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THE STIRLING ENGINE WITH ONE ADIABATIC CYLINDER

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LIST OF SYMBOLS

a,b,c	Dummy variables
k	Ratio of power piston to displacer piston swept volume (V_p/V_e)
M	Total mass of gas
M_c	Instantaneous mass of gas in compression space
M_e	Instantaneous mass of gas in expansion space
P	Instantaneous gas pressure
P_o	Initial gas pressure (at the beginning of Phase 1 and the end of Phase 4)
P_1	Gas pressure at the end of Phase 1
P_2	Gas pressure at the end of Phase 2
P_3	Gas pressure at the end of Phase 3
R	Gas constant
T_c	Gas temperature in the isothermal compression space
T_{ch}	Temperature of compression (cold-end) heat exchanger
T_e	Gas temperature in the expansion space
T_{eh}	Temperature of expansion (hot-end) heat exchanger
T_{2e}	Expansion-space gas temperature at the end of Phase 2
T_{3e}	Expansion-space gas temperature at the end of Phase 3
T_{4e}	Expansion-space gas temperature near the end of Phase 4
V	Instantaneous cylinder volume
V_e	Volume swept by displacer piston
V_p	Volume swept by power piston
x	A dummy variable
γ	Specific heat ratio of gas
η	Efficiency (W_{out}/W_{in})
ρ	Gas density
τ	Ratio of cooler to heater temperature (T_{ch}/T_{eh})
τ_2	A temperature ratio, T_{ch}/T_{2e}
τ_3	A temperature ratio, T_{ch}/T_{3e}
τ_4	A temperature ratio, T_{ch}/T_{4e}

THE STIRLING ENGINE WITH ONE ADIABATIC CYLINDER

C. D. West

This report shows that integration around the P-V loop of a Stirling-like cycle with an adiabatic expansion or compression space is possible through careful application of the ideal gas laws. The result is a set of closed-form solutions for the work output, work input, and efficiency for ideal gases. Previous analyses have yielded closed-form solutions only for machines in which all spaces behave isothermally, or that have other limitations that simplify the arithmetic but omit important aspects of real machines. The results of this analysis, although still far removed from the exact behavior of real, practical engines, yield important insights into the effects observed in computer models and experimental machines. These results are especially illuminating for machines intended to operate with fairly small temperature differences. Heat pumps and low-technology solar-powered engines might be included in this category.

1. INTRODUCTION

The ideal Stirling cycle has a four-cornered P-V diagram in which the basic processes of compression, heating, expansion, and cooling take place one at a time, and each process is an isothermal one. The working gas is expanded at a relatively high temperature and compressed at a lower temperature (Fig. 1). Analytical expressions (closed-form solutions) for the pressure variations and for the power output, power input, and efficiency are fairly easy to find in this case. The efficiency, of course, is simply Carnot efficiency for a machine operating between the same temperatures.

Real Stirling machines, although similar in principle, usually operate on a cycle that is significantly different from this ideal one; the piston movements are usually more or less continuous, so that the basic processes merge with expansion and compression taking place while the gas is distributed between the high- and low-temperature spaces, and the cylinders are often more nearly adiabatic than isothermal. This report is primarily concerned with the effects of an adiabatic cylinder but will

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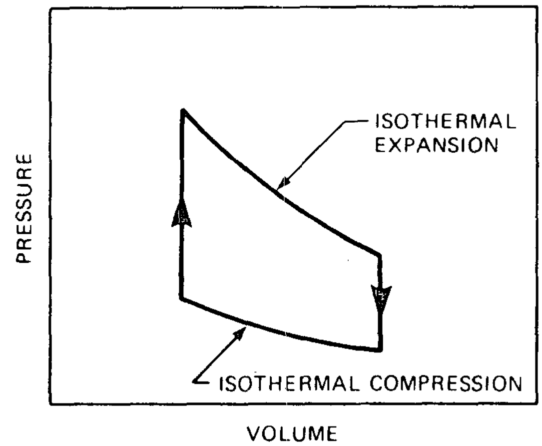

Fig. 1. The ideal Stirling cycle has constant-temperature compression and expansion phases separated by transfer of gas, without volume change, between a hot and a cold space.

begin with a brief discussion of past work on the analysis of Stirling machines and their departures from ideality.

The analytical problem posed by the continuous, merged processes was finally solved in the 1870s, half a century after the invention of the Stirling engine, when Gustaf Schmidt integrated the equations relating pressure and volume for sinusoidal piston movements. This integration yielded a closed-form expression for the power output of the cycle that has since been rewritten in many ways but is still usually known as the Schmidt equation. A useful format is shown below:

$$W = \pi P_{max} V_T \frac{1 - \tau}{1 + \kappa} \frac{\delta \sin \theta}{1 + \sqrt{1 - \delta^2}}$$

THE SCHMIDT EQUATION IS A CLOSED FORM SOLUTION FOR THE POWER OUTPUT OF AN ISOTHERMAL STIRLING ENGINE. IT IS DECEPTIVELY SIMPLE IN APPEARANCE.

The apparent simplicity of this equation is deceiving; when written in terms of the basic machine parameters it takes on quite a different appearance, as shown on the next page. Note the true complexity of the

$$W = \frac{\pi P_{mean} V_c (1 - T_c/T_e) \sin \alpha}{1 + \sqrt{1 - \dfrac{T_c^2/T_e^2 + 2T_cV_c \cos \alpha/(T_eV_e) + V_c^2/V_e^2}{\left\{\dfrac{T_c}{T_e} + \dfrac{V_c}{V_e} + \dfrac{2V_{de}T_c}{V_eT_e} + \dfrac{2V_rT_c \log(T_e/T_c)}{V_e(T_e - T_c)} + \dfrac{2V_{dc}}{V_e}\right\}^2}}}$$

$$\times \frac{1}{\dfrac{T_c}{T_e} + \dfrac{V_c}{V_e} + \dfrac{2V_{de}T_c}{V_eT_e} + \dfrac{2V_rT_c \log(T_e/T_c)}{V_e(T_e - T_c)} + \dfrac{2V_{dc}}{V_e}}$$

WHEN WRITTEN IN TERMS OF THE BASIC ENGINE PARAMETERS, THE SCHMIDT EQUATION IS SEEN TO BE RATHER COMPLEX.

Schmidt equation, for it is a measure of the difficulties encountered in searching for closed-form solutions to all but the most basic of approximations to a real Stirling machine. The Schmidt equation, in fact, is apparently the only case that has been solved, and even simple departures from pure sinusoidal motion, such as might be introduced by the use of connecting rods that are not infinitely long, lead tc integrations that have no known solutions in terms of elementary functions and can only be performed numerically. This is no longer a great burden, as pocket calculators can now perform such integrations with a single keystroke.

The Schmidt equation also allows for the effect of "dead volume," that is, for those gas spaces that are not swept by the pistons at any time during the cycle. These spaces usually represent the regenerator and the heat exchangers used for heating and cooling, as well as the volume remaining in the cylinders at the top dead center piston positions. Accounting for the dead volume leads to a more realistic representation because in most real Stirling machines the cylinder spaces do not behave isothermally, and so heat exchangers external to the cylinders are needed to add and remove the cyclic heat. Regeneration, which clearly requires a regenerator of finite volume, is needed in any case if high efficiency is sought.

The Schmidt analysis retains the assumption of isothermal processes so that cycle efficiency is equal to the efficiency of a Carnot machine operating between the same temperature limits, even though during the expansion and compression phases, part of the gas is hot and part of it is cold.

If one or more of the cylinders is not isothermal, no solutions have apparently been published to the work integral ($\oint$ PdV) except by numerical integration. The necessary equations were set up by Finkelstein[1] some 90 years after the publication of Schmidt's results, and they represented a major advance, opening the way for computer models of the nonisothermal Stirling cycle. There were seven simultaneous equations, of which two were differential equations. The system could be reduced algebraically to two simultaneous differential equations that, however, could only be solved numerically. Many computer codes for doing so now exist.

An analytical (closed-form) solution for the power output, but not the power input and efficiency, of a simplified, three-cornered model of a Stirling machine with one adiabatic cylinder has been published[2] and allows for the effect of dead volume. It provides a convenient physical explanation for some of the effects known from computer simulations and from experiment but is recognized to be an even less accurate representation of reality than a four-cornered cycle. A different simplification was used by Rallis and Urielli,[3] who assumed that all the gas (not only in the cylinders, but also in the heat exchangers) behaved adiabatically during the expansion and compression phases.

This report is an analysis of a four-cornered analog of a Stirling cycle with an adiabatic expansion space; it yields closed-form solutions for the power output, power input, and efficiency. Although the movements of the displacer and power pistons are separated from each other in the analysis, the expansion of the gas is allowed to take place simultaneously with its transfer between the hot and cold cylinders during the expansion phase.

Whether or not there is any point in pursuing such analyses is a reasonable question; after all, computer time is readily available and anyway, of what practical application is a solution that permits only one cylinder to be adiabatic?

The second part of the question is most easily answered. Several schemes[4–6] exist for rendering the cylinders of a real machine isothermal (or nearly so) that involve the fitting of many fins or small tubes to the displacer piston and allowing liquid to penetrate their interstices during part of the cycle. Without using exotic materials, this method is limited to the rather low temperatures at which commonly available liquids are thermally stable and have an acceptably low vapor pressure. Thus, in practice, this method can only be used at the cold end of the displacer, and therefore only the compression space is isothermalized, leaving gas in the expansion space to behave nearly adiabatically.

With regard to the first part of the question, if a closed-form solution could be found, it would certainly have great value. First, an explicit formula (however complex) is easily portable from one computing system or code to another. Second, it can be evaluated to almost any desired degree of precision without a noticeable increase in computing time and cost. Third, no stability or other numerical problems are involved in the evaluation.

The potential saving in computing time is a very important practical aspect, because the cost of computing the many hundreds of cases that may be required in the course of an optimization search is high enough with present codes to impose a limitation on such exercises. "Second order" codes — those that compute the output and efficiency of an idealized, lossless machine and then subtract the effects of the various known power and thermal loss mechanisms one by one — seem to be the most suitable for optimization searches,[7] and these are just the kind of codes that would benefit (in terms of computational speed and accuracy) if closed-form solutions for the basic cycle parameters could be developed.

Even if a complete analytical solution of the equations proves to be impossible, the attempt to find one may lead to equations that, although they require numerical solution, are faster or easier to evaluate than the present systems of differential equations.

Finally, physical insights into the processes involved in an adiabatic cylinder machine, insights that may not be gained simply from perusal of computer printouts, are offered by the closed-form solutions.

This report, then, comprises another step toward the desirable goal of an equivalent to the Schmidt equations that can be applied to adiabatic cylinder machines. The results are more advanced than we now have, but they are not as complete as we would like. Nor do they reveal whether that eventual goal is achievable even in principle. But they do give some analytical insight into the effects associated with the combination of isothermal heat exchangers and adiabatic cylinder spaces, and they do provide a picture that helps to explain some of the trends and effects predicted by the computer models.

2. ISOTHERMAL CYLINDERS

To demonstrate the physical significance of the procedure to be used, first consider the standard case of a machine — such as the one shown diagrammatically in Fig. 2 — in which the expansion and compression spaces

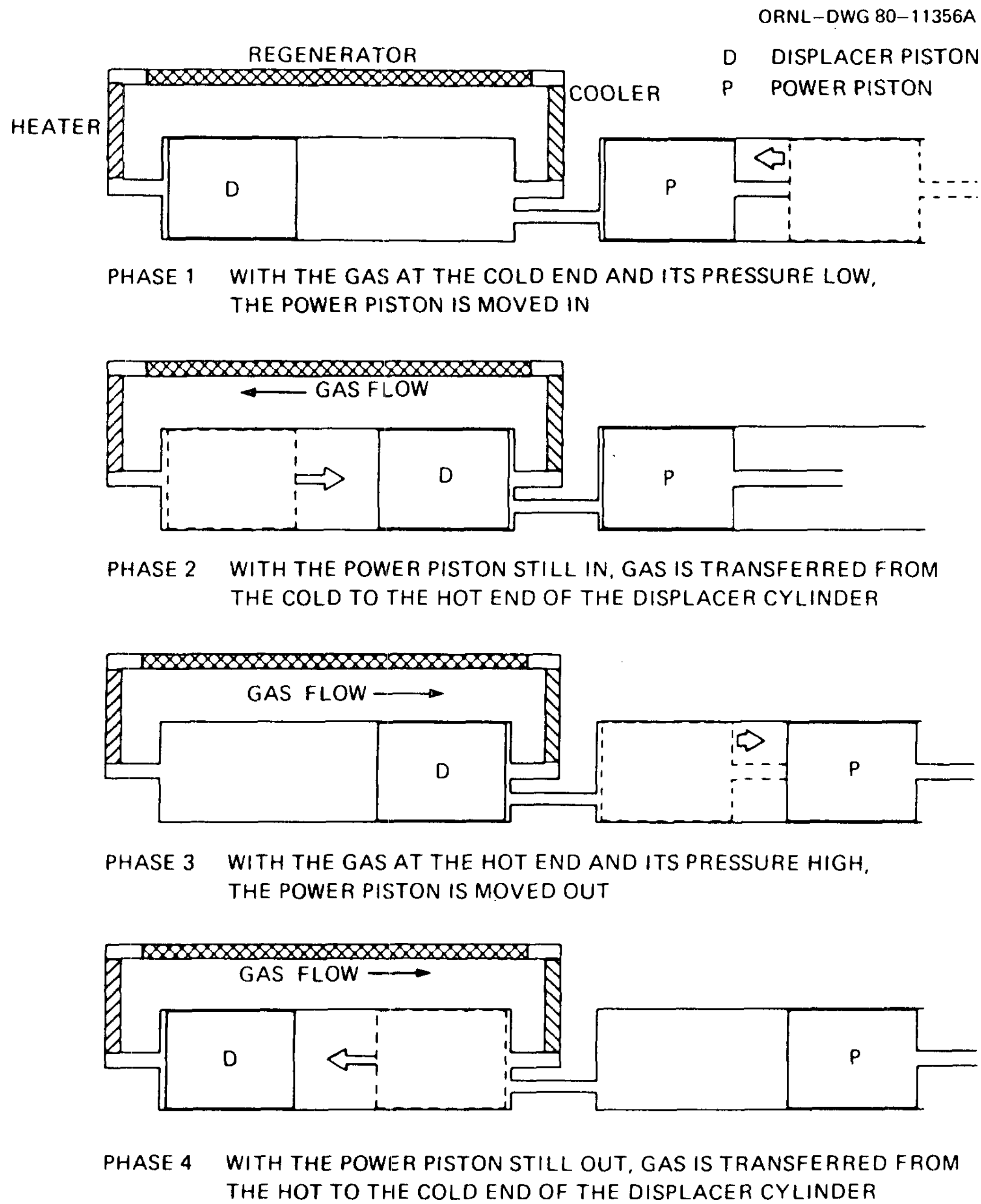

Fig. 2. Basic operation of a Stirling engine.

are both isothermal. The starting point, labelled 0 in the P–V diagram shown in Fig. 3, has all the gas in the (cold) compression space and the power piston fully withdrawn. During the first phase of the cycle, the power piston is moved in, thus compressing the gas, but the displacer piston is stationary. Next, the displacer is moved rightwards, thus displacing all the gas into the expansion cylinder. During this, the second phase, no net volume change occurs, and the action is therefore represented on the P–V diagram by a vertical line. The power piston is now moved all the way out, under the influence of the higher pressure resulting from some (initially, all) of the gas being at the temperature of the hot end; this is Phase 3. Finally, the displacer piston only is moved leftwards, thereby returning all of the gas to the compression space; at the end of this fourth phase, the initial conditions again exist, and the cycle is complete. The connecting tubes are assumed to be of negligibly small volume.

An evaluation of the power input and output during each of these phases follows.

<u>Phase 1</u>. The volume of the expansion cylinder does not change; thus, no work is done. The work in the compression cylinder, all done on the

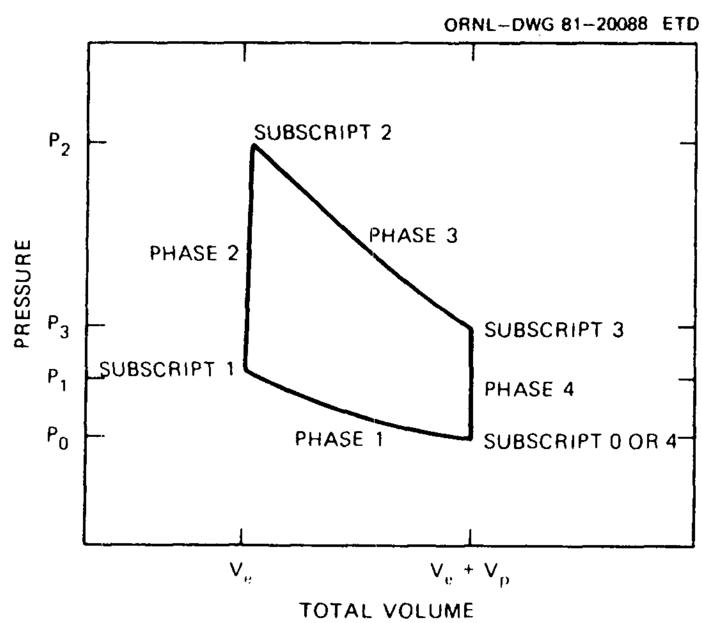

Fig. 3. Nomenclature used in the cycle analysis.

power piston, is denoted by W_{P_1}, and

$$W_{P_1} = \int_{\substack{P=P_o \\ V=V_e+V_p}}^{\substack{P=P_1 \\ V=V_e}} PdV ,$$

where V is the compression space volume, and P is the pressure at some stage during this phase. From the ideal gas equation we have

$$\frac{PV}{T_c} = \frac{P_o(V_e + V_p)}{T_c} = \frac{P_oV_e(1 + k)}{T_c} ,$$

where k is defined as the ratio between power piston and displacer piston swept volume. Therefore,

$$P = P_oV_e(1 + k)\,\frac{1}{V} ; \tag{1}$$

$$\therefore \quad W_{P_1} = P_oV_e(1 + k) \int_{V_e+V_p}^{V_e} \frac{dV}{V} = P_oV_e(1 + k) \log \frac{V_e}{V_e + V_p} ;$$

$$W_{P_1} = -P_oV_e(1 + k) \log (1 + k) . \tag{2}$$

The pressure in the system at the end of this phase is P_1, where from Eq. (1),

$$P_1 = P_o\,(1 + k) . \tag{3}$$

<u>Phase 2</u>. The power piston does not move, but work is done on the displacer piston in the expansion space (and an equal but opposite amount is done on the compression end). The work at the expansion end is W_{e_2}, where

$$W_{e_2} = \int_{\substack{P=P_1 \\ V=0}}^{\substack{P=P_2 \\ V=V_e}} PdV$$

and

$$\frac{PV}{T_e} + \frac{P(V_e - V)}{T_c} = \frac{P_oV_e(1 + k)}{T_c}$$

$$\therefore \quad P = \frac{P_oV_e(1 + k)}{T_c} \; \frac{1}{V/T_e + (V_e - V)/T_c} = \frac{P_oV_e(1 + k)}{V(\tau - 1) + V_e} \; ; \tag{4}$$

$$\therefore \quad W_{e2} = P_oV_e(1 + k) \int_0^{V_e} \frac{dV}{V(\tau - 1) + V_e} = \frac{P_oV_e(1 + k)}{\tau - 1}$$

$$\times \log \frac{V_e(\tau - 1) + V_e}{V_e} \; ;$$

$$W_{e2} = - \frac{P_oV_e(1 + k)}{1 - \tau} \log \tau \; . \tag{5}$$

The pressure in the system at the end of this phase is P_2 where, from Eq. (4)

$$P_2 = \frac{P_oV_e(1 + k)}{V_e(\tau - 1) + V_e} = \frac{P_o(1 + k)}{\tau} \; . \tag{6}$$

This is the highest pressure reached during the cycle.

Phase 3. Beginning with this higher pressure the power piston is moved outwards. The displacer piston does not move, and no work is done on either end of it. The work on the power piston is W_{p3}, where

$$W_{p3} = \int_{\substack{P=P_2 \\ V=0}}^{\substack{P=P_3 \\ V=V_p}} PdV$$

and

$$\frac{PV_e}{T_e} + \frac{PV}{T_c} = \frac{P_oV_e(1 + k)}{T_c} \;;$$

$$\therefore \quad P = \frac{P_oV_e(1 + k)}{T_c} \; \frac{1}{V_e/T_e + V_e/T_c} = \frac{P_oV_e(1 + k)}{V + \tau V_e} \;; \tag{7}$$

$$\therefore \quad W_{p_3} = P_oV_e(1 + k) \int_0^{V_p} \frac{dV}{V + \tau V_e} = P_oV_e(1 + k) \log \frac{V_p + \tau V_e}{\tau V_e} \;;$$

$$W_{p_3} = P_oV_e(1 + k) \log \frac{\tau + k}{\tau} \;. \tag{8}$$

The pressure in the system at the end of this phase is P_3 where, from Eq. (7),

$$P_3 = \frac{P_oV_e(1 + k)}{V_p + \tau V_e} = \frac{P_o(1 + k)}{\tau + k} \;. \tag{9}$$

<u>Phase 4</u>. In the final phase, with the power piston stationary, the displacer piston is moved leftward, returning all the gas to the compression space. The work on the expansion end of the displacer piston, W_{e_4}, is given by

$$W_{e_4} = \int_{\substack{P=P_3\\ V=V_e}}^{\substack{P=P_o\\ V=0}} PdV$$

and

$$\frac{PV}{T_e} + \frac{P(V_e - V)}{T_c} + \frac{PV_p}{T_c} = \frac{P_oV_e(1 + k)}{T_c} \;;$$

$$\therefore \quad P = \frac{P_oV_e(1 + k)}{T_c} \frac{1}{V/T_e + (V_e - V)/T_c + V_p/T_c}$$

$$= \frac{P_oV_e(1 + k)}{V(\tau + 1) + V_e(1 + k)} ; \qquad (10)$$

$$\therefore \quad W_{e4} = P_oV_e(1 + k) \int_{V_e}^{0} \frac{dV}{V(\tau - 1) + V_e(1 + k)} = \frac{P_oV_e(1 + k)}{\tau - 1}$$

$$\times \log \frac{V_e(1 + k)}{V_e(\tau - 1) + V_e(1 + k)} ;$$

$$W_{e4} = P_eV_o \frac{1 + k}{1 - \tau} \log \frac{\tau + k}{1 + k} . \qquad (11)$$

The pressure at the end of this phase is equal to the initial pressure P_o.

Work output, input, and efficiency. The mechanical work output over the whole cycle, W_{out}, is the sum of the work done on the power piston during Phases 1 and 3.

$$W_{out} = W_{p1} + W_{p3}$$

$$= -P_oV_e \frac{1 + k}{1 - \tau} \left[\log (1 + k) - \log \frac{\tau + k}{\tau}\right]$$

$$W_{out} = P_oV_e(1 + k) \log \frac{\tau + k}{\tau(1 + k)} . \qquad (12)$$

According to Rios and Smith[8] the power input (heat) is equal to the work done at the hot-space end of the displacer piston. This work is all done during Phases 2 and 4. Thus,

$$W_{in} = W_{e_2} + W_{e_4}$$

$$= -P_oV_e \frac{1 + k}{1 - \tau} \left(\log \tau - \log \frac{\tau + k}{1 + k} \right)$$

$$W_{in} = P_oV_e \frac{1 + k}{1 - \tau} \log \frac{\tau + k}{\tau(1 + k)} \ . \tag{13}$$

The efficiency η is the ratio of output to input power and is easily obtained from Eqs. (12) and (13):

$$\eta = \frac{W_{out}}{W_{in}} = \frac{P_oV_e(1 + k) \log \frac{\tau + k}{\tau(1 + k)}}{P_oV_e \frac{(1 + k)}{(1 - \tau)} \log \frac{\tau + k}{\tau(1 + k)}} \ ;$$

$$\eta = 1 - \tau \ . \tag{14}$$

This is equal to the Carnot efficiency, as we should expect.

3. ADIABATIC EXPANSION CYLINDER

The analysis of this new case (Fig. 3) allows the expansion cylinder to behave adiabatically but assumes that the heat exchangers and regenerator, although perfectly isothermal, are of negligible volume.

Phase 1. This phase is identical to the isothermal analysis, because the expansion cylinder volume remains at zero during this phase and thus plays no part — adiabatic or isothermal — in gas behavior. The results may therefore be copied from Eqs. (2) and (3):

$$W_{p1} = -P_o V_e (1 + k) \log (1 + k) , \tag{15}$$

and

$$P_1 = P_o (1 + k) . \tag{16}$$

Phase 2. Although this phase of the process is described as "heating at constant volume," note that not all the gas in that volume undergoes the same heating process. Indeed, over any portion of this phase only the gas passing through the heater at that time is being heated. The gas that passed through the heater earlier in this phase, and is already in the adiabatic expansion cylinder, receives no further energy from the heat source, except that which is carried in by new gas entering the cylinder.

The gas entering the expansion cylinder as the displacer is moved rightwards does so at the temperature T_{eh} of the hot-end heat exchanger. The gas in the cylinder will generally be at a higher temperature than T_{eh} because it will have been compressed by the rise in pressure as the displacer movement proceeds in an adiabatic space. The gas in the expansion cylinder is continually being compressed and mixed with fresh incoming gas at temperature T_{eh} as the displacer movement proceeds. The mixing of gas at two different temperatures is an irreversible process, so the system no longer has Carnot efficiency.

When the displacer has moved rightwards through a volume V and then moves through a further volume dV, the mass of gas leaving the isothermal cold space, dM_c, is given by

$$dM_c = \frac{PdV - (V_e - V)dP}{RT_{ch}} .$$

The first term on the right hand side is simply the mass of gas contained in the volume dV at pressure P and temperature T_{ch}. The second term is the change in mass of gas contained in a volume $V_e - V$ (the volume remaining in the cold cylinder) when the pressure is increased by dP.

The corresponding equation for the mass of gas entering the hot cylinder, dM_e, is slightly more complicated, because this space is adiabatic:

$$dM_e = \frac{PdV + VdP/\gamma}{RT_{eh}} .$$

This equation was a basis of the numerical cycle analysis done by Qvale and Smith[9] and is also a result contained, although in a somewhat disguised form, in the original Finkelstein equations for nonideal Stirling cycles.

Physically, the first term on the right hand side represents the mass of a volume dV of gas when it leaves the heater at a temperature T_{eh} and pressure P.

The second term represents the change of mass in a volume V when the pressure changes by dP: it is really a combination of the fractional increase in density — dP/P — caused by the pressure increase and the fractional decrease in density — dT/T_{eh} — caused by the adiabatic increase in temperature. The temperature increase is given by the usual relationship

$$\gamma \frac{dT}{T} = (\gamma - 1) \frac{dP}{P} ,$$

and the net result is a fractional increase in density given by

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} = \frac{dP}{P} \left(1 - \frac{\gamma - 1}{\gamma}\right) = \frac{1}{\gamma} \frac{dP}{P} .$$

The total mass of gas in the system is constant. Therefore, the mass leaving the cold cylinder must equal the mass entering the hot cylinder (i.e., $dM_e = dM_c$) or

$$\frac{PdV - (V_e - V)dP}{RT_{ch}} = \frac{PdV + VdP/\gamma}{RT_{eh}}$$

$$\therefore \quad PdV\left(\frac{1}{T_{ch}} - \frac{1}{T_{eh}}\right) = dP\left(\frac{V_e}{T_{ch}} + \frac{V}{\gamma T_{eh}} - \frac{V}{T_{ch}}\right);$$

$$\therefore \quad \frac{dP}{P} = \frac{(1-\tau)dV}{V_e - V(1-\tau/\gamma)} \;. \tag{17}$$

Integrating from the starting point of this phase, when the displacer is fully leftward and $P = P_1$, we find

$$\int_{P=P_1}^{P=P} \frac{dP}{P} = \int_{V=0}^{V=V} \frac{(1-\tau)dV}{V_e - V(1-\tau/\gamma)} \;.$$

Both sides can be integrated, yielding a relationship between P and V:

$$\log P/P_1 = \frac{1-\tau}{1-\tau/\gamma} \log \frac{V_e}{V_e - V(1-\tau/\gamma)} = \frac{\gamma(1-\tau)}{\gamma-\tau} \times \log \frac{V_e}{V_e - V(1-\tau/\gamma)} \;,$$

and

$$P = P_1 \left(\frac{V_e}{V_e - V(1-\tau/\gamma)}\right)^{\gamma(1-\tau)/(\gamma-\tau)} \;. \tag{18}$$

The work done at the hot end of the displacer piston during this phase is

W_{e2}, where

$$W_{e2} = \int_{\substack{P=P_1\\V=0}}^{\substack{P=P_2\\V=V_e}} PdV \; .$$

Substitute for P from Eq. (18)

$$W_{e2} = P_1 \int_0^{V_e} \left(\frac{V_e}{V_e - V(1 - \tau/\gamma)}\right)^{\gamma(1-\tau)/(\gamma-\tau)} dV$$

$$= P_1 \int_0^{V_e} \left(\frac{1}{1 - V(1 - \tau/\gamma)/V_e}\right)^{\gamma(1-\tau)/(\gamma-\tau)} dV \; .$$

Despite its rather overwhelming appearance, this integral is an elementary one of the form $(a + bx)^c dx$ and is easily performed, with the aid of a handbook of common integrals[10] if necessary. The general result is

$$\int (a + bx)^c dx = \frac{(a + bx)^{c+1}}{b(c + 1)} \; ;$$

in this case

$$a = 1 \; ,$$

$$b = -(1 - \tau/\gamma)/V_e \; ,$$

$$c = -\gamma(1 - \tau)/(\gamma - \tau) \; ,$$

$$\therefore \quad W_{e2} = P_1 \left[\frac{[1 - V(1 - \tau/\gamma)/V_e]^{-\gamma(1-\tau)/(\gamma-\tau)+1}}{-\dfrac{(1 - \tau/\gamma)}{V_e} \cdot [-\gamma(1 - \tau)/(\gamma - \tau) + 1]}\right]_0^{V_e}$$

$$= -P_1 V_e \frac{\gamma}{\tau(\gamma - 1)} \left[\left(\frac{\tau}{\gamma}\right)^{\tau(\gamma-1)/(\gamma-\tau)} - 1\right] ,$$

and by substituting from Eq. (16) for P_2, we can write W_{e_2} in terms of the initial conditions:

$$W_{e_2} = -P_o V_e (1 + k) \frac{\gamma}{\tau(\gamma - 1)} \left[\left(\frac{\tau}{\gamma}\right)^{\tau(\gamma-1)/(\gamma-\tau)} - 1 \right] . \qquad (19)$$

By making the substitution $x = \tau(\gamma - 1)/(\gamma - \tau)$, Eq. (19) can be rewritten as

$$W_{e_2} = -P_o V_e (1 + k) \frac{[(\tau/\gamma)^x - 1]}{(1 - \tau)x} .$$

As $\gamma \to 1$ and the gas behavior in the cylinder becomes the same whether the space is considered adiabatic or isothermal, $x \to 0$:

$$\underset{\gamma \to 1}{\text{limit}} \; W_{e_2} = \underset{x \to 0}{\text{limit}} \; -P_o V_e \frac{(1 + k)\tau^x - 1}{(1 - \tau)x} .$$

It is known (e.g., Ref. 11) that

$$\underset{x \to 0}{\text{limit}} \; \frac{a^x - 1}{x} = \log_e \alpha ,$$

and therefore

$$\underset{\gamma \to 1}{\text{limit}} \; W_{e_2} = -P_o V_e \; (1 + k) \frac{\log \tau}{1 - \gamma} .$$

This expression is, as we should expect, in exact agreement with the value given by Eq. (5) for an all-isothermal machine.

The pressure in the system at the end of this phase is P_2 where, from Eq. (18),

$$P_2 = P_1 \left[\frac{V_e}{V_e - V_e(1 - \tau/\gamma)} \right]^{\gamma(1-\gamma)/(\gamma/\tau)}$$

$$= P_o (1 + k) \left(\frac{\gamma}{\tau}\right)^{\gamma(1-\tau)/(\gamma-\tau)} . \qquad (20)$$

To carry out the calculation of pressure and work output during the next phase of the cycle, we need to know the temperature of the gas in the expansion cylinder at the end of Phase 2, T_{e2}.

From the ideal gas law

$$\frac{P_2 V_e}{T_{2e}} = \frac{P_o V_e (1 + k)}{T_{ch}} ,$$

$$\therefore \quad T_{2e} = \frac{P_2 \, T_{ch}}{P_o (1 + k)}$$

$$= \left[\frac{P_o (1 + k)}{P_o (1 + k)} \left(\frac{\gamma}{\tau} \right)^{\gamma(1-\tau)/(\gamma-\tau)} \right] T_{ch}$$

$$= \left(\frac{\gamma}{\tau} \right)^{\gamma(1-\tau)/(\gamma-\tau)} T_{ch} . \tag{21}$$

At this stage, a new variable can be defined to simplify the equations:

$$\tau_2 = \left(\frac{\tau}{\gamma} \right)^{\gamma(1-\tau)/(\gamma-\tau)} . \tag{22}$$

When $\tau \to 1$ (i.e., when the heater and cooler are kept at the same temperature), $\tau_2 \to 1$. Furthermore, when $\gamma \to 1$, so that the gas behavior is the same whether the space is considered isothermal or adiabatic, then $\tau_2 \to \tau$, and the new variable fills the same role as the actual heat exchanger temperature ratio. The variable τ_2 represents the actual heat exchanger temperature ratio modified by the effect of the adiabatic space.

With this substitution, we can rewrite the equations for W_{e2}, P_2, and T_{2e} in more convenient and compact forms:

$$W_{e2} = P_o V_e (1 + k) \, \frac{\gamma\left[1 - \tau_2^{\frac{\tau(\gamma-1)}{\gamma(1-\tau)}}\right]}{\tau(\gamma - 1)} , \tag{23}$$

$$P_2 = P_0 \frac{(1 + k)}{\tau_2} , \tag{24}$$

$$T_{2e} = \frac{T_{ch}}{\tau_2} = \frac{\tau}{\tau_2} T_{eh} . \tag{25}$$

Equation (25) may be thought of as a physical definition of the modified temperature parameter τ_2.

<u>Phase 3</u>. With the displacer stationary at the cold end, the power piston is withdrawn so that the compression-space volume increases from zero to V_p. During this phase, gas is always leaving the expansion cylinder; the gas remaining in the expansion cylinder therefore does not receive any admixture of gas from the outside, so its behavior is simply adiabatic. If the temperature of the gas in the hot cylinder at any moment during this phase is T_e, then from the usual adiabatic gas law

$$\frac{T_e}{T_{2e}} = \left(\frac{P}{P_2}\right)^{1-1/\gamma}$$

$$\therefore \quad T_e = T_{2e}\left(\frac{P}{P_2}\right)^{1-1/\gamma} . \tag{26}$$

The change of mass in the hot cylinder as the pressure falls can be calculated by the same methods used during the Phase 2 analysis, with the additional simplification that volume of the expansion space does not change during Phase 3:

$$dM_e = \frac{V_e dP}{\gamma R T_e} .$$

During this phase the pressure is always falling, that is, dP is actually a negative quantity.

The compression space is isothermal, and the gas in it is always at temperature T_{ch}, so that calculation of the increase of mass at the cold end of the machine is easy:

$$dM_c = \frac{PdV}{RT_{ch}} + \frac{VdP}{RT_{ch}} .$$

Once again, the mass leaving the expansion space must all enter the compression space. Therefore

$$\frac{V_e dP}{\gamma RT_e} + \frac{PdV}{RT_{ch}} + \frac{VdP}{RT_{ch}} = 0$$

$$\therefore \quad PdV = -dP\left(V + \frac{V_e T_{ch}}{\gamma T_e}\right) . \tag{27}$$

From the ideal gas law

$$\frac{PV_e}{T_e} + \frac{PV}{T_{ch}} = \frac{P_2 V_e}{T_{2e}} \tag{28}$$

$$\therefore \quad V = \frac{T_{ch}}{P}\left(\frac{P_2 V_e}{T_{2e}} - \frac{PV_e}{T_e}\right) = \frac{T_{ch}}{P}\left(\frac{P_2}{T_{2e}} - \frac{P}{T_e}\right) V_e .$$

First, substitute for T_e from Eq. (26)

$$V = \frac{T_{ch}}{P}\left[\frac{P_2}{T_{2e}} - \frac{P}{T_{2e}}\left(\frac{P_2}{P}\right)^{1-1/\gamma}\right] V_e = \frac{T_{ch}}{T_{2e}}\left[\frac{P_2}{P} - \left(\frac{P_2}{P}\right)^{1-1/\gamma}\right] V_e . \tag{28}$$

Then, substitute for T_{2e} from Eq. (25)

$$V = \tau_2 \left[\frac{P_2}{P} - \left(\frac{P_2}{P}\right)^{1-1/\gamma}\right] V_e . \tag{29}$$

Now, substitute this expression for V into Eq. (27):

$$PdV = -dP\left[\frac{\tau_2 P_2}{P} - \tau_2\left(\frac{P_2}{P}\right)^{1-1/\gamma} + \frac{T_{ch}}{\gamma Te}\right]V_e \ .$$

Substitute for T_e from Eq. (26)

$$PdV = -dP\left[\tau_2\left(\frac{P_2}{P}\right) - \tau_2\left(\frac{P_2}{P}\right)^{1-1/\gamma} + \frac{T_{ch}}{\gamma T_{2e}}\left(\frac{P_2}{P}\right)^{1-1/\gamma}\right]V_e \ .$$

Finally, substitute for T_{2e} from Eq. (25)

$$PdV = dP\left[\frac{\gamma - 1}{\gamma}\left(\frac{P_2}{P}\right)^{1-1/\gamma} - \frac{P_2}{P}\right]\tau_2 V_e \ . \tag{30}$$

The work done on the power piston during this phase is W_{p3}, where

$$W_{p3} = \int_{\substack{P=P_2\\V=0}}^{\substack{P=P_3\\V=V_p}} PdV \ ,$$

and by substituting for PdV from Eq. (30) this integral becomes

$$W_{p3} = \tau_2 P_2 V_e \int_{P_2}^{P_3}\left(\frac{\gamma - 1}{\gamma}\frac{P^{1/\gamma - 1}}{P_2^{\ 1/\gamma}} - \frac{1}{P}\right)dP$$

$$= \tau_2 P_2 V_e \left[\frac{\gamma - 1}{\gamma}\frac{\gamma P^{1/\gamma}}{P_2^{\ 1/\gamma}} - \log P\right]_{P_2}^{P_3}$$

$$= \tau_2 P_2 V_e \left\{(\gamma - 1)\left[\left(\frac{P_3}{P_2}\right)^{1/\gamma} - 1\right] - \log\frac{P_3}{P_2}\right\} \ .$$

Substitute for P_2 from Eq. (24)

$$W_{p3} = P_o V_e (1 + k) \left[(\gamma - 1) \left(\frac{P_3^{\,1/\gamma}}{P_2^{\,1/\gamma}} - 1 \right) - \log \frac{P_3}{P_2} \right] . \tag{31}$$

Returning to Eq. (29) and writing $V = V_p$ to find the pressure/volume relationship at the end of Phase 3 yields

$$V_p = \tau_2 \left[\frac{P_2}{P_3} - \left(\frac{P_2}{P_3} \right)^{1-1/\gamma} \right] V_e$$

$$= \tau_2 \frac{P_2}{P_3} V_e \left[1 - \left(\frac{P_3}{P_2} \right)^{1/\gamma} \right] ;$$

$$\therefore \quad \left(\frac{P_3}{P_2} \right)^{1/\gamma} - 1 = - \frac{P_3 V_p}{\tau_2 P_2 V_e} = - \frac{k}{\tau_2} \frac{P_3}{P_2} . \tag{32}$$

Substitute this into Eq. (31)

$$W_{p3} = -P_o V_e (1 + k) \left[(\gamma - 1) \frac{k}{\tau_2} \frac{P_3}{P_2} + \log \frac{P_3}{P_2} \right] . \tag{33}$$

To find P_3/P_2 we numerically solve Eq. (32), that is,

$$\left(\frac{P_3}{P_2} \right)^{1/\gamma} + \frac{k}{\tau_2} \left(\frac{P_3}{P_2} \right) = 1 . \tag{34}$$

Only in special cases is it possible to solve this equation analytically.

The gas temperature in the expansion cylinder at the end of this phase, T_{3e}, may be found by substituting P_3/P_2 for P/P_2 in Eq. (26) and then using the expression for T_{2e} in Eq. (25):

$$T_{3e} = T_{2e}\left(\frac{P_3}{P_2}\right)^{1-1/\gamma} = \frac{T_{ch}}{\tau_2}\left(\frac{P_3}{P_2}\right)^{1-1/\gamma} . \tag{35}$$

A new temperature ratio τ_3 can be defined for convenience such that

$$\tau_3 = \frac{T_{ch}}{T_{3e}} = \tau_2\left(\frac{P_2}{P_3}\right)^{1-1/\gamma} . \tag{36}$$

With the aid of these relationships, Eq. (33) can be rewritten in terms of the gas temperature rather than gas pressure, a form that may sometimes be more convenient.

$$W_{p3} = -P_oV_e(1 + k)\left[(\gamma - 1)\,\frac{k}{\tau_2}\left(\frac{\tau_2}{\tau_3}\right)^{\gamma/(\gamma-1)} + \log\left(\frac{\tau_2}{\tau_3}\right)^{\gamma/(\gamma-1)}\right] . \tag{37}$$

<u>Phase 4</u>. We now have enough information [in Eqs. (15) and (33)] to calculate the work output from the power piston for the cycle, but to calculate the efficiency we need the work input as well; therefore, we must continue the analysis through Phase 4. The gas in the expansion space continues to behave adiabatically, so we may continue to apply Eq. (26); therefore

$$T_e = T_{2e}\left(\frac{P}{P_2}\right)^{1-1/\gamma} = \frac{T_{ch}}{\tau_2}\left(\frac{P}{P_2}\right)^{1-1/\gamma} .$$

From the ideal gas law

$$\frac{PV}{T_e} + \frac{P(V_e - V)}{T_{ch}} + \frac{PV_p}{T_{ch}} = \frac{P_oV_e(1 + k)}{T_{ch}}$$

$$\therefore \quad PV\left(\frac{1}{T_e} - \frac{1}{T_{ch}}\right) = \frac{(P_o - P)V_e(1 + k)}{T_{ch}} ;$$

$$\therefore \quad V = P_oV_e(1 + k)\,\frac{1/P_o - 1/P}{1 - T_{ch}/T_e} \ . \tag{39}$$

The work done in the expansion cylinder during this phase is W_{e4}, where

$$W_{e4} = \int_{\substack{P=P_3\\V=V_e}}^{\substack{P=P_o\\V=0}} PdV \ .$$

Integrate by parts

$$W_{e4} = -P_3V_e - \int VdP$$

and substitute for V from Eq. (39)

$$W_{e4} = -P_3V_e - P_oV_e(1 + k)\int \frac{1/P_o - 1/P}{1 - T_{ch}/T_e}\,dP$$

$$= -P_oV_e(1 + k)\left(\frac{P_3}{P_o(1 + k)} + \int \frac{1/P_o - 1/P}{1 - T_{ch}/T_e}\,dP\right)$$

$$= -P_oV_e(1 + k)\left(\frac{P_3}{P_2}\,\frac{P_2}{P_o(1 + k)} + \int \frac{1/P_o - 1/P}{1 - T_{ch}/T_e}\,dP\right) \ .$$

Substitute for P_2 from Eq. (24)

$$W_{e4} = -P_oV_e(1 + k)\left[\frac{1}{\tau_2}\left(\frac{P_3}{P_2}\right) + \int \frac{1/P_o - 1/P}{1 - T_{ch}/T_e}\,dP\right] \ . \tag{40}$$

The integral is most conveniently evaluated by using expansion-space gas temperature T_e as the primary variable:

$$\int \frac{1/P_o - 1/P}{1 - T_{ch}/T_e}\,dP = \frac{1}{P_o}\int \frac{dP}{1 - T_{ch}/T_e} - \int \frac{dP}{P(1 - T_{ch}/T_e)}$$

and for an adiabatic gas

$$\frac{dP}{P} = \frac{\gamma}{\gamma - 1}\,\frac{dT_e}{T_e}\ ;$$

$$\int = \frac{1}{P_o} \int \frac{dP}{1 - T_{ch}/T_e} - \frac{\gamma}{\gamma - 1} \int_{T_{3e}}^{T_{4e}} \frac{dT_e}{T_e - T_{ch}}$$

$$\int = \frac{1}{P_o} \int \frac{dP}{1 - T_{ch}/T_e} - \frac{\gamma}{\gamma - 1} \log \frac{T_{4e} - T_{ch}}{T_{3e} - T_{ch}}\ . \tag{41}$$

The variable T_{4e} is the temperature of the small amount of gas remaining in the expansion space as the displacer piston approaches its extreme leftward position. At the time, the gas pressure approaches P_o. Therefore, from Eq. (38)

$$\frac{T_{4e}}{T_{2e}} = \left(\frac{P_o}{P_2}\right)^{1-1/\gamma}\ .$$

Substitute for P_o/P_2 from Eq. (24), and substitute for T_{2e} from Eq. (25):

$$\frac{T_{4e}}{T_{ch}} = \frac{1}{\tau_2}\left(\frac{\tau_2}{1 + k}\right)^{1-1/\gamma}\ ,$$

and we may define a new temperature ratio τ_4 such that

$$\tau_4 = \frac{T_{ch}}{T_{4e}} = \tau_2\left(\frac{1 + k}{\tau_2}\right)^{1-1/\gamma}$$

$$= (1 + k)^{1-1/\gamma}\,\tau_2^{\,1/\gamma} = (1 + k)^{1-1/\tau}\left(\frac{\tau}{\gamma}\right)^{\frac{1-\tau}{\gamma-\tau}}\ . \tag{42}$$

The right hand term in Eq. (41) can now be rewritten in terms of the temperature ratios τ_3 and τ_4 and the result substituted back into Eq. (40):

$$W_{e4} = -P_o V_e (1 + k) \left[\frac{1}{\tau_2} \frac{P_3}{P_2} - \frac{\gamma}{\gamma - 1} \right.$$

$$\left. \times \log \frac{\tau_3 (1 - \tau_4)}{\tau_4 (1 - \tau_3)} + \frac{1}{P_o} \int \frac{dP}{1 - T_{ch}/T_e} \right] . \quad (43)$$

The remaining integral looks more straightforward than it is:

$$\frac{1}{P_o} \int_{\substack{P=P_3 \\ T=T_{3e}}}^{\substack{P=P_o \\ T=T_{4e}}} \frac{dP}{1 - T_{ch}/T_e} = \frac{1}{P_o} \int \frac{T_e}{T_e - T_{ch}} \, dP$$

$$= \frac{1}{P_o} \left(\int \frac{T_e - T_{ch}}{T_e - T_{ch}} \, dP + \int \frac{T_{ch}}{T_e - T_{ch}} \, dP \right)$$

$$= \frac{1}{P_o} \left(\int_{P_3}^{P_o} dP + T_{ch} \int \frac{dP}{T_e - T_{ch}} \right)$$

$$= 1 - \frac{P_3}{P_o} + \frac{T_{ch}}{P_o} \int \frac{dP}{T_e - T_{ch}}$$

$$= 1 - \frac{P_3}{P_2} \frac{P_2}{P_o} + \frac{T_{ch}}{P_o} \int \frac{dP}{T_e - T_{ch}} . \quad (44)$$

Substitute for P_2/P_o from Eq. (24) and insert the resulting form of

Eq. (44) back into Eq. (43):

$$W_{e4} = -P_oV_e(1 + k)\left[\frac{1}{\tau_2}\left(\frac{P_3}{P_2}\right) + 1 - \left(\frac{P_3}{P_2}\right)\frac{(1 + k)}{\tau_2} - \frac{\gamma}{\gamma - 1}\right.$$

$$\left. \times \log\frac{\tau_3(1 - \tau_4)}{\tau_4(1 - \tau_3)} + \frac{T_{ch}}{P_o}\int\frac{dP}{T_e - T_{ch}}\right]$$

$$= -P_oV_e(1 + k)\left[1 - \frac{k}{\tau_2}\left(\frac{P_3}{P_2}\right) - \frac{\gamma}{\gamma - 1}\log\frac{\tau_3(1 - \tau_4)}{\tau_4(1 - \tau_3)}\right.$$

$$\left. + \frac{T_{ch}}{P_o}\int\frac{dP}{T_e - T_{ch}}\right] . \qquad (45)$$

From Eq. (38) we have

$$\frac{\tau_2 T_e}{T_{ch}} = \left(\frac{P}{P_2}\right)^{1-1/\gamma}$$

$$\therefore \quad P = P_2\left(\frac{\tau_2 T_e}{T_{ch}}\right)^{\gamma/(\gamma-1)} ;$$

$$\therefore \quad dP = \frac{\gamma}{\gamma - 1} P_2\left(\frac{\tau_2 T_e}{T_{ch}}\right)^{\gamma/(\gamma-1)}\frac{dT_e}{T_e} .$$

Substitute for P_2 from Eq. (24)

$$dP = \frac{\gamma}{\gamma - 1}\frac{1 + k}{\tau_2} P_o\left(\frac{\tau_2}{T_{ch}}\right)^{\gamma/(\gamma-1)} T_e^{1/(\gamma-1)}dT_e .$$

We can use this relationship to evaluate the integral in Eq. (45):

$$\frac{T_{ch}}{P_o} \int \frac{dP}{T_e - T_{ch}} = \frac{\gamma}{\gamma - 1} (1 + k) \left(\frac{\tau_2}{T_{ch}}\right)^{1/(\gamma-1)} \int_{T_{3e}}^{T_{4e}} \frac{T_e^{1/(\gamma-1)} dT_e}{T_e - T_{ch}} . \tag{46}$$

Under certain circumstances, the integral can be expressed in terms of elementary functions. Fortunately, these circumstances include the cases of most interest in this analysis. For an ideal monatomic or diatomic gas, $\gamma = 5/3$ or $7/5$, respectively; the exponent of T_e in the numerator of the integral is then 3/2 or 5/2, and the integral may be evaluated using reduction formulae given in tables of standard integrals (see Ref. 10).

For a monatomic gas the integral becomes

$$\frac{\gamma}{\gamma - 1} (1 + k) \left(\frac{\tau_2}{T_{ch}}\right)^{1/(\gamma-1)} \int_{T_{3e}}^{T_{4e}} \frac{T_e^{1/(\gamma-1)} dT_e}{T_e - T_{ch}} = \frac{5}{2} \frac{(1 + k)\tau_2^{3/2}}{T_{ch}^{3/2}} \int_{T_{3e}}^{T_{4e}} \frac{T_e \sqrt{T_e} dT_e}{T_e - T_{ch}}$$

$$= \frac{5(1 + k)\tau_2^{3/2}}{2T_{ch}^{3/2}} \left[2T_e^{1/2} \left(\frac{T_e}{3} + T_{ch}\right) + T_{ch}^2 \int \frac{dT_e}{\sqrt{T_e}(T_e - T_{ch})} \right]_{T_{3e}}^{T_{4e}}$$

$$= \frac{5}{2} \frac{(1 + k)\tau_2^{3/2}}{T_{ch}^{3/2}} \left[\frac{2}{3} \left(T_{4e}^{3/2} - T_{3e}^{3/2}\right) + 2T_{ch} \left(T_{4e}^{1/2} - T_{3e}^{1/2}\right) + T_{ch}^{3/2} \right.$$

$$\left. \times \log \frac{-T_{ch} - T_{4e} + 2\sqrt{T_{ch} T_{4e}}}{-T_{ch} - T_{3e} + 2\sqrt{T_{ch} T_{3e}}} - T_{ch}^{3/2} \log \frac{T_{4e} - T_{ch}}{T_{3e} - T_{ch}} \right]$$

$$= \frac{5}{2} (1 + k)\tau_2^{3/2} \left[\frac{2}{3} \left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}}\right) + \frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}} \right.$$

$$\left. + \log \frac{\tau_3(1 - \tau_4^{1/2})^2}{\tau_4(1 - \tau_3^{1/2})^2} - \log \frac{\tau_3(1 - \tau_4)}{\tau_4(1 - \tau_3)} \right] .$$

Substitute this back into Eq. (45), while setting $\gamma = 5/3$:

$$W_{e4} = -P_oV_e(1 + k)\left\{1 - \frac{k}{\tau_2}\left(\frac{P_3}{P_2}\right) - \frac{5}{2}\left[1 + (1 + k)\tau_2^{3/2}\right]\right.$$

$$\times \log\frac{\tau_3(1 - \tau_4)}{\tau_4(1 - \tau_3)} + 5(1 + k)\tau_2^{3/2}\left[\log\frac{\tau_3^{1/2}(1 - \tau_4^{1/2})}{\tau_4^{1/2}(1 - \tau_3^{1/2})}\right.$$

$$\left.\left. + \left(\frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}}\right) + \frac{1}{3}\left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}}\right)\right]\right\} ;$$

$$W_{e4} = -P_oV_e(1 + k)\left\{1 - \frac{k}{\tau_2}\,\frac{P_3}{P_2} - 5[1 + (1 + k)\tau_2^{3/2}]\right.$$

$$\times \log\frac{(1/\tau_4 - 1)^{1/2}}{(1/\tau_3 - 1)^{1/2}} + 5(1 + k)\tau_2^{3/2}\left[\log\frac{1/\tau_4^{1/2} - 1}{1/\tau_3^{1/2} - 1}\right.$$

$$\left.\left. + (1/\tau_4^{1/2} - 1/\tau_3^{1/2}) + (1/\tau_4^{3/2} - 1/\tau_3^{3/2})/3\right]\right\} . \quad (47)$$

Equation (47) is valid for a monatomic gas only.

For a diatomic gas the integral in Eq. (46) becomes

$$\frac{\gamma}{\gamma - 1}(1 + k)\left(\frac{\tau_2}{T_{ch}}\right)^{1/(\gamma-1)}\int_{T_{3e}}^{T_{4e}}\frac{T_e^{1/(\gamma-1)}dT_e}{T_e - T_{ch}} = \frac{7}{2}\,\frac{(1 + k)\tau_2^{5/2}}{T_{ch}^{5/2}}$$

$$\times \int_{T_{3e}}^{T_{4e}}\frac{T_e^2\sqrt{T_e}}{T_e - T_{ch}}\,dT_e$$

$$= \frac{7(1+k)\tau_2^{5/2}}{2T_{ch}^{5/2}} \left[2T_e^{1/2} \left(\frac{T_e^2}{5} + \frac{T_e T_{ch}}{3} + T_{ch}^2 \right) \right.$$

$$\left. + T_{ch}^3 \int \frac{dT_e}{T_e (T_e - T_{ch})} \right]_{T_{3e}}^{T_{4e}}$$

$$= \frac{7}{2} \frac{(1+k)\tau_2^{5/2}}{T_{ch}^{5/2}} \left[\frac{2}{5} (T_{4e}^{5/2} - T_{3e}^{5/2}) + \frac{2T_{ch}}{3} (T_{4e}^{3/2} - T_{3e}^{3/2}) + 2T_{ch}^2 \right.$$

$$\left. x\ (T_{4e}^{1/2} - T_{3e}^{3/2}) + \log \frac{-T_{ch} - T_{4e} + 2\sqrt{T_{ch}\ T_{4e}}}{-T_{ch} - T_{3e} + 2\sqrt{T_{eh}\ T_{3e}}} - \log \frac{T_{4e} - T_{ch}}{T_{3e} - T_{ch}} \right]$$

$$= \frac{7}{2} (1+k)\ \tau_2^{5/2} \left[\frac{2}{5} \left(\frac{1}{\tau_4^{5/2}} - \frac{1}{\tau_3^{5/2}} \right) + \frac{2}{3} \left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}} \right) \right.$$

$$\left. + 2 \left(\frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}} \right) + \log \frac{\tau_3 (1 - \tau_4^{1/2})^2}{\tau_4 (1 - \tau_3^{1/2})^2} - \log \frac{\tau_3 (1 - \tau_3)}{\tau_4 (1 - \tau_3)} \right].$$

Substitute this back into Eq. (45), while setting $\gamma = 7/5$:

$$W_{e4} = -P_o V_e\ (1+k) \left\{ 1 - \frac{k}{\tau_2} \frac{P_3}{P_2} - \frac{7}{2} \left(1 + (1+k)\tau_2^{5/2} \right) \log \frac{\tau_3 (1 - \tau_4)}{\tau_4 (1 - \tau_3)} \right.$$

$$+ 7\ (1+k)\tau_2^{3/2} \left[\log \frac{\tau_3^{1/2} (1 - \tau_4)^{1/2}}{\tau_4^{1/2} (1 - \tau_3)^{1/2}} + \frac{1}{5} \left(\frac{1}{\tau_4^{5/2}} - \frac{1}{\tau_3^{5/2}} \right) \right.$$

$$\left. \left. + \frac{1}{3} \left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}} \right) + \frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}} \right] \right\} \ ;$$

$$W_{e4} = -P_oV_e(1 + k)\left\{1 - \frac{k}{\tau_2}\frac{P_3}{P_2} - 7[1 + (1 + k)\tau_2^{5/2}]\right.$$

$$\times \log\frac{(1/\tau_4 - 1)^{1/2}}{(1/\tau_3 - 1)^{1/2}} + 7(1 + k)\tau_2^{5/2}\left[\log\frac{1/\tau_4^{1/2} - 1}{1/\tau_3^{1/2} - 1}\right.$$

$$+ (1/\tau_4^{1/2} - 1/\tau_3^{1/2}) + (1/\tau_4^{3/2} - 1/\tau_3^{3/2})/3$$

$$\left.\left. + (1/\tau_4^{5/2} - 1/\tau_3^{5/2})/5\right]\right\} . \quad (48)$$

Equation (48) is valid for a diatomic gas only.

Work output, input, and efficiency. The work output is the sum of the work done on the power piston during Phases 1 and 3, which may be obtained from Eqs. (15) and either (33) or (37).

From Eqs. (15) and (33)

$$W_{p1} + W_{p3} = W_{out} = -P_oV_e(1 + k)\log(1 + k)$$

$$- P_oV_e(1 + k)\left[(\gamma - 1)\frac{k}{\tau_2}\frac{P_3}{P_2} + \log\frac{P_3}{P_2}\right] ;$$

$$W_{out} = P_oV_e(1 + k)\left[\log\frac{P_2}{(1 + k)P_3} - (\gamma - 1)\frac{k}{\tau_2}\frac{P_3}{P_2}\right] . \quad (49)$$

In evaluating Eq. (49), the temperature ratio τ_2 must first be calculated from Eq. (22), and P_3/P_2 is then calculated by solving Eq. (34).

According to Rios and Smith[8] the heat input to the heat exchanger is equal to the indicated work done in the adjacent adiabatic cylinder. The work input is therefore the sum of the work done in the expansion space during Phases 2 and 4. These values may be taken from Eqs. (19) and either (47) or (48).

For a monatomic gas, the work input is

$$W_{in} = P_o V_e (1 + k) \left\{ \frac{5}{2\tau} [1 - (3\tau/5)^{2\tau/(5-3\tau)}] - 1 + \frac{k}{\tau_2} \frac{P_3}{P_2} + \frac{5}{2} \right.$$

$$\times [1 + (1 + k)^{3/2} \tau_2] \log \frac{1/\tau_4 - 1}{1/\tau_3 - 1} - 5 (1 + k)^{3/2} \tau_2^{3/2}$$

$$\left. \times \left[\log \frac{1/\tau_4^{1/2} - 1}{1/\tau_3^{1/2} - 1} + \left(\frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}} \right) + \frac{1}{3} \left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}} \right) \right] \right\} . \quad (50)$$

For a diatomic gas, the work input is

$$W_{in} = P_o V_e (1 + k) \left\{ \frac{7}{2\tau} [1 - (5\tau/7)^{2\tau/(7-5\tau)}] \right.$$

$$- 1 + \frac{k}{\tau_2} \frac{P_3}{P_2} + \frac{7}{2} \times [1 + (1 + k)^{3/2} \tau_2] \log \frac{1/\tau_4 - 1}{1/\tau_3 - 1}$$

$$- 7 (1 + k) \tau_2^{3/2} \left[\log \frac{1/\tau_4^{1/2} - 1}{1/\tau_3^{1/2} - 1} + \left(\frac{1}{\tau_4^{1/2}} - \frac{1}{\tau_3^{1/2}} \right) \right.$$

$$\left. \left. + \frac{1}{3} \left(\frac{1}{\tau_4^{3/2}} - \frac{1}{\tau_3^{3/2}} \right) + \frac{1}{5} \left(\frac{1}{\tau_4^{5/2}} - \frac{1}{\tau_3^{5/2}} \right) \right] \right\} . \quad (51)$$

In evaluating Eq. (50) or (51), τ_2 is obtained from Eq. (22), and P_3/P_2 is calculated by solving Eq. (34). The temperature ratios τ_3 and τ_4 are then calculated from Eqs. (36) and (42).

The efficiency is simply W_{out}/W_{in}, obtained from Eqs. (49) and (50) or (51) as appropriate.

4. SOME CONCRETE EXAMPLES AND PHYSICAL INTERPRETATIONS

Consider first an engine in which the displacer and power piston swept volumes are equal, so that the compression ratio (i.e. the ratio of maximum to minimum volume) is 2:1. Pressures and volumes will be expressed in terms of the initial pressure and displacer swept volume, respectively (i.e., $P_o = 1$, $V_e = 1$, and $V_p = 1$).

The P-V diagram of the all-isothermal engine is easily calculated; the vertices of the diagram are defined by Eqs. (3), (6), and (9). For a similar engine with an adiabatic expansion cylinder, the vertices are defined by Eqs. (16), (20), and (34). A particular example, with a heat exchanger temperature ratio of 3:1, corresponding to hot and cold and temperatures of ~650 and 35°C, respectively, is given in Table 1 and plotted, for a monatomic working gas, in Fig. 4. Note that the P-V diagram shows that a three-cornered approximation of its shape can be quite a good one, especially for the adiabatic cylinder engine, because the pressure does not change very much during the final displacement phase.

Table 1. Cycle pressure and power output

Variable	Isothermal cylinders	Adiabatic expansion cylinder	
		Diatomic	Monatomic
P_0	1.00	1.00	1.00
P_1	2.00	2.00	2.00
P_2	6.00	7.02	7.65
P_3	1.50	1.37	1.31
W_{out}	1.39	1.33	1.28

Note also that the adiabatic expansion cylinder machine with a monatomic working gas gives about 10% less output, but has almost 35% higher peak-to-peak pressure, than the all-isothermal one when both machines have

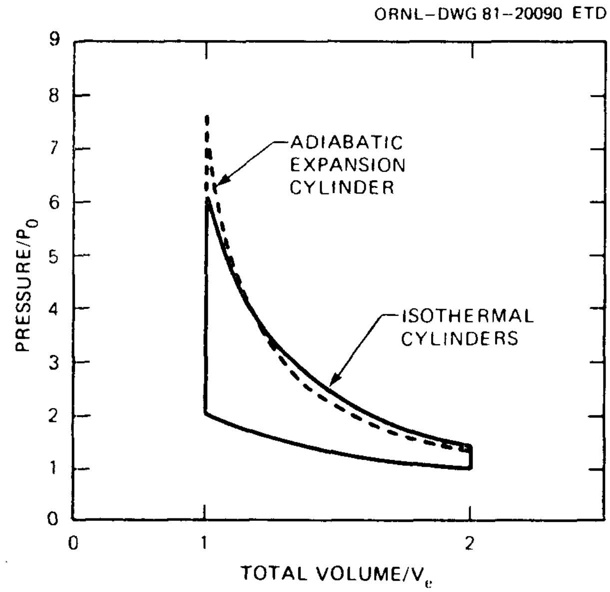

Fig. 4. Adiabatic and isothermal P-V loops.

the same initial pressure and volume (i.e., the same mass of working fluid).

Figure 5 shows the work output as a function of heater temperature, assuming that the cold-end temperature is 35°C, comparing the all-isothermal case and the adiabatic expansion cylinder machine. The work output of the isothermal machine does not fall to zero until the temperature difference between hot and cold ends has fallen to zero, but the adiabatic machine then has a negative output (i.e., work must be done to drive it). At some higher temperature (85°C in this case) the power output from the adiabatic machine will be zero. At higher temperatures still, it will be positive.

We can calculate the temperature difference at which the work output will fall to zero by setting the left hand side of Eq. (49) equal to zero:

$$\log \frac{1}{(1 + k)(P_3/P_2)} = (\gamma - 1)(k/\tau_2)(P_3/P_2) \ . \qquad (52)$$

Substitute for $(k/\tau_2)(P_3/P_2)$ from Eq. (34)

$$\log \frac{1}{(1 + k)(P_3/P_2)} = (\gamma - 1)[1 - (P_3/P_2)^{1/\gamma}] \ . \qquad (53)$$

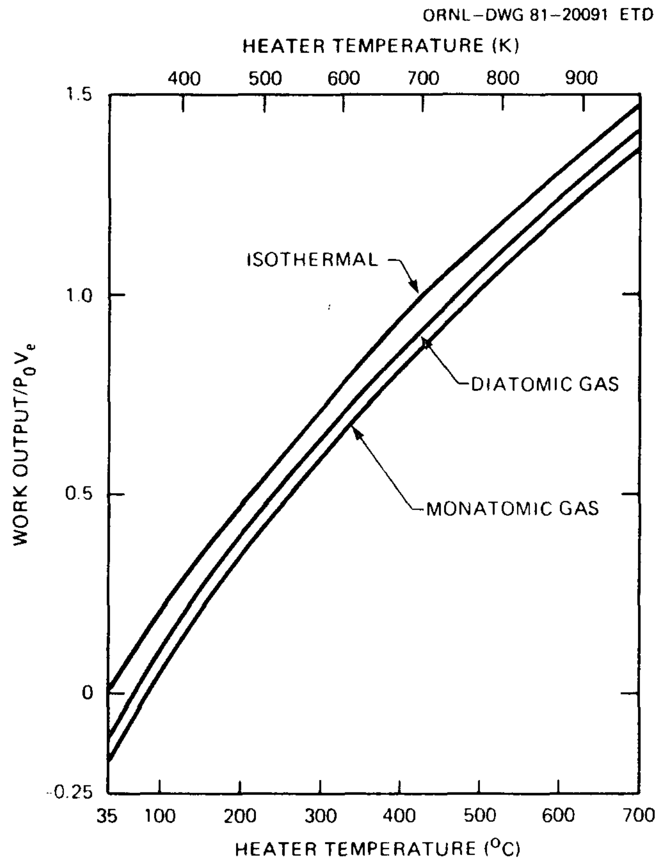

Fig. 5. Work output vs heater temperature.

This equation can be solved for P_3/P_2 and the result substituted into Eq. (34) to give the value of τ_3 for which the work output is zero. Finally, this value of τ_2 can be inserted into Eq. (22), which is then solved to give the heat exchanger temperature ratio τ for zero output. Some results are given in Table 2 and plotted in Fig. 6.

To understand the reason for the negative work output of the adiabatic machines at small temperature ratios, consider how the P-V diagram changes as the heater temperature is reduced (Fig. 7). At a moderate heater temperature, for example 350°C, the work output is positive. At a somewhat lower temperature — 170°C in this case — the extra reduction of pressure in the expansion stroke due to adiabatic cooling lowers the pressure during Phase 3 as far as the original pressure. The cycle becomes three-cornered, but the work output is still positive throughout the cycle. With a lower heater temperature (such as the 85°C example

Table 2. Heat exchanger temperature for zero work output

	Temperature ratio	Heater temperature[a] (°C)	Temperature difference[b] (°C)
Monatomic gas			
k = 0.5	0.910	65	30
k = 1.0	0.861	85	50
k = 1.5	0.828	99	64
Diatomic gas			
k = 0.5	0.940	55	20
k = 1.0	0.905	67	32
k = 1.5	0.882	76	41

[a]Assuming a cooler temperature of 35°C.

[b]Difference between heater and cooler temperatures.

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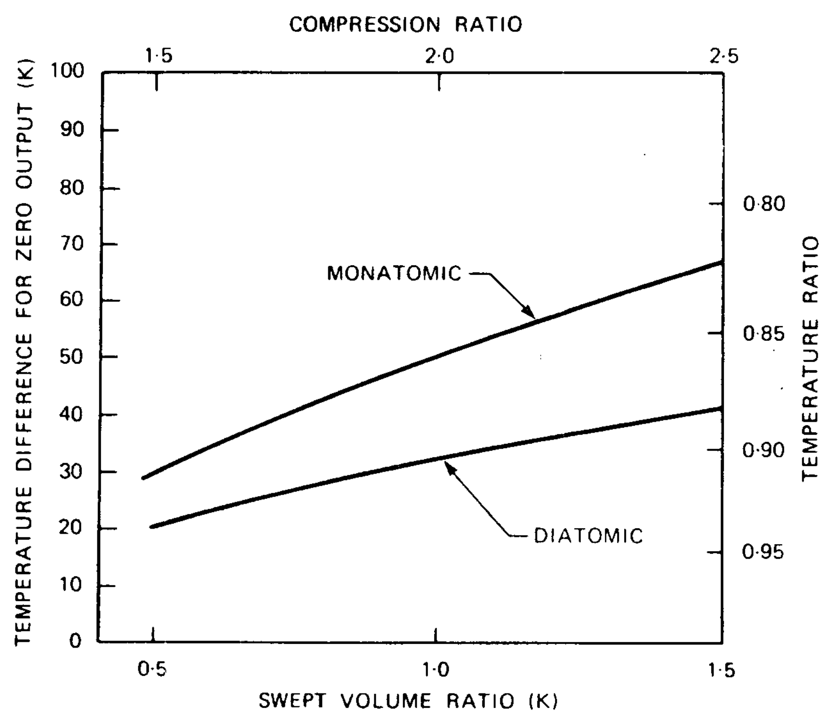

Fig. 6. Heater temperature at which work output falls to zero.

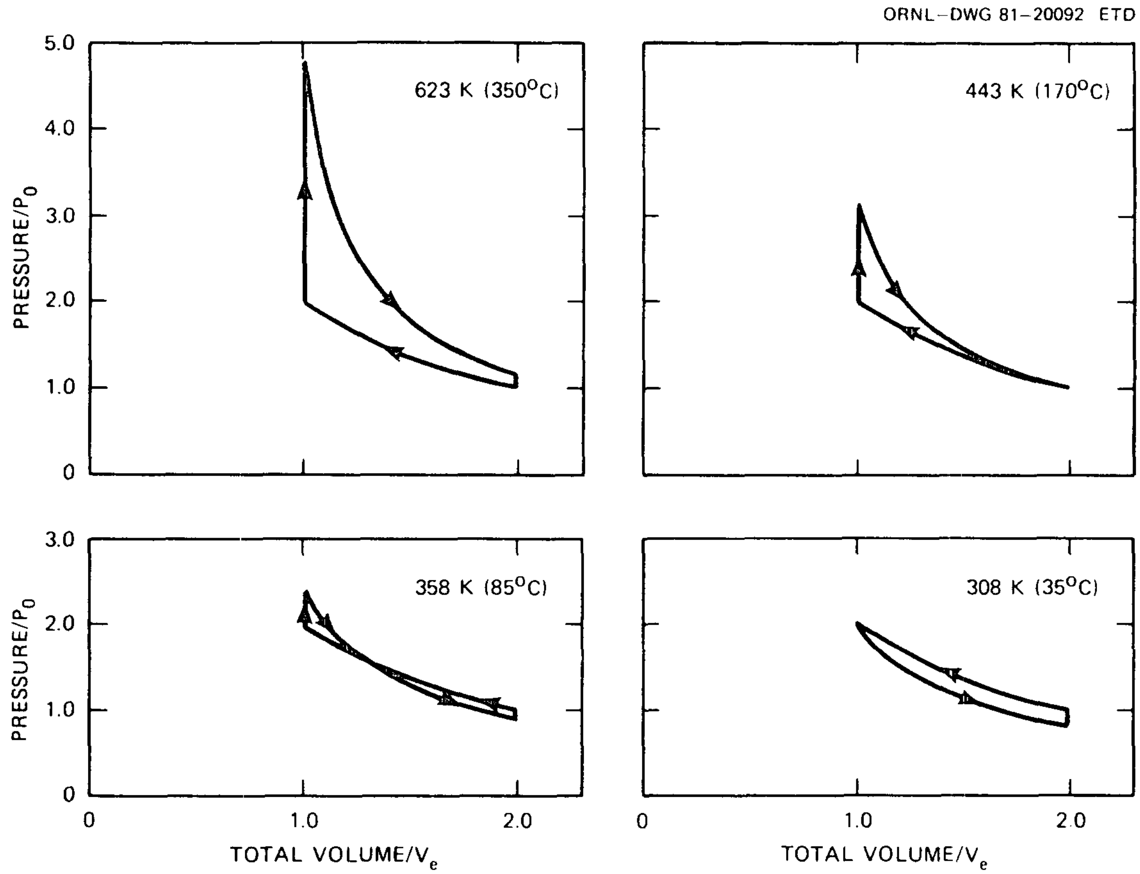

Fig. 7. P–V diagrams for various heater temperatures.

shown), the pressure on the expansion stroke actually falls below the initial pressure, and in part of the cycle the loop is traversed in a counter-clockwise direction, therefore requiring a work input. In the example shown for 85°C, this negative work exactly compensates the work output available from the first part of the cycle, leaving zero net output. When the heater temperature is lowered further still, the area of the negative work portion of the loop exceeds that of the positive portion, and instead of the machine running as an engine, a net work input is needed to drive it.

For a machine operating across a small temperature difference then, the work output (which may be negative) is the relatively small difference between the larger quantities represented by the two parts of the loop traversed in opposite senses. Consequently, the net output will be rather sensitive to even relatively small additional irreversibilities in the cycle. The same problem may occur in a Stirling machine operated as a heat pump.

The effect of adiabatic heating and cooling is greater, the greater the compression ratio (i.e., the greater the ratio of power piston to displacer swept volume). The compression ratio is simply equal to 1 + k, and Fig. 8 shows the work output per cycle as a function of compression ratio for a monatomic gas. As expected, increasing the power-piston swept volume gives a larger output when the heater temperature is reasonably high.

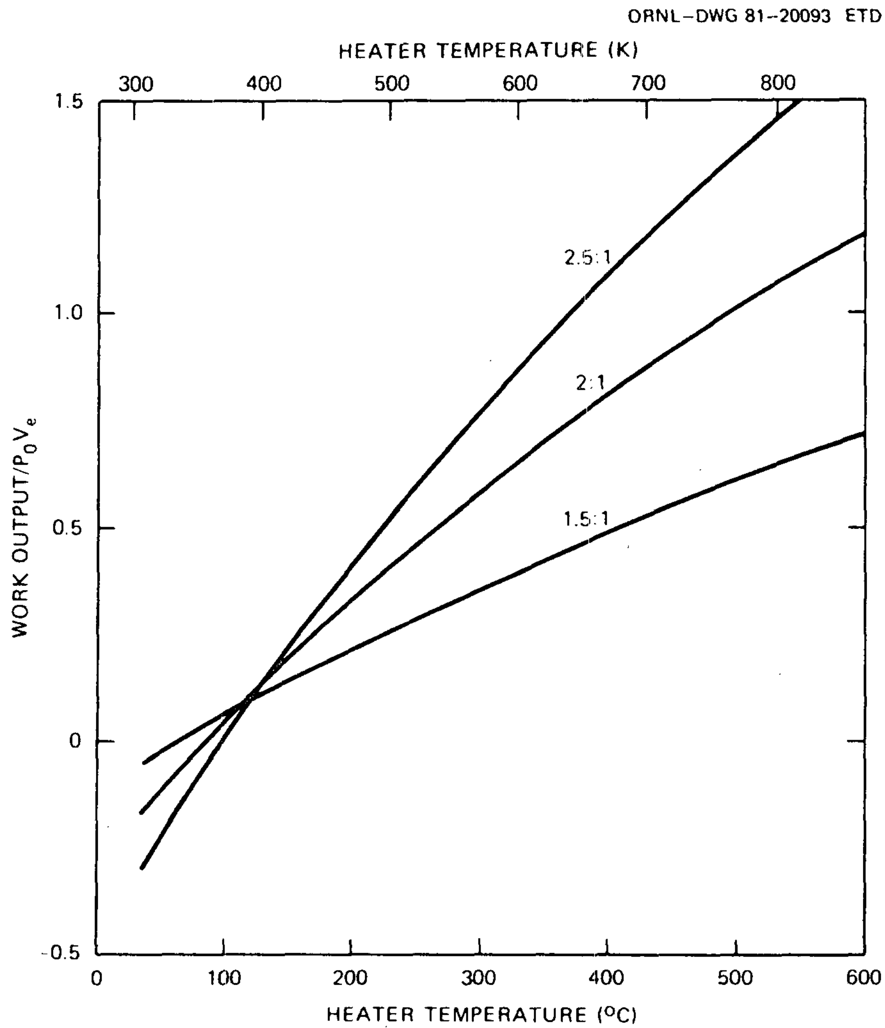

Fig. 8. Work output vs heater temperature with various compression ratios.

More surprisingly, however, the work output is actually reduced by increasing the power-piston swept volume (and thus the mass of working fluid) when the temperature difference is small. This result has important implications for machines, such as heat pumps or engines designed to operate with flat-plate solar collectors, that must operate with a small temperature difference between heater and cooler.

Consider next the heat input required. Figure 9 shows that the heat input required is higher for an adiabatic expansion-cylinder machine than for an all-isothermal one. The increase is 10–15% (if the working fluid is monatomic) when the displacer and power-piston swept volumes are equal.

The increase in heat input and the reduction in work output combine to drive down the efficiency. Figure 10 shows that the ideal efficiency is considerably lower for the adiabatic expansion cylinder machine than

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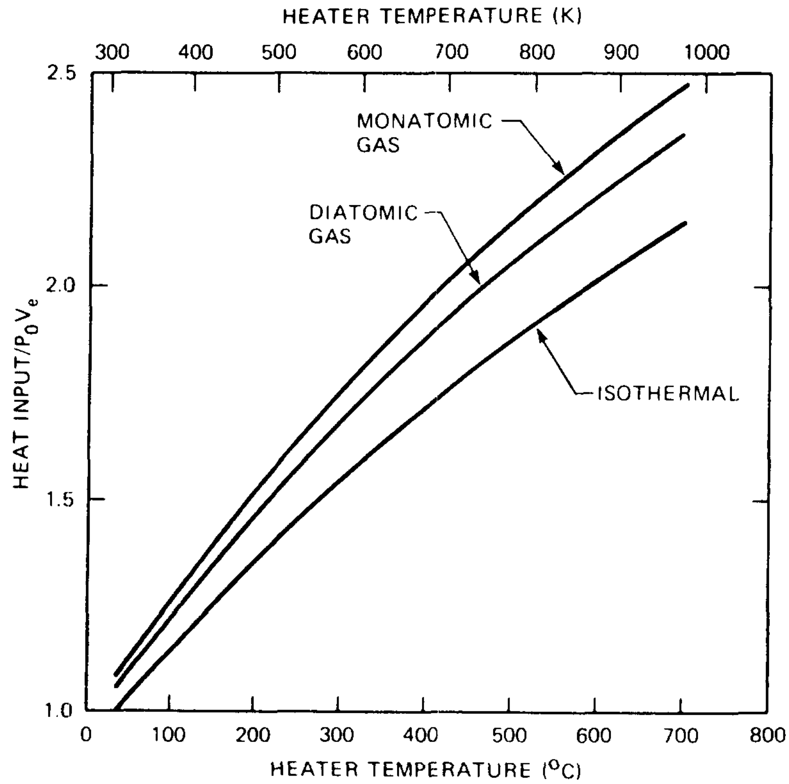

Fig. 9. Heat input for isothermal and adiabatic cases.

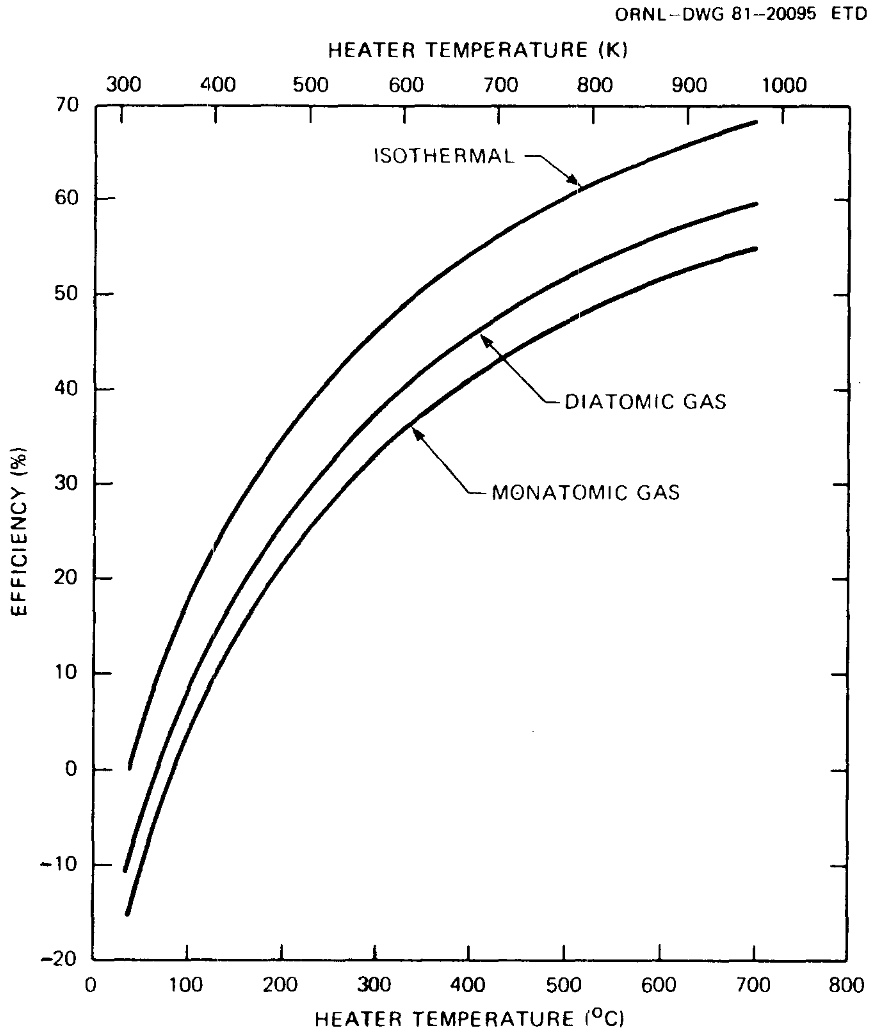

Fig. 10. Efficiency for isothermal and adiabatic cases.

for the isothermal one even in this ideal case where the practical losses (e.g., flow losses or transient heat transfer losses) are ignored. The reduction is naturally proportionately larger at the smaller temperature differences, and at high compression ratios it is also absolutely larger for small temperature differences, at least for a monatomic gas (Fig. 11). Again, there are important implications for machines operating across small temperature differences.

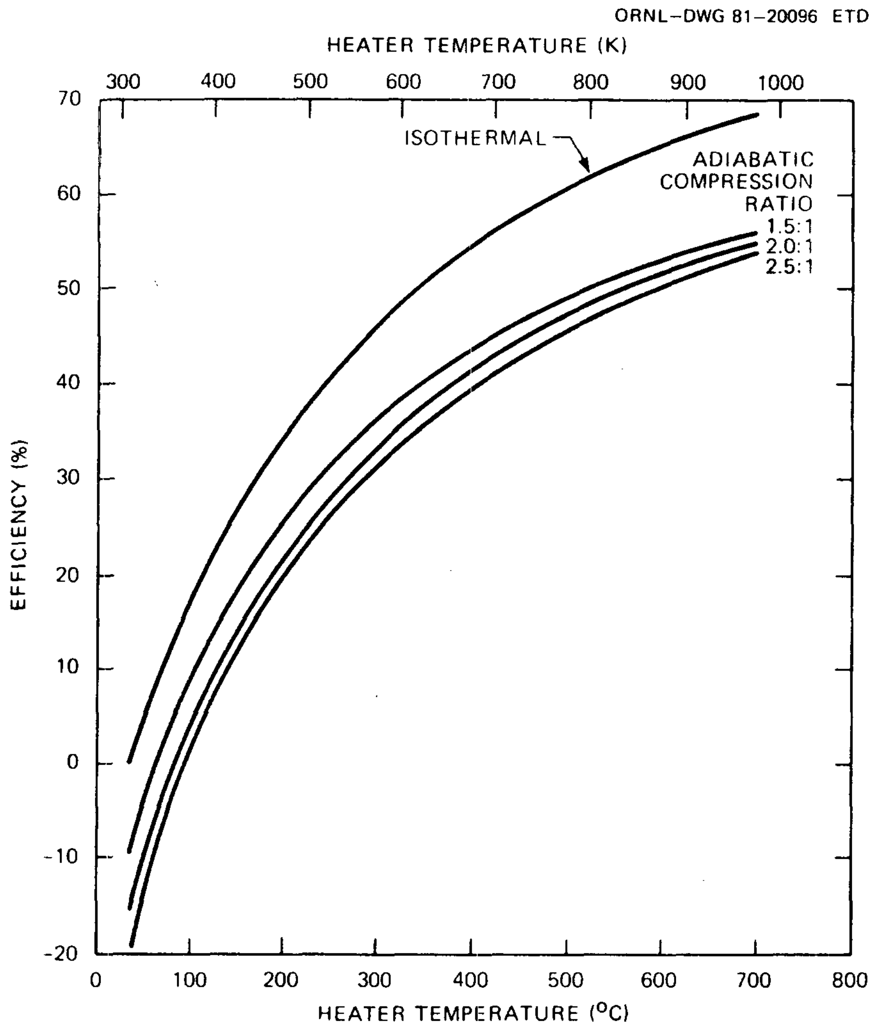

Fig. 11. Efficiency for various compression ratios (monatomic gas).

5. CONCLUSIONS

Integration around the P-V loop of a Stirling-like machine with an adiabatic cylinder is possible. The problems of such a task arise mainly from the complications caused by having a mixture of adiabatic spaces and isothermal spaces exchanging gas with each other throughout the cycle. The actual cycle treated is not that of a real Stirling engine, which usually has a near-sinusoidal variation of cylinder volume with time, but it is much more realistic in some important respects than the isothermal cylinder approximation that is usually made to facilitate analysis of the Stirling cycle.

The effects of the adiabatic cylinder include a reduction in power output and thermal efficiency but an increase in the heat input required, for a given quantity of working fluid. These effects are most marked when the temperature difference between the heater and cooler is relatively small. The peak pressure in the engine is increased considerably when the expansion cylinder behaves adiabatically rather than isothermally.

For small heater-to-cooler temperature differences, the P-V loop is still fairly large for the adiabatic cylinder case (unlike the all-isothermal machine, for which the loop closes up completely at zero temperature difference) but consists of a figure-eight shape with the two portions traversed in opposite senses. The net work output is the difference between these two larger areas, which explains quite clearly why the output of an adiabatic expansion cylinder machine is so sensitive to even small additional irreversibilities when the temperature difference is small.

The results clarify some of the difficulties that have been encountered or that are to be expected in operating Stirling engines or heat pumps with a low temperature difference between the hot and cold heat exchangers. The results also help to provide a physical background and explanation for some of the effects that are routinely observed when running computer models of Stirling engines, models that invariably use numerical integration of the gas behavior equations in adiabatic spaces. The cycle analyzed is a version of the ideal Stirling cycle that does not allow for the near-sinusoidal piston movements of most real machines, but the same general effects and trends must be present in both cases, although their magnitude will probably be reduced by the merging of different parts of the cycle that takes place when the piston movements are continuous. The magnitude of the adiabatic effects will be considerably reduced, although they will neither be eliminated nor (probably) qualitatively changed, when allowance is made for the presence of the heat exchanger volumes; preliminary work shows that finite-volume heat exchangers can be included in this type of analysis, and a future report may attempt to do so.

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Appendix

ADIABATIC COMPRESSION SPACE

Although less interesting practically than the adiabatic expansion cylinder machine, the case of an engine with an adiabatic compression space turns out to be much simpler mathematically. Some of the results are derived in this Appendix, using the same nomenclature as the main text.

The analysis can be conveniently begun at Point 2 on the P-V loop when all of the gas is at the temperature T_{eh} of the isothermal expansion space.

<u>Phase 3</u>. The process is similar to that for the case already treated, except that the gas in the expansion cylinder is always at temperature T_{eh}.

$$\frac{V_e dP/\gamma}{T_{eh}} + \frac{VdP/\gamma + PdV}{T_{ch}} = 0 \tag{A.1}$$

$$\therefore \quad (\tau V_e + V)\ dP/\gamma = -PdV\ ;$$

$$\therefore \quad \int_{P_2}^{P} \frac{dP}{P} = -\gamma \int_0^V \frac{dV}{\tau V_e + V}\ ;$$

$$\therefore \quad \log P/P_2 = \log \left(\frac{\tau V_e}{\tau V_e + V}\right)^{\gamma}\ ;$$

$$\therefore \quad P = P_2 \left(\frac{\tau V_e}{\tau V_e + V}\right)^{\gamma}\ . \tag{A.2}$$

The work done, W_{p3}, is given by

$$W_{p3} = \int_0^{V_p} PdV = P_2 \int_0^{V_p} \left(\frac{\tau V_e}{\tau V_e + V}\right)^{\gamma} dV$$

$$W_{p_3} = P_2 V_e \frac{\tau}{\tau - 1}\left[1 - \left(\frac{\tau^{\gamma-1}}{\tau + k}\right)\right] . \tag{A.3}$$

The pressure at the end of this phase, P_3, may be obtained by substituting $V = V_p$ into Eq. (A.2):

$$P_3 = P_2 \left(\frac{\tau}{\tau + k}\right)^{\gamma} . \tag{A.4}$$

Note that when the heater and cooler are at the same temperature, $\tau = 1$, the pressure ratio during the expansion stroke is simply $(1 + k)^{\gamma}$, as it would be if all the gas were expanded adiabatically. The gas temperature in the expansion cylinder, however, remains at T_{eh} during the expansion, and so is higher than it would have been if the gas had all expanded adiabatically. To offset this, the gas temperature in the compression space at the end of this stroke must be lower than it would have been in a purely adiabatic expansion. In fact, if the temperature in the compression space at the end of the stroke is T_{c_3}, then (for $\tau = 1$)

$$\frac{P_2 V_e}{T_{eh}} = \frac{P_3 V_e}{T_{eh}} + \frac{P_3 V_p}{T_{c_3}} .$$

Substitute for P_3 from Eq. (4) and substitute $T_{eh} = T_{ch}$:

$$T_{c_3} = T_{ch} \frac{k}{(1 + k)^{\gamma} - 1} . \tag{A.5}$$

<u>Phase 4</u>. The displacer piston is moved leftwards through a volume V, moving gas from the expansion cylinder into the compression cylinder, which it enters with the temperature T_{ch} imparted by the cooler.

$$\frac{(V_e - V)dP/\gamma - PdV}{T_{eh}} + \frac{(V_p + V)dP/\gamma + PdV}{T_{ch}} = 0 \tag{A.6}$$

$$\therefore \quad \frac{dP}{P} = \frac{-\gamma dV}{\frac{\tau + k}{1 - \tau} V_e + V} ;$$

integrate both sides, and

$$\log P/P_3 = \log \left(\frac{1}{1 + \frac{1-\tau}{\tau + k} \frac{V}{V_e}} \right)^{\gamma} ;$$

$$\therefore \quad P = P_3 \left(\frac{1}{1 - \frac{1-\tau}{\tau + k} \frac{V}{V_e}} \right)^{\gamma} .$$

Substitute for P_3 from Eq. (A.4)

$$P = P_2 \left(\frac{\tau}{\tau + k + (1 - \tau)\ V/V_e} \right)^{\gamma} . \tag{A.7}$$

The pressure at the end of this phase, P_o, is obtained by substituting $V = V_e$, corresponding to the displacer in its far leftward position with all the gas in the compression space

$$P_o = P_2 \left(\frac{\tau}{1 + k} \right)^{\gamma} . \tag{A.8}$$

<u>Phase 1</u>. The gas, now all in the compression space at an initial pressure P_o, is compressed adiabatically by moving the power piston inwards. From the usual adiabatic gas laws

$$P(V_e + V)^{\gamma} = P_o(V_e + V_p^{\gamma}) \tag{A.9}$$

$$\therefore \quad P = P_o \left(\frac{V_e\ (1 + k)}{V_e + V} \right) .$$

Substitute for P_o from Eq. (A.8)

$$P = P_2 \left(\frac{\tau V_e}{V_e + V} \right)^{\gamma} . \tag{A.10}$$

The work done on the power piston during this phase is W_{p1}, where

$$W_{p1} = \int_{\substack{P=P_2 \\ V=V_p^o}}^{\substack{P=P_1 \\ V=0}} PdV = P_2 \ (\tau V_e)^\gamma \int_{V_p}^{0} \frac{dV}{(V_e + V)^\gamma} ,$$

and

$$W_{p1} = -P_2 V_e \frac{\tau^\gamma}{\gamma - 1} \left[1 - \frac{1}{(1 + k)^{\gamma-1}} \right] . \tag{A.11}$$

The pressure at the end of this phase, P_1, is calculated by substituting $V = 0$ into Eq. (A.10)

$$P_1 = P_2 \tau^\gamma . \tag{A.12}$$

Work output and P-V loop. The work output W_{out} is the sum of the work done on the power piston during Phases 3 and 1, given by Eqs. (A.3) and (A.11):

$$W_{out} = W_{p3} + W_{p1} = \frac{P_2 V_e}{\gamma - 1} \left\{ \tau \left[1 - \left(\frac{\tau}{\tau + k} \right)^{\gamma-1} \right] - \tau^\gamma \left[1 - \left(\frac{1}{1 + k} \right)^{\gamma-1} \right] \right\} . \tag{A.13}$$

With the aid of Eq. (A.13) we can also calculate the temperature ratio for which the power output falls to zero. The work output $W_{out} = 0$ when

$$\tau \left[1 - \left(\frac{\tau}{\tau + k} \right)^{\gamma-1} \right] = \tau^\gamma \left[1 - \left(\frac{1}{1 + k} \right)^{\gamma-1} \right]$$

or

$$\tau^{1-\gamma}\left[1-\left(\frac{\tau}{\tau+k}\right)^{\gamma-1}\right]=1-\left(\frac{1}{1+k}\right)^{\gamma-1}\ ;$$

that is, when

$$\frac{1}{\tau^{\gamma-1}}-\frac{1}{(\tau+k)^{\gamma-1}}=1-\frac{1}{(1+k)^{\gamma-1}}\ . \qquad \text{(A.14)}$$

By inspection, this equality is fulfilled when $\tau = 1$, that is, when there is zero temperature difference between the heater and cooler. When the temperature difference is positive, the work output is positive. Table A.1 lists the ideal output for an all-isothermal machine and an adiabatic compression-space machine when both have the same mass of monatomic working fluid and the same cooler temperature (35°C), such that $MRT_{ch} = 1$. When the temperature difference is moderately low (less than about 400 K), the adiabatic machine gives more output than the all-isothermal one.

The vertices of the P-V loop are given by expressions (A.4), (A.8), and (A.12) for the pressure at the end of each phase. The shape of the P-V loop may be obtained from Eqs. (A.2) and (A.10) relating the pressure and volume during the expansion and compression phases, respectively. The constant volume transfers are represented by vertical lines joining the vertices P_3, P_0 and P_1, P_2.

Table A.1. Comparison of power output for all-isothermal and adiabatic compression-space engines

T_{eh} (°C)	Output power	
	Isothermal	Adiabatic
35	0.00	0.00
100	0.10	0.13
200	0.24	0.28
300	0.36	0.38
400	0.47	0.48
500	0.56	0.55
600	0.65	0.61
700	0.73	0.66

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