Fusion Method For Physical Systems Based On Physical Laws

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2.3 Fuser for Non-Smooth Laws

3. Conclusions

Reference for Non-Smooth Laws:

1. N. S. V. Rao, D. B. Reister, J. Barhen, Information Fusion in Physical Systems Using Physical Laws, Symposium on Energy Engineering Sciences, 2000, Argonne, IL, http://saturn.epm.ornl.gov/ nrao

Physical System

System Parameters: $P(z) = (p_1, p_2, \ldots, p_n)$

Parameter: $p_i(z)$ is measured and/or estimated:

(i) a_i **measurements**: measured by an instrument

$$m_i(z) = \{m_{i,1}(z), m_{i,2}(z), \ldots, m_{i,a_i}(z)\}$$

(ii) b_i **estimators**: computational method based on measurements

$$e_i(z) = \{e_{i,1}(z), e_{i,2}(z), \ldots, e_{i,b_i}(z)\}$$

In a nutshell:

Every parameter is either measured or estimated

– It's real values may not be known

Fusion Problem

For each parameter p_i, obtain a fuser f_i such that
$f_i(m_i(z), e_i(z))$ is "close" to $p_i(z)$.

Fuser for all parameters: $f = (f_1, \ldots, f_n)$

Expected error of fused is

$$E(f) = \sum_z \int C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n},$$

where $C[.]$ is a cost function.

Example: Classifiers:
Parameter $p_i \in \{0, 1\}$ is the actual class,
Fuser outputs Boolean vectors: $f = (f_1, \ldots, f_n) \in \{0, 1\}^n$
Fuser Error is

$$\begin{aligned} &C[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))] \\ &\quad = [f_1(m_1(z), e_1(z)) \oplus p_1] + \ldots + [f_n(m_n(z), e_n(z)) \oplus p_n] \end{aligned}$$

– simply, the number of parameters misclassified by fuser.

<u>Best Fuser</u>

Choose f_i from class $\mathcal{F}_i$ such as linear space, neural network.

<u>Best Expected Fuser:</u>

$f^* \in \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ one with least $E(.)$.

<u>Present Formulation:</u> Error distributions are not known
Measurements are given

$$\{<(m_1(z), e_1(z)), \ldots, (m_n(z), e_n(z))>: z = 1, \ldots, s\}.$$

Question: can we compute a close approximation to f^* ?

Note: Similar problems have been solved
but this problem is not amenable to conventional approaches

Traditional Paradigm

Cost Function is Direct:
Typically, training data is given for p_i:
– can employ a cost function such as

$$C[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))]$$
$$= [f_1(m_1(z), e_1(z)) - p_1]^2 + \ldots + [f_n(m_n(z), e_n(z) - p_n)]^2$$

Method of Empirical Risk Minimization:
Minimize empirical error by $\hat{f}$

$$\hat{E}(f) = \sum_z C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right],$$

instead of

$$E(f) = \sum_z \int C\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n},$$

Result: $E(\hat{f})$ can be shown to be close to $E(f^*)$ either
(i) as sample size goes to infinity (Asymptotic consistency), or
(ii) with high probability $E(\hat{f}) \leq E(f^*) + \epsilon$.

Difficulty: Traditional paradigm is not applicable:
every parameter p_i is either measured or estimated
– No actual parameter values are known

Physical Laws

Physical Law: relates the actual parameters
Consider the form

$$L[p_1(z), p_2(z), \ldots p_n(z)] = 0$$

Monotonicity Condition:
for any y_1, y_2, $|y_1| \leq |y_2|$, we have

$$\begin{aligned} &|L[p_1(z), \ldots, p_i(z) + y_1, \ldots, p_n(z)]| \\ &\quad \leq |L[p_1(z), \ldots, p_i(z) + y_2, \ldots, p_n(z)]|. \end{aligned}$$

Informally, accurate estimators have lesser or equal error compared to less-accurate estimators.

Note: we only consider time-invariant or position-invariant laws
$L[.]$ itself does not dependent on z

Example

Scenario: Known mass m subjected to a constant force f in a friction-free environment.

Physical Law: in this case is $f = ma$, and employ

$$L[f, a] = (f - ma)^2 = 0$$

Sensors: given
one force sensor, and two acceleration sensor.

Force measurements: $m_{1,1}(z) = f + \epsilon$, for some ϵ,
Acceleration measurements:
$m_{2,1} = a + \delta$
$m_{2,2} = 0.7a$
where δ is a small normally distributed error.

Physical Laws are not Exactly Satisfied:

$$L[m_{1,1}, m_{2,1}] = (\epsilon - m\delta)^2$$

$$L[m_{1,1}, m_{2,2}] = (\epsilon + 0.3ma)^2$$

Physical Law Violations:

For each parameter,
– choose a single estimator or measurement $\hat{p}_i$:
closeness of $L[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)]$ to 0
determines how closely the law is satisfied.

Observation:
Violation of physical law can be considered an error measure

Best Set of Measurements/Estimators

Basic Set: S

For each parameter p_i, choose one measurement or estimator $\hat{p}_i$: The error due to S is

$$\hat{E}(S) = \sum_z L[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)]$$

Random error of S:

$\hat{S}$: has least random error:

$$\hat{E}(\hat{S}) = \min_S \hat{E}(S).$$

Expected error of S:

$$E(S) = \sum_z \int L\left[\hat{p}_1(z), \hat{p}_2(z), \ldots, \hat{p}_n(z)\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n}$$

Best Sensor Set:

S^* has least expected error: $E(S^*) = \min_S E(S^*)$.

– among $\prod_{i=1}^{n}(a_i + b_i)$ possible basic sets

Example – Cntd.

Physical Laws not Exactly Satisfied:

$$L[m_{1,1}, m_{2,1}] = (\epsilon - m\delta)^2$$

$$L[m_{1,1}, m_{2,2}] = (\epsilon + 0.3ma)^2$$

Condition: $a > 0$, and $\epsilon \geq m\delta$; $|\delta| \leq |0.3a|$,
we have

$$L[m_{1,1}, m_{2,1}] \leq L[m_{1,1}, m_{2,2}],$$

As a result, for large values of a: better choice is $m_{2,1}$
otherwise $m_{2,2}$ is a better

Fusion Rule Estimation

Expected error of fuser f:

$$E(f) = \sum_z \int L\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right] dP_{m_1,\ldots,m_n|p_1,\ldots,p_n}$$

Optimal Expected Fuser: f^* has least expected error:
– $E(f)$ cannot be computed if the error distributions are not known,
– hence f^* is not computable.

Solution: compute $\hat{f}$ that minimizes the empirical cost

$$\hat{E}(f) = \sum_z L\left[f_1(m_1(z), e_1(z)), \ldots, f_n(m_n(z), e_n(z))\right],$$

based on of measurements

$$\{<(m_1(z), e_1(z)), \ldots, (m_n(z), e_n(z))>: z = 1, \ldots, s\}.$$

Summary of Performance:
We provide methods to ensure $E(f^*) \leq E(S^*)$,

$$E(\hat{f}) < E(\hat{S}),$$

– with a specified probability
– based entirely on the measurements, and
– without any knowledge of underlying distributions.

Isolation Property

Definition: Fuser class

$$\mathcal{F}_i = \{f_i(y) : \Re^{a_i+b_i} \mapsto \Re\},$$

for $y = (y_1, \ldots, y_{(a_i+b_i)})$, has the *isolation property* if it contains the function $\tau_j(y) = y_j$ for all $j = 1, 2, \ldots, (a_i + b_i)$.

Consequences of Isolation Property:
If each $\mathcal{F}_i$ satisfies the isolation property, we have

$$E(f^*) \leq E(S^*)$$

and

$$\hat{E}(\hat{f}) \leq \hat{E}(\hat{S}).$$

Examples:
1. Linear combinations
2. Linearized feedforward networks

Lipschitz Property

For any $g : [-A, A]^d \mapsto \Re$, let

$$\| g(r) \|_\infty = \sup_{r \in [-A,A]^d} |g(r)|.$$

Definition:
A $g(y) : [-A, A]^d \mapsto \Re^a$ is *Lipschitz* with constant k_g if for all $y_1, y_2 \in [-A, A]^d$, we have

$$\| g(y_1) - g(y_2) \|_\infty \leq k_g \| y_1 - y_2 \|_\infty .$$

Examples:
1. sigmoid neural networks are Lipschitz with the constant specified by the parameters of the network;
2. linear combination are Lipschitz

Example - Cntd

Fuser Class:: $f_i(y) = w_1 y_1 + \ldots + w_k y_k$, for $w_i \in [-W, W]$
– isolation property is trivially satisfied
– Lipschitz constant of f_i is W.

Linear Fuser

$$f_2(m_{2,1}, m_{2,2}) = w_1 m_{2,1} + w_2 m_{2,2}.$$

For the choice $w_1 = 0.3$ and $w_2 = 1$, we have $f_2(m_{2,1}, m_{2,2}) = a + 0.3\delta$, and

$$L[m_{1,1}, f_2(m_{2,1}, m_{2,2})] = (\epsilon - 0.3m\delta)^2,$$

– always smaller than $L[m_{1,1}, m_{2,1}]$, and
– is smaller than $L[m_{1,1}, m_{2,2}]$ for small $\delta \leq a$ with probability one

Asymptotic Convergence Result
Smooth Laws and Fusers

Physical Law: is Lipschitz

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: Lipschitz

General Result: Irrespective of the sensor distributions

$$E(\hat{f}) \to E(f^*),$$

as $s \to \infty$.

Informal Statement: error of $\hat{f}$ approaches optimal error

Furthermore:
If $\mathcal{F}_i$ satisfies the isolation property for all i, we have

$$E(\hat{f}) \to E(f^*) \leq E(S^*)$$

as $s \to \infty$.

Informal Statement: $\hat{f}$ approaches f^* which can have much smaller error than S^*.

<u>Finite Sample Result</u>
<u>Smooth Laws and Fusers</u>

Physical Law: Lipschitz

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: Lipschitz

<u>General Result:</u> Given sufficiently large sample,
we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon\right] \leq \delta,$$

irrespective of the sensor distributions.

Informally, with probability $1 - \delta$, the cost of the sample-based solution is within ϵ of the lowest achievable cost

Comments on Result
Smooth Laws and Fusers

Distribution-free finite sample result:

– depends only on measurements and

– does not depend on the sensor distributions.

Best possible result:

– stronger results such as showing $\delta = 0$ is not possible,

since f^* depends on a distribution (which is not finite-dimensional) and $\hat{f}$ depends on a *finite* sample.

Analytically justifies the method even for small sample sizes

Comments on Conditions
Smooth Laws and Fusers

Physical Law:

– Smoothness conditions are reasonable

– Lipschitz condition is satisfied for a number of physical laws, although not always guaranteed.

Fuser Class:

Isolation and Lipschitz properties are satisfied in

– linear combinations with bounded coefficients and

– piecewise linear feedforward networks

Finite Sample Result – Better than Best Sensor set
Smooth Laws and Fusers

Under isolation property for all $\mathcal{F}_i$

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$
$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally speaking, error of the computed fuser $\hat{f}$
(i) is not likely to be much higher than that of the best basic set, and
(ii) could be much smaller.

Theorem

Theorem 1 *Consider that the physical law is Lipschitz with constant k_L, and parameters, estimators and measurements are bounded such that $p \in [-C, C]^n$, $m_i \in [-A, A]^{a_i}$, and $e_i \in [-B, B]^{b_i}$. Let each fuser class $\mathcal{F}_i$ be Lipschitz with constant k_{f_i}. Let*

$$d = \sum_{i=1}^{n} (a_i + b_i)$$

and

$$k = k_L \max(k_{f_1}, k_{f_2}, \ldots, k_{f_n}).$$

Then given a sample size

$$s = \frac{512k(A+B)}{\epsilon^2} \left[d \ln \left(\frac{32k(A+B)}{\epsilon} \right) + \ln(8/\delta) \right],$$

we have

$$\mathbf{P} \left[E(\hat{f}) - E(f^*) > \epsilon \right] \leq \delta,$$

irrespective of the sensor distributions. Furthermore, $E(\hat{f}) \to E(f^*)$, as $s \to \infty$.

Corollary

Corollary 1 *Let $\mathcal{F}_i$ satisfy the isolation property for all $i = 1, 2, \ldots, n$. Under the same conditions as Theorem 1, we have following conditions satisfied.*

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$

$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Bounded Variation Property

One-dimensional function $h : [-A, A] \mapsto \Re$.
Partition of $[-A, A]$: Set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$-A = x_0 < x_1 < \ldots < x_n = A$$

$\mathcal{P}[-A, A]$: all possible partitions

Bounded-Variation: $g : [-A, A] \mapsto \Re$ is of bounded variation:
if there exists M such that for any $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$\sum(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M.$$

Multiple Dimensions:
A function $g : [-A, A]^d \mapsto \Re$ is of bounded variation if it is so in each of its input variable for every value of the other input variables.

Useful facts:
(i) not all continuous functions are of bounded variation,
e.g. $g(x) = x \cos(\pi/(2x))$ for $x \neq 0$ and $g(0) = 0$;
(ii) differentiable functions on compact domains are of bounded variation;
(iii) absolutely continuous functions, which include Lipschitz functions, are of bounded variation.

<u>Example</u>

Scenario: of H_2O heated in a container:
p_1: temperature
$p_2 \in \{0, 1\}$: state, $\quad p_2 = 0$: liquid
$\quad p_2 = 1$: steam

T_0: boiling temperature under this condition.

Physical law: : $p_2 = 0$ if $p_1 < T_0$ and $p_2 = 1$ otherwise.

$$L[p_1, p_2] = p_2 1_{\{p_1 < T_0\}} + (p_2 - 1) 1_{\{p_1 \geq T_0\}} = 0,$$

where the indicator function 1_C is:
1 if condition C is true
0 if condition C is false
– Not Lipschitz in p_1

<u>Motivation:</u> For non-smooth physical laws
To address the cases typified by the above $L[.]$
we consider the class of functions with bounded variation

<u>Pseudo-Dimension</u>

Fuser Class: With finite pseudo-dimension

For $\mathcal{G} = \{g : X \mapsto \Re\}$ and $S = \{x_1, x_2, \ldots, x_m\} \subseteq X$:

<u>Pseudo-shattering:</u> S is pseudo-shattered by F
if there are real numbers r_1, r_2, ..., r_m
such that for each $b \in \{0,1\}^m$ there is a function g_0 in $\mathcal{G}$ with

$$\mathrm{sgn}(f_b(x_i) - r_i) = b_i$$

for $1 \leq i \leq m$.

<u>Pseudo-Dimension:</u> $\mathcal{G}$ has pseudo-dimension d: maximum cardinality of a subset S of X that is pseudo-shattered by $\mathcal{G}$

<u>Examples:</u> Function classes with known pseudo-dimension
1. Feedforward sigmoidal neural networks
2. Vector spaces

Asymptotic Convergence Result

Physical Law: bounded variation

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: bounded pseudo-dimension

General Result: Irrespective of the sensor distributions

$$E(\hat{f}) \to E(f^*),$$

as $s \to \infty$.

Informal Statement: error of $\hat{f}$ approaches optimal error

Furthermore:
If $\mathcal{F}_i$ satisfies the isolation property for all i, we have

$$E(\hat{f}) \to E(f^*) \leq E(S^*)$$

as $s \to \infty$.

Informal Statement: $\hat{f}$ approaches f^* which can have much smaller error than S^*.

Finite Sample Result

Physical Law: Bounded Variation
– includes Lipschitz laws as subclass

Boundedness: parameters, estimators and measurements are bounded

Fuser Class: finite psuedo-dimension

General Result: Given sufficiently large sample,
we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon\right] \leq \delta,$$

irrespective of the sensor distributions.

Informally, with probability $1 - \delta$, the cost of the sample-based solution is within ϵ of the lowest achievable cost

Comments on Result

Distribution-free finite sample result:

– depends only on measurements and

– does not depend on the sensor distributions.

Best possible result:

– stronger results such as showing $\delta = 0$ is not possible,

since f^* depends on a distribution (which is not finite-dimensional) and $\hat{f}$ depends on a *finite* sample.

Analytically justifies the method even for small sample sizes

Comments on Conditions

Physical Law:

– Allows for discontinuities
– Allows for discrete-valued parameters

Fuser Class:

Isolation and pseudo-dimension properties are satisfied in
– linear combinations with bounded coefficients and
– piecewise linear feedforward networks

Finite Sample Result – Better than Best Sensor set

Under isolation property for all $\mathcal{F}_i$

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta$$

$$\mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally speaking, error of the computed fuser $\hat{f}$
(i) is not likely to be much higher than that of the best basic set, and
(ii) could be much smaller.

Theorem

Theorem 2 *Consider that the physical law is of bounded variation such that* $|L(p)| \leq M_L$ *for all* p*. Let parameters, estimators and measurements are bounded. Let each fuser class* $\mathcal{F}_i$ *have finite pseudo-dimension* d_i*, and each fuser function* g *be bounded such that* $|g(.)| \leq M$ *for all parameters. Let* $d = \sum_{i=1}^{n} d_i$*. Then given a sample of size*

$$s = \frac{256 M_L^2}{\epsilon^2} \left[4d \ln\left(\frac{128 e M}{\epsilon}\right) + (n+1)\ln(4/\delta) \right],$$

we have

$$\mathbf{P}\left[E(\hat{f}) - E(f^*) > \epsilon \right] \leq \delta,$$

irrespective of the sensor distributions.
Furthermore, $E(\hat{f}) \to E(f^*)$*, as* $s \to \infty$*.*

Corollary

Corollary 2 *Let $\mathcal{F}_i$ satisfy the isolation property for all $i = 1, 2, \ldots, n$. Under the same conditions as Theorem 2, we have following conditions satisfied.*

$$\mathbf{P}\left[E(\hat{f}) - E(S^*) > \epsilon\right] \leq \delta \quad \text{and} \quad \mathbf{P}\left[E(\hat{f}) - E(\hat{S}) > \epsilon\right] \leq \delta.$$

Informally, error of the fuser $\hat{f}$ is not likely to be much higher than that of the best basic set, and could be much smaller.

Methane Hydrates Problem

Gas hydrates: crystalline substances composed of water and gas, in which gas molecules are contained in cage-like lattices formed by solid water.

Gas hydrates are present in permafrost regions and beneath the sea in sediment of outer continental margins.

Alternative to conventional gas as a fuel:
– methane hydrates
– expected to be available in volumes exceeding known estimates for conventional gas reserves.

Challenging Problem: predict prescence of hydrates using measurements collected at wells

Methane Hydrates Problem - Measurements

Measurements:
At each well, using a suite of sensors
– density,
– neutron porosity,
– acoustic transit-time,
– electric resistivity
collected at various depths in the well

Our focus: estimation of the *porosity* at various depths.

Data:
3045 sets of measurements each collected at different depths in a single well.

Methods to estimate porosity: based on
– neutron measurements ($\hat{\phi}_1$),
– density measurements ($\hat{\phi}_2$),
– fluid velocity equation ($\hat{\phi}_3$),
– acoustic travel time based on S-wave ($\hat{\phi}_4$),
– time-average equation based on P-wave ($\hat{\phi}_5$),
– Wood's equation ($\hat{\phi}_6$)

Methane Hydrates Problem - Physical Law

Well-established physical law: relates:

ϕ: porosity
ρ: density
ψ: hydrate concentration

$$L[\phi, \psi, \rho] = (\phi[\rho_m - (1-\psi)\rho_w + \psi\rho_h] - \rho + \rho_m)^2 = 0,$$

where ρ_m, ρ_w, and ρ_h are known constants.

We employ:
– one measurement for density $\hat{\rho}$
– a single estimator $\hat{\psi}$ for the hydrate concentration using the Archie's equation – six estimates of porosity ϕ

Problem Addressed:
– we assume that the estimators $\hat{\rho}$ and $\hat{\psi}$ are reasonably accurate, and
– objective is to fuse the porosity estimates.

Methane Hydrates Problem - Estimators

Error due to a set of estimators $\hat{\phi}$, $\hat{\psi}$, and $\hat{\rho}$ is given by

$$\begin{aligned}\hat{E}(\hat{\phi},\hat{\psi},\hat{\rho}) &= \sum_{z=1}^{3045} L[\hat{\phi},\hat{\psi},\hat{\rho}] \\ &= \left(\hat{\phi}(z)[\rho_m - (1-\hat{\psi}(z))\rho_w + \hat{\psi}(z)\rho_h] - \hat{\rho}(z) + \rho_m\right)^2 .\end{aligned}$$

Error estimates for individual estimators:

estimator	$\hat{E}(.)$
$\hat{\phi}_1$	1.454920
$\hat{\phi}_2$	1.407088
$\hat{\phi}_3$	1.400968
$\hat{\phi}_4$	1.400967
$\hat{\phi}_5$	1.728479
$\hat{\phi}_6$	1.410965
fused estimate	0.079374

In terms of $\hat{E}(.)$, the best estimator for porosity is $\hat{\phi}_4$.

Fuser

Fuser based on linear combination

$$\hat{\phi}_F = w_7 + \sum_{i=1}^{6} w_i \hat{\phi}_i,$$

where $(w_1, \ldots, w_7) \in \Re^7$ is the weight vector.

Computation: Weight vector to minimize the error

$$\hat{E}(\hat{\phi}_F) \;=\; \sum_{z=1}^{3045} L\left[w_7 + \sum_{i=1}^{6} w_i \hat{\phi}_i(z), \hat{\psi}(z), \hat{\rho}(z)\right].$$

Solution:

w_1	1.198060
w_2	2.871261
w_3	-0.264511
w_4	-7.243817
w_5	2.254488
w_6	-6.330654
w_7	2.458406

Results

– Error achieved by $\hat{\phi}_F$ is about 20 times better than best estimator $\hat{\phi}_4$.

– More dramatic performance of the fuser was observed when smaller subsets of well log data were utilized.

Perspective:

Since the magnitude of the error depends critically on the exact form of $L[.]$, the improvement must be interpreted in the context,

Computation:

The computation is performed using the netlib library routines for solving the least square problems.

Conclusions

- New Paradigm for Sensor Fusion:
  - Physical systems - physical laws can be used for fusion
  - Each parameter is sensed and/or estimated by different sensors and methods
  - Provable performance - finite sample and asymptotic guarantees
  - Fused system better than best set
  - Physical laws considered - smooth and non-smooth
- Work in progress
  - Projective fusers — better than best subset of sensors ?
    — beyond isolation property.
  - Time-varying physical laws - can they be used for fusion ?
  - More general physical laws and fusion rules
  - Applications - methane hydrates, ballistic objects.