf{u}_\perp = \hat{\mathbf{z}}\times\nabla\phi/B_0$ and ϕ obtained from the Poisson equation, i.e.

$$\frac{Dn}{Dt} = 0 \ , \qquad \nabla_\perp^2\phi = 4\pi e n \ , \tag{1}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u}_\perp\cdot\nabla$, $-e$ is the electron charge, and the constant magnetic field is $\mathbf{B} = B_0\hat{\mathbf{z}}$.

THE MODIFIED DRIFT POISSON MODEL AND FLUID ANALOGY

The drift-Poisson model is based on an assumption of strictly two-dimensional behavior of the plasma in the trap. However, equilibrium computations [9–12] show that in general there is curvature in the electrostatic sheaths at the ends of a plasma. Based on this observation, we have introduced a new model [9] in order to explain the discrepancy between theory and experiment. The equations of this model, which we call the *modified drift-Poisson model*, are derived by integrating the three dimensional continuity equation including parallel compression $\partial(nu_z)/\partial z$ in z over $-L(r,\theta,t) < z < L(r,\theta,t)$. This treatment is consistent with the fact that the plasma is independent of z in the region $-L < z < L$ (neglecting the width of the sheaths at the ends, which is of order the Debye length). This leads to $Dn/Dt + n\left[u_z(L) - u_z(-L)\right]/L = 0$. Using the kinematic relations $u_z(r,\theta,L) = (D/Dt)\,L$, $u_z(r,\theta,-L) = -(D/Dt)\,L$ we find

$$\frac{D}{Dt}\left(nL\right) = 0 \ . \tag{2}$$

The quantity nL is the *line integrated density*, and Eq. (2) is the equation for conservation of charge in plasma columns aligned with $\mathbf{B}$. In this approximation, all quantities are independent of z for $-L < z < L$ (except $u_z \sim z$). Therefore the Poisson equation takes the same form as in Eq. (1), and the velocity $\mathbf{u}_\perp = \hat{\mathbf{z}}\times\nabla\phi/B_0$ is also unchanged. The plasma length is of the form $L(r,\theta,t) = L_0(r) + \tilde{\Lambda}[\phi(r,\theta,t) - \phi_0(r)]$, where $\phi_0(r)$ is the equilibrium potential, and $\tilde{\Lambda}$ allows for the possibility of a moving boundary; $\tilde{\Lambda}$ is in general a linear functional obtained by matching at $z = L, z = -L$ to the vacuum region $|z| > L(r,\theta,t)$. Equation (2) can be written in in dimensionless variables in the form

$$\frac{D}{Dt}\nabla_\perp^2\phi + \frac{\nabla_\perp^2\phi}{L}\left[-\frac{L_0'}{r}\frac{\partial\phi}{\partial\theta} + \frac{D}{Dt}\tilde{\Lambda}\right] = 0 \ , \tag{3}$$

where D/Dt is as before the $\mathbf{E}\times\mathbf{B}$ convective derivative. We assume a conducting wall at $r = r_w = 1$.

There is an analogy between the modified drift-Poisson equations and the shallow water equations [13] of geophysical fluid dynamics, in which the line integrated density nL corresponds to the potential vorticity, and the first term (the curvature term) in the brackets in Eq. 3 is the analog of either topography variation or of latitude variation in the Coriolis force in the β-plane approximation [13]. Specifically, for a cylindrical tank with a sloping bottom at $z = -H_0[1 - \Delta(r)]$ the equation analogous to Eq. (2) states the conservation of potential vorticity q in a frame rotating at constant angular frequency Ω_0:

$$\frac{Dq}{Dt} = 0 \ , \qquad q \equiv \frac{\zeta + 2\Omega_0}{h} \ . \tag{4}$$

Here, $h(r, \theta, t) = \eta(r, \theta, t) - H_0[1 - \Delta(r)]$ includes the topography variation as well as the motion of the free surface at $z = \eta$ and D/Dt is the convective derivative with the geostrophic flow $\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla\psi$. The streamfunction is given by $\psi = g\eta/(2\Omega_0)$, g is the gravitational acceleration, and ζ is the vorticity $\nabla_\perp^2 \psi$ in the rotating frame. Expanding [14] for $Ro << 1, \Delta(r) << 1$ (Ro is the Rossby number) and ignoring the constant $2\Omega_0$ we obtain the approximation

$$\frac{D}{Dt}\left[\nabla_\perp^2 \psi - k_R^2 \psi + 2\Omega_0 \Delta(r)\right] = 0 \ , \tag{5}$$

where the term proportional to $k_R^2 \equiv 4\Omega_0^2/gH_0$ is due to the free boundary, and and k_R^{-1} is called the *Rossby deformation radius.* Expanding near some radius $r > 0$, $\Delta(r)$ contributes a term linear in r. This is called the β-plane approximation. Expanding near $r = 0$ gives the γ-plane approximation, in which the leading order variation of $\Delta(r)$ is proportional to r^2. We will explore this analogy further in the context of an $m = 1$ instability in the γ-plane approximation. Other aspects of this analogy are explored in detail in [14].

MODEL

Equilibrium computations for Penning-Malmberg traps show that it is appropriate to parameterize the equilibrium length $L_0(r)$ of the plasma by

$$L_0(r) = L_0(0)\left(1 - \kappa r^2\right) \ , \tag{6}$$

with the curvature parameter κ typically positive [9]. We also consider density profiles with two parameters relating to the degree of hollowness and to the plasma radius relative to the wall radius. We assume the form [15]

$$n_0(r) = n_0(0)\left[1 - (r/r_p)^2\right]^2 \left[1 + (\mu + 2)\,(r/r_p)^2\right] \tag{7}$$

for $0 < r \leq r_p$ and zero otherwise, with r_p the plasma radius. The density at $r = 0$ is $n_0(0)$ and μ is the hollowness parameter. (The density profile is hollow if $\mu > 0$; in this case the $\mathbf{E} \times \mathbf{B}$ rotation profile $\Omega(r)$ is also hollow.)

LINEARIZATION AND FREE BOUNDARY PERTURBATION

Linearizing Eq. (3), we obtain

$$(\omega - m\Omega(r))\,\nabla_\perp^2\tilde{\phi} + \frac{mn_0'}{r}\,\tilde{\phi} = -\frac{mn_0L_0'}{rL_0}\,\tilde{\phi} - n_0(\omega - m\Omega(r))\,\tilde{\Lambda}[\tilde{\phi}]/L_0\ , \tag{8}$$

where we have assumed the normal mode dependence $\tilde{\phi} \sim e^{im\theta - i\omega t}$ and ω is the complex frequency. This equation is the conventional Rayleigh equation if the terms on the right due to variation of L are zero.

The free boundary linear perturbation $\tilde{\Lambda}[\tilde{\phi}]$ is computed by matching to the vacuum region at $z = L$, $z = -L$. In Ref. [9] we showed that by requiring continuity of the normal as well as tangential components of the electric field $\mathbf{E}$ at $z = L_0(r) + \tilde{\Lambda}[\tilde{\phi}]$ one obtains the following equation for $\tilde{\Lambda}$:

$$n_0(r)\tilde{\Lambda} = \left[1 + L_0'(r)^2\right]\left(\partial_z\tilde{\phi}\right)_e\ , \tag{9}$$

where the subscript e denotes the external region just to the right of $z = L_0(r)$. The matching conditions at the ends have been expressed in an alternate way [16]. Following this approach, we write the equilibrium density in the limit of zero Debye length as $n(r,z) = n_0(r)\Theta(L_0(r) - z)$, where Θ is a step function. Then, $\tilde{n}$ has a singular component due to axial motion $\tilde{n}_s = -\tilde{\Lambda}\partial_z n(r,z)$ given by

$$\tilde{n}_s = \tilde{\Lambda} n_0(r)\delta(z - L_0(r))\ . \tag{10}$$

Thus, the perturbed density $\tilde{n}$ is associated with a surface charge even though the full nonlinear density is bounded, consistent with continuity of the full electric field $\mathbf{E}$. Defining $\Psi \equiv [z - L_0(r)]/\sqrt{1 + L_0'(r)^2}$, such that $\nabla\Psi$ is the unit outward normal to the curved equilibrium plasma boundary at $\Psi = 0$, we write $\tilde{n}_s$ in terms of the distribution $\tilde{\sigma}$ of surface charge at $z = L_0(r)$ as

$$\tilde{n}_s = \tilde{\sigma}\delta(\Psi)\ , \qquad \tilde{\sigma} = \frac{\tilde{\Lambda} n_0(r)}{\sqrt{1 + L_0'(r)^2}}\ . \tag{11}$$

Relating the jump in the normal derivative $\left[\nabla\Psi\cdot\nabla\tilde{\phi}\right]_{z=L_0(r)}$ to $\tilde{\sigma}$ and requiring continuity of the tangential derivative of $\tilde{\phi}$, we obtain

$$\left[\partial_z\tilde{\phi}\right]_{z=L_0(r)} = \frac{\tilde{\sigma}}{\sqrt{1 + L_0'(r)^2}}\ . \tag{12}$$

Using $\partial_z\tilde{\phi} = 0$ in the plasma $z < L_0(r)$, we recover Eq. (9) from Eqs. (11), (12), showing that the two formulations are identical.

To express $\left(\partial_z \tilde{\phi}\right)_e$ at $z = L_0(r)$ in terms of $\tilde{\phi}$ at $z = L_0(r)$ involves solving Laplace's equation in the vacuum region $L_0(r) < z < L_0(0) + b$, where b is the length of the end-cap, and with Neumann boundary conditions at $z = L_0(0) + b$, a reasonable approximation to open boundary conditions. In general, this can be done in terms of a Green's function. However, for $b << r_w = 1$, assuming $L_0(0) - L_0(r) = O(b)$, there is a differential approximation [9]:

$$\left(\partial_z \tilde{\phi}\right)_e = b f(r)\, \nabla_\perp^2 \tilde{\phi}\,, \tag{13}$$

where $f(r) \equiv 1 + [L_0(0) - L_0(r)]/b$. Substituting Eqs. (9),(13) into Eq. (8), using Eq. (6), and neglecting terms of order κ^2, we find $f(r) = 1 + \kappa r^2/\eta$,

$$(1 + \eta + \kappa r^2)[\omega - m\Omega(r)]\nabla_\perp^2 \tilde{\phi} + \frac{m}{r}\left[n_0' - \frac{2\kappa r}{1 - \kappa r^2} n_0\right]\tilde{\phi} = 0\ , \tag{14}$$

where $\eta \equiv b/L_0(0)$.

For $\tilde{\Lambda} = 0$ in Eq. (8), i.e. ignoring free surface effects, and using Eq. (6), we find

$$(\omega - m\Omega)\, \nabla_\perp^2 \tilde{\phi} + \frac{m}{r}\left[n_0' - \left(\frac{2\,\kappa\, r}{1 - \kappa\, r^2}\right) n_0\right]\tilde{\phi} = 0\ . \tag{15}$$

Note that along with $\eta = 0$ the term κr^2 in the factor multipling $\nabla_\perp^2 \tilde{\phi}$ in Eq. (14) is dropped, because it is included there under the assumption $\eta \sim \kappa$. The two significant parameters in Eq. (15) are the hollowness μ of Eq. (7) and κ.

RESULTS WITH CURVATURE

For $\kappa > 0$ and $\eta = 0$, i.e. using Eqs. (15), and with (7) for $\mu > 0$ (hollow), a mode with a positive growth rate γ is found. For fixed μ and $\kappa \to 0$, γ scales as $\kappa^{2/3}$ for κ small, and $\omega_r = \Omega_{max}$. The fractional power and threshold $\kappa = 0$ are associated with a boundary layer near Ω_{max}, the maximum of $\Omega(r)$. The results $\omega_r = \Omega_{max}$ at marginal stability is in agreement with the experiments [4]. We have found that ω_r decreases slowly from Ω_{max} as κ is increased, giving two resonant radii (where $\Omega(r) = \omega_r$). The perturbation is localized within the radius where $\Omega(r)$ is maximum, showing the self-shielding property observed in the experiments [4]. For $\kappa < 0$ the mode is stable ($\gamma = 0$) with a real frequency ω_r that increases above Ω_{max} as $|\kappa|$ increases, i.e. the mode becomes nonresonant. More details are given in Ref. [9].

For $\mu, \kappa \to 0$, there are scaling properties due to the localization of the mode inside the radius r_Ω where $\Omega(r)$ is maximum. For μ, κ small but $\kappa/\mu \sim 1$, and using the localization $r \sim r_\Omega \sim \mu^{1/2}$ we find the scaling [9]

$$\gamma/\mu^2 = \Gamma(\kappa/\mu)\ . \tag{16}$$

The scaled growth rate γ/μ^2 as function of κ/μ from Eq. (15) is shown in Fig. –(a) for five values of μ. The results are in agreement with the scaling of Eq. (16). From

the scaling $\gamma \sim \kappa^{2/3}$ as $\kappa \to 0$ and these results we conclude $\gamma \approx 5 \times 10^{-3}\kappa^{2/3}\mu^{4/3}$ for $\kappa << \mu$. From Fig. –(a) we observe further that the marginal stability point to the right satisfies $\kappa/\mu \approx 1.55$. This marginal stability curve is shown in Fig. –(b). Note that the marginal stability curve is nearly linear even for κ, μ of order unity.

The linearized equation (15) for $\tilde{\Lambda} = 0$ has a modified Rayleigh criterion. The usual derivation [2] is easily generalized to show that there is a sufficient condition for stability: the equilibrium is stable if the line integrated density $n_0(r)L_0(r)$ is monotonic. This condition is satisfied for κ sufficiently large since the length $L_0(r)$ [c.f. Eq. (6)] is a decreasing function or r. The sufficient condition from the modified Rayleigh criterion as well as the actual marginal stability curve $\kappa/\mu = 1.55$ (c.f. Fig. –(b)).

Equilibrium computations show that κ increases as the length decreases, and that values $\kappa \sim 1$ can be obtained for short plasmas [9]. Thus, the linear results shown on Figs. predict stability for short plasmas. This stabilization has been observed in experiments [17].

RESULTS WITH CURVATURE AND FREE BOUNDARY

In Figure 2–(a) we show the growth rate γ obtained from Eq. (14) as a function of η for $\mu = 3$ and for seven equally spaced values of κ between 0 and 0.35. For these values of η the differential approximation of Eq. (13) used in Eq. (14) is adequate. These values of γ/ω_r are in reasonable agreement with experiments [4]. We find that the behavior as a function of η is similar to the behavior as a function of κ.

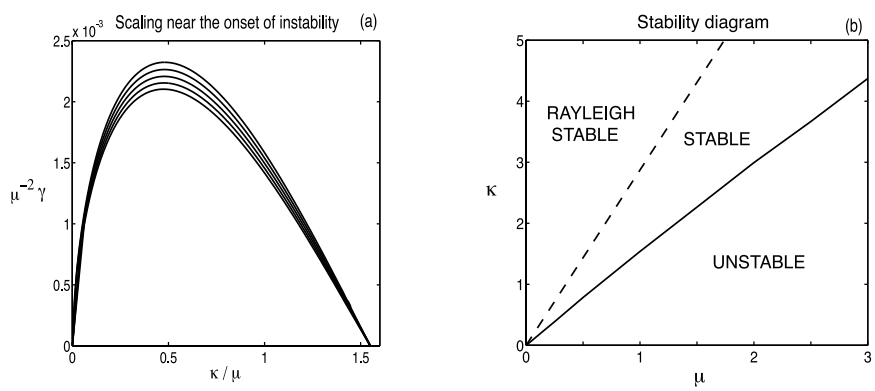

FIGURE 1. (a) Scaled growth rate γ/μ^2 as function of κ/μ, for $\mu = 0.05$ (top), 0.10, 0.15, 0.20, and 0.25 (bottom). The overlap of the curves for different values of μ indicates that in the limit $(\kappa, \mu) \to 0$, $\gamma(\kappa, \mu)$ converges to the self-similar form $\mu^2\Gamma(\kappa/\mu)$ of Eq. (16). (b) Stability diagram μ, κ. The solid curve is the stability boundary computed numerically, and satisfies $\kappa/\mu < 1.55$ for $\mu, \kappa << 1$, with little variation for $\kappa, \mu \sim 1$. The dashed line is the modified Rayleigh criterion $[n_0(r)L_0(r)]' < 0$ stability boundary, namely $\kappa/\mu < 2.87$ for $r_p = 0.59$.

In particular, $\gamma \sim \eta^{2/3}$ for $\kappa = 0, \eta > 0$, and the mode is stable and nonresonant ($\omega_r > \Omega_{max}$) for $\kappa = 0, \eta < 0$.

The linearized equation in the presence of curvature and free boundary effects in the differential approximation Eq. (14) satisfies the same modified Rayleigh criterion as for $\eta = 0$, namely $(n_0 L_0)' < 0$: in the usual derivation [2] $n_0'(r)$ is replaced by $(n_0 L_0)'/[L_0(1+\eta+\kappa r^2)]$. However, the observed stabilization for large η and $\kappa = 0$ [9] is not predicted by the modified Rayleigh criterion. Note also that the modified Rayleigh criterion applies to *all* modes with $m \neq 0$. We conclude that the usual diocotron modes with $|m| > 1$ can in principle be *stabilized* by sufficiently large curvature. We return to this point in the next section.

RESULTS FOR $M = 2$ MODES

We have obtained results for $m = 2$ modes in the presence of curvature and free boundary, i.e. κ and η. First, we have studied the case $\eta = 0$, with $r_p = 0.50$ and $0 < \Delta < 0.06$. ($\Delta \equiv 1/\mu$ is the hollowness parameter of Ref. [15].) For κ between 0 and 0.2, we found that the growth rate relative to that obtained in Ref. [15] changes by less than 0.1%, the real frequency changes by less than 1%, and the marginal stability point $\Delta \approx 0.05$ does not change measurably. In Fig. 2–(b) we show the growth rate γ, scaled to the $m = 1$ stable diocotron mode frequency $\omega_1 = \Omega(r = r_w)$, as in Ref. [15] for $\kappa = 0$, $0 < \eta < 0.1$ and Δ in the above range. These results show that in this range of η the growth rate of $m = 2$ modes increases by about a factor of two and the relative change in the real frequency changes by a factor up to 10%, i.e. $\Delta\omega_r/\omega_r \sim \eta$. The absolute changes $\Delta\gamma$ and $\Delta\omega_r$ are comparable. However, the marginal stability value of Δ increases by only

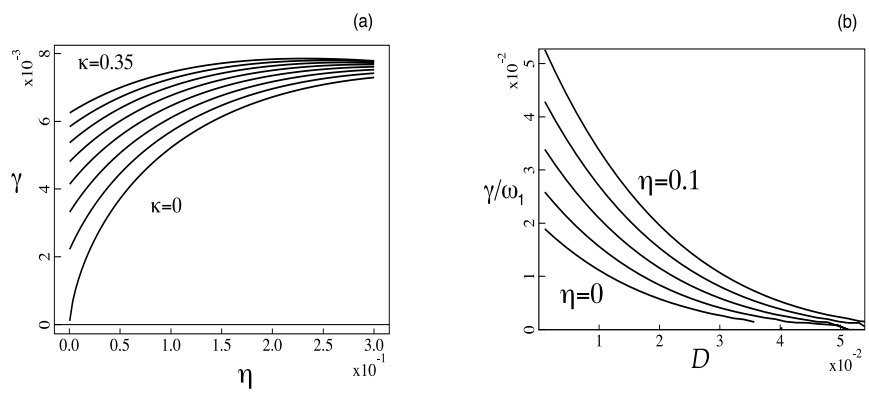

FIGURE 2. (a) Dependence of the growth rate γ on η according to Eq. (14), for seven equally spaced values of κ between 0 and 0.35 for $\mu = 3$. (b) Growth rate γ for the $m = 2$ mode, relative to the $m = 1$ diocotron frequency $\omega_1 = \Omega(r = r_w)$. Results are plotted for $r_w = 0.50, \kappa = 0$ as a function of the hollowness parameter $\Delta = 1/\mu$ for five equally spaced values of the free boundary parameter η between 0 and 0.1.

a small amount. As described in Ref. [15], these modes are destabilized by a small population of resonant particles with

$$\gamma \sim \left[\partial_r n_0(r)\right]_{r=r_s} \ ; \tag{17}$$

here r_s is the resonant surface, where $\omega_r = 2\Omega(r_s)$. For $\kappa > 0, \eta = 0$ and for $\kappa > 0, \eta \sim \kappa$ similar analyses show

$$\gamma \sim \partial_r \left[n_0(r) L_0(r)\right]|_{r=r_s} \ , \qquad \gamma \sim \frac{\partial_r \left[n_0(r) L_0(r)\right]}{1+\eta+\kappa r^2}|_{r=r_s} \ , \tag{18}$$

respectively. Because of this local resonant nature their growth rates are very small ($\gamma/\omega_r \sim$ a few times 10^{-3} [15].) The results with $\kappa > 0$, or the results with $\eta > 0$ of Fig. 2–(b), show that the conclusion of very small growth rates still holds. Moreover, these modes for $\eta = \kappa = 0$ have a very large critical μ (small Δ) for instability, which does not change substantially for reasonable values of κ and η. Furthermore, the possible stabilization for large κ is irrelevant for the $m = 2$ modes; they are unstable only for a large degree of hollowness $\Delta < 0.05$ [15] (consistent with Fig. 2–(b)), i.e. $\mu > 20$, so that an unrealistically large curvature, $\kappa \sim \mu$ would be required for the modified Rayleigh criterion to be satisfied. Such values of κ are not observed in equilibria [9]. Indeed, the representation (6) is invalid for $\kappa > 1/r_p^2$. It is anticipated that the $m > 2$ modes behave similarly to those for $m = 2$.

For comparison with these $m = 2$ results, we summarize the $m = 1$ results of Figs. with increasing μ and fixed κ: as κ is increased (other parameters as in these figures), there is a critical hollowness $\mu = \mu_c$ for $m = 1$ instability, namely $\mu_c = \kappa/1.55$. (It is noteworthy that μ_c goes to *zero* as $\kappa \to 0$.) As μ increases, the growth rate is of order $10^{-2}\kappa^2$ at $\mu \approx 2\kappa$ (with a scaling $\Omega_{max} \sim 1$. For $\mu = 3$ and $r_p = 0.59$, we have $\Omega_{max} = 0.61$, $\omega_1 = 0.13$.) For larger μ the growth rate continues to increase as $\gamma \approx 5 \times 10^{-3}\kappa^{2/3}\mu^{4/3}$. For these parameters and $\kappa \sim \mu \sim 1$, γ/ω_1 can be of order 0.1. These growth rates are much larger than the typical growth rates for the resonant $m = 2$ modes. Also, μ_c, the critical μ for destabilization, is typically much smaller, and μ_c goes to zero as κ goes to zero.

M=1 RESULTS IN ANALOGOUS FLUIDS

Based on the above results, showing instability for $m = 1$ modes in Penning-Malmberg traps with hollow density profiles, and the analogy with geophysical fluid dynamics, we have investigated the $m = 1$ stability properties of rotating fluids with topography variation and a free surface. Linearizing Eq. 5, we obtain

$$(\omega - m\Omega(r)) \left(\nabla_\perp^2 \tilde{\psi} - k_R^2 \tilde{\psi}\right) + \frac{m}{r}\left[\zeta_0'(r) - r k_R^2 \Omega(r) + \kappa r\right] \tilde{\psi} = 0 \ . \tag{19}$$

Here, $\zeta_0(r)$ is the equilibrium vorticity and $\Omega(r)$ is the equilibrium rotation velocity, both in the frame rotating at Ω_0. We have taken the relative topography $\Delta(r) =$

$\sigma r^2/r_w^2$, where $\sigma << 1$ and r_w is the radius of the tank, and defined $\kappa \equiv 4\sigma\Omega_0/r_w^2$. The relative topography $\Delta(r)$ includes the equilibrium parabolic deformation of the surface due to the rigid rotation Ω_0.

If Fig. 3–(a) we show the growth rate γ as a function of κ for three values of k_R. Note that instability occurs for $\kappa < 0$. There is a zero threshold in κ for $k_R = 0$, with $\gamma \sim \kappa^{2/3}$ for small κ, and stabilization is observed for large κ, similar to the plasma case. (Again there is a modified Rayleigh criterion, giving for example with $k_R = 0$ a sufficient condition for stability if $\zeta_0(r) + 2\Omega\Delta(r)$ is decreasing, i.e. if $\zeta_0'(r)+\kappa r$ is negative.) In Fig. 3–(b) we show γ as a function of k_R for six values of κ. The growth rate increases with k_R and $|\kappa|$. For $\kappa = 0$, γ goes to zero at $k_R = 0.065$. As in the plasma case, the real frequency is found to be Ω_{max} for $\kappa = k_R = 0$ and to decrease as k_R increases or $|\kappa|$ increases, i.e. when there is instability. The mode structure is similar to that in the plasma case. Specifically the mode has the self-shielding property, that the perturbation goes to zero rapidly just outside the radius where $\Omega(r)$ is maximum. Since the total height is $h = H_0[1 - \Delta(r)]$ for zero free boundary perturbation η, we see that instability occurs when the total height is an *increasing* function of r, $h'(r) = -H_0\Delta'(r)$. The fact that the stability criterion is opposite to that in the plasma case is traced to the fact that the potential vorticity in the lab frame is ζ/h, whereas the line integrated density in the plasma case, which plays the role of potential vorticity, has the form nL.

COMPARISON WITH EXPERIMENTS

As discussed, the theoretical results for $m = 1$ are in good agreement with the earlier experiments [4]. Further experiments have recently been done [18] over a wider range of parameters, and this agreement appears to be holding up. Specifically, the observed behavior in agreement is: (i) the scaling $\gamma \sim \kappa^{2/3}$ for small κ; (ii)

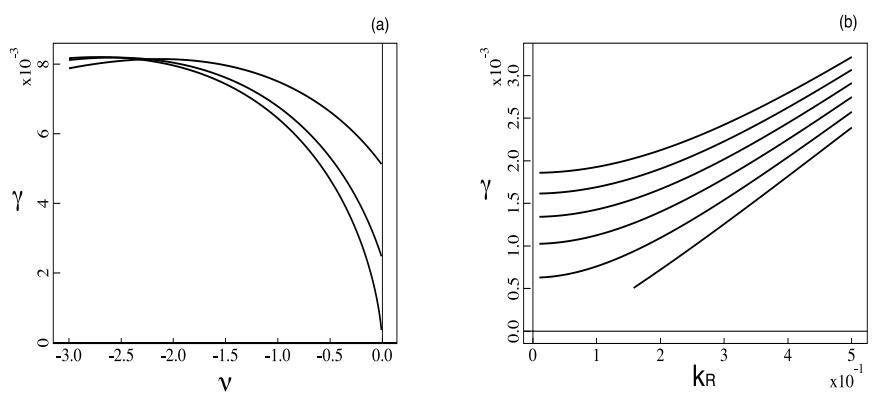

FIGURE 3. (a) Growth rate for the $m = 1$ mode in the γ-plane approximation from Eq. 19 as a function of κ for three equally spaced values of k_R between 0 and 1.0. (b) Growth rate for the $m = 1$ mode in the γ-plane approximation from Eq. 19 as a function of k_R for six equally spaced values of κ between 0 and -0.1.

the stabilization for short plasmas (large κ); (iii) the decrease of the real frequency for increasing positive κ; (iv) stability for $\kappa < 0$ with real frequency that increases above Ω_{max} with $|\kappa|$; (v) the self-shielding property. There appears to be a discrepancy in γ of about a factor of two, but it is possible that this may be resolved by a Green's function treatment that is not restricted to $b/r_w << 1$. (The conditions for the validity of the differential approximation of Eq. (13) are only qualitatively satisfied for the parameters of the experiments.) Also, Eq. (14) has the property that the perturbed density $\tilde{n} = \nabla_\perp^2 \tilde{\phi}$ should be zero where $[n_0(r)L_0(r)]'$ is zero. In the experiments [4], [18], $|\tilde{n}|$ does indeed have a minimum there, but the value appears to be positive. This possible discrepancy should also disappear in a Green's function treatment.

In conclusion, inclusion of terms related to end curvature and free boundary has a substantial effect on the $m = 1$ mode, destabilizing it for arbitrarily small hollowness μ (as long as κ is small enough.) The growth rates are relatively large. The zero threshold in κ and the scaling $\gamma \sim \kappa^{2/3}$ for small κ (and similar behavior for small η, measuring the free boundary effect) are traced to the fact that there is a boundary layer at Ω_{max}. The effects on the slowly growing resonant $m = 2$ modes (and probably on modes with $m > 2$) are weaker. In addition to these effects on known modes, the curvature leads to a new class of stable modes, analogous to the Rossby waves of geophysical fluid dynamics. These waves are treated in detail in Ref. [14].

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