Pergamon

Chaos, Solitons & Fractals Vol. 9, No. 7, pp. 1183–1198, 1998
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Printed in Great Britain
0960–0779/98 $19.00 + 0.00

PII: S0960-0779(97)00136-7

Non-Lipschitz Approach to Quantum Mechanics

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(Accepted 21 July 1997)

Abstract—An attempt to reconcile quantum mechanics with Newton's laws represented by the non-Lipschitz formalism has been made. As a proof-of-concept, a line of equally spaced atoms was studied. It appeared that enforcement of atom incompressibility required relaxation of the Lipschitz condition at the points of contact. This, in turn, led to fractional powers and discreteness of values of the basic parameters including energy and action, and finally, to the uncertainty relationship between positions and velocities. In addition to that, the relaxation of the Lipschitz condition caused instability of velocity with respect to small changes of the atom position, and that introduced an element of randomness in the system behavior. It was shown that the only model for the probability evolution which incorporates all the new properties of the motions is the Schrödinger equation. This means that quantum mechanics can be derived from Newton's laws if an unnecessary mathematical restriction—the Lipschitz condition—is removed from the mathematical formalism. Non-local properties of the model, as well as spin-effects and relativistic corrections are discussed. © 1998 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The governing equations of classical dynamics can be derived from Lagrange equations, from variational principles, or directly from Newton's laws of motion, and they may be presented in various equivalent forms. However, there is one mathematical restriction on all such forms: the differential equations describing a dynamical system

$$\dot{x}_i = v_i(x_1, x_2, \ldots, , x_n)\, i = 1, 2, \ldots, n \tag{1}$$

must satisfy the Lipschitz condition, which expresses that all the derivatives

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty \tag{2}$$

must be bounded. This mathematical restriction guarantees the uniqueness of the solution to equation (1), subject to fixed initial conditions, and that makes it the most attractive for a mathematical treatment. However, there is a certain price to be paid for such a mathematical convenience since in many cases the condition (2) is not compatible with the physical nature of motions. A detailed analysis of such cases for dissipative systems (with application to irreversibility in thermodynamics and to theory of turbulence) was presented in our earlier publications [1–4].

In this paper we will discuss only non-dissipative (Hamiltonian) systems with the emphasis on motions in the domain of an atomic scale.

2. THE PROBLEM FORMULATION

In order to trivialize our analysis, we will start with a line of equally spaced identical atoms of the mass m and the radius r_0. The two-atom potential will be presented in the simplest form:

$$V = \frac{1}{2}\gamma^2(\ell - x)^2 \tag{3}$$

where x is the distance between the atoms along the center line, and ℓ is the distance between the atom centers when they do not interact, Fig. 1.

The requirement of incompressibility of the atoms leads to the conditions:

$$\dot{x} = v \to 0 \text{ at } x \to 0 \tag{4}$$

and

$$\left|\frac{\partial V}{\partial x}\right| \to \infty \text{ at } x \to 0. \tag{5}$$

Then, as follows from the energy conservation

$$E = \frac{m\dot{x}^2}{2} + V = Const \tag{6}$$

and the condition (4):

$$V(0) = \frac{1}{2}v^2\ell^2 = E \tag{7}$$

where E is the total energy.

The only way to reconcile the conditions in equations (5) and (7) is to introduce an additional term in equation (3)

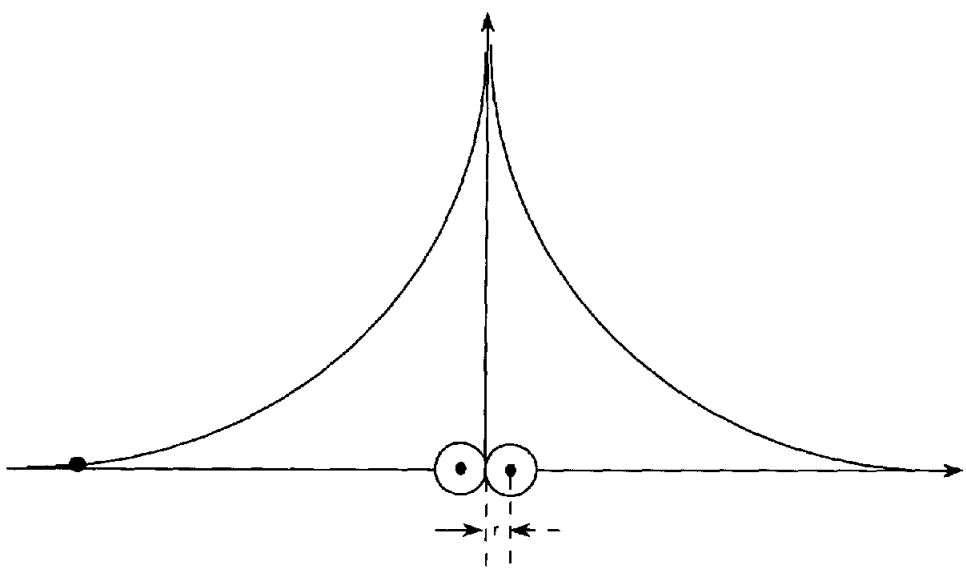

Fig. 1.

$$\tilde{V} = E\tilde{x}^{\alpha}, \tilde{x} = \frac{x}{r} \tag{8}$$

where r is a constant of the order of the radius of the atom, and α is a dimensionless positive constant:

$$0 < \alpha < 1 \tag{9}$$

Indeed, then:

$$\lim \tilde{V}(\tilde{x} \to 0) = 0, \text{ but } \lim \left| \frac{\mathrm{d}\tilde{V}}{\mathrm{d}\tilde{x}} \right| (\tilde{x} \to 0) = \lim |\alpha \tilde{x}^{\alpha - 1}| = \infty. \tag{10}$$

Obviously the condition (9) violates the Lipschitz condition for $\mathrm{d}\tilde{V}|\mathrm{d}\tilde{x}$ at $\tilde{x} \to 0$ in equation (10).

Now the potential (3) takes the form:

$$V = E(1 - \gamma\tilde{x}^2 - \tilde{x}^{\alpha}), \gamma = \frac{r^2}{\ell^2} < 1. \tag{11}$$

The additional term $\mathrm{d}\tilde{V}/\mathrm{d}\tilde{x}$ in the expression for the force:

$$F = \frac{\mathrm{d}V}{\mathrm{d}x} = -(2\gamma\tilde{x} + \alpha\tilde{x}^{\alpha - 1}) \tag{12}$$

dominates over the other terms when $x \to 0$, but it rapidly vanishes when $x \sim \ell$. In other words, the term in equation (8) in the expression for the potential (10) reconciles the conditions in equations (5) and (6) without changing the motion in the 'classical' domain $x \sim \ell$; but as will be shown below, it brings fundamental changes into motions on the atomic scale $x \sim r$.

Before deriving the equation of motion, we have to discuss in more detail the structure of the constant α. Since the two-atoms potential must be a symmetric function of the atom coordinates, it can be described only by an even function:

$$V(x) = V(-x), \tag{13}$$

and this requires that α must be a fraction with an even numerator and an odd denominator:

$$\alpha = \frac{2m}{2n+1}, m, n = 1, 2 \ldots. \tag{14}$$

It will be shown later that actually m in equation (14) must be an odd number. But now we will proceed with the equation of motion which immediately follows from the energy conservation (6):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = \pm\sqrt{\frac{2E}{m}\left(1 - \frac{V}{E}\right)} = \pm\sqrt{\frac{2E}{m}}\sqrt{\gamma\tilde{x}^2 + \tilde{x}^{\alpha}} \tag{15}$$

Obviously, on the atomic scale, i.e., within the domain where

$$x \sim r \tag{16}$$

equation (15) can be simplified since

$$\gamma\tilde{x}^2 = \frac{r^2}{\ell^2}\tilde{x}^2 < \tilde{x}^{\alpha}. \tag{17}$$

Introducing a dimensionless velocity

$$\tilde{v} = v\sqrt{\frac{m}{2E}} \tag{18}$$

one finally arrives at a very simple governing equation within a small region around $\tilde{x} = 0$:

$$\tilde{v} = \pm\tilde{x}^{\alpha/2}, 0 < \tilde{x} \sim 1. \tag{19}$$

Before proceeding with the analysis of equation (19), one has to note that as follows from equation (4),

$$\text{Sign}\,\tilde{v} = \text{Sign}\,\tilde{x}\,\text{at}\,x \to 0, \tag{20}$$

i.e., $\tilde{v}$ must be an odd function of $\tilde{x}$. This condition can be enforced by requiring that m in equation (14) is an odd number, and therefore, the final form of equation (14) is:

$$\alpha = \frac{2(2m-1)}{4n+1};\ n,m = 1,2,\cdots \text{etc.}\ m \leq n. \tag{21}$$

Introducing, for convenience, another constant:

$$\beta = \frac{\alpha}{2} = \frac{2m-1}{4m+1};\ n,m = 1,2,\ldots \text{etc.}\ m \leq n \tag{22}$$

we rewrite equation (19) in the most simple form:

$$\tilde{v} = \pm\tilde{x}^{\beta}, 0 < \tilde{x} \sim 1 \tag{23}$$

Two signs of the velocity in equations (15), (19), and (23) correspond to motions in opposite directions starting from the same position.

Now we can verify the boundedness of the fundamental invariants of motion when $\tilde{x} \to 0$.

The impulse of the force F:

$$I = \int_0^t F\,\mathrm{d}t = -\alpha\int_0^{\tilde{x}} \tilde{x}^{\alpha-1}\frac{\mathrm{d}x}{v} = -\alpha r\sqrt{\frac{m}{2E}}\int_0^{\tilde{x}} \tilde{x}^{(\alpha/2)-1}\,\mathrm{d}\tilde{x} = -2r\sqrt{\frac{m}{2E}}\tilde{x}^{\beta} \to 0\,\text{at}\,\tilde{x} \to 0 \tag{24}$$

vanishes with $\tilde{x}$ despite the unboundedness of the force F itself.

The action

$$S = \int_0^t (E-2V)\,\mathrm{d}t = -E\int_0^t (1-2\tilde{x}\alpha)\frac{\mathrm{d}x}{v} = -r\sqrt{\frac{mE}{2}}\int_0^{\tilde{x}} (1-2\tilde{x}^{\alpha})\tilde{x}^{-(\alpha/2)}\,\mathrm{d}\tilde{x}$$

$$= -r\sqrt{\frac{mE}{2}}\int_0^{\tilde{x}} (\tilde{x}^{-\beta}-2\tilde{x}^{\beta})\,\mathrm{d}\tilde{x} = -r\sqrt{\frac{mE}{2}}[(1-\beta)\tilde{x}^{1-\beta}-2(1+\beta)\tilde{x}^{1+\beta}] \to 0\,\text{at}\,\tilde{x} \to 0 \tag{25}$$

also vanishes with $\tilde{x}$.

Finally, one can find the period during which the motion approaches the point $\tilde{x} = 0$:

$$T = \sqrt{2m}\int_0^{\tilde{x}} \frac{\mathrm{d}x}{\sqrt{E-V(x)}} = r\sqrt{\frac{2m}{E}}\int_0^{\tilde{x}} \tilde{x}^{-\beta}\,\mathrm{d}\tilde{x} = r\sqrt{\frac{2m}{E}}(1-\beta)\tilde{x}^{1-\beta} \to 0\,\text{at}\,\tilde{x} \to 0. \tag{26}$$

3. QUANTS OF ENERGY AND UNCERTAINTY RELATIONSHIP

Let us turn to equations (18), (21) and (23) and express the kinetic energy of the motion W as a function of n and m:

$$W = E\tilde{x}^{[2(2m-1)/4n+1]}; n,m = 1,2,\ldots \text{etc.}, m \leq n. \tag{27}$$

As follows from equation (27), the energy W can take only discrete values and change by finite steps, or quants, since n and m are natural numbers. The smallest quant of energy found from the condition

$$m = 1, n \to \infty, \tag{28}$$

vanishes since

$$\Delta W = W_n - W_{n-1} = E[\tilde{x}^{2/(4(n-1)+1)} - \tilde{x}^{2/(4n+1)}] \to 0 \tag{29}$$

However, if one introduces action S_n for each value of energy W_n:

$$S_n = \frac{n}{\nu} W_n, \nu = Const, [\nu] = \frac{1}{\text{sec}} \tag{30}$$

then

$$\min \Delta S = \frac{E}{\nu} \lim_{n\to\infty} [n\tilde{x}^{2/(4n+1)} - (n-1)\tilde{x}^{2/(4n-3)}] = \frac{E}{\nu} h = 2\pi\hbar \tag{31}$$

Thus, we have arrived at a new universal constant h which represents the minimum quant of action S_n, and is to be considered as a fundamental characteristic of any system on the atomic scale.

In order to relate quants of action to quants of energy, another fundamental characteristic, the frequency ν had to be introduced.

Since we are trying to avoid making use of any preliminary knowledge about quantum mechanics, and instead, are deriving quantum mechanics from Newton's laws, we will not discuss here the physical consequences of the existence of $\hbar$, ν and n, but rather restrict ourselves by a comment that these constants can be identified with the Planck constant, frequency of electromagnetic waves, and quantum number, respectively.

Let us turn again to equation (27) and rewrite it in the following form:

$$W = E\tilde{x}^{2\beta} = E\tilde{x}^{\beta}\tilde{v} \tag{32}$$

Then the quants of energy and action are, respectively:

$$\Delta W = E\Delta(\tilde{x}^{\beta}\tilde{v}) \tag{33}$$

and

$$\Delta S = \frac{1}{\nu}[E\Delta(\tilde{x}^{\beta}\tilde{v})] \tag{34}$$

whence

$$\lim_{n\to\infty} \Delta(\tilde{x}^{\beta}\tilde{v}) = \lim \Delta(\tilde{x}^{1/2}\tilde{v}) = \frac{h\nu}{E} \tag{35}$$

and therefore,

$$\Delta(\tilde{x}^{1/2}\tilde{v}) \geq \frac{hv}{E} \tag{36}$$

In other words, the accuracy to which the dimensionless product $\tilde{x}^{1/2}\tilde{v}$ can be defined, in principle, cannot exceed the constant vh/E. Qualitatively equation (36) is similar to Heisenberg's uncertainty principle [3], however, quantitatively it looks different, and the reason for that is the following: in the Heisenberg formulation, the position and the velocity are considered as random variables and their uncertainties are presented in the form of standard deviations; on the contrary, in our analysis the position and the velocity are discrete, but still deterministic variables, and their uncertainties do not yet have any probabilistic structure. It is also interesting to note that the inequality (36) is not symmetric with respect to $\tilde{x}$ and $\tilde{v}$ since $\tilde{x}$ is under the square root. This asymmetry will be discussed later in connection with instability of velocities with respect to changes in positions.

Let us briefly summarize the results of this section: it has been demonstrated that enforcement of incompressibility of atoms leads to non-Lipschitz potential (11) which, in turn, creates a discreteness of possible values of kinetic energy and action. It was shown that the smallest quant of action is non-zero, and it can be identified with the Planck constant, while two additional invariants, v and n suggest that the motion on the atomic scale acquires some wave-like properties. Finally, the uncertainty relationship between position and velocity—similar to that postulated by Heisenberg—was derived.

4. QUANTUM DOMAIN

Let us turn now to a detailed analysis of equation (23) plotted in Fig. 2. The most remarkable property of the function (23) is the existence of a point $\tilde{x}_0 \neq 0$ which cuts the curve into two qualitatively different parts: for $\tilde{x} > \tilde{x}_0$ the curve has a smooth 'classical' form; but for $\tilde{x} < \tilde{x}_0$ the smoothness is lost, the velocity gradient grows sharply becoming unbounded at $\tilde{x} \to 0$.

The dividing point can be found from the condition that the curvature at $\tilde{x}_0$ has its maximum:

$$\frac{\tilde{v}''(\tilde{x}_0)}{[1+(\tilde{v}')^2]^{3/2}} = \frac{\beta(\beta-1)\tilde{x}_0^{\beta-2}}{(1+\beta^2\tilde{x}_0^{2\beta-2})^{3/2}} \to \max_{\tilde{x}_0} \tag{37}$$

whence

$$\tilde{x}_0 = \left[\frac{\beta^2(1-2\beta)}{2-\beta}\right]^{[1/(2(1-\beta))]}, 0 < \beta < \frac{1}{2} \tag{38}$$

It should be emphasized that the existence of the dividing point $\tilde{x}_0 \neq 0$ is a consequence of the relaxation of the Lipschitz condition: indeed,

$$\tilde{x}_0 = 0 \tag{39}$$

for the classical case when $\beta = 1/2$, i.e., $n \to \infty$.

It is easily verifiable that the function (38) has a maximum

$$\tilde{x}_{00} = \max_{\beta} \tilde{x}_0 = 0.017 \tag{40}$$

at $\beta = 0.3441$, i.e., at $n = m = 2$, and it vanishes at $\beta = 0$ and $\beta = \frac{1}{2}$. (41)

One should recall that the basic arguments in equation (38) are n and m rather than β, and therefore, the separating point $\tilde{x}_0$ can take only discrete values.

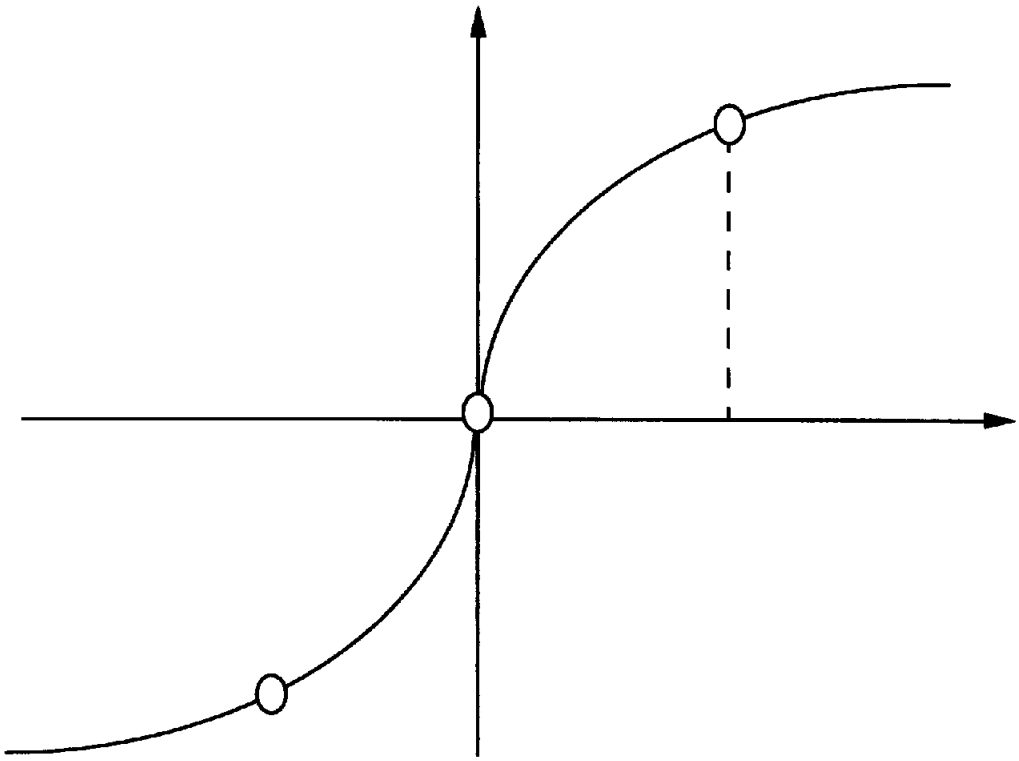

Fig. 2.

The main property of the region

$$0 \leq \tilde{x} \leq \tilde{x}_0 \tag{42}$$

is that the velocity gradients

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tilde{x}} = \beta \tilde{x}^{\beta - 1} \to \infty \text{ at } \tilde{x} \to 0 \tag{43}$$

in there are extremely large becoming unbounded at $\tilde{x} = 0$.

In our further discussions we will call this region a quantum domain.

5. EQUATIONS OF MOTIONS IN QUANTUM DOMAIN

Let us turn again to equation (23) and find the equation of motion subject to the initial conditions

$$\tilde{x} = \tilde{x}_0 \text{ at } \tilde{t} = 0, t = r\sqrt{\frac{2m}{E}}\,\tilde{t} \tag{44}$$

when the motion starts within the quantum domain and directed toward its origin $\tilde{x} = 0$.

Then taking the sign minus in equation (23)

$$\frac{d\tilde{x}}{\tilde{x}^{\beta}} = -d\tilde{t} \tag{45}$$

one easily finds:

$$\frac{1}{(1-\beta)}(\tilde{x}_0^{1-\beta}-\tilde{x}^{1-\beta}) = \tilde{t} \tag{46}$$

i.e.,

$$\tilde{x} = (\tilde{x}_0^{1-\beta}-(1-\beta)\tilde{t}^{1/(1-\beta)}. \tag{47}$$

The period during which the motion approaches the reflection point $\tilde{x} = 0$ follows from equation (46) and it coincides with the value (26) found earlier.

However, we have some problems with the reflected motion which is obtained from equation (23) with the sign plus:

$$\tilde{x} = \pm(1-\beta)\tilde{t}^{1/(1-\beta)}. \tag{48}$$

First of all, equation (48) is equipped with two signs since the fraction $\frac{1}{1-\beta} = \frac{4n+1}{2(n+1)}$ has an even demoninator. Clearly the positive (negative) sign corresponds to the reflection of the right (left) atom from the left (right) one.

Secondly, the motion (48) is fundamentally irreversible: indeed, as follows from equation (48)

$$\tilde{x}(-t) = i\tilde{x}(t), i = \sqrt{-1}. \tag{49}$$

This means that the backward motion does not exist at all.

Thirdly, equations (47) and (48) represent a regular solution to equation (23). However, there also exists a singular solution:

$$\tilde{x} \equiv 0 \tag{50}$$

which can be verified by the direct substitution of equation (40) into equation (23). (Obviously the coexistence of three different solutions to the same differential equation subject to the same initial conditions is caused by the relaxation of the Lipschitz condition at $\tilde{x} = 0$).

We will now show that the singular solution (40) possesses a very remarkable property: it is extremely stable with respect to changes in incoming (negative) velocities, and extremely unstable with respect to changes to outcoming (positive) velocities. In order to demonstrate that, linearize the governing equation (23) with respect to a point $\tilde{x}_* > 0$:

$$\frac{d\tilde{x}}{dt} = \pm a\tilde{x}+0(\tilde{x}^k), a = \beta\tilde{x}_*^{\beta-1} > 0, k = 2,3,\ldots \tag{51}$$

Then

$$\tilde{x} = \varepsilon e^{a\tilde{t}}, \varepsilon < 1 \tag{52}$$

In linear approximation, the constant a characterizes the rate of stability ($a<0$) or instability ($a>0$), and here

$$|a| \to \infty \text{ as } \tilde{x}_0 \to 0. \tag{53}$$

Therefore, the solution (50) changes its stability to instability at the moment of reflection when the sign in equation (23) switches from minus to plus.

It can be easily verified that the motions described by the regular solutions (47) and (48) are neutrally stable with respect to uncertainties in position $\tilde{x}$ since

$$\frac{\partial \tilde{x}}{\partial \tilde{x}_0} = (\tilde{x}_0^{1-\beta} \pm (1-\beta)\tilde{t})^{\beta/(1-\beta)}, \quad \text{and} \quad \frac{\partial \tilde{x}}{\partial \tilde{x}_0} \to 0 \text{ at } \tilde{t}, \tilde{x}_0 \to 0. \tag{54}$$

Indeed, as follows from equation (54), a small error in the coordinate $\tilde{x}$ decreases (in equation (47)) or increases (in equation (48)) with the rate of $\tilde{t}^{\beta/(1-\beta)}$ which is slower than an exponential rate, and that makes the solutions (47) and (48) Lyapunov neutrally stable.

However, the velocities of the same motions are extremely sensitive to the errors in $\tilde{x}$. Indeed

$$v = \pm(\tilde{x}_0^{1-\beta} \pm (1-\beta)\tilde{t})^{\beta/(1-\beta)} \tag{55}$$

and therefore,

$$\frac{\partial \tilde{v}}{\partial \tilde{x}_0} = \pm \frac{\beta}{(1-\beta)^2}(\tilde{x}_0^{1-\beta} \pm (1-\beta)\tilde{t})^{-[(1-2\beta)/(1-\beta)]}. \tag{56}$$

Equation (56) has a remarkable property:

$$\left|\frac{\partial \tilde{v}}{\partial \tilde{x}_0}\right|_{t\to 0} = \frac{\beta}{(1-\beta)^2}\tilde{x}_0^{-(1-2\beta)} \to \infty \text{ at } \tilde{x}_0 \to 0. \tag{57}$$

In other words, infinitesimal errors in positions cause unbounded errors in velocities. Such an instability is much stronger than the Lyapunov instability: it has the same power as the Hadamard instability which can occur in partial differential equations [13]. In our case, the instability is caused by the violation of the Lipschitz condition, and we will call it non-Lipschitz instability.

At this point, we can justify the term 'extremely unstable' applied in connection with equation (51): actually it means non-Lipschitz unstable, i.e., unstable in the sense of equation (57).

One can verify that small errors in velocities cause small errors in position, i.e., positions and velocities are not equal with respect to the relationships between their uncertainties.

Let us now summarize the results of this section. If the motion starts within the quantum domain and directed toward the point of collision $\tilde{x} = 0$, it is described by the regular solution to equation (23) given by equation (47). When the motion approaches $\tilde{x} = 0$, it switches to the singular solution (50) which is non-Lipschitz stable with respect to incoming velocity disturbances, and this switch is irreversible. But since the same singular solution is non-Lipschitz unstable with respect to outcoming velocity disturbances, the motion switches to the reflection branch of the regular solution (48), and this switch is also irreversible.

Thus, two branches, (47) and (48), of the regular solution are separated by the singular solution (50), and that causes the fundamental irreversibility of the motion (see equation (49)).

It should be emphasized that this irreversibility does not cause any loss of energy, and that is guaranteed by equation (6). We will stress again that both motions (47) and (48) are neutrally stable, but their velocities are non-Lipschitz unstable with respect to small errors in positions, and this asymmetry affects the uncertainty relationship (36).

6. EMERGENCE OF RANDOMNESS

Almost at the very beginning we have noticed some abnormalities in relationship between the position $\tilde{x}$ and the corresponding velocity $\tilde{v}$: it started with equation (23) which generates unbounded velocity gradients at $\tilde{x} \to 0$, and led to dynamical instability of the velocities (55). In Section 4 we have defined the quantum domain:

$$0 < \tilde{x} \leq \tilde{x}_0 \tag{58}$$

(where $\tilde{x}_0$ is given by equation (38)), within which the velocity gradient is extremely high (it becomes unbounded at $\tilde{x} \to 0$). Hence, for a given $\tilde{x}$ from equation (58), the corresponding velocity can take any values from $\tilde{v}(\tilde{x}_0)$ to $\tilde{v}(0) = 0$, i.e., it actually becomes random. It should be emphasized that the randomness is qualitatively different from the uncertainty defined by equation (36): the former is triggered by the instability, while the latter is emerged from the discreteness of the energy (although both of these phenomena represent non-Lipschitz effects).

Since we are dealing with a Hamiltonian system (indeed, the total energy E in equation (6) represents the Hamiltonian H), there are two possible scenarios for the evolution of randomness. The classical scenario is presented by the Liouville–Gibbs equation:

$$\frac{\partial f}{\partial t} + \{f, H\} = 0 \tag{59}$$

where f is the joint probability density as a function of generalized positions q, momenta p_i, and time t, and $\{\}$ is the Poisson's brackets:

$$\{f, H\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right). \tag{60}$$

The quantum scenario is described by the Schrödinger equation:

$$-\frac{\hbar}{\mathrm{i}} \frac{\partial \psi}{\partial t} = H\psi, \hbar = \frac{h}{2\pi}, \mathrm{i} = \sqrt{-1} \tag{61}$$

where ψ is a complex wave function representing a probability amplitude depending on coordinates q_1 and time t, and H is the Hamiltonian operator.

The choice between these two models of the probability evolution should be based upon their compatibility with two constraints which have been derived in Section 3: existence of discrete levels of energies (27), and the uncertainty relationship equation (36).

Let us start with the Schrödinger equation (61). Firstly, as shown by Heisenberg [1], the uncertainty principle

$$\delta p \, \delta x \geq \frac{h}{2} \tag{62}$$

(which is a statistical version of equation (36)) directly follows from equation (61). Secondly, the operator H in equation (52) has a discrete spectrum of real eigenvalues, and that perfectly fits into the discrete spectrum of energies (27).

On the other hand, the first scenario expressed by equation (59) can be immediately disqualified on the basis that it does not have a mechanism to preserve the constraint equations (27) and (36).

Thus the only model which describes the structure of microworld and uniquely follows from Newton's laws is the Schrödinger equation. In this context, the equations of motion (47) and (48) following from the governing equation (23) describe a deterministic (but unstable) microstructure behind the corresponding Schrödinger equation in the same way in which the dynamical equations of a random walk describe those of the Fokker–Planck equation [7, 9, 11].

7. DISCUSSION AND CONCLUSION

During more than six decades, quantum mechanics enjoyed an unprecedented success which overshadowed some conceptual concerns that it is based upon postulated laws which are not

only underivable from Newton's laws, but are fundamentally different from the latter. At the same time, there always was a strong belief that the mathematical foundation of Newtonian dynamics is perfect and unshakable. We first started having doubts regarding the issue of the perfection and completeness of this formalism when we tried to describe the effect of the snap of a whip; it appears that if the Lipschitz condition at the free end is preserved, one arrives at a unique, smooth, stable, but useless solution which does not describe any snap; however, if the Lipschitz condition is relaxed, then an additional, singular solution represents a strong cumulative effect which simulates the snap [1]. This example suggested that there are some physical phenomena which are incompatible with the Lipschitz condition. Since then, in several works [2–4] we have demonstrated that the removal of physically unjustifiable Lipschitz conditions in dissipative systems leads to fundamentally new phenomena such as terminal attractors and repellers, dynamical simulators of the Fokker–Planck equation, etc. The non-Lipschitz formalism helped to shed a new light on the irreversibility in thermodynamics and the origin of turbulence [2]. It was shown that the new mathematical formalism of Newtonian dynamics remained fully compatible with Newton's laws, and, at the same time, it preserved all the previous results of the classical (Lipschitz) version.

Actually the usefulness of the non-Lipschitz approach to dissipative dynamical systems gave a motivation for application of the same approach to conservative (Hamiltonian) systems, and in particular, for an attempt to reconcile quantum mechanics with Newton's laws represented by the non-Lipschitz formalism. For a proof-of-concept, we have selected a motion of a line of equally spaced identical atoms. We noticed a little inconsistency in conventional treatment of such a system: the atom incompressibility has never been incorporated into the mathematical formalism. It appears that the enforcement of the incompressibility of atoms required relaxation of the Lipschitz condition at the points of contact. This, in turn, led to fractional powers and discreteness of values of the basic variables, including energy and action, and finally, to the famous uncertainty principle. In addition to that, the violation of the Lipschitz condition caused instability of velocity with respect to small changes of position, and that introduced an element of randomness in the system behavior. It was shown that the only model for the probability evolution which incorporates all the new properties of the motion, is the Schrödinger equation, while the governing equation (23) of this motion represents a deterministic (but unstable) microstructure behind the corresponding Schrödinger equation.

Considering equation (23) as a hidden variables model, we will discuss the Bell theorem which states that local hidden variables are not compatible with quantum mechanics [6]. So, is the model (23) local? We will show that it is not. But first of all, one should recall that some models of classical dynamics, and in particular, those which include geometrical or kinematical (nonholonomic) constraints, are also non-local. Indeed, in an incompressible fluid, pressure is found not from a constitutive equation, but rather from a geometrical constraint $\operatorname{div} v = 0$ as a Lagrange multiplier, and therefore, it depends upon the global distribution of the velocity v over the whole volume. A similar effect takes place in a two-dimensional version of the model discussed above. Indeed, by extending equation (4) to the following:

$$\dot{x} = v_n \to 0, \quad \dot{y} = v_\tau \to 0 \text{ at } x, y \to 0 \tag{63}$$

i.e. by requiring a no-slip condition at the point of contact between two atoms, one must introduce a spin with an angular velocity ω to enforce equation (63):

$$\omega = -\frac{v_\tau}{r_0} \; at \; x = 0. \tag{64}$$

From the dynamical viewpoint, the no-slip condition (63) means that the atoms are represented

by rough spheres, and therefore, the "friction" force F_f which is proportional to the normal force F in equation (12), must be unbounded at $x = 0$.

$$|F_f| \to \infty \text{ at } x \to 0. \tag{65}$$

But, for a rotating sphere

$$F_f = \frac{2}{5} m r_0 \dot{\omega} = \frac{2}{5} m r_0 \frac{\mathrm{d}\omega}{\mathrm{d}x} v_n \propto \frac{\mathrm{d}\omega}{\mathrm{d}\tilde{x}} \tilde{x}^{\beta}. \tag{66}$$

Hence, the behavior of ω in a small neighborhood of the point-of-contact $x = 0$ is characterized by the condition:

$$\left| \frac{\mathrm{d}\omega}{\mathrm{d}\tilde{x}} \tilde{x}^{\beta} \right| \to \infty \text{ at } \tilde{x} \to 0. \tag{67}$$

In order to reconcile equations (64) and (67), one can utilize a non-Lipschitz term similar to those in equation (8):

$$\omega = -\frac{v_\tau}{r_0}\left(1 - \tanh x^{\gamma}\right), \quad \gamma = \frac{2m-1}{4n+3} < 1-\beta, \tag{68}$$

where γ is selected as a fraction whose value is the closest to the fraction (1-β).

Equation (68) represents a kinematical constraint which vanishes for $\tilde{x} \to \infty$ while satisfying both the conditions (64) and (67). This constraint introduces a non-locality in description of motion since the value of the spin is found not from the corresponding (local) dynamical equation

$$I_z \dot{\omega} = M, \tag{69}$$

where I_z is the moment of inertia, but rather from the global kinematical constraint (68), while the moment M in equation (69) plays the role of the reaction of this constraint.

Since equation (68) has the same structure as equation (23), the angular velocity ω can take only discrete values.

Strictly speaking, equation (23) can be rewritten in a form similar to (68) if there is no potential field, i.e. $V = 0$:

$$\tilde{v} = \pm \tanh \tilde{x}^{\beta},$$

since $\tanh \tilde{x} \cong \tilde{x}$ for small $\tilde{x}$, but

$$\tanh \tilde{x} \to 1 \text{ for } \tilde{x} \to \infty$$

i.e. the contact effect gradually vanishes.

The analog between equations (68) and (70) can be carried on further if one considers equation (70) as a constraint which requires that the velocity must gradually vanish as $x \to 0$. Indeed, then the non-Lipschitz force introduced in equation (12) will appear as a reaction of this constraint, and that in turn will introduce the corresponding non-Lipschitz potential (8). Now we can combine these two types of constraints in the following one: the relative velocity between two particles must gradually vanish as they approach the point of contact. If, for instance, these particles are represented by two identical rough spheres, the combined (non-slip) constraint can be formulated as:

$$(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j) \cdot (\tilde{\mathbf{v}}_i + \tilde{\mathbf{v}}_j) - \tanh \tilde{\mathbf{r}}_{ij}^{\beta} = 0 \tag{70}$$

$$(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j) \times (\tilde{\mathbf{v}}_i + \tilde{\mathbf{v}}_j) - \tanh \tilde{\mathbf{r}}_{ij}^{\gamma} = 0 \tag{71}$$

where $\tilde{\mathbf{v}}_i$ is the dimensionless velocity at the point of contact, $\tilde{\mathbf{r}}_{ij}$ is the dimensionless shortest

distance between the particle surfaces, and $\tilde{\mathbf{r}}_i$ is the dimensionless radius-vector of the center of the ith particle, while β and γ are expressed by equations (22) and (68), respectively.

Now we are ready to consider a general case of N interacting particles which are represented by incompressible rough spheres of radii a_i and masses m_i.

The kinetic energy of such a system is:

$$W = \frac{1}{2}\sum_{i=1}^{N}\left(m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i + \frac{4}{5}m_i a_i^2\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_i\right) \tag{72}$$

where $r_i(x_i, y_i, z_i)$ is the radius-vector of the center of ith particle, and $\omega_i(\varphi_i, \psi_i, \dot{\varphi}_i, \dot{\psi}_i, \dot{\theta}_i)$ is the vector of the angular velocity (spin) of ith particle.

The potential energy of the system for the case of the simplest gravitational attraction is:

$$V = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} G(1-\delta_{ij})\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{73}$$

where G is the gravitational constant, and δ_{ij} is the Kronecker delta.

However, the generalized coordinates r, φ_i, ψ_i, and θ_i in equation (72)equation (73) are not independent: they are supposed to provide no-slip contacts between the particles, and therefore, they should satisfy the constraints (70) and (71).

We will rewrite these constraints in a more general form since they were formulated for only two interacting particles with no potential field.

Firstly, in the presence of a potential field ($V \neq 0$), the effect of the constraint must be localized within a small neighborhood of the point of contact since on a distance the corresponding non-Lipschitz potential sharply decreases and it can be ignored in comparison to the potential V (see equation (11)equation (17)).

Secondly, if a particle can have contacts with several different particles, the constraint imposed by the closest particle must dominate over the others.

Both of these conditions can be implemented by introducing special compatibility parameters μ into v equation (70)equation (71).

$$\mu_i^{(1)}\sum_{j=1}^{n} v_{ij}[(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j)\cdot(\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_j) - \tanh\tilde{\mathbf{r}}_{ij}^{\beta}] = 0, \tag{74}$$

$$\sum_{j=1}^{n} v_{ij}[(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j)\times(\tilde{\mathbf{v}}_i - \tilde{\mathbf{v}}_j) - \tanh\tilde{\mathbf{r}}_{ij}^{\gamma}] = 0, \tag{75}$$

where

$$\mu_i = \begin{cases} 1 & \text{if } V \neq 0 \text{ and } r_{ij} \leqslant r_i^0, \\ 0 & \text{otherwise} \end{cases} \qquad v_{ik} = \begin{cases} 1 & \text{if } r_{ik} = \max_k r_{ij} \\ 0 & \text{otherwise} \end{cases}$$

and $\tilde{r}_i^0$ is the distance on which the non-Lipschitz potential can be ignored in comparison to the potential field V.

Equations (74) and (75) can be written down in the following compressed form:

$$\sum_{S=1}^{N} a_{ks}\dot{q}_s = 0, \quad k = 1, 2, \ldots, N', \tag{76}$$

where $\dot{q}_s$ are generalized velocities $\dot{x}_x, \dot{y}_x, \dot{z}_i, \dot{\varphi}_x, \dot{\psi}_x, \dot{\theta}_s$ and a_{ks} are the functions of the corresponding generalized coordinates, and N' is the number of constraints.

Now the governing equations for the system of N interacting particles can be presented by the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_{k=1}^{N'} \lambda_k a_{ks}, \quad s = 1, 2, \ldots, n. \tag{77}$$

Here $L = W - V$ is the Lagrangian, and λ_k are the Lagrangian multipliers representing reactions of the non-holonomic (kinematical) constraints (72) and (73).

Equations (76) and (77) form a coupled nonlinear system of $N+N'$ differential equations with respect to $N+N'$ variables.

The basic properties of this system will be outlined below.

Firstly, on the atomic scale, the system is not Hamiltonian (because of the non-holonomous constraints (76)); however, since equation (77) are homogeneous, the total energy is still preserved:

$$W + V = H = Const.$$

Secondly, on the atomic scale, the description of motion is non-local. Indeed, the reactions of constraints λ_k which describe forces and torques cannot be found from any local constitutive equations; instead, the whole global picture of the motion is needed to find them.

Thirdly, on the atomic scale, the system possesses the same qualitative properties (quantization of energy, non-Lipschitz instability and randomness) as those described for equation (23).

Forthly, the system has singularities at the points of contact where accelerations and reactive forces become unbounded. In order to avoid that, one can turn to the impulse–momentum version of the Lagrange equations [7], since both impulses and momenta are bounded (see equations (23) and (24)).

Fifthly, as follows from equation (71), the kinematical constraints have a long memory: spins of any two particles being in contact ones will be correlated "forever", and that describes the effect known as the Einstein–Podolsky–Rosen paradox.

It should be stressed that, generally speaking, equations (76) and (77) can be studied independently on the corresponding Schrödinger equation in the same way in which the dynamical equations simulating random walk can be studied directly without the corresponding Fokker–Planck equation. In this connection, we cannot exclude a possibility that investigation of quantum systems directly by equations (76) and (77) may lead to new, more subtle effects which could not be captured by the Schrödinger equation. In order to illustrate that, we will make some additional comments to these equations.

A solution to equation (77) defines a two-parametric family of trajectories and spins of the particles, respectively:

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_0, v_0, t, n, m), \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{r}_0, \boldsymbol{\omega}_0, t, n, m), \tag{78}$$

$$\mathbf{r}_0, \mathbf{r}(t=0), v_0 = \dot{n}(t=0), \boldsymbol{\omega}_0 = \boldsymbol{\omega}(t=0),$$

which depend upon discrete values of n and m via the parameters β and γ (see equations (22), (68), (72) and (73)).

We will now show that β and γ can be interpreted as random variables. Indeed, as pointed out in Section 6, for a given $\tilde{x}$ from the domain (58), the corresponding velocities can take any values from $\tilde{v}(\tilde{x}_0)$ to $\tilde{v}(0)$ as a result of the non-Lipschitz instability described by equation (57), and that makes the velocities random. But for a fixed $\tilde{x}$, different velocities mean different β (see equation (23), i.e. β can actually take all the values of the form (22) in the interval (38). However, not all of these values of β are equally probable. Indeed, consider a state $\tilde{x} = 0$ which is defined with an error $\Delta\tilde{x} < 1$. then, according to equation (36):

$$(\Delta \mathrm{x})^{2\beta} \geq \hbar \text{ where } \hbar = \frac{v\hbar}{E},$$

i.e. the minimum error $\Delta\tilde{x}_{\min}$

$$0 < \Delta\tilde{x}_{\min} = \tilde{\hbar}^{(1/2\beta)} < 1 \tag{79}$$

is bounded by the condition that $0 < \beta < 1/2$.

If one assumes that all the errors $\Delta\tilde{x}$ are equally probable, then the probability distribution for β can be obtained from equation (79):

$$p(\beta) = \frac{\tilde{\hbar}^{(1/2\beta)}|\ln\tilde{\hbar}|}{4\tilde{\hbar}\beta^2}. \tag{80}$$

Considering β and γ as random variables with the distributions (80), one can find from the solution (78) the probability p^* that a selected particle r_0^* at a fixed time t^* is located at the point r^*. In terms of the Schrödinger equation, this probability is equal to the square of the modules of the wave function. (At this stage, the wave function itself does not appear at all.)

In addition to that, one can find a probability P^{**} that a selected particle r_0^* at a fixed time t^{**} is located at the point r^* having the spin ω^*, and that cannot be found from the Schrodinger equation.

In some particular cases, the probabilities p^* and p^{**} can be time-independent. In order to enforce this condition, one has to restrict the free choice of the parameters n and m in such a way that only certain values of them are allowed. These values will define the discrete stationary levels of energy of the system.

It is worth mentioning that since equation (77) are nonlinear, the proposed model reinforces the link between quantum mechanics and nonlinear phenomena discussed in [8]. Moreover, due to Newtonian formulation of quantum mechanics, the relativistic effects can be incorporated by introducing the corresponding correction factor to the mass in equations (72) and (73).

It should be emphasized that the non-Lipschitz formalism of Newton laws is not only unified mechanics, but it also eliminates some "infinity-paradoxes" in quantum theory. In order to illustrate this, consider spherical waves as a solution to the Maxwell equation with respect to the electric potential φ [9]:

$$\frac{\partial^2}{\partial r^2}(r\varphi) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\varphi) = 0, \tag{81}$$

i.e.

$$\varphi(r, t) = \frac{1}{r} f(ct \pm r). \tag{82}$$

One immediately faces the point-charge problem with the inward wave:

$$\varphi(r, t) \to \infty \, at \, r \to 0 \tag{83}$$

unless the solution (82) is cut off at a small value

$$r = r_0, r_0/l < 1 \tag{84}$$

where l represents the classical (non-quantum) length scale.

However, the condition (84) does not represent a non-point charge: the solution (82) "does not know" about this charge and goes through it to infinity.

A way out of this situation is very simple if one relaxes the Lipschitz condition at $r = r_0$. Indeed, consider a solution to equation (81) subject to the following boundary conditions:

$$\varphi(r_0, t) = 0, \frac{\partial\varphi}{\partial r}(r_0, t) \to \infty \tag{85}$$

(compare with the conditions (4) and (5)).

It can be presented as

$$\varphi(r,t) = \frac{f(ct \pm r)}{r\left(1 - \frac{r_0}{r}\right)^{\chi - 1}}, \quad \chi < 1, \quad 1 - \chi = \frac{2m-1}{2n} \tag{86}$$

$$n, m = 1, 2, \ldots, \text{etc.}\, m < n \tag{87}$$

Indeed, in the classical scale ($r \sim l \gg r_0$) it coincides with the solution (82), and for $r = r_0$ it satisfies the boundary conditions (85). On the quantum scale ($r = r_0$) equation (86) has the quantum properties similar to those of equation (27).

Thus, it turns out that the micro-world is not as mysterious as it seems on the first sight: after all, it is still based upon Newton's laws.

Acknowledgements—The research described in this paper was performed by the Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology and was sponsored by the National Aeronautics and Space Administration, Office of Space Science and Technology.The author thanks Dr. J. Zbilut for encouragement to write this paper.

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