Quantum Evolution as a Nonlinear Markov Process

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Abstract

Based upon the Madelung version of the Schrödinger equation, quantum evolution in real space has been interpreted as a nonlinear Markov process in which future state depends upon present state and present probability distribution . From this viewpoint, the concepts of nonlocality, and entanglement have been discussed. The concept of hidden statistics have been introduced as an alternative way to formulate and interpret quantum formalism as well as to solve the Schrödinger equation. The problem of incompleteness of the Schrödinger description as well as physical meaning of the quantum potential is discussed.

1. *Introduction* The main-stream activity in quantum mechanics is associated with the Schrödinger equation which describes linear evolution of a wave function in Hilbert space. In this paper we depart from that tradition and turn to the equivalent equations in real space introduced by Madelung [1] and describing a particle of mass m in a potential $V(r)$:

$$\frac{\partial \rho}{\partial t} + -\frac{\hat{E}r}{\hat{E}m} - S\hat{\tilde{-}} = 0, \tag{1}$$

$$\frac{\bullet S}{\bullet t} + \frac{1}{2m}(-S)^2 + V - \frac{\hbar^2}{2m}\frac{-^2\sqrt{r}}{\sqrt{r}} = 0 \tag{2}$$

Here ρ and S are the components of the wave function $j = \sqrt{r}\, e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π.

We will demonstrate that analysis of Eqs. (1), (2) illuminates some special aspects of quantum evolution which could be overlooked in the Schrödinger equation.

2. *Quantum evolution without nonlocalities.* We will start analysis of Eqs. (1) and (2) with a special class of quantum states for which:

$$\nabla^2 \sqrt{r} \quad 0, \; -{}^3S \quad 0, -{}^3V \quad 0 \quad at \; t = 0 \tag{3}$$

One can verify that evolution of Eqs. (1), (2) preserves the condition (3), and therefore, the system reduces to

$$\frac{\partial \rho}{\partial t} + -{}^3\frac{\hat{\mathrm{E}} r}{\hat{\mathrm{E}} m} - S\hat{\tilde{-}} = 0, \tag{4}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = 0 \tag{5}$$

since the initial conditions (3) ensure no higher derivatives exist (in the Taylor expansion) and thus S and V depend only on derivatives up through second order.

Symbolically this system is identical to a classical system described by the Hamilton-Jacobi equation (5) where the phase S stands for the action. Eq. (4) is identical to the Liouville equation which describes the evolution of probabilities due to random initial conditions in Eq. (5). It should be emphasized that we are talking about quantum evolution whose mathematical description is identical to some "equivalent" classical system. Note: although it is true that the coupled equations (1) and (2) can be obtained from the Schrödinger equation [2] that is not the procedure in this paper.

Let us emphasize the main mathematical properties of the system (4), (5).

Firstly, the evolution of probability in Eq. (4) is linear. Indeed, Eq. (5) does not depend upon ρ, and therefore ∇S in Eq. (4) can be considered as a function of r and t, but not of ρ

Secondly, the system is of hyperbolic type. This can be established by starting with Eq. (5) which does not depend upon Eq. (4). Introducing weak discontinuities of S and applying a conventional linearization technique with respect to $(\nabla S)^2$ one obtains

$$\left[\frac{\partial S}{\partial t}\right] + \frac{-S}{m}[-S] = 0 \quad (\text{since } [V] = 0) \tag{6}$$

and taking into account the kinematical conditions of compatibility at the front of discontinuity, one obtains:

$$\left[\frac{\partial S}{\partial t}\right] = -\lambda_s[-S], \quad \left[-\lambda_s + \frac{-S}{m}\right][-S] = 0, \quad i.e.,$$

$$\lambda_s = \frac{-S}{m} \quad if \quad [-S] \neq 0 \tag{7}$$

where λ_s is the speed of propagation of the weak discontinuities of the phase S normal to the front of the discontinuity, and [ ] is the difference of the value of a quantity on two sides of a discontinuity surface.

A similar approach can be applied to Eq. (4) under condition that the fronts of discontinuities ρ and S do not coincide. Then on the front of ρ: $[-S] = 0$, $[-\nabla^2 S] = 0$, and, as follows from Eq. (4)

$$\lambda_\rho = \frac{\nabla S}{m} = \lambda_s \tag{8}$$

where λ_ρ is the speed of propagation of the weak discontinuities of the quantum probability density ρ.

It should be noticed that if the fronts of weak discontinuities of ρ and S coincide, then $[S] \neq 0$ and $\nabla^2 S \to \infty$, i.e., this case is physically unrealistic.

Thus, all the disturbances in the system (4), (5) propagate as traveling waves with the finite velocity $\nabla S/m$,and therefore, the quantum system under the condition (3) do not have non-localities. In other words, nonlocality is not a necessary property of a quantum system (by a nonlocality we understand an instantaneous action on a distance)..

Thirdly, symbolically one can introduce new variables via the "Hamiltonian"

$$H = \frac{1}{m}(-S)^2 + V \tag{9}$$

as

$$\frac{dx_i}{dt} = \frac{\bullet H}{\bullet \frac{\bullet S}{\bullet x_i}}, \quad \frac{dp_i}{dt} = -\frac{\bullet H}{\bullet x_i}, \; p_i = \frac{\bullet S}{\bullet x_i} \tag{10}$$

Since Eq. (4) (which is a quantum analog of the Liouville equation) describes the evolution of quantum probability, the system (10) (which is a quantum analog of the Hamiltonian canonical equations) describes the evolution of the variables behind the probability evolution, i.e., "hidden" statistics.

Thus, the Madelung equations (1),(2) reveal that in a very special case (see Eqs. (3)) the quantum system can be local, the evolution of quantum probabilities can evolve linearly in real space, and such systems can be characterized by symbolic hidden statistics.

3. *Mathematical origin of quantum nonlocality.* Removing the restrictions (3), one arrives at the general case of quantum evolution which now is described by the Madelung equations in their original form (1),(2). The main difference between these equations and their simplified version (4),(5) is due to the so called quantum potential which is the last term in Eq. (2): this term changes the Madelung equations from hyperbolic to parabolic type. This change is similar to transition from the Euler's to the Navier-Stokes equations due to viscosity.

Now Eqs. (6) and (7) are not applicable any more since the weak discontinuities of ρ and S are impossible. Indeed, in order the second derivatives $\nabla^2 r$ and $-{}^2S$ to exist, the first derivatives $\nabla\rho$ and $-S$ must be continuous, (except in pathological cases) i.e.,

$$[\nabla\rho] \equiv 0, \; [-S] \; 0 \tag{11}$$

That means that any disturbance of ρ or S in one point of space instantaneously transmitted to the whole space, and this is the mathematical origin of nonlocality. But is this a unique property of quantum evolution? Obviously, not. Any parabolic equation (such as Navier-Stokes equations, or Fokker-Planck equation) has exactly the same nonlocal properties. However, the difference between the quantum and classical nonlocalities is in their physical interpretation. Indeed, the Navier-Stokes equations are

derived from simple laws of Newtonian mechanics, and that is why a physical interpretation of nonlocality is very simple: if a fluid is incompressible, then the pressure plays the role of a reaction to the geometrical constraint $\nabla \cdot v = 0$, and it is transmitted instantaneously from one point to the whole space (the Pascal law). One can argue that the incompressible fluid is an idealization, and that is true. However, it does not change our point: such a model has a lot of engineering applications, and its nonlocality is well understood. The situation is different in quantum mechanics since the Schrödinger equation has never been derived from Newtonian mechanics: it has been postulated. That is why the physical origin of the same mathematical phenomenon cannot be reduced to simpler concepts such as "forces": it should be accepted as an attribute of the Schrödinger equation.

4. *Quantum entanglement.* Quantum entanglement is a very special phenomenon which follows from quantum nonlocality. In order to illustrate it using the Madelung equation, consider two electrons in the field of a positive atomic nucleus and rewrite Eqs. (1),(2) for the system of two electrons:

$$\frac{\partial \rho}{\partial t} + \nabla_1 \cdot \left(\frac{\rho}{m} \nabla_1 S \right) + \nabla_2 \cdot \left(\frac{\rho}{m} \nabla_2 S \right) = 0 \tag{12}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left[(\nabla_1 S)^2 + (\nabla_2 S)^2 \right] + V - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} = 0 \tag{13}$$

where

$$V = -\sum_{i=1}^{2} \frac{\mathrm{e}^2 \mathrm{e}_o}{\sqrt{x_i^2 + y_i^2 + \mathrm{z}_i^2}} + \frac{\mathrm{e}^2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (\mathrm{z}_1 - \mathrm{z}_2)^2}}, \tag{14}$$

$\rho(r_1, r_2)$ is joint probability density for both electrons, ∇ is the Laplacian operator in which the differentiation is with respect to all six coordinates of the two electrons x_i, y_i, z_i $(i = 1, 2)$, and e_0 is the number of protons. It is assumed that the mass of the electron is negligible compared to that of the nucleus.

Due to the second term in Eq. (14), the motions of the electrons are correlated, i.e.,

$$\rho(r_1, r_2) \neq \rho_1(r_1)\rho_2(r_2) \tag{15}$$

Therefore, because of the parabolicity of the system (12), (13), any disturbance of the conditional probability density $\rho_1(r_1, r_2)$ of the first electron instantaneously effects the conditional probability density $\rho_2(r_1, r_2)$ of the second electron, and that represents the quantum entanglement.

In our example, two electrons were entangled due to the interaction via the potential (15). This interaction vanishes with increase of the distance between them. However, there has been discovered another class of quantum states which are entangled in the course of their simultaneous creation (under the requirement that some attributes such as spin or polarization, be conserved), or as a result of natural evolution of composite quantum system. In that type of entanglement, the interactions do not change with the distance at all. Although a physical machinery of such a phenomenon is not yet understood, its mathematical origin is still in the parabolicity of the Madelung (and the Schrödinger) equations.

5. *Quantum evolution and nonlinear Markov processes.* So far we have not found a fundamental and unique property of quantum mechanics which would distinguish it from classical mechanics. Indeed, as pointed out above, the nonlocality (at least in terms of mathematical formalism) occurs in classical models as well; the same can be said about wave interference, or correlated stochastic processes.

But let us turn to the type of Markov processes which appear in classical and quantum mechanics. In Newtonian world, all Markov processes are characterized by linear evolution of probabilities (see the Fokker-Planck or the Liouville equations in which the coefficients can depend only upon time or space coordinates, but not upon the probability itself). Actually the linear evolution of Markov processes in classical physics was proved by Kolmogorov under some insignificant mathematical restrictions. The fact

that all such processs eventually approach ergodic states with maximum entropy is in agreement with the second law of thermodynamics.

The situation in quantum mechanics is different. Firstly one has to realize that all the quantum evolutions are represented by Markov processes since in parabolic equations future state depends only upon the present one. Secondly, although a quantum Markov process is linear with respect to the wave function, it is nonlinear with respect to the probability density ρ (see the Madelung equations (1), (2)). That nonlinearity brings fundamental changes in physical interpretation of quantum mechanics, and this interpretation will be discussed below.

6. *The concept of hidden statistics.* Consider a system of nonlinear Langevin equations.

$$\dot{v}_i = f_i(\{v\}) + g_{ij}(\{v\})\Gamma_j(t),\ i = 1,2,\ldots n \tag{16}$$

where $\Gamma_j(t)$ is the Langevin (stochastic) force:

$$<\Gamma_j(t)> = 0,\ <\Gamma_i(t)\Gamma_j(t')> = 2\delta_{ij}\delta(t-t') \tag{17}$$

and v is a random variable. Here the Einstein summation convention is used.

The solution to Eqs. (16) is a stochastic process whose probability evolution $\rho(\{v\},t)$ satisfies the Fokker-Planck equation [4] :

$$\frac{\partial\rho}{\partial t} = \left[-\sum_{i=1}^{n}\frac{\partial}{\partial v_i}D_i^{(1)}(\{v\}) + \sum_{i,j=1}^{n}\frac{\partial^2}{\partial v_i\partial v_j}D_{ij}^{(2)}(\{v\})\right]\rho \tag{18}$$

where

$$D_i^{(1)} = f_i(\{v\}) + g_{kj}(\{v\})\frac{\partial}{\partial v_k}g_{ij}(\{v\}) \tag{19}$$

$$D_{ij}^{(2)} = g_{ik}(\{v\})g_{jk}(\{v\}). \tag{20}$$

One can associate the Langevin equations (16) with the hidden statistics behind the corresponding Fokker-Planck equation (18). Indeed, Eqs. (16) simulate randomness while Eq. (18) manipulates with the values of its probability.

It should be emphasized that although the Langevin equations are nonlinear (with respect to $\{v\}$), the Fokker-Planck equation is linear (with respect to ρ).

Suppose that the diffusion coefficients (20) in Eq. (18) depend upon the probability ρ, i.e.,

$$D_{ij}^{(2)} = D_{ij}^{(2)}(\{v\}, r) \tag{21}$$

Obviously that makes Eq. (18) nonlinear. But what happens now with the Langevin equations (16), i.e., with the hidden statistics? As follows from Eq. (20), at least some coefficients g_{ik} must depend upon probability ρ, and Eqs. (16) are modified to:

$$\dot{v}_i = f_i(\{v\}) + g_{ij}(\{v\}, r)\,G_j(t) \tag{22}$$

One can see that now the hidden statistics cannot be studied separately from the nonlinear version of Eq. (18): there is a master-slave relationship between Eqs. (18) and (22), and that brings a fundamental difference between linear and nonlinear Markov processes. Indeed, according to Eqs. (22), the next step of the particle performing a stochastic process is based not only upon its present state, but also upon the probability distribution over all possible present states. (Obviously the process is still Markovian since past is not involved).

7. *Hidden statistics of quantum evolution.* Continuing the line of argumentation presented in the previous section, let us pose the following problem: find the system of differential equations which would describe the evolution of hidden statistics behind the Madelung equation in the same way in which the slave version of the Langevin equation (22) describes the hidden statistics behind the nonlinear version of the Fokker-Planck equation (18) with the diffusion (21). For that purpose, turn to Eq (1) and compare it with the Liouville equation

$$\frac{\partial \rho'}{\partial t} + -\frac{\hat{E}\,r\,\hat{C}}{\hat{E}\,m} - S\,\hat{C} = 0 \tag{23}$$

for the classical system of Hamiltonian equations:

$$\frac{dx_i'}{dt} = \frac{\partial H'}{\partial p_i}, \frac{dp_i'}{dt} = -\frac{\partial H'}{\partial x_i}, H' = \frac{1}{m}(\nabla S')^2 + V, \frac{\partial S'}{\partial x_i} = p_i' \tag{24}$$

Here S' is the true classical action governed by the Hamilton-Jacobi equation

$$\frac{\partial S'}{\partial t} + \frac{1}{2m}(\nabla S')^2 + V = 0 \tag{25}$$

and H', x_i', p_i' are classical Hamiltonian, coordinates and momenta, respectively.

The relationship between the equations (23) and (24) is the following: if the variables $x_i'(t)$ and $p_i'(t)$ in Eq. (24) are given initial conditions

$$x_i'(t=0) = \overset{o}{x}_i(x), \quad p_i'(t=0) = \overset{o}{p}_i \tag{26}$$

where $\overset{o}{x}_i'(x)$ run over a random set of values with the probability density

$$\rho'(\{\overset{o}{x}'\}, t=0) = \overset{o}{r} \tag{27}$$

then the evolution of ρ' will be governed by Eq. (23). One should notice that Eqs. (24) do not depend upon ρ': they can be solved based only upon the initial conditions (26). At the same time the evolution of probability density ρ' is linear since $\nabla S'$ does not depend upon ρ' (see Eq. (25)).

Let us now return to the quantum case and write down the following system of differential equations:

$$\frac{dx_i}{dt} = \frac{\partial \tilde{H}}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial \tilde{H}}{\partial x_i}, \tilde{H} = \sum p_i^2 + V(\{x\}) - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\tilde{\rho}}}{\sqrt{\tilde{\rho}}} \tag{28}$$

where $\tilde{r}(\{x\}, t)$ is an arbitrary function to be determined below, and $\tilde{H}(\{x\}, \{p\})$ is an auxiliary functon obtained from the classical Hamiltonian H' (see Eq. (24)).
Obviously

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{\partial \tilde{H}}{\partial x_i} + \frac{\partial \tilde{H}}{\partial \sqrt{\tilde{r}}}\frac{\partial \sqrt{\tilde{r}}}{\partial x_i} + \frac{\partial \tilde{H}}{\partial \nabla^2\sqrt{\tilde{r}}}\frac{\partial \nabla^2\sqrt{\tilde{r}}}{\partial x_i} \quad . \tag{29}$$

One can now introduce another auxiliary function $\tilde{S}(\{x\}, t)$ which follows from $\tilde{H}$:

$$\frac{\partial \tilde{S}}{\bullet t} = -\tilde{H}. \tag{30}$$

For any first order system of ordinary differential equations (not necessarily Hamiltonian) one can find the corresponding Liouvill-Gibbs equation which describes the evolution of probability density ρ triggered by random initial conditions. In case of Eqs. (28) the Liouville-Gibbs equation will be represented by the following equation:

$$\frac{\partial \rho}{\partial t} + -\frac{\hat{\nabla} r}{m} - \tilde{S}\hat{\tilde{\ }} = 0 \tag{31}$$

Here $\rho(\{x\})$ implicitly depends upon the arbitrary function $\tilde{r}$ (see Eqs. (29) and (30). Let us choose this function such that

$$\tilde{r} = r \tag{32}$$

Then, as follows from Eq. (2), (29) and (30)

$$\tilde{S} = S \tag{33}$$

where $S/\hbar$ is the quantum phase.

Now we can formulate the final result:

If the probability evolution of a quantum system is described by Eq. (1) and (2), then the hidden statistics behind these equations is described by the system of stochastic differential equations with respect to position x_i and momentum p_i of the particles:

$$\frac{dx_i}{dt} = \frac{\bullet \tilde{H}}{\bullet p_i}, \frac{dp_i}{dt} = -\frac{\bullet \tilde{H}}{\bullet x_i}, \tag{34}$$

where

$$\tilde{H} = p^2 + V(\{x\}) - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{35}$$

while the source of stochasticity is in the randomness of the initial conditions. For instance, if the initial momentum p_i in Eqs. (34) are known exactly, then, according to the uncertainty principle, the positions x_i can be formulated only probabilistically, and Eqs. (34) become stochastic.

However we should emphasize a fundamental difference between the classical and quantum cases. In the latter, the hidden statistics described by the stochastic system (34) depends upon the Madelung equation (1),(2) (see Eq. 35)), i.e., the system (34) forms a slave equation (similar to the case of Eqs. (18),(21) and (22). This means that Eqs. (34) formally cannot be solved separately from Eqs. (1),(2). At the same time, Eqs. (1),(2) can be solved independently from Eqs. (34) and then their solution

$$\rho = \rho(\{x\}, t) \tag{36}$$

should be substituted in Eqs. (34) via the "Hamiltonian" $\tilde{H}$ given by Eqs. (35). After that Eqs. (34) can be solved, and their solution will represent the positions $x_i(t)$ and momenta $p_i(t)$ as stochastic processes whose evolution of probability satisfies the Madelung equations (1),(2).

However, there is an alternative way to solve Eqs. (34) if one recalls that the probability density $\rho(\{x\}, t^*)$ at any fixed instant t^* can be found from a statistical analysis of the ensemble of random values of $\{x\}$ at t^* that is if one combines step by step solution of Eq. (34) with the statistical analysis of $\{x\}$ at each time step. Then Eqs. (34) can be solved independently from Eqs. (1),(2). This is one of possible strategies:
Turn to Eq. (34), fix the initial momenta

$$p_i(t=0) = \overset{o}{p}_i \tag{37}$$

and introduce a random set of the initial positions

$$x_i(t=0) = \overset{o}{x}_i(\xi) \tag{38}$$

then find the corresponding probability density

$$\rho(t=0) = \overset{o}{r}(\{x\}) \tag{39}$$

by performing statistical analysis of the set (38).

Next substitute $\overset{o}{p}'_i, \overset{o}{x}_i(\xi)$ and $\overset{o}{r}(\{x\})$ into Eq. (35) to find $\tilde{H}(t=0) = \tilde{H}^o\left(\{\overset{o}{x}\}, \{\overset{o}{p}\}, \overset{o}{r}\right)$ and determine the values of x_i and p_i at $t + \Delta t$ from Eq. (34) selecting Δt sufficiently small. After similar statistical analysis of the set $x_i(\Delta t)$, the next value of $\rho(\{x\}, \Delta t)$ can be found etc. Eventually this process will

reconstruct the hidden stochastic process behind the Madelung equations, as well as the probability density $\rho(\{x\},t)$ without invoking Eqs. (1),(2).

Another approach to solution of Eqs. (34) can be based upon successive approximations: start with an arbitrary $\rho_{(1)}$ and find the solution to Eqs. (34); perform the statistical analysis of this solution and recalculate $\rho = \rho_{(2)}$; continue this process until $\rho_{(n-1)} = r_{(n)}$ (the conditions of an assumed convergence will not be discussed here).

Thus, the model of hidden statistics offers an alternative way of representation and interpretation of quantum formalism as well as a new approach to the solution of the Madelung (and therefore, the Schrödinger) equations. In many respects, such a solution by stochastic simulations has certain advantages over calculations of probabilities using partial differential equations. Such advantages were discussed [5] in connection with solutions to the Fokker-Planck equation.

8. *Incompleteness of Schrödinger equation*. Let us take a closer look at the relationships between equations (34) and equations (1), (2) which represent the quantum version of the Hamiltonian equations and the Liouville equation respectively. As in the classical case, here the Madelung equations can be uniquely reconstructed from the "Hamiltonian" equations. However, the inverse is not true: there is an infinite number of "Hamiltonian" equations which correspond to the same Madelung equation. (Loosely speaking, the "Hamiltonian" equations can be associated with a vector field while the Madelung equations – with the scalar field of its divergency). But does it mean that the "Hamiltonian" equations provide more complete description of the underlying quantum system? The answer depends on the axiomatics of quantum theory: if the Schrödinger equation is postulated as the most complete model of quantum world, then all the modifications of the "Hamiltonian" equations must be disregarded, being considered as physically identical; however, if the "Hamiltonian" equations are postulated as the prime model, then they provide more complete description of quantum evolution than the Schrödinger equation (although they are in full agreement with the latter). In principle, this dilemma has to be resolved by experiments. Nevertheless, one can also appeal to the intrinsic logic of the quantum mechanics. Indeed, in classical world the Hamiltonian

equations are the prime model which provides the most complete description of the classical evolution, while the Liouville equation describes only certain aspects of this evolution. Then, why should one switch over the roles of the "Hamiltonian" and Madelung (i.e., "Liouville") equations as soon as the quantum potential changes from zero to non-zero? On the contrary, based upon the correspondence principle, the preference should be given to the smoothest transition from quantum to classical theory, and therefore, one should accept the second solution, i.e., to postulate equation (34).

Let us qualitatively illustrate phenomena which may exist in quantum evolution without being detected by the Schrödinger equation. For that purpose we will turn to the classical case and consider the Liouville equation. This equation describes evolution of the volume in phase space, and it is insensitive to the redistribution of growth along different coordinates. But this redistribution is responsible for such a fundamental phenomenon as chaos which can be detected by the Hamiltonian system (but cannot be detected by the Liouville equation). The same type of quantum phenomena can be missed by the Schrödinger equation.

9. *Kinematical invariants of motion*. Returning to Eq. (34)

$$\dot{x}_i = \frac{\partial \tilde{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \tilde{H}}{\partial x_i}, \quad i,j = 1,2,\ldots 3n \tag{40}$$

and equating the mixed derivatives (under the assumption of the existence and continuity of the second order partial derivatives).

$$\frac{\partial^2 \tilde{H}}{\bullet x_i \bullet x_j} = \frac{\bullet^2 \tilde{H}}{\bullet x_j \bullet x_i} \tag{41}$$

one obtains

$$\frac{\partial \dot{p}_i}{\partial x_j} = \frac{\partial \dot{p}_j}{\partial x_i}, \quad i, j = 1, 2, \ldots 3n \tag{42}$$

The significance of these kinematical relationships between the momenta and the position of n particles is in the fact that they are valid for both classical and quantum systems since the quantum potential (as well as the classical potential) was eliminated by Eq. (41).

If one associates Eq. (40) with the Euler's equations for incompressible fluid, then Eqs. (42) can be associated with the Helmholtz equations for the conservation of vorticity.

10. *Physical meaning of quantum potential*. Continuing further the analogy with the Euler's equations, one can try to associate quantum potential with the reaction of a kinematical constraint in the same way in which pressure in incompressible fluid is the react of the condition of incompressibility

$$\nabla \bullet v \geq 0 \tag{43}$$

where v is fluid velocity.

The only kinematical constraint (imposed by the particle-wave duality) in quantum mechanics is the uncertainty relationship:

$$\Delta x \Delta p \quad \hbar/2 \tag{44}$$

where Δx and Δp are the standard deviations of x and p as random variables.

As pressure implements the constraint (43) in Euler's equation, the quantum potential implements the constraint (44) in the Madelung equations (1), (2). Indeed, it can be proven that the Madelung equations satisfy the uncertainty relationship (44) due to the quantum potential. However, there are situations when the relationship (44) is trivially satisfied (in case of inquality in (44)), and then the quantum potential is not

needed (see Eqs. (3), (4) and (5)). Similarly, in the case of inequality in the condition (43) atomized fluid), pressure is equal to zero.

In order to illustrate our point, consider the one-dimensional motion of a particle $x(\xi,t)$. For that case, Eq. (34) reduces to the form:

$$m \frac{\partial^2 x(\xi,t)}{\partial t^2} = -\frac{\partial V(x)}{\partial x} + \frac{\hbar^2}{2m}\frac{\partial}{\partial \xi}\left[\frac{1}{\sqrt{\rho_x}(\xi)}\frac{\partial^2 \sqrt{\rho_x}(\xi)}{\partial \xi^2}\right] \tag{45}$$

where m is mass, V is classical potential, and the last term is the gradient of the quantum potential representing a force F_q. Here $\rho_x(\xi)$ is the probability distribution of x over its possible values ξ. Without the last term, Eq. (45) would represent a second Newton's law applied to an infinite number of independent samples of a particle forming a continuum $x(\xi)$. The last term F_q in Eq. (45) introduces an additional force which depends upon the probability distribution of x over ξ, and thereby it couples motions of all possible samples $x(\xi)$.

Let us show that this coupling protects quantum evolution from a violation of the constraint imposed by the uncertainty relationship (44). Indeed assume that for $\xi \to \xi_*$, x is approaching a sharp value, i.e.,

$$\rho(\xi \to \xi_*) = \delta(\xi \to \xi_*) \tag{46}$$

where δ is a Dirac function

Then

$$F_q \begin{cases} -\infty & at\ \xi < \xi_* \\ +\infty & at\ \xi > \xi_* \end{cases} \tag{47}$$

i.e., this additional quantum force diffuses the sharp value of x by pushing the samples $x = x < x_*$ *and* $x = x > x_*$ in opposite directions, i.e., away from the sharp value. An example of this can be seen in the computer graphics of Fig. 3.1 in [2] .

This example illustrates the interpretation of quantum potential as a reaction of the kinematical constraint (44).

11. Quantum intelligence: The properties of nonlinear Markov processes described above follow from the mathematical formalism, and do not depend upon their physical nature. Therefore, since quantum evolution represents a nonlinear Markov process, one can give the following interpretation of the distinctive property of quantum evolution: a quantum particle "chooses" the next step based not only upon its present position (as a classical particle performing a stochastic motion) but also upon the probability distribution over all the possible present positions. In other words, a quantum particle has more complete information about its present state. This property has a striking similarity with evolution of living systems [6,7] . In order to illustrate that, we will start with the following example: consider a small physical particle in a state of random migration due to thermal energy, and compare its diffusion, or physical random walk, with a biological random walk performed by a wild-type bacterium which can be associated with the simplest biological particle called a monad. The fundamental difference between these two types of motions can be detected in the probability space: the probability evolution of the physical particle (which can be associated with the Fokker-Planck equation) is always linear, and it has only one attractor – a stationary stochastic process where the motion is trapped. On the contrary, a typical probability evolution of a biological particle is nonlinear: it can have many different attractors, but eventually each attractor can be departed without any "help" from outside. However, such a behavior violates the second law of thermodynamics unless there is another "hidden" object which interacts with the monad. In order to find this object, one has to turn to mathematical psychology which postulates that a human (as well as other living systems) possesses a self-image and interacts with it. In terms of the mathematical formalism of classical physics, the self-image can be represented by the probability evolution (mental dynamics) associated with

the corresponding random walk (motor dynamics). Then the interaction between the monad and its self-image is implemented by the feedback from mental to motor dynamics, and that makes the probability evolution nonlinear. Actually the deviation from linear evolution expresses the "free-will" of the monad. From the physical viewpoint, the self-image is an external object, and that reconciles biological random walk with the second law of thermodynamics.

Since a biological particle possesses the self-image, it can predict future expected values of its parameters, and by interacting with this image, change the expectations if they are not consistent with the objectives. In other words, the nonlinear probability evolution makes a biological particle "smarter" than the physical one in dealing with the future and in approaching its object.

However, it should be emphasized that here the model of nonlinear random walk is phenomenological: the feedback between the nonlinear version of the Fokker-Planck equation and the Langevin equation simulates an extremely complex biochemical machinery of signal processing and incorporates it into the formalism of nonlinear dynamics. On the contrary, in quantum evolution the nonlinearity of Markov process is genuine: it follows from the axioms of quantum physics.

Obviously we do not suggest that the similarity between quantum evolutions and evolutions of simple living systems is deeper than only on the level of mathematical formalism. However, this comparison illuminates a distinctive property of quantum evolutions as nonlinear Markov processes.

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