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Chaotic Scattering Without Chaos—A Counterexample

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Abstract—A scattering system is commonly called irregular or chaotic if it has a scattering function which is singular on a fractal. The only previously known mechanism generating such an irregular scattering function rests on the existence of a chaotic bound set in the closure of the set of asymptotically free trajectories. Therefore, it has been generally assumed that the measurement of an irregular scattering function implies the existence of a chaotic bound dynamics which is responsible for the fractal set of singularities. In this paper a counterexample is constructed yielding an integrable and smooth Hamiltonian which has orbiting resonances on a Cantor set. This counterexample shows that the existence of a chaotic bound set is not a necessary condition for irregular scattering. It is possible to have irregular scattering without chaotic bound motion. Copyright ©1996 Elsevier Science Ltd.

1. INTRODUCTION

Irregular or chaotic scattering—both terms are used interchangeably in the literature—is nowadays commonly defined as the 'irregular' situation where some scattering function is singular on a fractal set. A scattering function in the present context is roughly any function which describes the behaviour of the outgoing scattering asymptote as a function of the incoming scattering asymptote.

Irregular scattering in this sense is also called chaotic because of two reasons. The first circumstantial reason is the scattering function's instability near the fractal set of singularities, a feature reminiscent of chaos in bound systems. The second hypothetical but crucial one is that the origin of the fractal singularity set is assumed to be a chaotic motion on the invariant bound set. This hypothesis was suggested by the known examples of irregular scattering. In other words, one has as a phenomenon an irregular scattering function, for which only one mechanism is known: a chaotic bound motion. But whereas the phenomenon is readily measurable by physical experiments the mechanism is often not so easily accessible. Even if there are no experimental barriers to get to the bound orbits one typically meets the difficulty that they form a set of measure zero.

The general, if often implicitly stated, conception has been that phenomenon and mechanism were more or less equivalent. The irregularity of a scattering function has been considered to be just the distinct fingerprint of an otherwise elusive form of bound chaos (cf., for example, [6]). Indeed, this conception seems to be at the bottom of the interchangeable usage of both terms, irregular and chaotic scattering.

The first indication that there were problems with this picture was found in 1990 in a paper by Chen *et al.* [1]. These authors showed that bound chaos can be invisible in those scattering maps whose dimensionality is smaller than the degrees of freedom of the Hamiltonian dynamics. If phase space has $N > 2$ degrees of freedom one needs a large enough fractal dimension D_c of the chaotic bound set, namely $D_c > 2N - 3$, in order to be able to see the chaos with positive probability in one-dimensional scattering maps as a fractal set of singularities. However, this was taken only as a caveat that sometimes one needed the full scattering transformation to see the fingerprints of chaos.

So there are fingers (chaotic sets of small fractal dimension) which leave only a weak fingerprint (singularities in higher dimensional scattering maps). But surely, where there is a fingerprint there must be a finger? In [3] Jung *et al.* constructed an infinite sequence of hard discs accumulating at one end and obeying a special scaling law such that the ensuing deflection function has an infinite and self-similar set of singularities although the bound set is not chaotic. However, since in their example the singularity set is not a fractal because it is only a countable set and chaos arises simultaneously with fractality when perturbing the configuration of scatterers, they concluded that as a criterion for 'true scattering chaos' one requires in addition that the singularity set is truly a fractal, i.e. in the language used here one needs a chaotic bound set.

In this paper it will be shown that the existence of a chaotic bound set is *not* a necessary condition for chaotic/irregular scattering. The relevance of this result for physics is that measuring an irregular or irregular looking scattering function need not say much about the mechanism generating the irregularities.

In the following Section 2 we are going to construct the counterexample and discuss its properties. The construction is based on the well-known phenomenon of orbiting resonance in central (radially symmetric) potentials. However, contrary to the known examples, our counterexample shows orbiting singularities not just on a set of isolated points but on a Cantor set. A discussion of properties and variants of the counterexample can be found in Section 3. In Section 4 we will more precisely formulate the notions used heuristically in this Introduction. The section provides largely background material to put our problem into context.

I would like to state here that this counterexample does not persist under smooth perturbations, so that it may be considered with some justification to be pathological. However, although a perturbed set of its singularities need not remain fractal, it looks approximately like a fractal of singularities down to resolutions depending on the size of the perturbation. Its scattering irregularity persists at least approximately. Indeed, there are many such potentials in the sense that for any given but large enough energy and in any C^0 neighbourhood of a given continuous potential vanishing at infinity, there is a smooth potential that shows for energy irregular scattering but not bound chaos.

It is an open question whether there are potentials for which irregular scattering without a chaotic bound set is a feature persistent under small smooth perturbations.

2. CONSTRUCTION OF THE COUNTEREXAMPLE

2.1. *The bound set for central potentials at scattering energies*

By transforming to polar coordinates one can identify the set of central (rotationally symmetric) smooth potentials defined on $\mathbb{R}^2$ and vanishing at infinity, with the set C of smooth symmetric functions defined on $\mathbb{R}$ and vanishing at infinity. Regard C as a subset of the Banach space of real functions vanishing at infinity (i.e. with topology induced by the C^0-supremum norm $||f||_\infty = \sup|f|$). The main result of this paper can now be formulated as follows.

Theorem 2.1. In C there is a dense set of potentials exhibiting irregular scattering which is *not* induced by chaotic bound motion.

The simplest way a potential V can trap particles occurs by spiralling orbits. This is not considered to be an example of irregular scattering as long as the initial conditions leading to trapping are isolated (cf. [2]). Furthermore, as the Hamiltonian is integrable, there is no chaos in these types of dynamics. The idea leading to a counterexample is to construct a smooth potential which shows orbiting on a Cantor set of initial conditions.

For a given smooth central potential V with compact support $\mathrm{supp}(V) \subset [0, b_{\max}]$, an asymptotic particle energy $E_0 \geq \sup V$ and an impact parameter $b \geq 0$ the total curvature of the trajectory, that is the historical angle, is given by $\theta_V(b)$ of equation (17). If the potential decreases fast enough and has no singularities of order between r^{-2} and r^{-4} then the only points where θ_V is not defined are those for which the integrand has a non-integrable singularity at the turning point. This is called an orbiting resonance (cf. Ref. [4]). In the following let $v(r) := 1 - V(r)/E_0$ and let $r_{\min}(b)$ be the classical turning point of a trajectory with impact parameter b.

Lemma 1. For a central potential $V \in C_0^2(\mathbb{R})$ with monotone increasing $r \mapsto r^2 v(r)$ and energy $E_0 > \sup V$ both the bound set of trajectories and the singularities of the deflection function θ_V are given by the orbiting resonances, i.e. by those angular momenta $l = b\sqrt{2E_0}$ for which $r^2 v(r)$ has a critical point (derivative 0) at $r_{\min}(b)$. The bound set at E_0 consists of the 'limit circles' (radii $r_{\min}(b)$) of the orbiting resonances.

Proof. Using the integrated equations of motion a spiralling orbit occurs where the deflection function (17) is divergent. Taylor expansion with Lagrange's remainder around the classical turning point yields with $r_{\min} := r_{\min}(b)$:

$$r^2 v(r) - b^2 = (2r_{\min} v(r_{\min}) + r_{\min}^2 v'(r_{\min}))(r - r_{\min}) + (v(\tilde{r}) + \tilde{r}v'(\tilde{r}) + \tilde{r}^2 v''(r)/2)(r - r_{\min})^2 \tag{1}$$

with $\tilde{r} \in]r_{\min}, r[$. As v' and v'' are bounded, the integral exists if and only if $w(r) := r^2 v(r)$ does *not* have a critical point at the turning point $r = r_{\min}(b)$. Otherwise the integrand has a non-integrable singularity $\sim 1/(r - r_{\min}(b))$. As the effective potential $V_{\mathrm{eff}}(r) = [l^2/(2r^2)] + V(r) = E_0 b^2/r^2 + V(r)$, $b := l/(2\sqrt{E_0})$ also has a critical point at $r_{\min}(b)$, there is also a bound circular orbit at radius $r_{\min}(b)$. However, there are no other bound orbits at energy E_0 because $w' \geq 0$ implies that the effective potential is decreasing left of $r_{\min}$, so that $r_{\min}$ is an inclination point of V_{eff}: as $V'_{\mathrm{eff}}(r) = -(E_0/r^3)(r\,w'(r) - 2(w(r) - b^2))$, we know that $V'_{\mathrm{eff}}(r) \leq 0$ left of $r_{\min}$ (i.e. $w(r) \leq b^2$), and also right of $r_{\min}$ because otherwise $r_{\min}$ was not reachable by a scattering orbit. $\square$

2.2. *A Cantor S-curve*

We are looking for a function $\sigma : [0, 1] \to [0, 1]$ with the following properties:

(1) the function σ is smooth and strictly increasing,

(2) on a Cantor set C the derivative $\sigma' = 0$.

Such a function will be called a *Cantor S-curve*. It will be identified as $\sigma(r) = r^2 v(r)$, thus leading to a potential $V_S(r) = E_0(1 - \sigma(r)/r^2)$ which shows orbiting for particles with energy E_0 on a Cantor set of impact parameters.

We construct σ in two steps.

First, we construct a 2-level C^∞ function h, i.e. a function which is constant everywhere but for a finite interval where it is strictly increasing. We use the standard smoothening function $\phi_\epsilon \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\phi_\epsilon) = [-\epsilon, \epsilon]$

$$\phi_\epsilon(x) := q\left(1 - \frac{x^2}{\epsilon^2}\right), \quad \text{with } q(x) := \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{2}$$

to define a function h

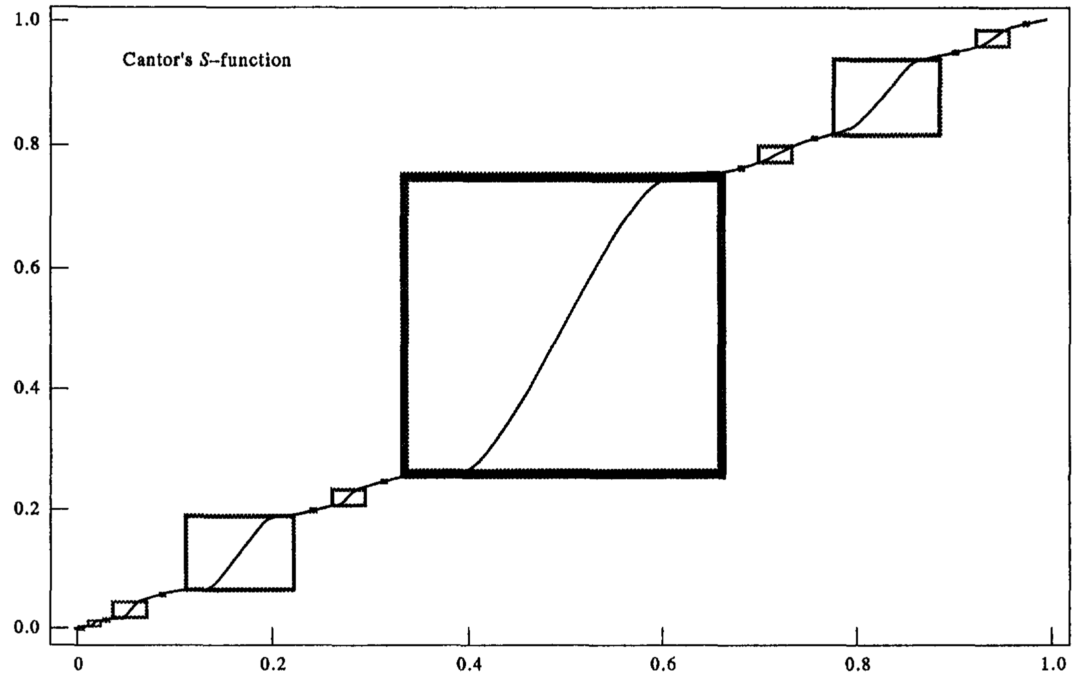

Fig. 1. A Cantor S-curve for a C^2 S-function h. The rectangles are the Cartesian product of the tremas of Cantor's middle-thirds set (abscissa) and of Cantor's middle-halves set (ordinate).

$$h(x) = \frac{\int_{-\infty}^{x} \phi_{1/2}(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \phi_{1/2}(y)\mathrm{d}y} - 1/2 \tag{3}$$

$$h(x) = \frac{\int_{-1/2}^{x} \exp(\frac{-1}{1-4y^2})\mathrm{d}y}{\int_{-1/2}^{1/2} \exp(\frac{-1}{1-4y^2})\mathrm{d}y} - 1/2 \quad \text{for } x \in [-1/2, 1/2] \tag{4}$$

which has the properties:

(1) $h: [-1/2, 1/2] \to [-1/2, 1/2]$ bijective and strictly increasing,

(2) h is smooth (C^∞),

(3) for all derivatives $n \in \mathbb{N}$: $h^{(n)}(-1/2) = 0 = h^{(n)}(1/2)$.

Next, similar to the construction of the Cantor function (devil's staircase), we set together scaled down versions of h to form a smooth staircase σ ('Cantor S-curve') from 0 to 1. The building blocks are affine images $2\lambda^n h(\mu^n(x - x_{s_1\ldots s_n})) + y_{s_1\ldots s_n}$ of the function h which map the tremas $I_{s_1\ldots s_n}$ of a measure 0 Cantor set with centre $x_{s_1\ldots s_n}$ to some interval $J_{s_1\ldots s_n}$ with centre $y_{s_1\ldots s_n}$. To get a smooth curve we have to make sure it is continuous in the Cantor points and that the left and right limits of the derivatives coincide in the Cantor points. Continuity can be achieved by avoiding vertical gaps between the building blocks. As the points of the Cantor set are accumulation points of the borders of the tremas a smooth function s will result if and only if $0 < \lambda < \mu < 1$, because then the maximum of any derivative over a generation n trema will tend to 0 for $n \to \infty$. A specific construction is given by

Proposition 1. Let $C_{1/3}$ be Cantor's middle-thirds set and $I_{s_1\ldots s_n}(C_{1/3})$ be the middle-thirds trema determined by its left boundary $\sum_{i=1}^{n} s_i 3^{-i}$, $(s_i)_{i=1}^{n-1} \subset \{0, 2\}$, $s_n = 1$. The function

$$\tilde{\sigma} : [0, 1] \setminus C_{1/3} \to [0, 1] \tag{5}$$

$$\tilde{\sigma}(x) = 2\,(4^{-n}) h[3^n(x - x_{s_1\ldots s_{n-1}})] + y_{s_1\ldots s_{n-1}} \quad \text{for } x \in I_{s_1\ldots s_{n-1}1} \tag{6}$$

$$x_{s_1\ldots s_{n-1}} = \sum_{i=1}^{n-1} s_i 3^{-i} + 3^{-n+1}/2 \tag{7}$$

$$y_{s_1\ldots s_{n-1}} = \sum_{i=1}^{n-1} \hat{s}_i 4^{-i} + 4^{-n+1}/2 \text{ with } \hat{s}_i = \begin{cases} s_i & \text{if } s_i \in \{0, 1\} \\ 3 & \text{if } s_i = 2 \end{cases} \tag{8}$$

can be smoothly extended to yield a Cantor S-curve $\sigma : [0, 1] \to [0, 1]$.

Proof. (1) Equation (6) defines a function $\tilde{\sigma}$ on $[0, 1] \setminus C_{1/3}$ which is continuously extensible to $[0, 1]$: Restricted to the single $C_{1/3}$-tremas $I_{s_1\ldots s_{n-1}}(C_{1/3})$ the function $\tilde{\sigma}$ is obviously continuous, even smooth. The image of $I_{s_1\ldots s_{n-1}1}(C_{1/3})$ is Cantor's middle-halves trema $I_{s_1\ldots s_{n-1}1}(C_{1/2})$ wearing the same label $s_1 \ldots s_{n-1}1$. This can be seen as follows. Whereas the Cantor's middle-thirds encoding is just the base 3 expansion of $x \in [0, 1]$ which is given by $(s_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} s_i 3^{-i}$, the encoding of Cantor's middle-halves is given by $(s_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} \hat{s}_i g_i$ with weights

$$g_1 = 1/4, \quad g_{i+1} = \begin{cases} g_i/4 & \text{if } s_i \in 0, 2 \\ g_i/2 & \text{if } s_i = 1 \end{cases} \tag{9}$$

and recoding $\hat{s}_i$ as given in the proposition. As $s_n = 1$ is the first appearance of the digit 1 in the tuple $s_1 \ldots s_n$ determining the trema $I_{s_1\ldots s_{n-1}1}(C_{1/3})$ with centre $x_{s_1\ldots s_{n-1}} = \sum_{i=1}^{n-1} s_i 3^{-i} + 3^{-n}(1+2)/2$ and length 3^{-n}, the trema $I_{s_1\ldots s_{n-1}1}(C_{1/2})$ has centre $y_{s_1\ldots s_{n-1}} = \sum_{i=1}^{n-1} \hat{s}_i 4^{-i} + 4^{-n}(\hat{1}+\hat{2})/2$ and length $(1/2)4^{-n+1} = 2(4^{-n})$. Thus equation (6) determines $\tilde{\sigma}$ as the affinely transformed h which maps trema $I_{s_1\ldots s_{n-1}1}(C_{1/3})$ strictly increasing onto the trema $I_{s_1\ldots s_{n-1}1}(C_{1/2})$. As both trema sequences $\{0, 2\}^* \to 2^{[0,1]}$, $s_1 \ldots s_{n-1} \mapsto I_{s_1\ldots s_{n-1}1}(C_{1/3})$ and $s_1 \ldots s_{n-1} \mapsto I_{s_1\ldots s_{n-1}1}(C_{1/2})$ define partitions of the unit interval minus the respective Cantor set and these partitions consist of ordered intervals, the function $\tilde{\sigma}$ maps $[0, 1] \setminus C_{1/3}$ strictly increasing onto $[0, 1] \setminus C_{1/2}$ with

$$\lim_{n\to\infty} \tilde{\sigma}\Big(\sum_{i=1}^{\infty} s_i 3^{-i}\Big) = \sum_{i=1}^{\infty} \hat{s}_i g_i \tag{10}$$

for $s_i \in \{0, 2\}$ so that $\sum s_i 3^{-i} \in C_{1/3}$.

(2) The continuous extension σ of $\tilde{\sigma}$ to $[0, 1]$ is smooth (C^∞): As σ is continuous and by construction smooth on the open set $[0, 1] \setminus C_{1/3}$ it is enough to check that for $x \in C_{1/3}$ we have for the mth derivatives $\lim_{x_i \to x_0} \sigma^{(m)}(x_i)$ exists for any sequence $(x_i) \subset [0, 1] \setminus C_{1/3}$. However, due to the different scalings of the tremas of Cantor's middle-thirds and middle-halves set we know that for $x_i \in I_{s_1\ldots s_{n-1}1}(C_{1/3})$ the derivative

$$0 \le \sigma^{(m)}(x_i) \le \frac{2(4^{-n})}{3^{-n}} \max_{x\in[0,1]} h^{(m)}(x) \underset{n\to\infty}{\to} 0. \tag{11}$$

For those Cantor points which are on the boundary of some trema we get from one side immediately derivative 0. $\square$

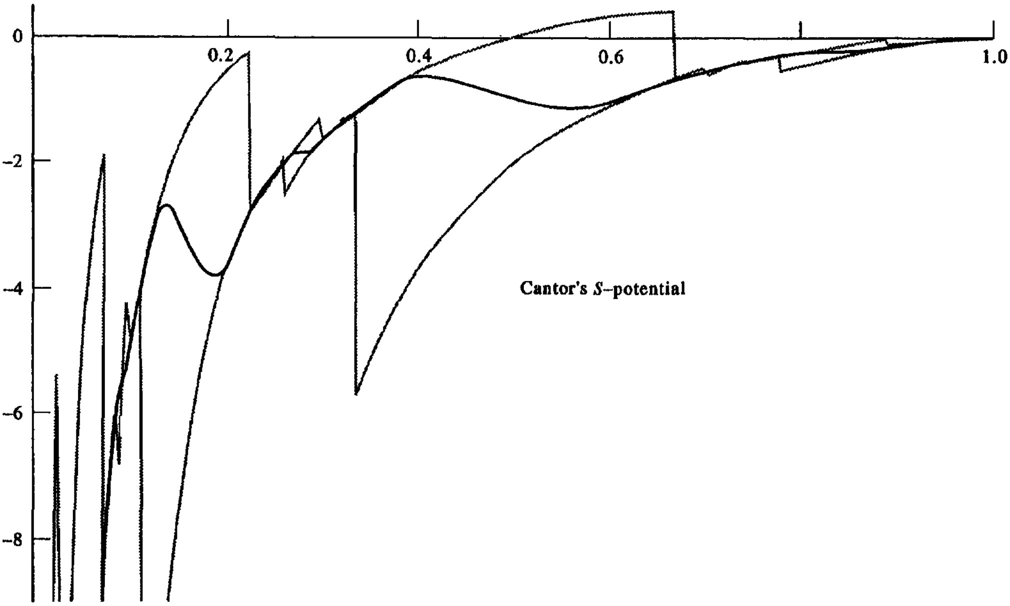

Fig. 2. A Cantor S-potential for the Cantor S-curve of Fig. 1. Additionally, the rectangles of Fig. 1 appear as images under the same transformation which maps the Cantor S-curve to its associated Cantor S-potential.

Figure 1 illustrates this construction. However, as the C^∞ h given by equation (3) contains a non-elementary integral, we chose

$$h(x) = \begin{cases} \frac{1}{2}[\exp(4x/(2x+1)) - 1] & \text{if } x \in [-1/2, 0] \\ \frac{1}{2}[\exp(4x/(2x-1)) - 1] & \text{if } x \in]0, 1/2] \end{cases} \tag{12}$$

which is only C^2 at 0. But this suffices for Lemma 1.

2.3. *A Cantor S-potential*

We are now in a position to give a potential which shows irregular scattering but lacks chaotic bound orbits. Setting $\sigma(r) = r^2 v(r)$ yields:

Proposition 2. Let σ be a Cantor S-curve and $E_0 > 0$. Then the potential

$$V_\sigma(r) = \begin{cases} E_0(1 - \sigma(r)/r^2) & \text{if } r \in]0, 1] \\ 0 & \text{if } r > 1 \end{cases} \tag{13}$$

yields a potential on the positive real axis which is continuous everywhere and smooth but at the boundary point $r = 1$ of its support $[0, 1]$. It shows measurably irregular scattering at energy E_0 but no bound chaos. The singularities occur for impact parameters b s.t. $b^2 \in C_{1/2}$.

Proof. Using impact parameter (angular momentum) and conjugate angle as coordinates a scattering map on the energy shell E_0 is given by $T : [0, 2\pi[\times\mathbb{R} \to [0, 2\pi[\times\mathbb{R}$, $T(\phi, b) = (\phi + \theta_{V_\sigma}(b) \bmod 2\pi, b)$. The deflection function $\theta := \theta_{V_\sigma}$ is by construction (Proposition 1) divergent to infinity on the Cantor set $\sqrt{C_{1/2}}$. Let $b_0^2 \in C_{1/2}$. Because of divergence at b_0 and

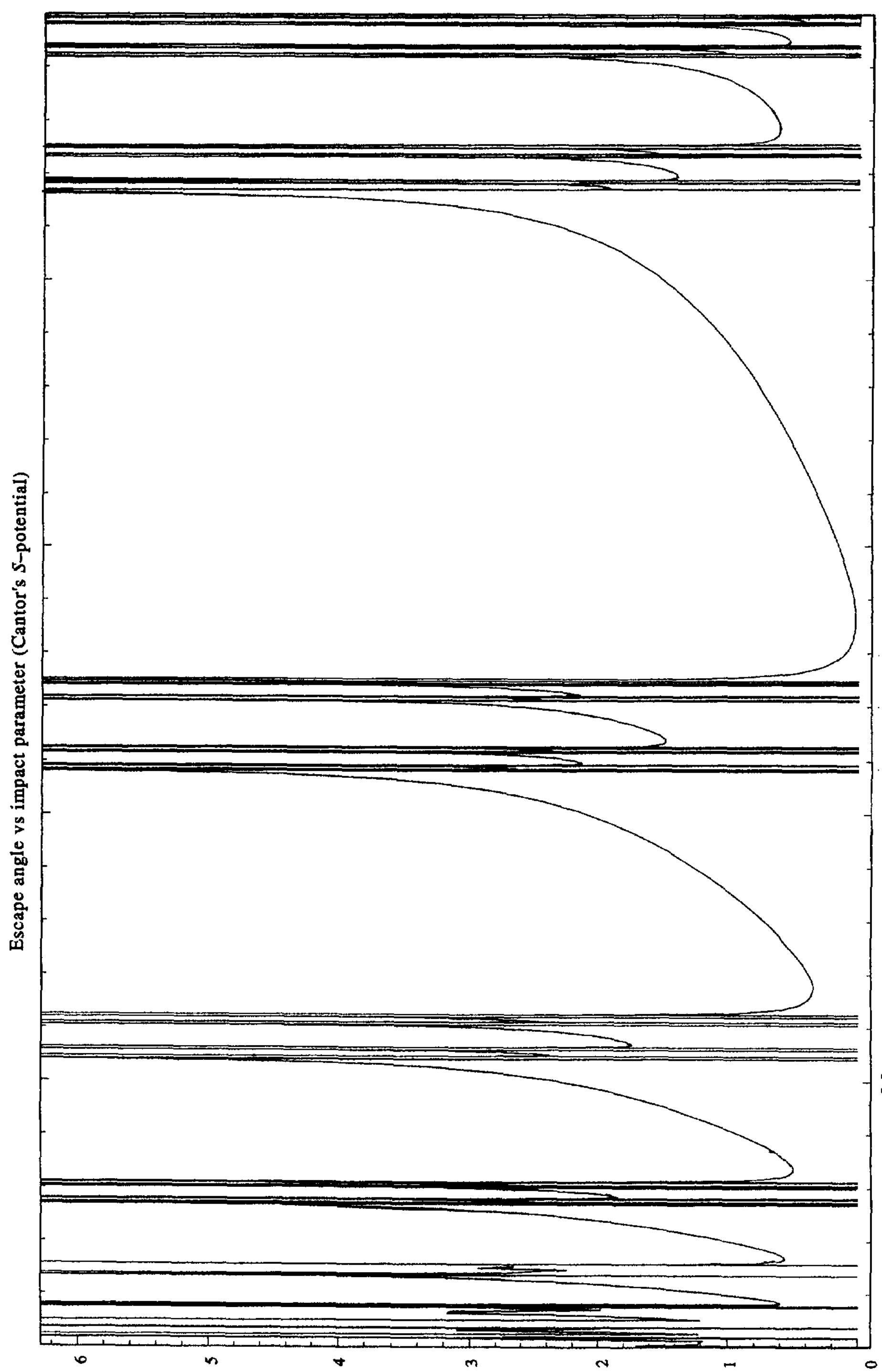

Fig. 3. The scattering function escape angle vs impact parameter of the Cantor S-potential of Fig. 2.

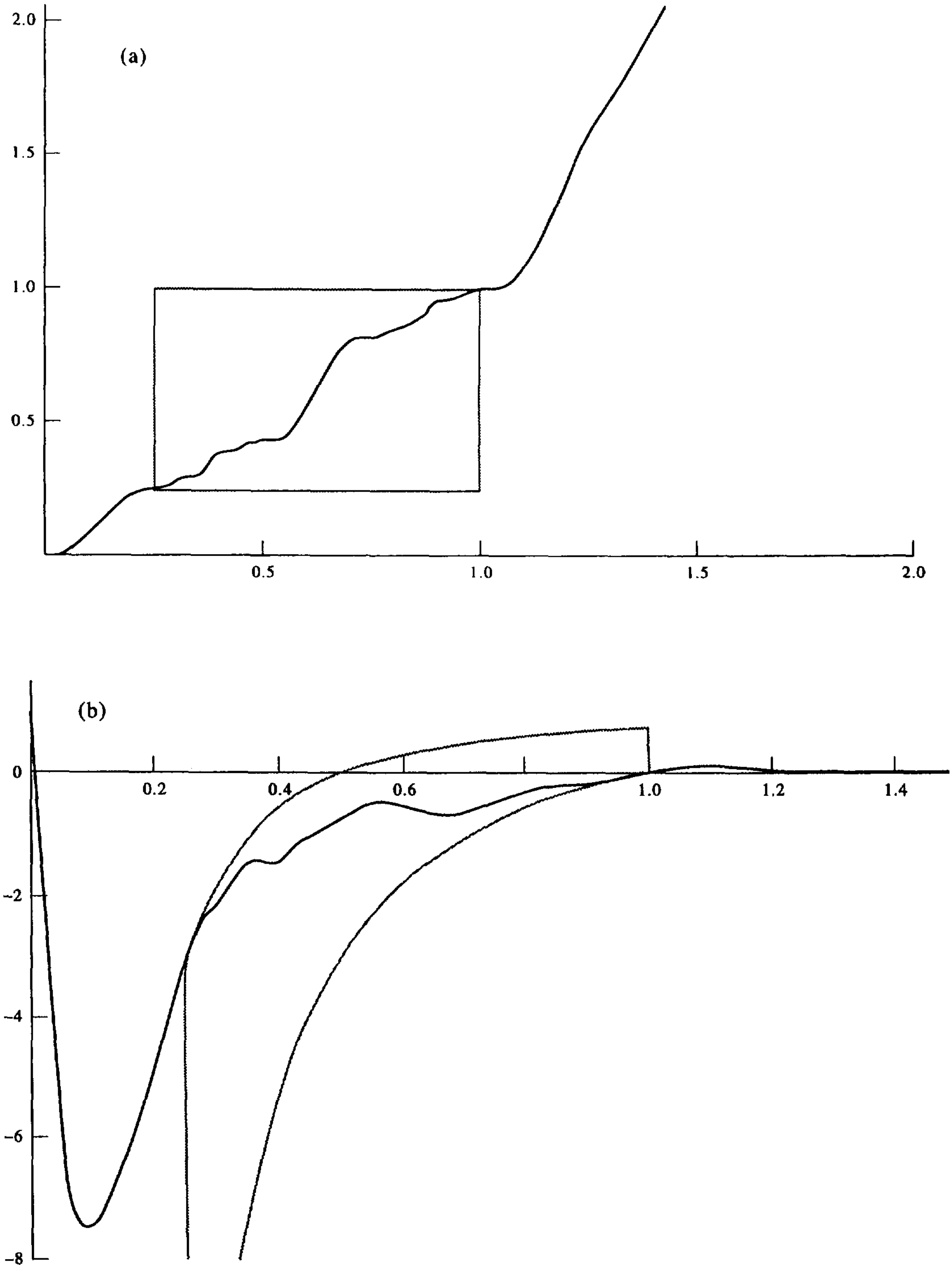

Fig. 4. (a) A variant of the Cantor S-curve leading to a smooth counterexample. The original Cantor S-curve appears as an affine image inside the rectangle. (b) The smooth potential corresponding to the variant of the Cantor S-curve of (a) together with the deformed rectangle.

continuity on the complement of $\sqrt{C_{1/2}}$ for $\delta > 0$ small enough there is $n > 0$ s.t. $\theta(b) > 2\pi$ for $|b - b_0| < \delta$ and $\theta(b') = 2\pi n$ for some b' with $|b' - b_0| < \delta$. Take another $0 < \delta' < \delta$ s.t. $\theta(b) > \theta(b') + \pi$ for $|b - b_0| < \delta'$ and $\theta(b'') = 2\pi n' + \pi/2$ for some $n' \geq n$ and some b'' with $|b'' - b_0| < \delta'$. Hence, for $K = \pi/2$ one has $A_\delta \supset ([0, 2\pi[\times\theta^{-1}([\theta(b'), \theta(b') + \pi/2]) \times ([0, 2\pi[\times\theta^{-1}([\theta(b''), \theta(b'') + \pi/2])$, which has positive measure. The same holds for S^{-1}, so that one also has a measurable mixing property. Lemma 1 shows explicitly that this irregularity is not induced by a chaotic bound motion. □

Figure 2 shows this potential (however for the more easily computable C^2-smooth h), whereas in Fig. 3 the scattering function $\theta_{V_\sigma} \bmod 2\pi$ is plotted.

Remark 2.1. (1) As usual for irregular scattering the staying time (time delay) has the same singularities as the deflection function. The time spent inside the potential's support $[0, 1]$ is

$$T(b) = \sqrt{2/E_0}\int_{r_{\min}(b)}^{1} \frac{r\,\mathrm{d}r}{\sqrt{\sigma(r) - b^2}}. \tag{14}$$

(2) One can easily proceed to construct a counterexample $V_{\hat\sigma}$ which is defined and smooth everywhere on the closed half axis $[0, \infty[$. To represent a central potential in $\mathbb{R}^2$ its derivatives must vanish at 0. All this can be achieved by affinely deforming σ to the domain $[1/4, 1]$, say, and then extending it smoothly to $\hat\sigma$ on $[0, \infty[$ in such a way that one can extend $\hat\sigma(r)/r^2$ smoothly to 0, where its derivatives should vanish and that for large enough r, $\hat\sigma(r)/r^2$ is constant, namely 1. For example,

$$\hat\sigma(r) := \begin{cases} (1/4)\,\psi_{1/8}(r - 1/8) & \text{if } 0 \leq r < 1/4 \\ (3/4)\,\sigma(4r/3 - 1/3) + 1/4 & \text{if } 1/4 \leq r \leq 1 \\ 1 + (r^2 - 1)\psi_{1/8}(r - 9/8) & \text{if } 1 \leq r < \infty \end{cases} \tag{15}$$

where ψ_ϵ is a smooth function from level 0 to 1 on the interval $[-\epsilon, \epsilon]$ given, for example [cf. equation (3)], by

$$\psi_\epsilon(x) = \frac{\int_{-\infty}^{x} \phi_\epsilon(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \phi_\epsilon(y)\mathrm{d}y}. \tag{16}$$

Figure 4 shows this construction.

To complete the proof of the main theorem we are finally going to show that in any neighbourhood (w.r.t. the C^0-supremum norm) of a continuous potential V vanishing at infinity one can find such a counterexample.

Proof. Let an $\epsilon > 0$ be given. As V vanishes at infinity, there is a compact interval $[0, b_{\max}[$ s.t. $|V| \leq \epsilon$ outside $[0, b_{\max}[$. Approximate (Weierstraß) the restriction $V|_{[0,b_{\max}[}$ by a polynomial P s.t. $||V - P||_{[0,b_{\max}[} \leq \epsilon/2$. Let $E_0 > \sup|V| + \sup|P|$ and as usual $v(r) = 1 - V(r)/E_0$, $w(r) = r^2 v(r)$, $p(r) = 1 - P(r)/E_0$, $q(r) = r^2 p(r)$. Then $v, p > 0$ and $w, q \geq 0$ have precisely one zero, namely at $r = 0$. As q is also a polynomial there is an interval $[0, r_*]$ s.t. the restriction $q'|_{]0,r_*[} > 0$ and $\min q|_{[r_*+\eta, b_{\max}]} > q(r_0)$ for any $\eta > 0$. Choose $0 < r_1 < r_*$ and let $0 < \delta < \epsilon/(2E_0 r_1^2)$. As $q|_{[r_1,r_*]}$ is uniformly continuous there is a finite n and a decomposition $0 < r_1 < r_2 < r_3 < r_4 < \ldots < r_n := r_*$ s.t. q varies on each interval $[r_i, r_{i+1}]$, $i = 1 \ldots n-1$, by at most δ. Define $\tilde q$ as follows. On $[0, r_1]$ let $\tilde q = q$; on $[r_1, r_2]$ let $\tilde q$ be a smooth function s.t. for all $n \geq 0$: $\tilde q^{(n)}(r_1) = q^{(n)}(r_1)$, $\tilde q(r_2) = q(r_2)$ and for all $n > 0$: $\tilde q^{(n)}(r_2) = 0$; on each $[r_i, r_{i+1}]$, $i = 2 \ldots n-1$ let $\tilde q$ be a Cantor S-curve between $q(r_i)$ and $q(r_{i+1})$; on $[r_{n-1}, r_n]$ let $\tilde q$ be a smooth interpolation between $\tilde q|_{[0,r_{n-1}]}$ and $q|_{[r_n,b_{\max}]}$; on $[r_n, b_{\max}]$ let $\tilde q = q$. By construction we have for $p = q/r^2$, $\tilde p = \tilde q/r^2$: $||p - \tilde p||_{[0,b_{\max}]} \leq \epsilon/(2E_0)$. Extend now $\tilde P = E_0(1 - \tilde p)$ to $[0, \infty[$ by smooth and monotonic interpolation between itself and the constant 0 function. Then, we conclude that $||V - \tilde P||_{[0,\infty]} \leq ||V - P||_{[0,b_{\max}]} + ||P - \tilde P||_{[0,b_{\max}]} \leq \epsilon$. Furthermore, as the values $\tilde q|_{[0,r_*]}$ assumes are smaller than each value assumed by $\tilde{q}|_{[r_*,\infty]}$, the critical points of $\tilde{q}|_{[0,r_*]}$ are turning points of suitable impact parameters $b \in [0, b_{\max}]$. As there is a Cantor set of such critical points irregular scattering occurs. Since the bounded motion in an integrable system is at most quasi-periodic, it is not chaotic. □

This completes the proof of the theorem.

3. PROPERTIES AND VARIANTS OF THE COUNTEREXAMPLE

(i) An arbitrarily small C^2-perturbation can remove all critical points of a Cantor S-curve because they are all inclination points. Therefore the constructed counterexample is not persistent under such perturbations. All singularities of a corresponding scattering function can vanish instantly, although they persist in an approximate way. The perturbed deflection function still shows peaks close to the former singularities.

(ii) By a simple modification of a Cantor S-curve one can construct a counterexample which is still smooth but has persistent singularities accumulating at a Cantor set. Modify an S-curve $h : [-1/2, 1/2] \to [-1/2, 1/2]$ to include one local minimum (with a non-vanishing second derivative) in the interior $]0, 1[^2$. This local minimum will certainly be assumed as a turning point and being above the left boundary point the boundary points of the tremas in the resulting modified but still smooth Cantor S-curve σ_M are not shadowed so that they are still assumed as turning points. However, contrary to the inclination points at the boundaries of the tremas, where the derivatives of all orders vanish, these local minima are stable with respect to small C^2-perturbations. They form a set in whose closure the original Cantor set of critical points lies. And, of course, they lead to singularities of scattering functions.

A small C^2-perturbation can still destroy irregularity, but at least an isolated set of singularities persists. The local minima in the higher generation tremas can be destroyed by smaller perturbations than the lower generation tremas with their deeper local minima. It is an open question whether a counterexample can be constructed which is persistent under small smooth perturbations, in the sense that the set of singularities of perturbed systems remains (or still contains) a Cantor set.

(iii) What happens to the counterexample when one changes the energy of the incoming particles? As $V(r) = E_0(1 - w(r)/r^2)$ with w fixed by construction, an energy change from E_0 to E_1 results in a new $w_{E_1}(r) = (1 - E_0/E_1)r^2 + (E_0/E_1)w(r)$. Hence an energy change can be regarded as a smooth perturbation of the potential V, so that point (ii) is applicable. A variation of energy $E \to E_0$ for the modified counterexample $w = \sigma_M$ leads to a transition to irregularity with the deflection function acquiring more and more (in the limit containing a Cantor set of) singularities.

This modification still lacks the feature of irregularity under small perturbations of energy. It is possible to do a Cantor S-construction for the effective potential at a fixed impact parameter and get irregularity in the energy instead of in the impact parameter but then one needs just the right impact parameter. The major open question is whether there are counterexamples showing irregular scattering without chaos on a whole energy interval.

4. BACKGROUND AND DEFINITIONS

First, we specify our definition of chaotic or irregular scattering for Hamiltonian systems. For this purpose we review some basic notions from classical mechanics.

4.1. *Basic notions*

Suppose on some manifold M a Hamiltonian system is given with Hamiltonian H and global Hamiltonian flow Φ_t on the cotangent bundle $T^*(M)$.

In the simple case of $M = \mathbb{R}^n$, interaction Hamiltonian $H = |p|^2/2 + V(x)$ (where $V \in C_0^\infty(\mathbb{R}^n)$) and reference Hamiltonian $H_0 = |p|^2/2$ with flows Φ_t, Φ_t^0, respectively, one knows (cf. [7]) that $\Omega_\pm := \lim_{t\to\pm\infty} \Phi_{-t} \circ \Phi_t^0$ exist pointwise on the open sets $D_\pm = T^*(\mathbb{R}^m) \setminus (\mathbb{R}^m \times \{0\})$. The Möller-transformations $\Omega_\pm$ are local canonical transformations (in particular, diffeomorphisms), whose ranges we write as $\mathcal{R}_\pm$, respectively. In the present case, the complements of their ranges are just the forward and backward bound sets, i.e. $C\mathcal{R}_\pm = \bigcup_n \{u = (x, p) \in T^*(\mathbb{R}^m) : ||\Phi_{\pm t} u|| < n \ \forall\, t > 0\}$. Furthermore, one has asymptotic completeness, i.e. the Liouville measure of the states which are (exclusively) either forward or backward free is 0.

The *scattering transformation* S is defined as the local canonical transformation $S = \Omega_+ \circ \Omega_-^{-1}$, $\mathcal{R}_- \to \mathcal{R}_+$. Any $f \circ S \circ g$, with observables $f \in C^\infty(\overline{\mathcal{R}_-}, \mathbb{R}^q)$ and a diffeomorphism g of an open $N \subset \mathbb{R}^r$ into $\mathcal{R}_-$ which is continuously extensible to $\bar{N}$, where $q, r \le 2 \dim(M)$, will now be called a *scattering function*. Observe that it is smooth by definition.

For $M = \mathbb{R}^2 \setminus 0$, $H = p^2/2 + V(|x|)$, V smooth with compact support $\operatorname{supp}(V) \subset [0, b_{\max}]$, $H_0 = p^2/2 = E_0$, the point transformation to polar coordinates generates a canonical transformation $(x, y, p_x, p_y) \mapsto (r, \phi, p_r, l)$ by which the Hamiltonian becomes $H' = p_r/2 + l^2/2 + V(r)$ and the scattering transformation $S: (r, \phi, p_r, l) \mapsto (r, \phi + \tilde{\theta}_V(l) \bmod 2\pi, p_r, l)$ with deflection function $\tilde{\theta}_V(l) = \theta_V(b)$, impact parameter $b = l/\sqrt{2E_0}$, $v(r) = 1 - V(r)/E_0$:

$$
\begin{aligned}
\theta_V(b) &= \pi - 2 \int_{r_{\min}(b)}^{\infty} \frac{b\,\mathrm{d}r}{r\sqrt{r^2 v(r) - b^2}} \\
&= 2 \arccos\left(\frac{b}{b_{\max}}\right) - 2 \int_{r_{\min}(b)}^{b_{\max}} \frac{b\,\mathrm{d}r}{r\sqrt{r^2 v(r) - b^2}}
\end{aligned} \tag{17}
$$

where $r_{\min}(b)$ is the classical turning point of the trajectory and is determined as the largest zero of the radicand $r^2 v(r) - b^2$.

4.2. *Irregular scattering*

The usual definition of irregular scattering (cf., for example, [5], Section 5.4) can now be formulated for $M \subset \mathbb{R}^n$ as follows.

Definition 4.1. A smooth map T of an open set $B \subset \mathbb{R}^r$ into some $\mathbb{R}^q$, with $r, q \le 2n$, is called *irregular* if

(1) T is not uniformly continuous and

(2) the set of singularities $\operatorname{Sing}(T)$, i.e. the set of points in the closure $\bar{B}$ to which T cannot be continuously extended, is a closed non-trivial nowhere dense set.†

One speaks of *irregular or chaotic (Hamiltonian) scattering*, if it has an irregular scattering function.

Observe that the presence of irregularity of scattering in spite of smoothness is a consequence of missing uniform continuity. If the scattering transformation S is defined on a dense set then the existence of an irregular scattering function is equivalent to S itself being irregular. This is sufficient because the identity function $f = g = Id$ will yield an irregular scattering function. It is necessary because under the assumptions above for f, g the set $\operatorname{Sing}(S) \supset g(\operatorname{Sing}(f \circ S \circ g))$ contains a closed non-trivial nowhere dense set.

Now, where are the fingerprints of chaos in chaotic (irregular) scattering? First, an irregular scattering transformation S shows by definition sensitive dependence at least on its singularity

† i.e. with empty interior, but containing a Cantor set.

set $\mathrm{Sing}(S)$ of initial conditions in a limit sense, i.e. since $\mathrm{Sing}(S)$ is closed there is a $K > 0$ s.t. for all $u_0 \in \mathrm{Sing}(S)$ and for all neighbourhoods N of u_0 there exist points $u_1, u_2 \in N$, s.t. $||S(u_1) - S(u_2)|| > K$. Second, if the inverse scattering transformation S^{-1} is also irregular then S shows also a kind of mixing property at least on a singularity set C of initial conditions in a limit sense, i.e. there is a $\tilde{K} > 0$ s.t. for all $v_0 \in S(C)$ and for all neighbourhoods N of v_0 there exist points $v_1, v_2 \in N$, s.t. $||S^{-1}(v_1) - S^{-1}(v_2)|| > K$.

However, the resulting sensitive dependence on initial conditions or mixing property are only noticeable by experiments if it occurs on a set of positive measure. One could call such systems *measurably irregular*.

4.3. *The chaotic bound set*

The set of bound states of a Hamiltonian system, which we also call *the bound or trapped set*, is defined as the intersection of the forward and backward bound sets $\mathcal{B}_\pm := C\mathcal{R}_\pm = \bigcup_n \{u = (x, p) \in T^*(\mathbb{R}^m) : ||\Phi_{\pm t} u|| < n \ \forall\, t > 0\}$, i.e. $\mathcal{B} := \mathcal{B}_+ \cap \mathcal{B}_-$. Apparently it is an invariant set. Any invariant subset B of the bound set is called *an invariant bound set*. It is called chaotic if the restricted flow $\Phi_t|_B$ is chaotic w.r.t. a given definition of chaos for bound systems. The known mechanism for irregular scattering is that a chaotic bound set B borders the forward or backward free set $\mathcal{R}_\pm$. We say that irregular scattering is *induced by chaotic bound motion* if the set of singularities $\mathrm{Sing}(S) \subset \overline{\mathcal{R}_+} \setminus \mathcal{R}_+$ is contained but for a trivial subset in a chaotic bound set B.

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