ON THE ORTHOGONALITY OF THE MACDONALD'S FUNCTIONS

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ABSTRACT. A proof of an orthogonality relation for the MacDonald's functions with identical arguments but unequal complex lower indices is presented. The orthogonality is derived first via a heuristic approach based on the Mehler-Fock integral transform of the MacDonald's functions, and then proved rigorously using a polynomial approximation procedure.

1. INTRODUCTION

Certain problems of mathematical physics arising in spheroidal or cylindrical domains, e.g. Laplace's equation, have solutions that involve MacDonald's functions [1] and conical functions [1] [2]. These functions which also enter integral transforms such as those of Kontorovich-Lebedev, and Mehler-Fock as kernels [3] find important applications in boundary value problems of electrostatics and elasticity [4]. These applications typically entail modeling material domains [5]-[7] or voids in material domains [8] with the appropriate continuous surfaces generated by fixing one of the coordinates in the chosen coordinate system [9].

The study of one such problem [10], where the probe of an Atomic Force Microscope (AFM) [11] was modeled as a hyperboloid of revolution, resulted in the following newly derived integral expansion for the Cartesian coordinate z [12]

$$z = -\pi z_0 \int_1^\infty \eta' \, d\eta' \int_0^\infty \frac{q \tanh \pi q}{\cosh \pi q} P^0_{-\frac{1}{2}+iq}(0) \left[P^0_{-\frac{1}{2}+iq}(\mu) - P^0_{-\frac{1}{2}+iq}(0) \right] \times P^0_{-\frac{1}{2}+iq}(\eta') P^0_{-\frac{1}{2}+iq}(\eta) \, dq, \tag{1.1}$$

where z_0 is a scale factor that defines the focal distance of the hyperboloid in the spheroidal (η, μ, φ) coordinate system, and the Ps denote the conical functions . This

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integral expansion comprises the key element in the study of the Coulomb interaction of the AFM's dielectric probe with a charged substrate or sample surface. Here the charge distribution can be the result of an applied potential difference between the probe and the sample, or a result of the sample being naturally charged [13]. A crucial step in proving this expansion is the validity of the following orthogonality relation for the MacDonald's functions of identical arguments but different complex lower indices

$$\frac{2}{\pi^2} q' \sinh(\pi q') \int_0^\infty \frac{K_{iq}(\alpha) K_{iq'}(\alpha)}{\alpha}\, d\alpha = \delta(q - q'), \tag{1.2}$$

where the Ks denote the MacDonald's functions. A similar orthogonality relation for the conical functions, also utilized in the proof of the expansion in (1.1), was first derived by Van Nostrand [14]. In short, the proof there involved considering the associated Legendre equation being satisfied by two linearly independent solutions. The differential equation was then manipulated and integrated, whereupon the orthogonality relation was derived by resorting to limiting considerations and asymptotic expansions of the conical functions. We also note that Titchmarsh [25] proves a dual orthogonality relation for the MacDonald's functions

$$\frac{2}{\pi^2\, \alpha} \int_0^\infty q \sinh(\pi q) K_{iq}(\alpha) K_{iq}(\alpha')\, dq = \delta(\alpha - \alpha'), \tag{1.3}$$

$\alpha,\, \alpha' > 0$.

The proof of the orthogonality relation in (1.2) is the aim of this paper. In this work, Section 2 defines the MacDonald's and the conical functions and presents a couple of relevant propositions concerning properties of these functions, followed by their proofs. In Section 3, the main section, a heuristic derivation for (1.2) based on an integral representation for the conical functions and the Mehler-Fock transform is outlined. Moreover, in Section 3, we introduce the Orthogonality Theorem which states (1.2) and also present its proof. We also note that a different approach to this orthogonality problem, based on the analytic properties of testing functions for distributions has been considered in [28].

2. MACDONALD'S AND CONICAL FUNCTIONS

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and define the MacDonald's functions ([1] or [3] page 354, 6-1-6) by

$$K_{iq}(\alpha) = \int_0^\infty e^{-\alpha \cosh \zeta} \cos(q\zeta)\, d\zeta, \qquad \text{where} \qquad \alpha, q > 0. \tag{2.1}$$

The above can also be written as (see [15], [16])

$$K_{iq}(\alpha) = \frac{\pi}{2i \sinh(\pi q)} [I_{-iq}(\alpha) - I_{iq}(\alpha)], \tag{2.2}$$

where $I_\nu(\alpha)$ are the modified Bessel functions

$$I_\nu(\alpha) = \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(r+\nu+1)} \left(\frac{\alpha}{2}\right)^{2r+\nu}.$$

For $\eta = \cosh\zeta$ and $\zeta \geq 0$, define the conical functions

$$P^m_{-\frac{1}{2}+iq}(\eta) = \frac{2^{m+1} Z^m_q \tanh^m \frac{\zeta}{2}}{\pi(2m-1)!!\cosh\frac{\zeta}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos\left[2q\sinh^{-1}\left(\sinh\frac{\zeta}{2}\cos y\right)\right]\sin^{2m} y}{\sqrt{1-\tanh^2\frac{\zeta}{2}\sin^2 y}}\,dy, \tag{2.3}$$

where $m \in \mathbb{N}_0$, $(-1)!! = 1$, $Z^0_q = 1$, and

$$Z^m_q = (-1)^m \left(q^2 + \frac{1}{4}\right)\left(q^2+\frac{9}{4}\right)\cdots\left(q^2 + \frac{(2m-1)^2}{4}\right), \tag{2.4}$$

with $Z^0_q = 1$ (see [17], [18], [1], [3]). For asymptotic expansions and numerical considerations regarding the conical functions see [19], [20] and the references therein.

We conclude this section by proving two results, which will be used throughout the rest of this paper.

Proposition 2.5. *For all $\alpha, q > 0$*

$$|K_{iq}(\alpha)| \leq \sqrt{\frac{\pi}{q\sinh(\pi q)}}\,e^{\alpha}, \tag{2.6}$$

and

$$|K_{iq}(\alpha)| \leq \sqrt{\frac{\pi}{2\alpha}}\,e^{-\alpha}. \tag{2.7}$$

Proof. From the property of the Gamma function [16], we have

$$\Gamma(r+1\pm iq) = (1\pm iq)(2\pm iq)\cdots(r\pm iq)\Gamma(1\pm iq) \overset{\text{def}}{:=} a_r(\pm q)\Gamma(1\pm iq),$$

with $a_0(q) = 1$. Noting that $|a_r(q)| = |a_r(-q)|$, (2.2) implies

$$|K_{iq}(\alpha)| \leq \frac{\pi}{2\sinh(\pi q)}\left(\frac{1}{|\Gamma(1-iq)|} + \frac{1}{|\Gamma(1+iq)|}\right)\sum_{r=0}^{\infty}\frac{1}{r!\,|a_r(q)|}\left(\frac{\alpha}{2}\right)^{2r}.$$

Now (2.6) follows from the inequality

$$\sum_{r=0}^{\infty}\frac{1}{r!\,|a_r(q)|}\left(\frac{\alpha}{2}\right)^{2r} \leq \sum_{r=0}^{\infty}\frac{1}{(r!)^2}\left(\frac{\alpha}{2}\right)^{2r} \leq \left[\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\alpha}{2}\right)^{r}\right]^2 = e^{\alpha},$$

together with the fact (see [15], [21])

$$|\Gamma(1\pm iq)| = \sqrt{\frac{\pi q}{\sinh(\pi q)}}. \tag{2.8}$$

Next, from (2.1), we have

$$|K_{iq}(\alpha)| \le \int_0^\infty e^{-\alpha(1+\frac{1}{2}\zeta^2)}\,d\zeta = \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha},$$

which proves (2.7). □

Proposition 2.9. *Fix $\eta_0 > 1$ and $m \in \mathbb{N}_0$. Then there exists a constant $C > 0$, depending only on m and η_0, such that*

$$\left|\frac{d^m}{d\eta^m} P^0_{-\frac{1}{2}+iq}(\eta)\right| \le C\, q^{2m} \tag{2.10}$$

for all $q > 0$ and $\eta \in [1, \eta_0]$.

Proof. The proof uses the following relation[1] for the conical functions.

$$\frac{d^m}{d\eta^m} P^0_{-\frac{1}{2}+iq}(\eta) = \left(\eta^2-1\right)^{-\frac{m}{2}} P^m_{-\frac{1}{2}+iq}(\eta), \tag{2.11}$$

where $\eta \ge 1$, $q > 0$, and $m \in \mathbb{N}_0$ (see [3] or [18], p. 334, (8.6.6)). Fix $q > 0$ and let

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2 y}}\,dy.$$

Noting that $\left|\frac{\tanh\frac{\zeta}{2}}{\sinh\zeta}\right| \le \frac{1}{2}$, for all $\zeta \ge 0$, (2.3) and (2.11) imply

$$\left|\frac{d^m}{d\eta^m} P^0_{-\frac{1}{2}+iq}(\eta)\right| \le \frac{2|Z^m_q|}{\pi(2m-1)!!} K(\tanh\frac{\zeta_0}{2}),$$

where $\eta = \cosh\zeta \in [1, \eta_0]$, $\zeta \in [0, \zeta_0]$, $\eta_0 = \cosh\zeta_0$. To complete the proof use (2.4) to conclude $|Z^m_q| \le C_m q^{2m}$, where $C_m > 0$ depends only on m. □

3. MAIN RESULT

We start with a heuristic derivation of the orthogonality relation. Consider the zero order Mehler-Fock transform of $e^{-\alpha\eta}$ as given by [3]

$$e^{-\alpha\eta} = \sqrt{\frac{2}{\pi\alpha}} \int_0^\infty q\tanh(\pi q) K_{iq}(\alpha) P^0_{-\frac{1}{2}+iq}(\eta)\,dq, \tag{3.1}$$

where $\eta = \cosh\zeta \in [1, \infty[$, $\alpha \ge 0$, and $q \ge 0$. Using the integral representation in (2.3) for the conical functions, the following integral representation for the zero order conical functions can be derived [3]

$$P^0_{-\frac{1}{2}+iq}(\eta) = 2^{\frac{1}{2}}\pi^{-\frac{3}{2}}\cosh(\pi q)\int_0^\infty e^{-\alpha\eta}\frac{K_{iq}(\alpha)}{\sqrt{\alpha}}\,d\alpha. \tag{3.2}$$

[1] We note here that, although indifferent in the present work, for the general form of the conical functions $P^m_{-\frac{1}{2}+iq}(z)$, and for $-1 < z < 1$, some references introduce a $(-1)^m$ multiplicative factor to the right hand side of (2.11) [2][p. 148].

Representing the latter integral as $\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty$, we substitute (3.2) in (3.1) and change the order of integration via Fubini's Theorem. This yields

(3.3)
$$P^0_{-\frac{1}{2}+iq}(\eta) = \frac{2}{\pi^2}\cosh(\pi q)\lim_{\varepsilon\to 0^+}\int_0^\infty q'\tanh(\pi q')P^0_{-\frac{1}{2}+iq'}(\eta)\,dq'\int_\varepsilon^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha.$$

Since η is arbitrary, (3.3) would lead us to expect that the expression in the left side of (1.2) is the Dirac distribution $\delta(q-q')$. In fact, this is the content of our main result.

Orthogonality Theorem . *For each $q, q' > 0$*
$$\frac{2}{\pi^2}q'\sinh(\pi q')\int_0^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha = \delta(q-q'),$$
where δ denotes the Dirac distribution.

Before giving the proof of the Orthogonality Theorem, we need a few preliminary results. Also, for $q, q', \epsilon > 0$, let
$$F(q,q';\epsilon) = \frac{2}{\pi^2}q'\sinh(\pi q')\int_\epsilon^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha.$$
Note that, by (2.7), the above integral converges for all $\epsilon > 0$; a fact which will be used throughout the paper.

Proposition 3.4. *If $q > 0$ and $n \in \mathbb{N}_0$, then*
$$q^{2n}\,\mathrm{sech}(\pi q) = \lim_{\epsilon\to 0^+}\int_0^\infty q'^{2n}\,\mathrm{sech}(\pi q')F(q,q';\epsilon)\,dq', \tag{3.5}$$

Proof. To simplify the notation, we put $P^0_q(\eta) = P^0_{-\frac{1}{2}+iq}(\eta)$. Also, throughout the proof, different positive constants will be denoted by C. Now, let $\eta \ge 1$ and $m \in \mathbb{N}_0$. We begin by proving the following two identities.
$$\sqrt{\frac{2}{\pi^3}}\cosh(\pi q)\int_0^\infty \left(\frac{d^m}{d\eta^m}\,e^{-\alpha\eta}\right)\frac{K_{iq}(\alpha)}{\sqrt{\alpha}}\,d\alpha = \frac{d^m}{d\eta^m}P^0_q(\eta) \tag{3.6}$$
and
$$\frac{d^m}{d\eta^m}\,e^{-\alpha\eta} = \sqrt{\frac{2}{\pi\alpha}}\int_0^\infty K_{iq}(\alpha)q\tanh(\pi q)\frac{d^m}{d\eta^m}P^0_q(\eta)\,dq, \quad \forall\alpha > 0. \tag{3.7}$$
For $m = 0$, the above identities are equations (3.1) and (3.2) mentioned above. Moreover, we only prove the case $m = 1$. The proof for $m \ge 2$ follows inductively and is similar to the one given for the case $m = 1$.

For $h \in \mathbb{R}$, (3.1) gives
$$\frac{P^0_q(\eta+h) - P^0_q(\eta)}{h} = \sqrt{\frac{2}{\pi^3}}\cosh(\pi q)\int_0^\infty f_h(\alpha)\,d\alpha, \tag{$\star$}$$

where

$$f_h(\alpha) = \frac{e^{-\alpha(\eta+h)} - e^{-\alpha\eta}}{h} \frac{K_{iq}(\alpha)}{\sqrt{\alpha}}.$$

Noting that $|\frac{e^{-\alpha(\eta+h)}-e^{-\alpha\eta}}{h}| \leq C|\alpha|$ for $|h|$ sufficiently small, it follows from (2.7) that $|f_h(\alpha)|$ is dominated by $C\, e^{-\alpha}$, which obviously belongs to $L^1(0,\infty)$. Letting $h \to 0$ in ($\star$), the Lebesgue Dominated Convergence Theorem implies (3.6) for the case $m = 1$.

Similarly, for $h \in \mathbb{R}$, (3.2) gives

$$\frac{e^{-\alpha(\eta+h)} - e^{-\alpha\eta}}{h} = \sqrt{\frac{2}{\pi\alpha}} \int_0^\infty g_h(q)\, dq,$$

where

$$g_h(q) = K_{iq}(\alpha) q \tanh(\pi q) \frac{P_q^0(\eta+h) - P_q^0(\eta)}{h}.$$

By (2.10), $|\frac{P_q^0(\eta+h)-P_q^0(\eta)}{h}| \leq Cq^2$ for $|h|$ sufficiently small. Thus it follows from (2.6) that $|g_h(q)| \leq Cg(q)$, where $g(q) = q^2\sqrt{\frac{q}{\sinh(\pi q)}}$. However, $g(q) = O(q^2)$ as $q \to 0$; and, $g(q) = O(q^{\frac{5}{2}} e^{-\pi q/2})$ as $q \to \infty$. Therefore, $g(q) \in L^1(0,\infty)$ and (3.7) follows from a similar argument given in the proof of (3.6).

Next, using (3.7) in (3.6) yields

$$\begin{aligned} \operatorname{sech}(\pi q) \frac{d^m}{d\eta^m} P_q^0(\eta) &= \frac{2}{\pi^2} \int_0^\infty \left(\int_0^\infty K_{iq'}(\alpha) q' \tanh(\pi q') \frac{d^m}{d\eta^m} P_{q'}^0(\eta)\, dq' \right) \frac{K_{iq}(\alpha)}{\alpha}\, d\alpha \\ &= \lim_{\epsilon \to 0^+} \frac{2}{\pi^2} \int_\epsilon^\infty \left(\int_0^\infty K_{iq'}(\alpha) q' \tanh(\pi q') \frac{d^m}{d\eta^m} P_{q'}^0(\eta)\, dq' \right) \frac{K_{iq}(\alpha)}{\alpha}\, d\alpha \end{aligned} \tag{3.8}$$

For each $\epsilon > 0$, one can use (2.6), (2.7), and (2.10) to obtain

$$\begin{aligned} &\int_\epsilon^\infty \int_0^\infty \left| q' \tanh(\pi q') \frac{K_{iq'}(\alpha) K_{iq}(\alpha)}{\alpha} \frac{d^m}{d\eta^m} P_{q'}^0(\eta) \right| dq'\, d\alpha \\ &\qquad \leq C \left(\int_\epsilon^\infty \frac{1}{\alpha^{\frac{3}{2}}}\, d\alpha \right) \cdot \left(\int_0^\infty {q'}^{2m} \sqrt{\frac{q'}{\sinh(\pi q')}}\, dq' \right) < \infty, \end{aligned}$$

where the same argument as the one given in the proof of (3.7) shows that the last integral in the above inequality is bounded. Therefore, we may apply Fubini's Theorem to (3.8):

$$\operatorname{sech}(\pi q) \frac{d^m}{d\eta^m} P_q^0(\eta) = \lim_{\epsilon \to 0^+} \int_0^\infty \operatorname{sech}(\pi q') \left(\frac{d^m}{d\eta^m} P_{q'}^0(\eta) \right) F(q, q'; \epsilon)\, dq', \tag{3.9}$$

for each $m \in \mathbb{N}_0$, $q > 0$, and $\eta \geq 1$. As a consequence of (2.3) and (2.11)

$$\left. \frac{d^m}{d\eta^m} P_q(\eta) \right|_{\eta=1} = \frac{2Z_q^m}{\pi(2m-1)!!} \int_0^{\frac{\pi}{2}} \sin^{2m} y\, dy = \frac{Z_q^m}{(2m)!!}.$$

Thus, letting $\eta = 1$ in (3.9) and using the fact that Z_q^m is an even polynomial in q of degree $2m$, the proof of the proposition follows easily from an inductive argument.

□

The next lemma is standard. However, for the sake of completeness, we have included a proof. Also recall that for a nonempty open set $\Omega \subseteq \mathbb{R}$, $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions whose supports are compact subsets of Ω.

Lemma 3.10. *Suppose $a > 0$ and let $\phi \in C_c^\infty(\mathbb{R})$. If $\delta > 0$, then there exists a polynomial P such that*

$$\left|\phi(x) - P(x)e^{-a|x|}\right| < \delta, \qquad \forall x \in \mathbb{R}.$$

Moreover, if ϕ is an even function, then P may be taken to be an even polynomial.

Proof. Let $w(x) = e^{-a|x|}$ on $\mathbb{R}$ and define the weighted L^2-space,

$$L_w^2 = \{f : \mathbb{R} \to \mathbb{C} | f \text{ is measurable}, \int_{-\infty}^{\infty} |f(x)|^2 w(x)\, dx < \infty\}.$$

It follows that the span of the set of polynomials $\{x^n\}_{n=0}^\infty \subset L_w^2$ is dense in L_w^2. A proof of this fact is similar to that in [23] [Sec. 21.64, p. 416] (where it is proved that the Hermite polynomials are complete in a weighted L^2-space).

Since $\phi \in C_c^\infty(\mathbb{R})$, we have $\phi'(x)e^{a|x|} \in L_w^2$. Thus, from the above fact, there is a polynomial Q such that

$$||\phi' e^{a|x|} - Q||_{L_w^2} = \int_{\mathbb{R}} \left|\phi'(x)e^{a|x|} - Q(x)\right|^2 e^{-a|x|}\, dx < \sqrt{\frac{a}{2}}\, \delta. \tag{3.11}$$

Now, let $P(x) = e^{a|x|} \int_{-\infty}^x Q(t)e^{-a|t|}\, dx$. Then P is the desired polynomial. To see this, let $x \in \mathbb{R}$. Then

$$\begin{aligned}\left|\phi(x) - P(x)e^{-a|x|}\right| &= \left|\int_{-\infty}^x \left(\phi'(t) - Q(t)e^{-a|t|}\right) dt\right| \\ &\le \int_{\mathbb{R}} \left|\phi'(x)e^{a|x|} - Q(x)\right| e^{-a|x|}\, dx \\ &\le \left(\int_{\mathbb{R}} e^{-a|x|}\, dx\right)^{\frac{1}{2}} ||\phi' e^{a|x|} - Q||_{L_w^2}.\end{aligned}$$

So the result follows from (3.11).

Suppose now that ϕ is even. With P as above write $P = P_e + P_o$ where P_e (P_o) is an even (odd) polynomial. Then

$$\left|P_o(x)e^{-a|x|}\right| = \frac{1}{2}\left|\phi(x) - P(x)e^{-a|x|} - \left(\phi(-x) - P(-x)e^{-a|-x|}\right)\right| < \delta,$$

on $\mathbb{R}$. This implies

$$\left|\phi(x) - P_e(x)e^{-a|x|}\right| \le \left|\phi(x) - P(x)e^{-a|x|}\right| + \left|P_o(x)e^{-a|x|}\right| < 2\delta,$$

on $\mathbb{R}$ and this proves the second part of Lemma 3.10.

□

Our next and final lemma contains two parts. Only part (ii) will be used in the proof of the Orthogonality Theorem. Part (i), however, is essential to get (ii) and it also contains a useful inequality with further application; therefore, it has been stated separately. Also $\|\cdot\|_\infty$ denotes the usual sup–norm.

Lemma 3.12. *Let $\beta > \frac{1}{2}$, and define*

$$F_\beta(q,q';\epsilon) = \frac{1}{[\cosh(\pi q')]^\beta}\, F(q,q';\epsilon)$$

(i) *For each $q>0$, there exists $C_q>0$ (depending only on q) such that*

$$|F_\beta(q,q';\epsilon)| \le C_q 2^\beta |\ln\frac{\epsilon}{2}| q'^{\frac{1}{2}} e^{-(\beta-\frac{1}{2})\pi q'}, \qquad \textit{for all} \qquad 0<\epsilon<2.$$

(ii) *If $\psi:(0,\infty)\to\mathbb{R}$ is differentiable and bounded on $(0,\infty)$, then there exists $C_{q,\beta}>0$ (depending only on q, β) such that*

$$\limsup_{\epsilon\to 0^+}\left|\int_0^\infty \psi(q')F_\beta(q,q';\epsilon)\,dq'\right| \le C_{q,\beta}\,\|\psi\|_\infty.$$

Proof. From the definition of $F(q,q';\epsilon)$

$$F_\beta(q,q';\epsilon) = G(q,q',\beta)\int_\epsilon^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha,$$

where

$$G(q,q',\beta) = \frac{2}{\pi^2}q'[\operatorname{sech}(\pi q')]^\beta \sinh(\pi q'). \tag{3.13}$$

Fix $q>0$. In all that follows, $0<\epsilon<2$ and different positive constants will be denoted by C (a pure numerical) or C_q (depending only on q). We estimate $F_\beta(q,q';\epsilon)$ $(\beta>\frac{1}{2})$ by considering the integrals $\int_2^\infty, \int_\epsilon^2$ separately in the definition of F_β. For the former integral, we have by (2.6), (2.7),

$$\begin{aligned}\left|G(q,q',\beta)\int_2^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha\right| &\le \frac{\sinh(\pi q')}{\pi[\cosh(\pi q')]^\beta}\sqrt{\frac{2q'}{\sinh(\pi q')}}\int_2^\infty \alpha^{-\frac{3}{2}}\,d\alpha \\ &= \frac{2}{\pi}[\operatorname{sech}(\pi q')]^\beta\sqrt{q'\sinh(\pi q')}.\end{aligned} \tag{3.14}$$

Next, we estimate $\int_\epsilon^2 \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha$. Recall from (2.2) that

$$K_{iq}(\alpha) = A(\alpha,q) + A(\alpha,-q),$$

where

$$A(\alpha,q) = \frac{\pi i}{2\sinh(\pi q)}I_{iq}(\alpha) = \frac{\pi i}{2\sinh(\pi q)}\frac{e^{iq\ln\frac{\alpha}{2}}}{\Gamma(1+iq)}\sum_{r=0}^\infty \frac{1}{r!a_r(q)}\left(\frac{\alpha}{2}\right)^{2r}, \tag{3.15}$$

with $a_r(q) = (1+iq)(2+iq)\cdots(r+iq)$ and $a_0(q)=1$.

Noting that $A(\alpha,-q) = \overline{A(\alpha,q)}$, it follows

$$K_{iq}(\alpha)K_{iq'}(\alpha) = 2\Re[A(\alpha,q)A(\alpha,q')] + 2\Re[A(\alpha,q)A(\alpha,-q')]. \tag{3.16}$$

Now for all real q_1, q_2 and $\alpha > 0$, (3.15) implies

$$A(\alpha, q_1)A(\alpha, q_2) = M(q_1, q_2)e^{i(q_1+q_2)\ln\frac{\alpha}{2}} \sum_{n=0}^{\infty} b_n(q_1, q_2)\left(\frac{\alpha}{2}\right)^{2n}, \tag{3.17}$$

where

$$M(q_1, q_2) = \frac{-\pi^2}{4\sinh(\pi q_1)\sinh(\pi q_2)\Gamma(1+iq_1)\Gamma(1+iq_2)}, \tag{3.18}$$

and

$$b_n(q_1, q_2) = \sum_{\substack{r+s=n\\ r,s\geq 0}} \frac{1}{r!s!a_r(q_1)a_s(q_2)}, \quad n \geq 0.$$

Moreover, for all $n \geq 0$ and real q_1, q_2, it is easily seen

$$|b_n(q_1, q_2)| \leq \sum_{\substack{r+s=n\\ r,s\geq 0}} \frac{1}{(r!s!)^2} \leq \sum_{\substack{r+s=n\\ r,s\geq 0}} \frac{1}{r!s!} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} = \frac{2^n}{n!}. \tag{3.19}$$

Thus, the series in (3.17) converges uniformly $\alpha > 0$ and all real q_1, q_2. Consequently,

$$\begin{aligned}\int_{\epsilon}^{2} \frac{A(\alpha, q_1)A(\alpha, q_2)}{\alpha}\, d\alpha &= M(q_1, q_2)\sum_{n=0}^{\infty} b_n(q_1, q_2)\int_{\epsilon}^{2} e^{i(q_1+q_2)\ln\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)^{2n}\frac{d\alpha}{\alpha}\\ &= M(q_1, q_2)\sum_{n=0}^{\infty} a_n(q_1, q_2, \epsilon),\end{aligned} \tag{3.20}$$

where

$$a_n(q_1, q_2, \epsilon) = \frac{b_n(q_1, q_2)}{2n + i(q_1+q_2)}\left[1 - e^{i(q_1+q_2)\ln\frac{\epsilon}{2}}\left(\frac{\epsilon}{2}\right)^{2n}\right], \quad n \geq 0. \tag{3.21}$$

Since $|2n + i(q+q')| > q$ for all $n \in \mathbb{N}_0$ and $q, q' > 0$, it follows from (3.20), (3.21), and (3.19) that

$$\left|\int_{\epsilon}^{2} \frac{A(\alpha, q)A(\alpha, q')}{\alpha}\, d\alpha\right| \leq |M(q, q')|\,\frac{2e^2}{q}. \tag{3.22}$$

For $q' > 0$ and $q \neq q'$, we have from (3.20)

$$\int_{\epsilon}^{2} \frac{A(\alpha, q)A(\alpha, -q')}{\alpha}\, d\alpha = M(q, -q')a_0(q, -q', \epsilon) + M(q, -q')\sum_{n=1}^{\infty} a_n(q, -q', \epsilon). \tag{3.23}$$

The reason for writing the first term of the above sum separately comes from the existence of the singularity as $q' \to q$. This fact becomes more clear throughout the rest of the proof and specially in the proof of part (ii). Now a similar argument as in (3.22), together with observation $|2n + i(q-q')| \geq 2n \geq 2$ for $n \geq 1$, implies

$$\left|M(q, -q')\sum_{n=1}^{\infty} a_n(q, -q', \epsilon)\right| \leq |M(q, q')|\sum_{n=1}^{\infty}\frac{2^n}{n!} < |M(q, q')|e^2. \tag{3.24}$$

Also from (3.21) and definition of $b_n(q,-q')$

$$\begin{aligned}\Re\left[M(q,-q')a_0(q,-q',\epsilon)\right] = \frac{\Im M(q,-q')}{q-q'}\left[\cos\left[(q-q')\ln\frac{\epsilon}{2}\right]-1\right] \\ - \Re M(q,-q')\frac{\sin\left[(q-q')\ln\frac{\epsilon}{2}\right]}{q-q'}.\end{aligned} \tag{3.25}$$

Using (3.18), we can estimate the first term in (3.25);

$$\left|\frac{\Im M(q,-q')}{q-q'}\right| \le |M(q,q')|\left|\frac{1}{q'-q}\left[\frac{\Gamma(1+iq')}{\Gamma(1-iq')}-\frac{\Gamma(1+iq)}{\Gamma(1-iq)}\right]\right| \le |M(q,q')|\delta_q, \tag{3.26}$$

where in the last inequality we have used the properties of the Gamma function to conclude

$$\delta_q = \sup_{q'>0}\left|\frac{1}{q'-q}\left[\frac{\Gamma(1+iq')}{\Gamma(1-iq')}-\frac{\Gamma(1+iq)}{\Gamma(1-iq)}\right]\right| < \infty.$$

Now putting all pieces together, (3.13), (3.16), (3.20), (3.23) yield

$$F_\beta(q,q';\epsilon) = H_1(q,q';\epsilon) + H_2(q,q')\frac{\sin\left[(q-q')\ln\frac{\epsilon}{2}\right]}{q-q'}, \tag{3.27}$$

where

$$\begin{aligned}H_1(q,q';\epsilon) = G(q,q',\beta)\Bigg(\int_2^\infty \frac{K_{iq}(\alpha)K_{iq'}(\alpha)}{\alpha}\,d\alpha + \\ 2\Re\left[M(q,q')\sum_{n=0}^\infty a_n(q,q',\epsilon)\right] + 2\Re\left[M(q,-q')\sum_{n=1}^\infty a_n(q,-q',\epsilon)\right] + \\ \frac{\Im M(q,-q')}{q-q'}\left(\cos\left[(q-q')\ln\frac{\epsilon}{2}\right]-1\right)\Bigg),\end{aligned} \tag{3.28}$$

and

$$H_2(q,q';\epsilon) = -G(q,q',\beta)\,\Re M(q,-q'). \tag{3.29}$$

From (2.8) and definition of $M(q_1,q_2)$ (see (3.18)), we have

$$|M(q,q')| \le \frac{\pi}{4}[qq'\sinh(\pi q)\sinh(\pi q')]^{-\frac{1}{2}}. \tag{3.30}$$

Using the simple observation $[\mathrm{sech}(\pi q')]^\beta\sqrt{q'\sinh(\pi q')} \le C\,2^\beta q'^{\frac{1}{2}}e^{-(\beta-\frac{1}{2})\pi q'}$, it follows from (3.30) and our obtained estimates (3.14), (3.22), (3.24), and (3.26) that

$$|H_1(q,q';\epsilon)|,\ |H_2(q,q')| \le C_q\,2^\beta q'^{\frac{1}{2}}e^{-(\beta-\frac{1}{2})\pi q'} \tag{3.31}$$

Finally, since $0<\epsilon<2$, (3.31) and (3.27) yield

$$|F_\beta(q,q';\epsilon)| \le C_q\left|\ln\frac{\epsilon}{2}\right| 2^\beta q'^{\frac{1}{2}}e^{-(\beta-\frac{1}{2})\pi q'}, \tag{3.32}$$

for all $q'>0$. This proves part (i).

To prove part (ii), let $\psi : (0,\infty) \to \mathbb{R}$ be differentiable and bounded on $(0,\infty)$. By (3.32) the integral $\int_0^\infty \psi(q')F_\beta(q,q';\epsilon)\,dq'$ is convergent. Using (3.27), we write

$$\begin{aligned}\int_0^\infty \psi(q')F_\beta(q,q';\epsilon)\,dq' &= \int_0^\infty \psi(q')H_3(q,q';\epsilon)\,dq' \\ &\quad + \int_{\frac{1}{2}q}^{\frac{3}{2}q} \psi(q')H_2(q,q')\frac{\sin\left[(q'-q)\ln\frac{\epsilon}{2}\right]}{q'-q}\,dq' \\ &\quad + \int_{\substack{|q'-q|>\frac{1}{2}q \\ q'>0}} \psi(q')H_2(q,q')\frac{\sin\left[(q'-q)\ln\frac{\epsilon}{2}\right]}{q'-q}\,dq' \\ &= I_1 + I_2 + I_3.\end{aligned}$$

From (3.31), it follows

$$|I_1 + I_3| \le C_q\, 2^\beta\left(1+\frac{2}{q}\right)||\psi||_\infty \int_0^\infty q'^{\frac{1}{2}}e^{-(\beta-\frac{1}{2})\pi q'}\,dq' \le C_{q,\beta}||\psi||_\infty,$$

In order to estimate I_2, let $u = q'-q$. Then

$$I_2 = \int_{-\frac{1}{2}q}^{\frac{1}{2}q} f(u)\frac{\sin|\ln\frac{\epsilon}{2}|u}{u}\,du,$$

where $f(u) = -\psi(u+q)H_2(q,u+q)$. Next, we use the fact that if $f:(a,b)\to\mathbb{R}$ $(a<0<b)$ is continuous and bounded such that $(f(u)-f(0))/u$ is also bounded in a neighborhood of $u=0$, then (see [24])

$$\lim_{R\to\infty}\int_a^b f(u)\frac{\sin(Ru)}{u}\,du = \pi f(0). \tag{3.33}$$

Now form (3.29) and (3.18) it follows that $(f(u)-f(0))/u$ is bounded near $u=0$. Thus, (3.33) and (3.31) imply

$$\lim_{\epsilon\to 0^+}|I_2| = \pi|f(0)| = \pi|\psi(q)H_2(q,q)| \le C_{q,\beta}\,||\psi||_\infty.$$

This completes the proof of part (ii). □

Finally, we are in the position to present a proof of our main result.

Proof of Orthogonality Theorem. We show that $F(q,q';\epsilon) \xrightarrow[\epsilon\to 0^+]{} \delta(q-q')$, in the distribution sense; more explicitly (see [22]),

$$\lim_{\epsilon\to 0^+}\int_0^\infty \phi(q')F(q,q';\epsilon)\,dq' = \phi(q), \qquad \text{for all} \quad \phi\in C_c^\infty(0,\infty). \tag{3.34}$$

Now let $\phi\in C_c^\infty(0,\infty)$ and suppose $\delta>0$. For $\frac{1}{2}<\beta<1$, clearly

$$\phi(q)e^{-(1-\beta)\pi|q|}\cosh(\pi q) \in C_c^\infty(0,\infty),$$

and we may extend this to an even function in $C_c^\infty(\mathbb{R})$. Then by Lemma 3.10 there exists an even polynomial P such that

$$\left|\phi(q)e^{-(1-\beta)\pi|q|}\cosh(\pi q) - P(q)e^{-(1-\beta)\pi|q|}\right| < \delta, \qquad \forall q \in \mathbb{R}.$$

Multiplying the above inequality by $[\cosh(\pi q)]^{\beta-1}e^{(1-\beta)\pi|q|}$ yields

$$|\phi(q) - P(q)\mathrm{sech}(\pi q)|[\cosh(\pi q)]^\beta < 2^{1-\beta}\delta, \qquad \forall q > 0. \tag{3.35}$$

Multiplying (3.35) by $[\mathrm{sech}(\pi q)]^\beta$ (< 1) yields

$$|\phi(q) - P(q)\mathrm{sech}(\pi q)| < 2^{1-\beta}\delta, \qquad \forall q > 0. \tag{3.36}$$

For the remainder of the proof fix $q > 0$. Then

$$\begin{aligned}\int_0^\infty \phi(q')F(q,q';\epsilon)\,dq' = \int_0^\infty & P(q')\mathrm{sech}(\pi q')F(q,q';\epsilon)\,dq' \\ &+ \int_0^\infty \psi(q')F(q,q';\epsilon)\,dq',\end{aligned} \tag{3.37}$$

where

$$\psi(q') = \phi(q') - P(q')\mathrm{sech}(\pi q').$$

Now (3.5) implies (since P is an even polynomial),

$$\lim_{\epsilon\to0^+}\int_0^\infty P(q')\mathrm{sech}(\pi q')F(q,q';\epsilon)\,dq' = P(q)\mathrm{sech}(\pi q),$$

for each $q > 0$. This and (3.36), (3.37) yield

$$\begin{aligned}\limsup_{\epsilon\to0^+}&\left|\phi(q) - \int_0^\infty \phi(q')F(q,q';\epsilon)\,dq'\right| \\ &\le 2^{1-\beta}\delta + \limsup_{\epsilon\to0^+}\left|\int_0^\infty \psi(q')F(q,q';\epsilon)\,dq'\right| \\ &= 2^{1-\beta}\delta + \limsup_{\epsilon\to0^+}\left|\int_0^\infty \psi(q')[\cosh(\pi q')]^\beta F_\beta(q,q';\epsilon)\,dq'\right|.\end{aligned} \tag{3.38}$$

By Lemma 3.12, as well as

$$\|\psi(q')[\cosh(\pi q')]^\beta\|_\infty < 2^{1-\beta}\delta,$$

(by (3.35)), the last term in (3.38) is at most $2^{1-\beta}C_q\delta$. Thus

$$\limsup_{\epsilon\to0^+}\left|\phi(q) - \int_0^\infty \phi(q')F(q,q';\epsilon)\,dq'\right| \le 2^{1-\beta}(1 + C_q)\delta.$$

Since $\delta > 0$ is arbitrary, we get (3.34). This proves the Theorem. □

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