A New Analogy Between Nonneutral Plasmas and Geophysical Fluid Dynamics

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Abstract.
We discuss an analogy between magnetically confined nonneutral plasmas and geophysical fluid dynamics. The analogy has its roots in the *modified drift Poisson model*, a recently proposed model that takes into account the plasma compression due to the variations of the plasma length [1]. The conservation of the line integrated density in the new model is analogous to the conservation of potential vorticity in the shallow water equations, and the variation of the plasma length is isomorphic to variations in the Coriolis parameter with latittude or to topography variations in the quasigeostrophic dynamics. We disuss a new class of linear and nonlinear waves that owe their existence to the variations of the plasma length. These modes are the analog of Rossby waves in geophysical flows.

INTRODUCTION

There is a well-known analogy between nonneutral plasmas confined in a Penning-Malmberg trap and two-dimensional inviscid fluid dynamics. In this analogy the plasma electrostatic potential and density correspond to the fluid streamfunction and vorticity respectively [2]. This analogy has proved to be particularly useful in the experimental study of various fluid dynamics problems using nonneutral plasmas, e.g. Ref. [3]. The goal of this paper is to study a new analogy between nonneutral plasmas and geophysical fluid dynamics. This analogy is based on the *modified drift-Poisson* system, a recently proposed model that generalizes the usual drift-Poisson equations by taking into account the variations of the plasma lenght [1]. The modified drift-Poisson model consists of the conservation of the line-integrated density and Poisson equation,

$$\frac{D}{Dt}(nL) = 0\,, \qquad \nabla^2\phi = 4\pi e n\,, \tag{1}$$

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where L is the plasma length, ϕ is the electrostatic potential, n is the plasma density, e the electron charge, and $D/Dt = \partial_t + \mathbf{u}\cdot\nabla$ with $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\phi/B_0$ the $\mathbf{E}\times\mathbf{B}$ drift velocity.

This model was originally proposed to resolve a controversy regarding the stability of the $m = 1$ diocotron mode. According to linear theory the $m = 1$ diocotron mode should be stable [2]. However, about ten years ago Driscoll found a robust exponential growth of $m = 1$ perturbations in hollow plasma density profiles [4]. The contradiction between this experimental result and theory has been a long-standing problem. In the modified drift-Poisson model the $m = 1$ instability is naturally explained by the destabilizing effect of the variations of the plasma length (positive curvature at the plasma sheats) that is accompained by the plasma density compression. The growth rate, frequency and mode structure of the $m = 1$ instability predicted by the modified drift-Poisson model are in good agreement with the experimental results reported in [4], and recent experiments [5] have further verified the prediction of this model.

In general the plasma length L depends on r, θ and t, and its exact expression involves a numerical Green's function calculation. However, for the present discussion we will assume $L = L_0(r)$. A more general model for L which incorporates free boundary effects on the plasma lenght without the need to compute the Green's function is discussed in [1]. For $L = L_0(r)$ the modified drift-Poisson model (1) becomes

$$\frac{\partial \nabla^2\phi}{\partial t} + \left\{\phi, \nabla^2\phi\right\} - \left(\frac{L_0'}{rL_0}\right)\, \frac{\partial\phi}{\partial\theta}\, \nabla^2\phi = 0\,, \tag{2}$$

where the prime denotes derivative with respect to r, and $\{f, g,\} \equiv 1/r\,(\partial_r f\,\partial_\theta g -$ $\partial_r f\,\partial_\theta g$. Writing $\phi = \phi_0(r) + \tilde{\phi}(r,\theta,t)$, and neglecting nonlinear terms in $\tilde{\phi}$ we get the linearized version of Eq. (1)

$$\frac{\partial \nabla^2\tilde{\phi}}{\partial t} + \left\{\phi_0, \nabla^2\tilde{\phi}\right\} + \left\{\tilde{\phi}, \nabla^2\phi_0\right\} - \left(\frac{L_0' n_0}{rL_0}\right)\, \frac{\partial\tilde{\phi}}{\partial\theta} = 0\,. \tag{3}$$

The precise form of $L_0(r)$ depends on the numerical solution of the plasma equilibrium equations. However, as discussed in Ref. [1], $L_0(r)$ can be fit to the parabola

$$L_0(r) = L_0(0)\left[1 - \kappa r^2\right]\,, \tag{4}$$

where $L_0(0)$ and the curvature κ depend on the equilibrium parameters. Typically $\kappa > 0$.

ANALOGIES WITH GEOPHYSICAL FLUID DYNAMICS

When the variation of the plasma length L is taken into account the two-dimensional density n is *not* conserved, and the analogy with the two-dimensional

Euler equation breaks down. However, there remains a new and interesting similarity with geophysical fluid dynamics. In addition to its intrinsic theoretical interest, this analogy is important from the perspective of modeling geophysical flows with nonneutral plasmas experiments in Penning-Malmberg traps.

To explain this similarity consider a uniform density, incompressible, rotating fluid, shown in Fig. **??**, with free surface $z = \eta(x, y, t)$, and bottom topography $z = -H_0[1 - \Delta(r)]$. This system is commonly used in geophysical fluid dynamics as the starting point in the development of simple models of the oceans and the atmosphere [6]. An important parameter in rotating fluid dynamics is the *Rossby number* defined as $Ro \equiv U/(2\Omega L)$ where U is a horizontal velocity scale, L is a horizontal length scale, and Ω is the magnitude of the rotation frequency.

The limit $Ro << 1$ is of particular interest in geophysical flows. In this case, because of the rapid rotation the horizontal velocity, $\mathbf{u}$, is to a good approximation independent of z. If in addition it is assumed that the scale of vertical motions is small compared to the scale of the horizontal motion we get the *shallow-water* model which implies the conservation of the *potential vorticity*, q,

$$\frac{Dq}{Dt} = 0 \, , \qquad q \equiv \frac{\zeta + f}{h} \, , \tag{5}$$

where ζ is the vorticity, $h = \eta + H_0(1 - \Delta)$ is the fluid depth, and $f = 2\Omega \sin\varphi$ is the Coriolis parameter with φ the latitute angle measured from the equator [6].

In the non-rotating (inertial) frame Eq. (5) reduces to $D(\zeta/h)/Dt = 0$ which is analogous to the conservation of the line integrated density in Eq. (1) if we identify the plasma density n with the vorticity ζ and the variations of the fluid depth h with the inverse of the plasma lenght $1/L$.

Topography variations. There are two sources of variability in (5), one due to the variations of the fluid depth $h = h(r, \theta, t)$ and the other due to the variations of Coriolis parameter $f = f(\varphi)$. Consider first the case when $f = f_0 = \text{constant}$ and

$$h = H_0 \left[1 - \Delta(r)\right] \, , \tag{6}$$

where we have neglected free surface effects and, as shown in Fig. [**?**], Δ is the topography variation. In the small Rossby number limit $\zeta/2f_0 \sim \Delta \sim R_o \ll 1$, we have

$$q = \frac{\zeta + f_0}{H_0\,(1 - \Delta)} \approx \frac{f_0}{H_0} \left[1 + \frac{\zeta}{f_0} + \Delta(r)\right] + O(R_o^2) \, . \tag{7}$$

In the *quasigeostrophic approximation* the advective derivative in (5) becomes $D/Dt = \partial_t + \mathbf{u} \cdot \nabla$, where $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi$ is the geostrophic velocity, which is the analogue of the $\mathbf{E} \times \mathbf{B}$ plasma drift velocity, ψ is the streamfucntion and $\zeta = \nabla^2\psi$. In this limit Eq. (5) becomes the *quasigeostrophic potential vorticity equation*:

$$\frac{\partial \nabla^2\psi}{\partial t} + \left\{\psi, \nabla^2\psi\right\} - \left(\frac{2\Omega\Delta'}{r}\right) \frac{\partial\psi}{\partial\theta} = 0 \, , \tag{8}$$

where, the prime denotes derivative with respect to r. Writting $\psi = \psi_0(r) + \tilde{\psi}(r,\theta,t)$, we get the linearized drift-Poisson model

$$\frac{\partial \nabla^2 \tilde{\psi}}{\partial t} + \left\{\psi_0, \nabla^2 \tilde{\psi}\right\} + \left\{\tilde{\psi}, \nabla^2 \psi_0\right\} - \left(\frac{2\Omega\Delta'}{r}\right) \frac{\partial \tilde{\psi}}{\partial \theta} = 0 \,. \tag{9}$$

Note that Eq. (2) is different from Eq. (8), but the linearized drift-Poisson model (3) is identical to the linearized quasigeostrophic equation (9) is we make the identification $\Delta \leftrightarrow 1/f_0 \int dr\, L_0'(r)\, n_0(r)/L_0(r)$. In particular for L_0 in (4) $\Delta \sim \Delta_0\,(1 - a\kappa r^2)$ where $a > 0$. That is, a positive plasma length curvature is analogue to a "mountain".

Coriolis parameter variation. Consider now the case in which there are no topography variations ($\Delta = 0$) but the Coriolis parameter varies with latitude $f = 2\Omega \sin\varphi$. Let φ_0 denote a reference latitute angle and write

$$f \approx 2\Omega \left[\sin\varphi_0 + \cos\varphi_0\, \delta\varphi - \sin\varphi_0\, (\delta\varphi)^2/2 \ldots\right] . \tag{10}$$

In this approximation the potential vorticity becomes

$$q = \frac{\zeta + f}{H_0} \approx \frac{1}{H_0}\left(f_0 + \zeta + \beta\, r - \gamma\, r^2 + \ldots\right) , \tag{11}$$

where β and γ are constants, and $r = \delta\varphi\, R$ with R the radius of the earth. At mid-latitiude $\cos\varphi_0 \neq 0$ and $q \approx f_0 + \zeta + \beta r$. This is the so-called β–plane approximation. However, near the poles $\cos\varphi_0 \approx 0$ and thus $q \approx f_0 + \zeta - \gamma r^2$ which is known as the γ–plane approximation [7,8]. From Eqs. (7) and (11) we have that the variations of the Coriolis parameter in the earth can be mimiced by topography variations if we identify $2\Omega\Delta'(r) \leftrightarrow \beta - 2\,\gamma\, r$. This identification is the guiding principle in the modeling of geophysical flows with laboratory experiments.

ROSSBY WAVES

According to (1) the variation of the plasmas lenght L induces a compression of the plasma density n. This compression provides the restoring mechanism of a new class of plasma waves which are the analog of Rossby waves in geophysical fluid dynamics.

Linear solutions As a simple illustration of these waves, consider a "top-hat" piece-wise constant density profile:

$$n_0 = \text{constant}\,, \qquad \phi_0 = \Omega/2\left[(r/r_p)^2 - 1\right] , \tag{12}$$

for $r < r_p$, and $n_0 = \phi_0 = 0$ for $r > r_p$. For this density distribution $\Omega = n_0/2$ for $r < r_p$, and $\Omega \sim 1/r$ for $r > r_p$. According to the linearized modified drift-Poisson model (3), for $r < r_p$

$$[\partial_t + (\Omega/r_p)\partial_\theta]\, \nabla^2\tilde{\phi} + \gamma\,\lambda(r)\,\partial_\theta\tilde{\phi} = 0\ , \qquad \gamma \equiv 4\kappa\,\Omega\ , \qquad \lambda = L_0(0)/L_0(r)\ . \quad (13)$$

Substituting $\tilde{\phi} = f(r)\, e^{i\,(m\,\theta - \omega t)}$ in (13) we get

$$f'' + \frac{1}{r} f' + \left[\lambda(r)\, D^2 - \frac{m^2}{r^2}\right] f = 0\,, \qquad D \equiv \sqrt{\frac{m\gamma}{m\Omega - \omega}}\ . \tag{14}$$

In the small curvature limit $\kappa \ll 1$, $\lambda \to 1$ and Eq. (14) reduced to Bessel's equation. Therefore, in this limit, we get the following linear solution for $r < r_p$:

$$\tilde{\phi}_< = A\, \mathrm{J}_m\,(Dr)\; e^{i(m\theta - \omega t)}\ , \tag{15}$$

where J_m is the Bessel function of order m, and A is a constant.

For the vacuum region, $r > r_p$, we simply have $\nabla^2\tilde{\phi} = 0$. The solution of this equation that vanishes at $r = 1$ (the boundary of the trap) is

$$\tilde{\phi}_> = B\, \left(r^{-m} - r^m\right)\; e^{i(m\theta - \omega t)}\ . \tag{16}$$

From the matching conditions $\phi_<(r_p) = \phi_>(r_p)$, and $\phi'_<(r_p) = \phi'_>(r_p)$ we get the linear dispersion relation:

$$\omega_{mn} = m\left[\Omega - (r_p/\rho_{mn})^2\; \gamma\right]\ , \tag{17}$$

where ρ_{mn} is determined by the solution of

$$\frac{J_m(\rho_{mn})}{J'_m(\rho_{mn})} = -\alpha_m\, \rho_{mn}\ , \qquad \alpha_m \equiv \frac{1}{m}\left(\frac{1 - r_p^{2m}}{1 + r_p^{2m}}\right)\ . \tag{18}$$

Note that according to (18) these linear waves propagate only in the one direction.

Nonlinear solutions To construct nonlinear traveling wave solutions of the modified drift-Poisson system consider

$$\phi = \frac{\omega}{2}\, r^2 + \psi(r, m\theta - \omega t)\ , \tag{19}$$

where ω is a constant. Substituing (19) into (2) we get

$$\left\{\psi, L_0\, \left[\nabla^2\psi + 2\,\omega/m\right]\right\} = 0\ . \tag{20}$$

The general solution of (20) is

$$L_0\left[\nabla^2\psi + 2\,\omega/m\right] = \mathcal{F}(\psi)\ , \tag{21}$$

where $\mathcal{F}$ is an arbitrary function of ψ. A solvable special case is

$$F(\psi) = L_0(0)\left[2\Omega - D^2\,\psi\right]\ , \tag{22}$$

where Ω is a constant and D is defined in (14). For this special case the solution (19) reduces to

$$\phi = \frac{\Omega}{2} r^2 + f(r)\, e^{i\,(m\,\theta - \omega t)} \ , \tag{23}$$

where $f(r)$ is determined by the solution of Eq. (14). As discussed before, in the small curvature limit $\kappa \ll 1$, Eq. (14) reduced to Bessel's equation and $f(r) = \mathrm{J}_m(Dr)$, The interior solution (23) has to be coupled to the solution of the vacuum equation $\nabla^2 \phi = 0$, whose solution is (16). From here the calculation proceeds as the linear problem presented before.

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