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Decentralized robust disturbance attenuation for a class of large-scale nonlinear systems

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Abstract

We solve the problem of decentralized H_∞ almost disturbance decoupling for a class of large-scale nonlinear uncertain systems in the absence of matching conditions. The method combines ideas from decentralized adaptive control and centralized nonlinear H_∞ control. We relax earlier assumptions on the uncertain time-varying interconnections which are demanded to be only bounded by general nonlinear functions in this work. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of large-scale systems has been motivated by their wide applicability to many practical systems such as power systems and spacecraft. Decentralized control schemes, whose design can be based on available local subsystem information and local performance criteria, are widely used – see [19] for instance. In this paper, we design the decentralized control law for a class of large-scale nonlinear systems whose normal forms exhibit triangular dependence on local coordinates. The objective of the design is to make the closed-loop interconnected system asymptotically stable while attenuating the effect of the disturbance.

Since various kinds of uncertainty commonly appear in the system models, either in the isolated subsystems or in the interconnections, adaptive and/or robust control of large-scale systems has received considerable attention during the last two decades. There are many control results concerning linear nominal systems with linear subsystem interconnections or interconnections satisfying linear bounds – see [19,1,3,6,21,22,5,16] and references cited therein. While most of the results need matching condition assumptions, [16] presents a totally decentralized control scheme for systems of arbitrary relative degree. In [21], avoidance of matching conditions is achieved by applying integrator backstepping. Note also that interconnections are commonly required to be not only linear, but also weak in the sense that they are dominated by subsystem stability margins [19,3,6,22,5]. An advance on the interconnection constraint was presented in [17,18] where the interconnection was extended to be bounded by any higher-order polynomial. Then in [4], a stabilizing design is given for a nonlinear time-varying system with the interconnection bounded by a general nonlinear function.

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However, the uncertainties in [17,18,4] require matching conditions. Recently in [9], based on the work of [17,18] and backstepping and tuning function design [12], a class of large-scale nonlinear systems in their decentralized strict feedback form was designed to be adaptively stable; the matching condition was relaxed and the time-invariant interconnections are bounded by higher-order polynomials. In this paper, we assume time-varying interconnections without matching conditions which are bounded by general nonlinear functions and develop a robust control design.

In contrast to the large number of decentralized adaptive control results, there are few results on decentralized H_∞ control. In [9,24,2,10], the decentralized linear H_∞ control technique was applied for linear interconnected systems with linear interconnections where disturbances preserve the interconnection structure. Centralized nonlinear H_∞ control has achieved considerable development since the initial result [20]. Especially in [14,8,10], a triangular system structure was exploited to explicitly construct the controller which overcomes the general difficulty of solving the Hamilton–Jacobi-type partial differential inequality. The so-called "nonlinear H_∞ almost disturbance decoupling problem" is solved in the sense that the designed internally stable system maintains arbitrary $\mathscr{L}_2$ gain from the disturbance input to output.

In this paper, we move to close the gap between decentralized and centralized nonlinear H_∞ control. We consider the decentralized robust H_∞ control problem for large-scale nonlinear systems and are motivated by [9]. The results combine the above-mentioned ideas from decentralized adaptive control and centralized nonlinear H_∞ control for structured systems. Uncertain interconnections are extended to be time-varying and bounded by general nonlinear functions. Disturbances enter each subsystem affinely with subsystem states interacting through their input vector fields. The stabilizer design consists of constructing a nonlinear storage function [23] and the interactions are effectively dominated by subsystem stability margins. A decentralized control law is recursively constructed and the internally stable closed-loop system maintains any given $\mathscr{L}_2$-gain from the disturbance input to the output.

The remainder of the paper is organized as follows. In Section 2, we give the system configuration and state the main problem and necessary assumptions. Then in Section 3, our main theorem is presented and a recursive design procedure is developed to obtain the explicit decentralized controller. In Section 4, simulations for a two-subsystem example are shown to demonstrate the performance of our controller. Finally the paper is concluded by brief remarks in Section 5.

Notations. The notation used in this paper is standard. $|\cdot|$ denotes the usual Euclidean norm for vectors. ∇f denotes the gradient $\partial f/\partial s$ of a real-valued function f. We say that $z:(0,T)\to\Re^k$ is in $\mathscr{L}_2(0,T)$ if $\int_0^T |z(t)|^2\,\mathrm{d}t<\infty$. A continuous function $\phi:\Re_+\to\Re_+$ is said to be of class $\mathscr{K}_\infty$ if it is strictly increasing and satisfies $\phi(0)=0$, and $\phi(s)\to\infty$ as $s\to\infty$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of any square matrix P. The arguments of a function will be omitted sometimes in the analysis when no confusion can arise.

2. Problem formulation

We consider in this paper a class of large-scale nonlinear systems S which are composed of the following single-input-single-output (SISO) subsystems S_i $(1\leqslant i\leqslant N)$:

$$\begin{aligned}
\dot z_i &= \begin{bmatrix} 0 & 1 & & 0\\ & & \ddots & \\ 0 & 0 & & 1\\ 0 & 0 & \dots & 0\end{bmatrix}\begin{bmatrix} z_{i1}\\ \vdots\\ z_{i,\kappa-1}\\ z_{i\kappa}\end{bmatrix} + \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\cdot\{\xi_{i1}+\gamma_{i0}(t,z_1,\dots,z_N)+p_{i0}(t,z_1,\dots,z_N)\omega_i\}\\
&\triangleq A_iz_i+B_i\cdot[\xi_{i1}+\gamma_{i0}(t,z_1,\dots,z_N)+p_{i0}(t,z_1,\dots,z_N)\omega_i],\\
\dot\xi_{i1} &= \xi_{i2}+\gamma_{i1}(t,z_1,\dots,z_N,\xi_{i1})+p_{i1}(t,z_1,\dots,z_N,\xi_{i1})\omega_i,\\
\dot\xi_{i2} &= \xi_{i3}+\gamma_{i2}(t,z_1,\dots,z_N,\xi_{i1},\xi_{i2})+p_{i2}(t,z_1,\dots,z_N,\xi_{i1},\xi_{i2})\omega_i,\\
&\ \vdots
\end{aligned}\tag{1}$$

$$\dot{\xi}_{i,n-\kappa} = u_i + \gamma_{i,n-\kappa}(t, z_1, \ldots, z_N, \xi_i) + p_{i,n-\kappa}(t, z_1, \ldots, z_N, \xi_i)\omega_i,$$

$$y_i = z_{i1},$$

where $z_i = (z_{i1}, \ldots, z_{i\kappa}) \in \Re^{\kappa}$, $z = (z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}})^{\mathrm{T}}$, $\xi_i = (\xi_{i1}, \ldots, \xi_{i,n-\kappa}) \in \Re^{n-\kappa}$, and $\xi = (\xi_1^{\mathrm{T}}, \ldots, \xi_N^{\mathrm{T}})^{\mathrm{T}}$, z and ξ are the state vectors, $u_i \in \Re$ is the control input, $\omega_i \in \Re^{m_i}$ is the disturbance input, $y_i \in \Re$ is the to-be-controlled output; the unknown functions γ_{il}, p_{il}, $(0 \leqslant l \leqslant n-\kappa)$ are locally Lipschitz in states and piecewise continuous in t, and $\gamma_{il}(t, 0, 0, \ldots, 0) = 0$.

Our class of systems (1) is motivated by the time-invariant decentralized strict feedback form in [9] where necessary and sufficient geometric conditions are given to characterize a subclass of systems (1) (see [7]). Disturbances are introduced to enter each subsystem affinely with subsystem states interacting through their input vector fields. Furthermore, we allow in this paper for a larger set of nonlinear interconnections, as shown in the following assumptions.

Assumption 1. There exist known smooth functions $a_{ilj}(\cdot), b_{ilj}(\cdot), \varphi_{ilj}(\cdot), \Phi_{ilj}(\cdot)$, with $\varphi_{ilj}(0) = 0$, $\Phi_{ilj}(0) = 0$, such that for each $0 \leqslant l \leqslant n-\kappa$ the uncertain interconnections satisfy

$$|\gamma_{il}(t, z_1, \ldots, z_N, \xi_{i1}, \ldots, \xi_{il}) - \gamma_{il}(t, 0, \ldots, 0, \xi_{i1}, \ldots, \xi_{il})| \leqslant \sum_{j=1}^{N} a_{ilj}(\xi_{i1}, \ldots, \xi_{il})\varphi_{ilj}(|z_j|), \tag{2}$$

$$|p_{il}(t, z_1, \ldots, z_N, \xi_{i1}, \ldots, \xi_{il}) - p_{il}(t, 0, \ldots, 0, \xi_{i1}, \ldots, \xi_{il})| \leqslant \sum_{j=1}^{N} b_{ilj}(\xi_{i1}, \ldots, \xi_{il})\Phi_{ilj}(|z_j|). \tag{3}$$

For the sake of simplicity, we let $a_{i0j} = b_{i0j} = 1$.

Assumption 2. There exist known smooth functions $\psi_l(\cdot), \Psi_{il}(\cdot)$, with $\psi_l(0)=0$, such that for each $0 \leqslant l \leqslant n-\kappa$,

$$|\gamma_{il}(t, 0, \ldots, 0, \xi_{i1}, \ldots, \xi_{il})| \leqslant \psi_l(|(\xi_{i1}, \ldots, \xi_{il})|), \tag{4}$$

$$|p_{il}(t, 0, \ldots, 0, \xi_{i1}, \ldots, \xi_{il})| \leqslant \Psi_{il}(|(\xi_{i1}, \ldots, \xi_{il})|). \tag{5}$$

Since $\gamma_{i0}(t, 0) = 0$, we can take $\psi_0 \equiv 0$; and Ψ_{i0} is a nonnegative constant.

The objective of our design is to find robust decentralized controllers in terms of local subsystem states and make the whole interconnected system globally uniformly asymptotically stable while arbitrarily attenuating the effect of the disturbance in the sense of $\mathscr{L}_2$ gain. A precise statement of this control problem is given below:

Problem of decentralized H_∞ almost disturbance decoupling. Find decentralized smooth state feedback controllers $u_i = u_i(z_i, \xi_i)$ such that, for any given positive constant μ, the closed-loop interconnected system satisfies the following dissipation inequality

$$\int_0^T |y|^2 \,\mathrm{d}t \leqslant \mu \int_0^T |w|^2 \,\mathrm{d}t + v(z(0), \xi(0)), \quad \forall \omega \in \mathscr{L}_2(0, T), \ \forall T \geqslant 0, \tag{6}$$

where v is a positive semidefinite function and $(z(0), \xi(0))$ is the initial condition. Furthermore, the origin is globally uniformly asymptotically stable (GUAS) if $w = 0$.

Remark 1. Nonlinear gain bounds on the interconnections are motivated by ideas used in recent stability results [13,11]. Nonlinear interconnection bounds with a matching condition were used in [9] where a saturation type of controller was designed to stabilize a class of uncertain time-varying nonlinear interconnected systems.

3. Main result

The following theorem states our main result:

Theorem 1. *Under Assumptions* 1 *and* 2, *the problem of decentralized* H_∞ *almost disturbance decoupling is solvable for system* (1).

The proof of Theorem 1 uses a combination of centralized H_∞ almost disturbance decoupling method [14], its robust version [10] and decentralized designs [4,9]. A stepwise procedure is presented by application of robust backstepping to the large-scale system (1).

Proof *Step* 0: We start by considering the z_i-subsystem with ξ_{i1} as the virtual control input. Choose the storage function:

$$V_i(z_i) = \phi_i(V_{i0}(z_i)), \tag{7}$$

where $\phi_i(\cdot)$ is a smooth $\mathscr{K}_\infty$ function and $V_{i0} = z_i^{\mathrm{T}} P_i z_i$ with P_i is a positive-definite symmetric matrix solving the algebric Riccati equation:

$$A_i^{\mathrm{T}} P_i + P_i A_i - 2\epsilon P_i B_i B_i^{\mathrm{T}} P_i + Q_i = 0, \tag{8}$$

where ϵ_i is a positive constant and Q_i is a positive-definite symmetric matrix.

Differentiating (7) along the solution of z_i-subsystem, we have

$$\dot{V}_i = \frac{\partial \phi_i}{\partial V_{i0}} \cdot 2z_i^{\mathrm{T}} P_i \cdot \{A_i z_i + B_i[\xi_{i1} + (\gamma_{i0}(t,z) - \gamma_{i0}(t,0)) + (p_{i0}(t,z) - p_{i0}(t,0))\omega_i + p_{i0}(t,0)\omega_i]\}. \tag{9}$$

Using the inequality $2ab \leqslant a^2 + b^2$, $(a, b \in \mathfrak{R})$ and Assumption 1, the interactions in the second term in the square bracket of (9) are decoupled which gives

$$\begin{aligned}
&\frac{\partial \phi_i}{\partial V_{i0}} 2z_i^{\mathrm{T}} P_i B_i (\gamma_{i0}(t,z) - \gamma_{i0}(t,0)) \\
&\quad \leqslant \left| \frac{\partial \phi_i}{\partial V_{i0}} \right| \cdot |2z_i^{\mathrm{T}} P_i B_i| \sum_{j=1}^{N} \varphi_{i0j}(|z_j|) \\
&\quad \leqslant \left| \frac{\partial \phi_i}{\partial V_{i0}} \right|^2 \cdot |z_i^{\mathrm{T}} P_i B_i|^2 \sum_{j=1}^{N} \delta_{j0} + \sum_{j=1}^{N} \delta_{j0}^{-1} \varphi_{i0j}^2(|z_j|).
\end{aligned} \tag{10}$$

Similarly, the third term in the square bracket of (9) turns to

$$\begin{aligned}
&\frac{\partial \phi_i}{\partial V_{i0}} 2z_i^{\mathrm{T}} P_i B_i (p_{i0}(t,z) - p_{i0}(t,0))\omega_i \\
&\quad \leqslant \sum_{j=1}^{N} \frac{1}{\tau_i} \left| \frac{\partial \phi_i}{\partial V_{i0}} \right|^2 \cdot |z_i^{\mathrm{T}} P_i B_i|^2 \Phi_{i0j}(|z_j|)^2 + N\tau_i |\omega_i|^2 \\
&\quad \leqslant \frac{1}{4\tau_i^2} \left| \frac{\partial \phi_i}{\partial V_{i0}} \right|^4 \cdot |z_i^{\mathrm{T}} P_i B_i|^4 \sum_{j=1}^{N} \varrho_{j0} + \sum_{j=1}^{N} \varrho_{j0}^{-1} \Phi_{i0j}(|z_j|)^4 + N\tau_i |\omega_i|^2,
\end{aligned} \tag{11}$$

where $\delta_{j0}, \varrho_{i0}, \tau_i$ are any positive constants.

For the last term of (9), which is time-varying, we apply the bound from Assumption 2 to give

$$\frac{\partial \phi_i}{\partial V_{i0}} 2z_i^{\mathrm{T}} P_i B_i p_{i0}(t,0)\omega_i \leqslant \frac{1}{d_{2i}} \left| \frac{\partial \phi_i}{\partial V_{i0}} \right|^2 |z_i^{\mathrm{T}} P_i B_i|^2 \Psi_{i0}^2 + d_{2i} |\omega_i|^2, \tag{12}$$

where d_{2i} is any positive constant.

Choose our virtual control $\xi_{i1} = \xi_{i1}^*(z_i)$ as

$$\xi_{i1}^*(z_i) = -\left\{ \epsilon_i B_i^{\mathrm{T}} P_i z_i + \frac{1}{2}\frac{\partial \phi_i}{\partial V_{i0}}(z_i^{\mathrm{T}} P_i B_i)\left(\sum_{j=1}^{N} \delta_{j0}\right) + \frac{1}{8\tau_i^2}\left(\frac{\partial \phi_i}{\partial V_{i0}}\right)^3 (z_i^{\mathrm{T}} P_i B_i)^3 \left(\sum_{j=1}^{N} \varrho_{j0}\right) \right.$$
$$\left. + \frac{1}{2d_{2i}}\left(\frac{\partial \phi_i}{\partial V_{i0}}\right)(z_i^{\mathrm{T}} P_i B_i)\Psi_{i0}^2 \right\}. \tag{13}$$

Notice $\xi_{i1}^*(0) = 0$.

When substituting (13) into (9) and applying the bounds (10)–(12), we get

$$\dot{V}_i \leqslant \frac{\partial \phi_i}{\partial V_{i0}}(-z_i^{\mathrm{T}} Q_i z_i) + \left[\sum_{j=1}^{N} \delta_{i0}^{-1}\varphi_{i0j}^2(|z_j|) + \sum_{j=1}^{N} \varrho_{j0}^{-1}\Phi_{i0j}^4(|z_j|)\right] + (N\tau_i + d_{2i})|\omega_i|^2$$
$$\triangleq -\alpha_{i0}(|z_i|) + \sum_{j=1}^{N} \beta_{i0j}(|z_j|) + \gamma_{i0}|\omega_i|^2. \tag{14}$$

By constructing virtual control ξ_{i1} according to (13) for the z_i-subsystem, we achieve a dissipation inequality (14) which implies that nett dissipation is determined by the dominance of a local state function α_{i0} over a state function $(\sum_{j=1}^{N} \beta_{i0j})$. Note that inequality (14) is only valid when $\xi_{i1} = \xi_{i1}^*$; otherwise, additional terms must be included. We leave the choice of ϕ_i till the final stage of the design problem. From here, we start our recursive backstepping procedure.

Step 1: Augment the z_i-subsystem with the ξ_{i1}-subsystem, and choose a storage function as

$$W_{i1}(z_i, \xi_{i1}) = V_i(z_i) + (\xi_{i1} - \xi_{i1}^*)^2. \tag{15}$$

Denote

$$-\dot{\xi}_{i1}^* = -\frac{\partial \xi_{i1}^*}{\partial z_i}(A_i z_i + B_i \xi_{i1}) - \frac{\partial \xi_{i1}^*}{\partial z_i} B_i[\gamma_{i0}(t,z) + p_{i0}(t,z)\omega_i]$$
$$\triangleq \vartheta_{i1}(z_i, \xi_{i1}) + \sigma_{i0}(z_i)[\gamma_{i0}(t,z) + p_{i0}(t,z)\omega_i]. \tag{16}$$

Differentiating W_{i1} along the solutions of the (z_i, ξ_{i1})-subsystem yields

$$\dot{W}_{i1} \leqslant -\alpha_{i0}(|z_i|) + \sum_{j=1}^{N} \beta_{i0j}(|z_j|) + \gamma_{i0}|\omega_i|^2 + 2\widetilde{\xi}_{i1}\left\{ \xi_{i2} + \left(\frac{\partial \phi_i}{\partial V_{i0}} z_i^{\mathrm{T}} P_i B_i + \vartheta_{i1}\right) \right.$$
$$+ \sum_{k=0}^{1} \sigma_{ik}[\gamma_{ik}(t,z,\xi_{i1}) - \gamma_{ik}(t,0,\xi_{i1})] + \sum_{k=0}^{1} \sigma_{ik}[p_{ik}(t,z,\xi_{i1}) - p_{ik}(t,0,\xi_{i1})]\omega_i$$
$$\left. + \gamma_{i1}(t,0,\xi_{i1}) + \sum_{k=0}^{1} \sigma_{ik}\ p_{ik}(t,0,\xi_{i1})\omega_i \right\}, \tag{17}$$

where we denote $\widetilde{\xi}_{i1} = \xi_{i1} - \xi_{i1}^*(z_i)$ and $\sigma_{i1} = 1$.

Since $\xi_{i1}^*(z_i)$ is smooth and $\xi_{i1}^*(0) = 0$, there exists a smooth function $\zeta_i(z_i)$ such that $\xi_{i1}^*(z_i) = z_i^{\mathrm{T}}\zeta_i(z_i)$. This together with the definition of $\widetilde{\xi}_{i1}$ gives

$$|\xi_{i1}| \leqslant \widetilde{\zeta}_i(z_i) \cdot (|\widetilde{\xi}_{i1}| + |z_i|), \tag{18}$$

where $\widetilde{\zeta}_i(z_i)$ is a smooth nonnegative function.

Applying the bounds from Assumption 2 to the second last term of (17), and using (18) and the fact that ψ_{i1} is smooth and $\psi_{i1}(0)=0$, we have

$$\begin{aligned}2\widetilde{\xi}_{i1}\gamma_{i1}(t,0,\xi_{i1}) &\leqslant 2|\widetilde{\xi}_{i1}|\cdot\psi_{i1}(|\xi_{i1}|)\\ &\leqslant 2|\widetilde{\xi}_{i1}|\bar{\psi}_{i1}(|\xi_{i1}|)|\xi_{i1}|\\ &\leqslant 2|\widetilde{\xi}_{i1}|\widetilde{\psi}_{i1}(z_i,\xi_{i1})(|\widetilde{\xi}_{i1}|+|z_i|)\\ &\leqslant 2|\widetilde{\xi}_{i1}|^2\widetilde{\psi}_{i1}(z_i,\xi_{i1})+\frac{1}{d_{1i}}|\widetilde{\xi}_{i1}|^2\widetilde{\psi}_{i1}^2(z_i,\xi_{i1})+d_{1i}|z_i|^2,\end{aligned}\tag{19}$$

where d_{1i} is a positive design parameter to be given later and $\widetilde{\psi}_{i1}$ is a smooth nonnegative function.

Similarly, applying Assumption 2, for any $d_{2i}>0$, the last term of (17) turns to

$$\begin{aligned}&2\widetilde{\xi}_{i1}\sum_{k=0}^{1}\sigma_{ik}\ p_{ik}(t,0,\xi_{i1})\omega_i\\ &\leqslant\frac{1}{d_{2i}}|\widetilde{\xi}_{i1}|^2\left[\sum_{k=0}^{1}\sigma_{ik}\Psi_{ik}(|\xi_{ik}|)\right]^2+d_{2i}|\omega_i|^2.\end{aligned}\tag{20}$$

Applying the bounds from Assumption 1 to the fourth and third last terms of (17), respectively, we obtain

$$\begin{aligned}&2\widetilde{\xi}_{i1}\sum_{k=0}^{1}\sigma_{ik}[\gamma_{ik}(t,z,\xi_{i1})-\gamma_{ik}(t,0,\xi_{i1})]\\ &\leqslant 2|\widetilde{\xi}_{i1}|\sum_{k=0}^{1}|\sigma_{ik}|\sum_{j=1}^{N}a_{ikj}(\xi_{i1})\varphi_{ikj}(|z_j|)\\ &\leqslant|\widetilde{\xi}_{i1}|^2\sum_{k=0}^{1}|\sigma_{ik}|^2\sum_{j=1}^{N}\delta_{jk}a_{ikj}^2(\xi_{i1})+\sum_{k=0}^{1}\sum_{j=1}^{N}\delta_{jk}^{-1}\varphi_{ikj}^2(|z_j|)\ ,\end{aligned}\tag{21}$$

$$\begin{aligned}&2\widetilde{\xi}_{i1}\sum_{k=0}^{1}\sigma_{ik}[p_{ik}(t,z,\xi_{i1})-\ p_{ik}(t,0,\xi_{i1})]\omega_i\\ &\leqslant 2|\widetilde{\xi}_{i1}|\sum_{k=0}^{1}|\sigma_{ik}|\sum_{j=1}^{N}b_{ikj}(\xi_{i1})\Phi_{ikj}(|z_j|)\ |\omega_i|\\ &\leqslant\frac{1}{4\tau_i^2}|\widetilde{\xi}_{i1}|^4\sum_{k=0}^{1}|\sigma_{ik}|^4\sum_{j=1}^{N}\varrho_{jk}b_{ikj}^4(\xi_{i1})+\sum_{k=0}^{1}\sum_{j=1}^{N}\varrho_{jk}^{-1}\Phi_{ikj}^4(|z_j|)+2N\tau_i|\omega_i|^2,\end{aligned}\tag{22}$$

where δ_{ik},ϱ_{ik} are positive constants.

Now, we can choose the virtual controller $\xi_{i2}=\xi_{i2}^*$ as

$$\begin{aligned}\xi_{i2}^*=-\frac{1}{2}\Bigg\{&c_{i1}\widetilde{\xi}_{i1}+\frac{\partial\phi_i}{\partial V_{i0}}2z_i^{\mathrm{T}}P_iB_i+2\vartheta_{i1}+2\widetilde{\xi}_{i1}\widetilde{\psi}_{i1}(z_i,\xi_{i1})+\frac{1}{d_{1i}}\widetilde{\xi}_{i1}\widetilde{\psi}_{i1}^2(z_i,\xi_{i1})\\ &+\frac{1}{d_{2i}}\widetilde{\xi}_{i1}\left[\sum_{k=0}^{1}\sigma_{ik}\Psi_{i1}(|\xi_{ik}|)\right]^2+\widetilde{\xi}_{i1}\sum_{k=0}^{1}\sigma_{ik}^2\sum_{j=1}^{N}\delta_{jk}a_{ikj}^2(\xi_{i1})+\frac{1}{4\tau_i^2}\widetilde{\xi}_{i1}^3\sum_{k=0}^{1}\sigma_{ik}^4\sum_{j=1}^{N}\varrho_{jk}b_{ikj}^4(\xi_{i1})\Bigg\},\end{aligned}\tag{23}$$

where c_{i1} is a positive constant.

Then (17) with the bounds (19)–(22) implies

$$\dot{W}_{i1} \leqslant -\alpha_{i1}(|z_i|) - C_{i11}\widetilde{\xi}_{i1}^2 + \sum_{j=1}^{N} \beta_{i1j}(|z_j|) + \gamma_{i1}|\omega_i|^2, \tag{24}$$

where

$$\alpha_{i1}(|z_i|) = \alpha_{i0}(|z_i|) - d_{1i}|z_i|^2, \tag{25}$$

$$\beta_{i1j}(|z_j|) = \beta_{i0j} + \sum_{k=0}^{1} \{\delta_{jk}^{-1}\varphi_{ikj}^2(|z_j|) + \varrho_{jk}^{-1}\Phi_{ikj}^4(|z_j|)\}, \tag{26}$$

$$\gamma_{i1} = \gamma_{i0} + 2N\tau_i + d_{2i}, \tag{27}$$

$$C_{i11} = c_{i1}. \tag{28}$$

Note that inequality (24) is only valid when $\xi_{i2} = \xi_{i2}^*$.

So from Step 1 the dissipation inequality (24) possesses the same property as described at the end of Step 0 with modified bound functions. We now show that this property remains while recursively argumenting the ξ_{il}-subsystem.

From *Step* 2 to *Step* $n-\kappa$, we can establish the following Claim (see the appendix for the proof).

Claim. *Given any index* $2 \leqslant \imath \leqslant n-\kappa$, *for the system* $S_{i\imath}$

$$\begin{aligned} \dot{z}_i &= A_i z_i + B_i[\xi_{i1} + \gamma_{i0}(t,z) + p_{i0}(t,z)\omega_i], \\ \dot{\xi}_{i1} &= \xi_{i2} + \gamma_{i1}(t,z,\xi_{i1}) + p_{i1}(t,z,\xi_{i1})\omega_i, \\ &\vdots \\ \dot{\xi}_{i\imath} &= \xi_{i,\imath+1} + \gamma_{i\imath}(t,z,\xi_{i1},\ldots,\xi_{i\imath}) + p_{i\imath}(t,z,\xi_{i1},\ldots,\xi_{i\imath})\omega_i, \end{aligned} \tag{29}$$

there exist $\imath+1$ *smooth functions*

$$\xi_{ik}^* = \xi_{ik}^*(z_i,\xi_{i1},\ldots,\xi_{i,k-1}), \xi_{ik}^*(0,0,\ldots,0) = 0, \quad 1 \leqslant k \leqslant \imath+1$$

such that in new coordinates

$$\widetilde{z}_i = z_i, \widetilde{\xi}_{ik} = \xi_{ik} - \xi_{ik}^*(z_i,\xi_{i1},\ldots,\xi_{i,k-1}), \quad 1 \leqslant k \leqslant \imath,$$

the storage function

$$W_{i\imath} = V_i + \sum_{k=1}^{\imath} \widetilde{\xi}_{ik}^2, \tag{30}$$

has time derivative, with $\xi_{i,\imath+1} = \xi_{i,\imath+1}^*$, *satisfying the dissipation inequality*

$$\dot{W}_{i\imath} \leqslant -\alpha_{i\imath}(|z_i|) - \sum_{k=1}^{\imath} C_{i\imath k}\widetilde{\xi}_{ik}^2 + \sum_{j=1}^{N} \beta_{i\imath j}(|z_j|) + \gamma_{i\imath}|\omega_i|^2, \tag{31}$$

where

$$\alpha_{i\imath}(|z_i|) = \alpha_{i,\imath-1}(|z_i|) - d_{1i}|z_i|^2, \tag{32}$$

$$\beta_{i\imath j}(|z_j|) = \beta_{i,\imath-1,j}(|z_j|) + \sum_{k=0}^{\imath} \{\delta_{jk}^{-1}\varphi_{ikj}^2(|z_j|) + \varrho_{jk}^{-1}\Phi_{ikj}^4(|z_j|)\}, \tag{33}$$

$$\gamma_{i\imath} = \gamma_{i,\imath-1} + (\imath+1)N\tau_i + d_{2i}, \tag{34}$$

$$C_{\imath\imath k} = C_{i,\imath-1,k} - d_{3i} \quad \textit{for } k = 1, \ldots, \imath - 1,$$

$$C_{\imath\imath k} = c_{ik} \quad \textit{for } k = \imath. \tag{35}$$

As shown in *Step* 1, the Claim holds for the system (29) with $\imath = 1$. At *Step* $n - \kappa$, by means of the Claim, a decentralized state feedback control

$$u_i = \xi^*_{i,\,n-\kappa+1}(z_i, \xi_i) \tag{36}$$

is iteratively built. For the closed-loop system S_i, if we denote the storage function as

$$W_i(z_i, \xi_i) \triangleq W_{i,\,n-\kappa}(z_i, \xi_i) = V_i + \sum_{k=1}^{n-\kappa} \widetilde{\xi}^2_{ik}, \tag{37}$$

we have

$$\dot{W}_i \leqslant -\alpha_i(|z_i|) - \sum_{l=1}^{n-\kappa} C_{il}\widetilde{\xi}^2_{il} + \sum_{j=1}^{N} \beta_{ij}(|z_j|) + \gamma_i|\omega_i|^2 \tag{38}$$

where

$$\alpha_i(|z_i|) = \frac{\partial \phi_i}{\partial V_{i0}}(z_i^{\mathrm{T}} Q_i z_i) - (n-\kappa)\, d_{1i}|z_i|^2, \tag{39}$$

$$\beta_{ij}(|z_j|) = \sum_{l=0}^{n-\kappa}\{(n-\kappa-l+1)\delta_{jl}^{-1}\varphi^2_{ilj}(|z_j|) + (n-\kappa-l+1)\varrho_{jl}^{-1}\Phi^4_{ilj}(|z_j|)\}, \tag{40}$$

$$\gamma_i = (n-\kappa+1)[\tfrac{1}{2}(n-\kappa+2)N\tau_i + d_{2i}], \tag{41}$$

$$C_{il} = c_{il} - (n-\kappa-l)\, d_{3i}. \tag{42}$$

Choose the positive parameter d_{3i} as

$$d_{3i} < \min_{1 \leqslant l \leqslant n-\kappa-1} \left\{ \frac{c_{il}}{n-\kappa-l} \right\} \tag{43}$$

so that $C_{il} > 0$.

Define a storage function for the whole interconnected system S as

$$W(z, \xi) = \sum_{i=1}^{N} W_i(z_i, \xi_i). \tag{44}$$

Then from (38), we know

$$\dot{W} \leqslant \sum_{i=1}^{N} \left\{ -\alpha_i(|z_i|) - \sum_{l=1}^{n-\kappa} C_{il}\widetilde{\xi}^2_{il} + \sum_{j=1}^{N} \beta_{ij}(|z_j|) + \gamma_i|\omega_i|^2 \right\}. \tag{45}$$

The idea now is to compensate for the effect of the other subsystem interactions using the subsystem stability margins.

Notice

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ij}(|z_j|) = \sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ji}(|z_i|). \tag{46}$$

For smooth functions vanishing at the origin $\varphi_{jli}(\cdot)$ and $\Phi_{jli}(\cdot)$, we have

$$\varphi_{jli}(|z_i|) = |z_i| \int_0^1 \nabla\varphi_{jli}(s|z_i|)\,\mathrm{d}s \leqslant |z_i|\widetilde{\varphi}_{jli}(|z_i|), \tag{47}$$

$$\Phi_{jli}(|z_i|) = |z_i| \int_0^1 \nabla\Phi_{jli}(s|z_i|)\,\mathrm{d}s \leqslant |z_i|\widetilde{\Phi}_{jli}(|z_i|), \tag{48}$$

where $\widetilde{\varphi}_{jli}(|z_i|), \widetilde{\Phi}_{jli}(|z_i|)$ are smooth and nondecreasing functions.

Observe that

$$\lambda_{\min}(P_i)|z_i|^2 \leqslant V_{i0}(z_i), \tag{49}$$

which gives

$$|z_i| \leqslant \sqrt{\frac{V_{i0}(z_i)+1}{\lambda_{\min}(P_i)}} \triangleq \eta_i(V_{i0}). \tag{50}$$

Note that the functions η_i are smooth positive functions.

By the nondecreasing property of functions $\widetilde{\varphi}_{jli}(\cdot)$ and $\widetilde{\Phi}_{jli}(\cdot)$, we have

$$\begin{aligned} &\widetilde{\varphi}_{jli}(|z_i|) \leqslant \widetilde{\varphi}_{jli} \circ \eta_i(V_{i0}), \\ &\widetilde{\Phi}_{jli}(|z_i|) \leqslant \widetilde{\Phi}_{jli} \circ \eta_i(V_{i0}). \end{aligned} \tag{51}$$

Now construct the derivative of $\phi_i(V_{i0})$ as

$$\begin{aligned} \frac{\partial\phi_i}{\partial(V_{i0})} = k_i &+ \frac{1}{\lambda_{\min}(Q_i)} \sum_{j=1}^{N}\sum_{l=0}^{n-\kappa}(n-\kappa-l+1)\delta_{il}^{-1}[\widetilde{\varphi}_{jli} \circ \eta_i(V_{i0})]^2 \\ &+ \frac{V_{i0}}{\lambda_{\min}(Q_i)\cdot\lambda_{\min}(P_i)} \cdot \sum_{j=1}^{N}\sum_{l=0}^{n-\kappa}(n-\kappa-l+1)\varrho_{il}^{-1}[\widetilde{\Phi}_{jli} \circ \eta_i(V_{i0})]^4, \end{aligned} \tag{52}$$

where k_i's are positive constants. It can be checked that by the construction (52) and $\phi_i(0)=0$, ϕ_i is a smooth $\mathscr{K}_\infty$ function.

Substitute (52) into (39) in view of (40), (46) and (51), and we have the dominance condition

$$\sum_{i=1}^{N}\alpha_i(|z_i|) \geqslant \sum_{i=1}^{N} D_i|z_i|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\beta_{ij}(|z_j|), \tag{53}$$

where $D_i = k_i\lambda_{\min}(Q_i) - (n-\kappa)d_{1i}$. Choose the design parameter d_{1i} as

$$d_{1i} < \frac{k_i\lambda_{\min}(Q_i)}{n-\kappa}, \tag{54}$$

so that $D_i > 0$.

Then from (45) and (53) we obtain

$$\dot{W} \leqslant \sum_{i=1}^{N}\left\{-D_i|z_i|^2 - \sum_{l=1}^{n-\kappa} C_{il}\widetilde{\xi}_{il}^2 + \gamma_i|\omega_i|^2\right\} \tag{55}$$

and it follows:

$$\dot{W} \leqslant \sum_{i=1}^{N}\{-D_i|z_{i1}|^2 + \gamma_i|\omega_i|^2\}. \tag{56}$$

Taking the integral along time t, the $\mathscr{L}_2$ gain from ω to y of the closed-loop system is obtained as

$$\int_0^T |y|^2\,\mathrm{d}t \leqslant \mu\int_0^T |w|^2\,\mathrm{d}t + v(z(0),\xi(0)), \tag{57}$$

where

$$\mu = \min_{1\leqslant i\leqslant N}\{\gamma_i\} \Big/ \max_{1\leqslant i\leqslant N}\{D_i\},$$
$$v(z(0),\xi(0)) = W(z(0),\xi(0)) \Big/ \max_{1\leqslant i\leqslant N}\{D_i\}. \tag{58}$$

In the absence of the disturbance, i.e. when $\omega_i = 0$, from (55), we have

$$\dot{W} \leqslant \sum_{i=1}^{N}\left\{-D_i|z_i|^2 - \sum_{l=1}^{n-\kappa} C_{il}\tilde{\xi}_{il}^2\right\} \triangleq -W_a(z,\xi) \leqslant 0. \tag{59}$$

From (37) and (44),

$$W(z,\xi) = \sum_{i=1}^{N}\left\{\phi_i(V_{i0}(z_i)) + \sum_{l=1}^{n-\kappa}\tilde{\xi}_{il}^2\right\} \tag{60}$$

is a continuously differentiable, positive definite and radially unbounded function and its derivative is negative definite. Hence we obtain the GUAS of the closed-loop system when $\omega_i = 0$. □

Remark 2. The result has points of connection with stability results for large-scale systems – see [15] for instance. We design a storage function W_i for ith subsystem S_i by applying backstepping which ends with a dissipation inequality (38). Then we choose storage function W for interconnected system S to be the sum of all subsystem storage functions W_i (see (44)). To get net energy dissipation for the system S, we dominate other subsystem interactions by the subsystem stability margins. This is done by carefully choosing the storage function ϕ_i of (7). In the special case when the interconnections are bounded by polynomials, ϕ_i is a polynomial. Thus the idea of dominance which appeared in early decentralized work (M-matrix design) [15] is implemented in this nonlinear systems design.

4. An illustrative example

We consider the following interconnected system composed of two subsystems:

$$S_1 : \begin{cases} \dot{z}_{11} = z_{12}, \\ \dot{z}_{12} = \xi_{11} + \sin(2t)z_{21}e^{z_{21}} + (1 + z_{11}z_{21})\omega_1, \\ \dot{\xi}_{11} = u_1 + \xi_{11}z_{21}\omega_1, \\ y_1 = z_{11}, \end{cases} \tag{61}$$

$$S_2 : \begin{cases} \dot{z}_{21} = \xi_{21} + z_{11}z_{12}\sin(\xi_{21}) + \omega_2, \\ \dot{\xi}_{21} = u_2 + \omega_2, \\ y_2 = z_{21}. \end{cases} \tag{62}$$

It can be checked that system S is in form (1) and satisfies Assumptions 1 and 2. We choose the design parameters $\epsilon_1 = 1; \epsilon_2 = 1; Q_1 = 2I; Q_2 = 2$ to get

$$P_1 = \begin{bmatrix} 2.8284 & 1.0000 \\ 1.0000 & 1.4142 \end{bmatrix}, \qquad P_2 = 1.$$

Other parameters are chosen to be $k_i = 8; \delta_{il} = \varrho_{il} = 0.1; \tau_i = 1; c_i = 0.1; d_{2i} = 2$ for $i = 1,2; l = 1,2$.

According to (52), we have

$$\frac{\partial\phi_1}{\partial(z_1^{\mathrm{T}}P_1z_1)} = 8 + 0.5\cdot 10\left[0.5\cdot\sqrt{\frac{(z_1^{\mathrm{T}}P_1z_1)+1}{0.8966}}\right]^2 + \frac{(z_1^{\mathrm{T}}P_1z_1)}{1.7932}\cdot 10\cdot\left[0.5\cdot\sqrt{\frac{(z_1^{\mathrm{T}}P_1z_1)+1}{0.8966}}\right]^4,$$
$$\frac{\partial\phi_2}{\partial(z_2^{\mathrm{T}}P_2z_2)} = 8 + 0.5\cdot 10\cdot\left[e^{\sqrt{(z_2^{\mathrm{T}}P_2z_2)+1}}\right]^2 + \frac{(z_2^{\mathrm{T}}P_2z_2)}{2}\cdot 10\cdot\left[0.5\cdot\sqrt{(z_2^{\mathrm{T}}P_2z_2)+1}\right]^4. \tag{63}$$

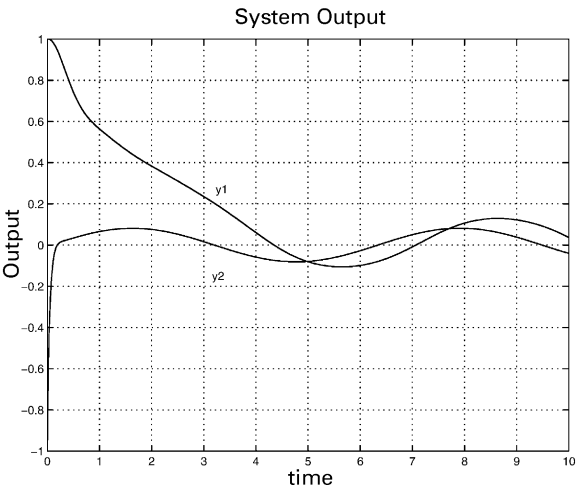

Fig. 1. Output of the closed-loop system.

Then in Step 0 by (13), we have

$$\begin{aligned}\xi_{11}^* &= -z_{11} - 1.4142z_{12} - 0.5 \cdot \frac{\partial \phi_1}{\partial (z_1^{\mathrm{T}} P_1 z_1)} \cdot (z_{11} + 1.4142z_{12}) \cdot 0.2 \\ &\quad -0.1250 \left(\frac{\partial \phi_1}{\partial (z_1^{\mathrm{T}} P_1 z_1)} \right)^3 \cdot (z_{11} + 1.4142z_{12})^3 \cdot 0.2 - 0.25 \frac{\partial \phi_1}{\partial (z_1^{\mathrm{T}} P_1 z_1)} \cdot (z_{11} + 1.4142z_{12}), \\ \xi_{21}^* &= -z_2 - 0.5 \frac{\partial \phi_2}{\partial (z_2^{\mathrm{T}} P_2 z_2)} \cdot z_2 \cdot 0.2 - 0.1250 \left(\frac{\partial \phi_2}{\partial (z_2^{\mathrm{T}} P_2 z_2)} \right)^3 \cdot z_2^3 \cdot 0.2 - 0.25 \frac{\partial \phi_2}{\partial (z_2^{\mathrm{T}} P_2 z_2)} \cdot z_2. \end{aligned} \tag{64}$$

Denote

$$\begin{aligned} \vartheta_1(z_1, \xi_{11}) &= -\frac{\partial \xi_{11}^*}{\partial z_{11}} z_{12} - \frac{\partial \xi_{11}^*}{\partial z_{12}} \xi_{11}, \\ \vartheta_2(z_2, \xi_{21}) &= -\frac{\partial \xi_{21}^*}{\partial z_{21}} \xi_{21}, \\ \sigma_{10}(z_1) &= -\frac{\partial \xi_{11}^*}{\partial z_{12}}, \\ \sigma_{20}(z_2) &= -\frac{\partial \xi_{21}^*}{\partial z_{21}}. \end{aligned} \tag{65}$$

So in Step 1 by (23) the robust decentralized control laws are constructed as

$$\begin{aligned} u_1 &= -\vartheta_1 - 0.1 \cdot (\xi_{11} - \xi_{11}^*) - \frac{\partial \phi_1}{\partial (z_1^{\mathrm{T}} P_1 z_1)} (z_{11} + 1.4142z_{12}) - 0.25 \cdot \sigma_{10}^2 \\ &\quad - 0.1250 \cdot (\xi_{11} - \xi_{11}^*)^3 \cdot (\sigma_{10}^4 \cdot 0.1 + 0.1), \end{aligned} \tag{66}$$

$$u_2 = -\vartheta_2 - 0.1 \cdot (\xi_{21} - \xi_{21}^*) - \frac{\partial \phi_2}{\partial (z_2^{\mathrm{T}} P_2 z_2)} z_2 - 0.25 \cdot (\sigma_{20} + 1)^2. \tag{67}$$

The system output and control input are shown in Fig. 1 and 2 which demonstrate a satisfactory disturbance attenuation performance of our closed-loop system. The initial condition used is $[1, 0, 0, -1, 0]$ and the disturbances are $\omega_1 = \omega_2 = \sin(t)$.

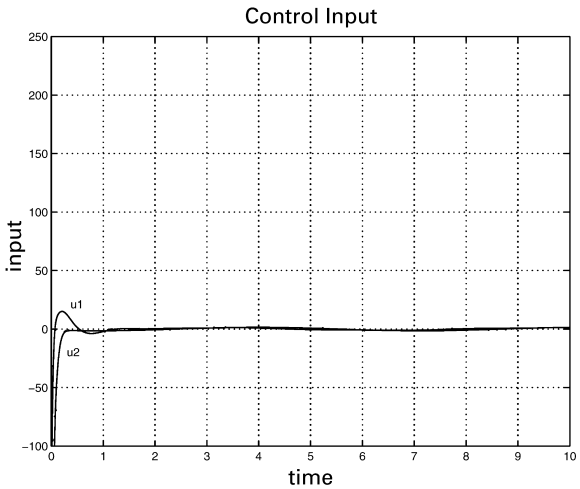

Fig. 2. Control input history.

5. Conclusions

The problem of decentralized H_∞ almost disturbance decoupling has been solved for systems (1). A stepwise procedure which applied backstepping was developed and a decentralized controller was explicitly constructed. Our results allow uncertain interconnections to be time-varying and bounded by general nonlinear functions. We design a storage function that dominates the uncertainties which are not necessarily weak. The designed closed-loop system achieves arbitrary disturbance attenuation in the sense of $\mathscr{L}_2$ gain, and in the absence of the disturbance the system is globally uniformly asymptotically stable. Simulations exhibit a good performance of a two-subsystem example.

Appendix. Proof of the claim

As stated previously, the claim holds for system S_{i1}. Assume that the claim is true for system $S_{i,l-1}$, i.e. $\imath = l-1$. We wish to prove the claim for $\imath = l$, i.e. system S_{il}.

Consider the storage function

$$W_{il} = V_i + \sum_{k=1}^{l} \widetilde{\xi}_{ik}^2. \tag{68}$$

Differentiating it along the solutions of the $(z_i, \xi_{i1}, \ldots, \xi_{il})$-subsystem in (29), we have

$$\begin{aligned}\dot{W}_{il} \leqslant & -\alpha_{i,l-1}(|z_i|) - \sum_{k=1}^{l-1} C_{i,l-1,k}\widetilde{\xi}_{ik}^2 + \sum_{j=1}^{N} \beta_{i,l-1,j}(|z_j|) + \gamma_{i,l-1}|\omega_i|^2 \\ & + 2\widetilde{\xi}_{il}[\xi_{i,l+1} + \Xi_{il}(z_i, \xi_{i1}, \ldots, \xi_{il}) + \Gamma_{il}(t, 0, \xi_{i1}, \ldots, \xi il) + E_{il}(t, 0, \xi_{i1}, \ldots, \xi_{il})\omega_i \\ & + F_{il}(t, z, \xi_{i1}, \ldots, \xi_{il}) + G_{il}(t, z, \xi_{i1}, \ldots, \xi_{il})\omega_i],\end{aligned} \tag{69}$$

where

$$\Xi_{il} = \frac{1}{2}\frac{\partial W_{i,l-1}}{\partial \xi_{i,l-1}} - \frac{\partial \xi_{il}^*}{\partial z_i}(A_i z_i + B_i \xi_{i1}) - \sum_{k=1}^{l-1} \frac{\partial \xi_{il}^*}{\partial \xi_{ik}} \xi_{i,k+1}, \tag{70}$$

$$\Gamma_{il} = \sum_{k=1}^{l} \sigma_{ik}\gamma_{ik}(t, 0, \xi_{i1}, \ldots, \xi_{ik}), \tag{71}$$

$$E_{il} = \sum_{k=0}^{l} \sigma_{ik}\, p_{ik}(t, 0, \xi_{i1}, \ldots, \xi_{ik}), \tag{72}$$

$$F_{il} = \sum_{k=0}^{l} \sigma_{ik}[\gamma_{ik}(t, z, \xi_{i1}, \ldots, \xi_{ik}) - \gamma_{ik}(t, 0, \xi_{i1}, \ldots, \xi_{ik})], \tag{73}$$

$$G_{il} = \sum_{k=0}^{l} \sigma_{ik}[\, p_{ik}(t, z, \xi_{i1}, \ldots, \xi_{ik}) - p_{ik}(t, 0, \xi_{i1}, \ldots, \xi_{ik})] \tag{74}$$

with

$$\sigma_{i0} = -\frac{\partial \xi_{i1}^*}{\partial z_i} B_i,$$

$$\sigma_{ik} = -\frac{\partial \xi_{il}^*}{\partial \xi_{ik}} \quad \text{when } k = 1, \ldots, l-1,$$

$$\sigma_{il} = 1. \tag{75}$$

To handle time-varying term Γ_{il}, firstly we notice that the mapping

$$\Theta : (z_i, \xi_{i1}, \xi_{i2}, \ldots, \xi_{il}) \mapsto (z_i, \widetilde{\xi}_{i1}, \widetilde{\xi}_{i2}, \ldots, \widetilde{\xi}_{il})$$

is a global diffeomorphism since its Jacobian matrix is lower triangular with all diagonal components equal to the constant one. And it preserves the origin. Therefore, Θ^{-1} is smooth and also preserves the origin. So

$$\psi_k(|(z_i, \xi_{i1}, \ldots, \xi_{ik})|) = \psi_k(|\Theta^{-1}(z_i, \widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{ik})|). \tag{76}$$

By a repeated application of operations used in (18), there exists a smooth function $\widetilde{\psi}_{ik}$ such that

$$\psi_k(|(\xi_{i1}, \ldots, \xi_{ik})|) \leqslant \widetilde{\psi}_{ik}(z_i, \widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{ik}) \cdot \left(|z_i| + 0 \sum_{\imath=1}^{k} |\widetilde{\xi}_{i\imath}| \right). \tag{77}$$

Using the bound from Assumption 2, there exists a smooth function $\Delta_{il}(\cdot)$ such that

$$\begin{aligned}
2\widetilde{\xi}_{il}\Gamma_{il}(t, 0, \xi_{i1}, \ldots, \xi_{il}) &\leqslant 2|\widetilde{\xi}_{il}| \left[\sum_{k=1}^{l} |\sigma_{ik}|\ \psi_k(|(\xi_{i1}, \ldots, \xi_{ik})|) \right] \\
&\leqslant 2|\widetilde{\xi}_{il}| \sum_{k=1}^{l} |\sigma_{ik}| \widetilde{\psi}_{ik}(z_i, \widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{ik}) \left(|z_i| + \sum_{\imath=1}^{k} |\widetilde{\xi}_{i\imath}| \right) \\
&\leqslant \left[\frac{1}{d_{1i}} \widetilde{\xi}_{il}^2 \left(\sum_{k=1}^{l} \sigma_{ik}\widetilde{\psi}_{ik} \right)^2 + d_{1i}|z_i|^2 \right] + \left[\widetilde{\xi}_{il}^2 \Delta_{il}(z_i, \widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{il}) + \sum_{k=1}^{l-1} d_{3i}|\widetilde{\xi}_{ik}|^2 \right] \\
&\triangleq \widetilde{\xi}_{il}^2 \widetilde{\Delta}_{il}(z_i, \widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{il}) + \left[d_{1i}|z_i|^2 + \sum_{k=1}^{l-1} d_{3i}|\widetilde{\xi}_{ik}|^2 \right],
\end{aligned} \tag{78}$$

where d_{3i} is a positive design parameter whose rule is given in (43).

Similarly, apply the bounds from Assumption 2 and 1 to the last three terms of (69),

$$2\widetilde{\xi}_{il}E_{il}(t,0,\xi_{i1},\ldots,\xi_{il})\omega_i \leqslant 2|\widetilde{\xi}_{il}|\left(\sum_{k=0}^{l}|\sigma_{ik}|\Psi_{ik}(|(\xi_{i1},\ldots,\xi_{ik})|)\right)|\omega_i|$$
$$\leqslant \frac{1}{d_{2i}}|\widetilde{\xi}_{il}|^2\left[\sum_{k=0}^{l}|\sigma_{ik}|\Psi_{ik}(|(\xi_{i1},\ldots,\xi_{ik})|)\right]^2 + d_{2i}|\omega_i|^2, \tag{79}$$

$$2\widetilde{\xi}_{il}F_{il}(t,z,\xi_{i1},\ldots,\xi_{il}) \leqslant 2|\widetilde{\xi}_{il}|\left(\sum_{k=0}^{l}|\sigma_{ik}|\sum_{j=1}^{N}a_{ikj}(|(\xi_{i1},\ldots,\xi_{ik})|)\varphi_{ikj}(|z_j|)\right)$$
$$\leqslant \widetilde{\xi}_{il}^2\sum_{k=0}^{l}\sum_{j=1}^{N}\sigma_{ik}^2\delta_{jk}a_{ikj}^2(|(\xi_{i1},\ldots,\xi_{ik})|) + \sum_{k=0}^{l}\sum_{j=1}^{N}\delta_{jk}^{-1}\varphi_{ikj}^2(|z_j|), \tag{80}$$

$$2\widetilde{\xi}_{il}G_{il}(t,z,\xi_{i1},\ldots,\xi_{il})\omega_i \leqslant 2|\widetilde{\xi}_{il}|\left(\sum_{k=0}^{l}|\sigma_{ik}|\sum_{j=1}^{N}b_{ikj}(|(\xi_{i1},\ldots,\xi_{ik})|)\varphi_{ikj}(|z_j|)\right)\omega_i$$
$$\leqslant \frac{1}{4\tau_i^2}\widetilde{\xi}_{il}^4\sum_{k=1}^{l}\sum_{j=1}^{N}\sigma_{ik}^4\varrho_{jk}b_{ikj}^4(|(\xi_{i1},\ldots,\xi_{ik})|) + \sum_{k=0}^{l}\sum_{j=1}^{N}\varrho_{jk}^{-1}\Phi_{ikj}^4(|z_j|)$$
$$+(l+1)N\tau_i|\omega_i|^2. \tag{81}$$

So we choose virtual controller $\xi_{i,l+1}=\xi_{i,l+1}^*$ as

$$\xi_{i,l+1}^* = -\frac{1}{2}\left\{c_{il}\widetilde{\xi}_{il} + 2\Xi_{il} + \widetilde{\xi}_{il}\widetilde{\Delta}_{il}(z_i,\widetilde{\xi}_{i1},\ldots,\widetilde{\xi}_{il}) + \frac{1}{d_{2i}}\widetilde{\xi}_{il}\left[\sum_{k=0}^{l}|\sigma_{ik}|\Psi_{ik}(|(\xi_{i1},\ldots,\xi_{ik})|)\right]^2\right.$$
$$\left. + \widetilde{\xi}_{il}\sum_{k=0}^{l}\sum_{j=1}^{N}\sigma_{ik}^2\delta_{jk}a_{ikj}^2(\xi_{i1},\ldots,\xi_{ik}) + \frac{1}{4\tau_i^2}\widetilde{\xi}_{il}^3\sum_{k=1}^{l}\sum_{j=1}^{N}\sigma_{ik}^4\varrho_{jk}b_{ikj}^4(\xi_{i1},\ldots,\xi_{ik})\right\}, \tag{82}$$

where c_{il} is a positive constant.

Substituting (82) into (69) and in view of bounds (78)–(81), we get

$$\dot{W}_{il} \leqslant -[\alpha_{i,l-1}(|z_i|) - d_{1i}|z_i|^2] - \left[\sum_{k=1}^{l-1}(C_{i,l-1,k} - d_{3i})\widetilde{\xi}_{ik}^2 + c_{il}\widetilde{\xi}_{il}^2\right]$$
$$+\sum_{j=1}^{N}\left[\beta_{i,l-1,j}(|z_j|) + \sum_{k=0}^{l}(\delta_{jk}^{-1}\varphi_{ikj}^2(|z_j|) + \varrho_{jk}^{-1}\Phi_{ikj}^4(|z_j|))\right] + [\gamma_{i,l-1} + (l+1)N\tau_i + d_{2i}]|\omega_i|^2. \tag{83}$$

With appropriate definitions of $\alpha_{il}, C_{ilk}, \beta_{ilj}$ and γ_{il}, we get (31) with $\imath = l$ which completes the proof of the Claim. □

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