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Abstract

A small perturbation at the interface of two immiscible liquids under perpendicularly applied electric fields can drive the system away from a basic state. This inability of a system to sustain its state is called *instability* and has been the subject of numerous studies of theoretical and practical importance. The classical linear stability analysis hinges on the assumption that a small perturbation can be decomposed into Fourier modes and that the evolution of the perturbation can be investigated by superposing the evolution of each individual mode. Since this framework is akin to a wave propagation analysis, it is not surprising that a dispersion equation is at the heart of linear stability analysis. This equation relates the complex wave frequency to the real wave number in a nontrivial way. The prime goal of stability analysis is to explore the values of frequencies that represent the boundary between stable and unstable states by solving the dispersion relation. Because this equation is complex, standard methods of solution for real algebraic equations cannot be applied without change. Therefore, we report on the necessary manipulations for numerically solving the dispersion equation in the form of two coupled real algebraic equations. Applications of the results to a system consisting of water, propanol, and toluene are examined.

1. Introduction

Stability of fluid interfaces under electric fields has been the subject of numerous investigations. Taylor and McEwan (1965) first studied the stability of the horizontal interface subjected to an initially perpendicular uniform electric field. They observed that the fluid interface located between two flat electrodes, kept at a suitable voltage difference, rose uniformly (Fig. 1) while the section of the interface not directly under the electrodes dropped. Beyond a certain voltage limit, an instability prevented the interface from retaining its original shape. The interface became agitated in light of the onset of a multitude of conical tips. Taylor and McEwan discussed an analytical model to predict the onset of the instability (critical voltage). However, the model neglected viscous and

interfacial electric shear forces. Despite these restricting assumptions, adequate agreement between theory and contrived experiments was achieved.

Melcher and Smith (1968) recognized that disparities between stability analysis and general experiments could be traced to the existence of shear stresses at the interface, owing to finite charge relaxation time. Therefore, their comprehensive work applied to generic cases of interfaces under shear stress. A dispersion relation resulting from linear stability theory was derived and studied for some key limiting cases, namely, zero shear-stress dynamics, infinite relaxation time limit for liquid-gas interfaces, and instantaneous relaxation limit. Melcher and Smith also conducted experiments to validate their analysis within the appropriate limits.

Dong et al. (2000) studied the interfacial behavior of a liquid-liquid interface under an applied electric field in an apparatus similar to that of Taylor and McEwan; the dimensions, however, were different. The cell had an inner height of 11.5 cm, an inner diameter of 4.8 cm, and a volume of 210 mL. Stainless steel mesh plates (4.4 cm in diameter) were used as electrodes and were placed parallel to the liquid-liquid interface. The lower electrode was connected to a high-voltage power supply, and the upper electrode was grounded. A positive- or negative-polarity DC voltage was applied by a DC power supply, which had a range from 0 to 30 kV, with a limit of 10 mA (Fig. 1). The liquid-liquid interface was formed by combining deionized water and isopropanol with toluene and isopropanol. As an electric field was applied, the interface became agitated. At first, the onset of instability was the same as that observed by Taylor and McEwan (and Melcher and Smith). However, as more voltage was applied, liquid columns formed. This phenomenon had been briefly reported by Terasawa (1983).

The stability analysis of the particular system used in Dong et al.'s experiments required a general solution of the dispersion relation presented by Melcher and Smith. Since solutions to the dispersion relation in its most general form are not available by means of tabulated functions, a numerical approach was adopted. Here we report on the algebra and calculus of complex variables necessary for obtaining systematic solutions via Newton's method.

The report is organized as follows. Section 2 explains why vertical displacement of the interface in Dong et al.'s experiments was not observed with the naked eye. Section 3 gives a background of dispersion relations and illustrates the relevance of purely dispersive modes. Section 3.2 provides an example of a Gaussian pulse propagating in different media, and Sect. 3.3 reviews Newton's method for the problem at hand. The results are collected in Sect. 4 (and in Tables 1 through 8), and the conclusions, in Sect. 5.

2. Interface Displacement

According to Taylor and McEwan (1965), the interface (Fig. 1) rose to a height determined by

$$\frac{eV_0^2}{2} = h^2 g \Delta r \left(h_0 - h\right) ,$$

where e is the electric permittivity, V_0 is the applied voltage, g is the gravity, and Δr is the density difference ($\Delta r = r_b - r_a$, where r_b is the mass density of the lower fluid and r_a is the mass density of the upper fluid). For the system under investigation, $\Delta r > 0$

(Dong et al. 2000, Table 1), and h and h_0 are positive quantities. In addition, $h_0 > h$ (Fig. 1). By rearranging the above equation, we have a third-degree polynomial,

$$h^3 - h_0 h^2 + \frac{eV_0^2}{2g\Delta r} = 0 \ .$$

Denoting $a_0 := \frac{eV_0^2}{2g\Delta r} > 0$, we solve for the three roots, h_1, h_2, and h_3 (Abramowitz 1964) of

$$h^3 - h_0 h^2 + a_0 = 0 \ ,$$

where

$$h_1 = s_1 + s_2 + \frac{h_0}{3},$$

$$h_2 = -\frac{1}{2}(s_1 + s_2) + \frac{h_0}{3} + \frac{i\sqrt{3}}{2}(s_1 - s_2),$$

$$h_3 = -\frac{1}{2}(s_1 + s_2) + \frac{h_0}{3} - \frac{i\sqrt{3}}{2}(s_1 - s_2),$$

$$q = -\frac{h_0^2}{9} \quad \text{, and} \quad r = \frac{2\,h_0^3 - 27a_0}{54},$$

$$s_1 = \left[r + \sqrt{q^3 + r^2}\right]^{\frac{1}{3}}, \quad \text{and}$$

$$s_2 = \left[r - \sqrt{q^3 + r^2}\right]^{\frac{1}{3}}.$$

We examined the interface displacement $h_0 - h$ for three values of a_0 shown in Table 1. These values reflect three levels of voltage, namely, 1, 1.5, and 2 kV, respectively, for system 3 of Dong et al. (Table 1) when the initial height h_0 is 4×10^{-2} m. In each case, the discriminant $q^3 + r^2$ is negative therefore the roots are real and distinct. The computed

values of h_2 were discarded since they were always negative, and the values of h_3 were unrealistically large (too close to h_0). We conclude that $h = h_1$ and present the values of the vertical interface displacement $h_0 - h$ in the right-most column of Table 1. Although the displacement increases with increasing a_0, the values predicted by theory for the Dong et al. systems are relatively small, <0.2 mm, and cannot be easily detected by the naked eye.

3. Dispersion Relation for Interfacial Perturbation Modes

The classical linear stability analysis decomposes a small arbitrary perturbation into independent Fourier modes $e^{i(kx-wt)}$, where i is the imaginary number, k is the wave number (real), x is the space coordinate, w is the time propagation frequency, and t is time. Therefore, stability analysis has the same framework as wave propagation theory. The next two sections are devoted to an introductory discussion of the so-called dispersion relation.

3.1. Relevance of $D(w, k) = 0$

The interaction between a medium and a perturbation determines the characteristics of the disturbance motion through the dispersion equation $D(w,k) = 0$, which is an implicit relation between the wave number k and the time propagation frequency w. This relation is readily obtained from the substitution of the Fourier mode $e^{i(kx-wt)}$ into the governing equation for the perturbation in the particular medium. Therefore, if the frequency is real (null imaginary part), the mode remains oscillatory

(i.e., it neither grows nor decays). However, if the frequency has a nonzero imaginary part, the mode will either grow exponentially (if the imaginary part is positive) or decay exponentially (if the imaginary part is negative). Oscillatory modes (real frequencies) are of particular relevance to stability analysis because they represent states of neutral and/or marginal stability. The collection of neutral states defines the dividing surface, in parameter space, between stable and unstable states. Finding the locus of this surface is the ultimate goal of stability analysis.

3.2. Example of Gaussian Pulse Dispersion

Consider the one-dimensional Gaussian pulse (formula) with unity amplitude propagating on the positive side of the x axis with phase speed $\frac{w}{k}$. The Fourier representation of this pulse is given by

$$\int_{-\infty}^{\infty} e^{\frac{-k^2}{4}} \; e^{i\,(kx-wt)} \, dk \, .$$

If a particular medium is nondispersive and nondissipative, all modes have the same phase speed and, therefore, $w = k$. This is the simplest possible dispersion relation. All modes travel with the same speed as indicated in Fig. 2. The evolution of the pulse in this case was computed from its Fourier representation by integrating via trapezoidal rule in the frequency space interval $(-10^4, 10^4)$ divided into 10^6 equal subintervals. Since all modes travel together, the shape of the pulse is not changed by the supporting medium.

However, if a medium is purely dispersive (frequency is real) with phase speed dependent on the wave number, say $w = k^3$, then modes with short wavelength (large

wave number) propagate faster than modes with long wavelength (small wave number). Figure 3 shows this effect where an initially Gaussian pulse quickly transforms into a train of waves as it propagates. The dispersive effect shown in the figure is a qualitatively accurate description of a pulse propagation on the surface of a deep pond caused by a falling stone. The outside crests disappear because they propagate with lower energy and new ones appear from behind.

3.3. Newton's Solution Method for $D(w,k) = 0$

Newton's method is a well-known iterative scheme for computing solutions of nonlinear real-valued algebraic equations (Dennis and Schnabel 1983). The simplest implementation of the method requires an initial guess for the values of the unknowns, the derivative of the algebraic equations with respect to the unknowns, and a residual tolerance for stopping the iterations.

Since w is a complex variable, $D(w,k) = 0$ implies that the real and imaginary parts of D must vanish. Therefore, the dispersion equation can be readily solved by Newton's method by decomposing

$$D(w,k) = u(x,y,k) + i\,v(x,y,k)$$

and forming the system

$$u(x,y,k) = 0\,,$$

$$v(x,y,k) = 0\,,$$

where x and y are the real and imaginary parts of w. Given the initial guess $\left(x^{(0)}, y^{(0)}\right)$ Newton's method then reduces to the iterative scheme

$$\begin{pmatrix} x^{(i)} \\ y^{(i)} \end{pmatrix} = \begin{pmatrix} x^{(i-1)} \\ y^{(i-1)} \end{pmatrix} - \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}^{-1} \begin{pmatrix} u^{(i-1)} \\ v^{(i-1)} \end{pmatrix}$$

for i = 1, 2, 3 . . . until $\left|x^{(i)} - x^{(i-1)}\right|$ and $\left|y^{(i)} - y^{(i-1)}\right|$ are less than a small tolerance.

4. Results

The particular dispersion relation employed in this study was derived by Melcher and Smith for a system of two horizontal immiscible fluid layers. Standard linear stability theory was used to perturb the equations of motion of a viscous incompressible fluid under an applied perpendicular electric field; the effects of surface tension and gravity were also taken into account. Electrical conductivity and permittivity of the fluids were included in the analysis so that the effect of charge relaxation could be investigated. Melcher and Smith did not perform a general analysis of their dispersion equation; however, they did examine particular limits, as mentioned in Sect. 1.

Here we show necessary manipulations of the dispersion equation for a general analysis via numerical methods. The dispersion relation was organized as follows:

$$D(w,k) = p_1 \cdot p_2 + p_3 \cdot p_4 = 0$$

$$p_1 = -(1 + t_1 + t_2 + t_3 + t_4)$$

$$p_2 = t_5 + t_6$$

$$p_3 = t_7 + t_8 + t_9$$

$$p_4 = t_{10} + t_{11} + t_{12}$$

where all values of t_i are nonlinear terms of w and k. To solve $D(w,k) = 0$, Newton's method (Sect. 3.3) requires the real and imaginary parts of D, namely, u and v,

respectively. These can be obtained by decomposing the individual terms t_i in their real and imaginary parts (Tables 2 and 3) so that

$$u = \mathrm{Re}(p_1)\ \mathrm{Re}(p_2) - \mathrm{Im}(p_1)\ \mathrm{Im}(p_2) + \mathrm{Re}(p_3)\ \mathrm{Re}(p_4) - \mathrm{Im}(p_3)\ \mathrm{Im}(p_4), \text{ and}$$

$$v = \mathrm{Re}(p_1)\ \mathrm{Im}(p_2) + \mathrm{Im}(p_1)\ \mathrm{Re}(p_2) + \mathrm{Re}(p_3)\ \mathrm{Im}(p_4) + \mathrm{Im}(p_3)\ \mathrm{Re}(p_4).$$

In addition, we have computed the derivative of all values of t_i with respect to x and y, where $w = x + i\,y$, so that the derivatives of u and v can be readily obtained as (see Tables 4 through 8)

$$\begin{aligned}\partial_\alpha u = {} & \partial_\alpha\mathrm{Re}(p_1)\ \mathrm{Re}(p_2) + \mathrm{Re}(p_1)\ \partial_\alpha\mathrm{Re}(p_2) - \partial_\alpha\mathrm{Im}(p_1)\ \mathrm{Im}(p_2) - \mathrm{Im}(p_1)\ \partial_\alpha\mathrm{Im}(p_2) + \\ & \partial_\alpha\mathrm{Re}(p_3)\ \mathrm{Re}(p_4) + \mathrm{Re}(p_3)\ \partial_\alpha\mathrm{Re}(p_4) - \partial_\alpha\mathrm{Im}(p_3)\ \mathrm{Im}(p_4) - \mathrm{Im}(p_3)\ \partial_\alpha\mathrm{Im}(p_4)\end{aligned}$$

$$\begin{aligned}\partial_\alpha v = {} & \partial_\alpha\mathrm{Re}(p_1)\ \mathrm{Im}(p_2) + \mathrm{Re}(p_1)\ \partial_\alpha\mathrm{Im}(p_2) + \partial_\alpha\mathrm{Im}(p_1)\ \mathrm{Re}(p_2) + \mathrm{Im}(p_1)\ \partial_\alpha\mathrm{Re}(p_2) + \\ & \partial_\alpha\mathrm{Re}(p_3)\ \mathrm{Im}(p_4) + \mathrm{Re}(p_3)\ \partial_\alpha\mathrm{Im}(p_4) + \partial_\alpha\mathrm{Im}(p_3)\ \mathrm{Re}(p_4) + \mathrm{Im}(p_3)\ \partial_\alpha\mathrm{Re}(p_4)\end{aligned}$$

where ∂_α denotes derivatives with respect to x and y. The derivatives of the real and imaginary parts of the p_i terms are obtained by differentiation of the t_i terms which we list in Tables 4 through 8.

5. Conclusions

This investigation reports on the complex-valued algebra and differentiation required for a general treatment of a dispersion equation obtained by Melcher and Smith in the stability analysis of a liquid-liquid system under a perpendicular electric field. This work has been used in Dong et al.'s experimental analysis, where Newton's method was applied to solve the dispersion relation in general limits.

We have also explained on theoretical grounds why in Dong et al.'s experiments a vertical displacement of the interface has not been observed by the naked eye. The

analysis follows Taylor and McEwan and predicts a vertical displacement < 0.2 mm for values of applied voltage up to 2 kV.

6. Acknowledgments

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8. Figures

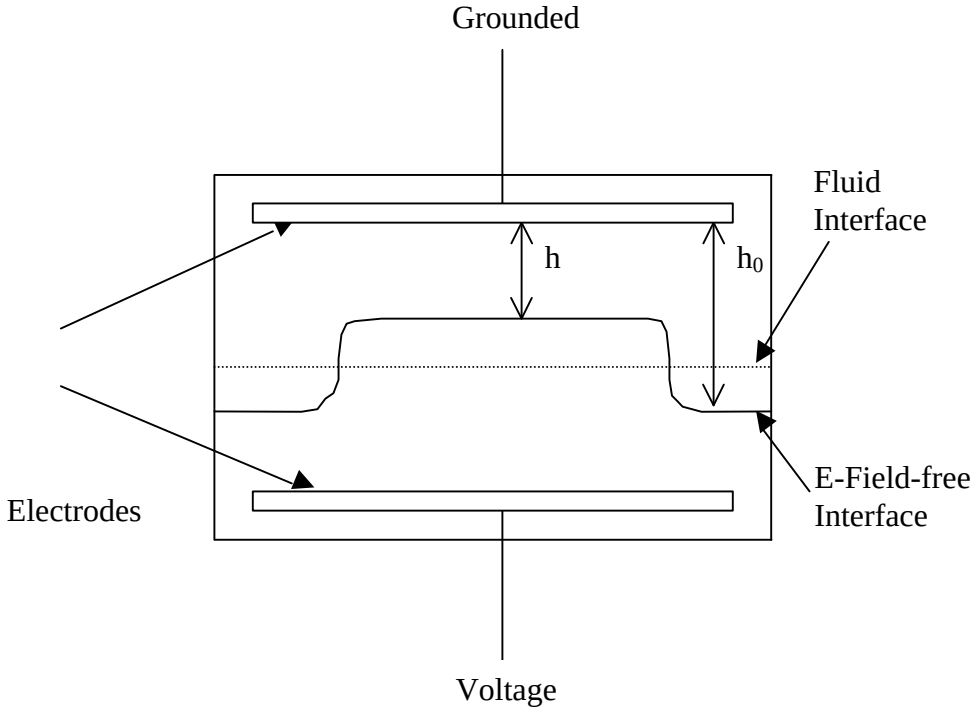

Fig. 1. Schematic of experimental apparatus (Dong et al. 2000).

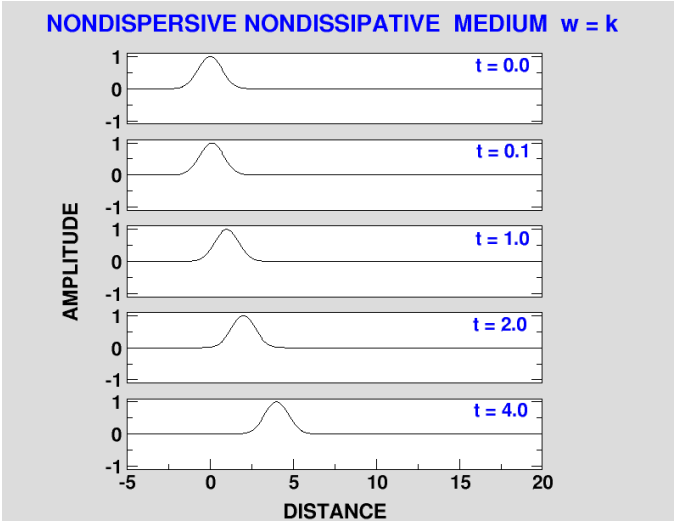

Fig. 2. Propagation of a single Gaussian pulse without change in shape at a constant speed.

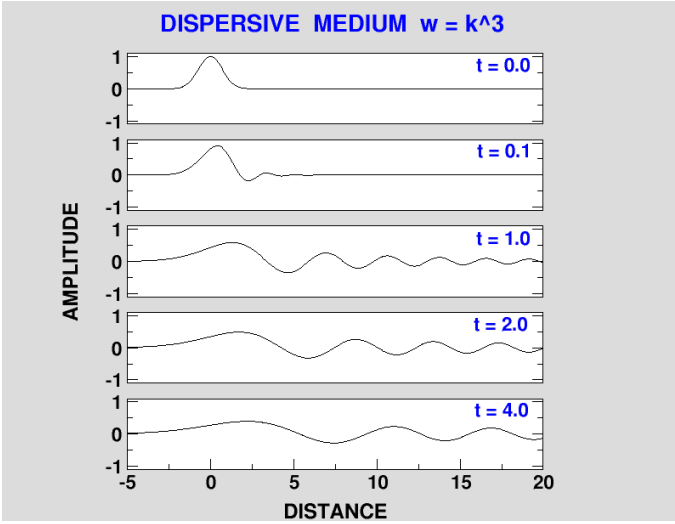

Fig. 3. Conversion of a single Gaussian pulse into a train of waves propagating with variable phase speed k^3.

Table 1. Interface displacement for three different levels of voltage (1.0, 1.5, and 2.0 kV; rows 1, 2 and 3 respectively) in system 3 of Dong et al. ($\varepsilon = 14.3 \times 8.854 \times 10^{-2}$ F/m, $g = 9.81$ m/s^2, and $\Delta\rho = 100$ kg/m^3).

Distance between interface and upper electrode	constant in section 2	interface displacement
h0 (m)	**a0 (m3)**	**h0 - h (mm)**
4.0×10^{-2}	6.45×10^{-8}	4.04×10^{-2}
4.0×10^{-2}	1.45×10^{-7}	9.10×10^{-2}
4.0×10^{-2}	2.58×10^{-7}	1.63×10^{-1}

Table 2. Values for t_i, separated into the real and imaginary parts. For q, f, and y see Table 3.

Term	Real	Imaginary
t_1	$gk(a_b - a_a)\dfrac{(x^2-y^2)}{(x^2+y^2)^2}$	$-2gk(a_b - a_a)\dfrac{xy}{(x^2+y^2)^2}$
t_2	$\dfrac{Tk^3}{(r_a+r_b)}\cdot\dfrac{x^2-y^2}{(x^2+y^2)^2}$	$\dfrac{-2Tk^3}{(r_a+r_b)}\cdot\dfrac{xy}{(x^2+y^2)^2}$
t_3	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)}\cdot\dfrac{(x^2-y^2)Re(f)+2xyIm(f)}{(x^2+y^2)^2}$	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)}\cdot\dfrac{(x^2-y^2)Im(f)-2xyRe(f)}{(x^2+y^2)^2}$
t_4	$2k^2(n_a a_a + n_b a_b)\dfrac{x}{(x^2+y^2)}$	$-2k^2(n_a a_a + n_b a_b)\dfrac{y}{x^2+y^2}$
t_5	$\left(\dfrac{a_a}{n_b}+\dfrac{a_b}{n_a}\right)\dfrac{(x^2-y^2)}{2k^2}$	$\left(\dfrac{a_a}{n_b}+\dfrac{a_b}{n_a}\right)\dfrac{xy}{k^2}$
t_6	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a+r_b)}\left[(x^2-y^2)Re(q)-2xyIm(q)\right]$	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a+r_b)}\left[2xyRe(q)+(x^2-y^2)Im(q)\right]$
t_7	$\dfrac{k[\![eE]\!]}{(r_a+r_b)}\cdot\dfrac{(x^2-y^2)Re(\Psi)+2xyIm(\Psi)}{(x^2+y^2)^2}$	$\dfrac{k[\![eE]\!]}{(r_a+r_b)}\cdot\dfrac{(x^2-y^2)Im(\Psi)-2xyRe(\Psi)}{(x^2+y^2)^2}$
t_8	$\dfrac{2k[\![m]\!]}{(r_a+r_b)}\cdot\dfrac{x}{x^2+y^2}$	$\dfrac{-2k[\![m]\!]}{(r_a+r_b)}\cdot\dfrac{y}{x^2+y^2}$
t_9	$2k(n_b a_b - n_a a_a)\dfrac{x}{x^2+y^2}$	$-2k(n_b a_b - n_a a_a)\dfrac{y}{x^2+y^2}$
t_{10}	$\left(\dfrac{a_b}{n_a}-\dfrac{a_a}{n_b}\right)\cdot\dfrac{(x^2-y^2)}{2k}$	$\left(\dfrac{a_b}{n_a}-\dfrac{a_a}{n_b}\right)\cdot\dfrac{xy}{k}$
t_{11}	$\dfrac{[\![m]\!]}{2kn_a n_b (r_a+r_b)}(x^2-y^2)$	$\dfrac{[\![m]\!]}{kn_a n_b (r_a+r_b)}xy$
t_{12}	$\dfrac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a+r_b)}\left[(x^2-y^2)Re(q)-2xyIm(q)\right]$	$\dfrac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a+r_b)}\left[2xyRe(q)+(x^2-y^2)Im(q)\right]$

Table 3. Values of q, f, and y

Re (q)	$\dfrac{[\![eE]\!]\,[s_a+s_b+x](e_a+e_b)}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
Im (q)	$\dfrac{-[\![eE]\!](e_a+e_b)\,y}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
Re (f)	$\dfrac{[x(e_a+e_b)+s_a+s_b][e_bE_bs_a-e_aE_as_b-[\![E]\!]e_ae_bx]-[\![E]\!](e_a+e_b)e_ae_by^2}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
Im (f)	$y\dfrac{(e_a+e_b)[e_aE_as_b-e_bE_bs_a]-(s_a+s_b)\,[\![E]\!]e_ae_b}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
Re (y)	$\dfrac{[x(e_a+e_b)+(s_a+s_b)][x(e_aE_a+e_bE_b)+E_as_a+E_bs_b]+(e_a+e_b)[(E_ae_a)+(E_be_b)]\,y^2}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
Im (y)	$y\dfrac{[\![E]\!](s_be_a-s_ae_b)}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$

Table 4. Derivatives of t_1, t_2, and t_3. For derivatives of q, f, and y, see Table 8.

$\partial_x \mathrm{Re}(t_1)$	$2gk(a_b - a_a)\,\dfrac{x(3y^2 - x^2)}{(x^2+y^2)^3}$
$\partial_y \mathrm{Re}(t_1)$	$2gk(a_b - a_a)\,\dfrac{y(y^2 - 3x^2)}{(x^2+y^2)^3}$
$\partial_x \mathrm{Im}(t_1)$	$2gk(a_b - a_a)\,\dfrac{y(3x^2 - y^2)}{(x^2+y^2)^3}$
$\partial_y \mathrm{Im}(t_1)$	$2gk(a_b - a_a)\,\dfrac{x(3y^2 - x^2)}{(x^2+y^2)^3}$
$\partial_x \mathrm{Re}(t_2)$	$\dfrac{2Tk^3}{(r_a + r_b)} \cdot \dfrac{x(3y^2 - x^2)}{(x^2+y^2)^3}$
$\partial_y \mathrm{Re}(t_2)$	$\dfrac{2Tk^3}{(r_a + r_b)} \cdot \dfrac{y(y^2 - 3x^2)}{(x^2+y^2)^3}$
$\partial_x \mathrm{Im}(t_2)$	$\dfrac{2Tk^3}{(r_a + r_b)} \cdot \dfrac{y(3x^2 - y^2)}{(x^2+y^2)^3}$
$\partial_y \mathrm{Im}(t_2)$	$\dfrac{2Tk^3}{(r_a + r_b)} \cdot \dfrac{x(3y^2 - x^2)}{(x^2+y^2)^3}$
$\partial_x \mathrm{Re}(t_3)$	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)} \cdot \dfrac{(x^2+y^2)\partial_x A - 4xA}{(x^2+y^2)^3}$ $\quad A = (x^2 - y^2)\,\mathrm{Re}(f) + 2xy\,\mathrm{Im}(f)$
$\partial_y \mathrm{Re}(t_3)$	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)} \cdot \dfrac{(x^2+y^2)\partial_y A - 4yA}{(x^2+y^2)^3}$ $\quad A = (x^2 - y^2)\,\mathrm{Re}(f) + 2xy\,\mathrm{Im}(f)$
$\partial_x \mathrm{Im}(t_3)$	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)} \cdot \dfrac{(x^2+y^2)\partial_x B - 4xB}{(x^2+y^2)^3}$ $\quad B = (x^2 - y^2)\,\mathrm{Im}(f) - 2xy\,\mathrm{Re}(f)$
$\partial_y \mathrm{Im}(t_3)$	$\dfrac{k^2[\![E]\!]}{(r_a+r_b)} \cdot \dfrac{(x^2+y^2)\partial_y B - 4yB}{(x^2+y^2)^3}$ $\quad B = (x^2 - y^2)\,\mathrm{Im}(f) - 2xy\,\mathrm{Re}(f)$

Table 5. Derivatives of t_4, t_5, and t_6. For derivatives of q, f, and y, see Table 8.

$\partial_x \mathrm{Re}(t_4)$	$2k^2(n_a a_a + n_b a_b)\dfrac{(y^2 - x^2)}{(x^2+y^2)^2}$
$\partial_y \mathrm{Re}(t_4)$	$-4k^2(n_a a_a + n_b a_b)\dfrac{xy}{(x^2+y^2)^2}$
$\partial_x \mathrm{Im}(t_4)$	$4k^2(n_a a_a + n_b a_b)\dfrac{xy}{(x^2+y^2)^2}$
$\partial_y \mathrm{Im}(t_4)$	$2k^2(n_a a_a + n_b a_b)\dfrac{(y^2 - x^2)}{(x^2+y^2)^2}$
$\partial_x \mathrm{Re}(t_5)$	$\left(\dfrac{a_a}{n_b} + \dfrac{a_b}{n_a}\right)\dfrac{x}{k^2}$
$\partial_y \mathrm{Re}(t_5)$	$-\left(\dfrac{a_a}{n_b} + \dfrac{a_b}{n_a}\right)\dfrac{y}{k^2}$
$\partial_x \mathrm{Im}(t_5)$	$\left(\dfrac{a_a}{n_b} + \dfrac{a_b}{n_a}\right)\dfrac{y}{k^2}$
$\partial_y \mathrm{Im}(t_5)$	$\left(\dfrac{a_a}{n_b} + \dfrac{a_b}{n_a}\right)\dfrac{x}{k^2}$
$\partial_x \mathrm{Re}(t_6)$	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a + r_b)}\left[(x^2-y^2)\partial_x \mathrm{Re}(q) + 2x\mathrm{Re}(q) - 2xy\partial_x \mathrm{Im}(q) - 2y\mathrm{Im}(q)\right]$
$\partial_y \mathrm{Re}(t_6)$	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a + r_b)}\left[(x^2-y^2)\partial_y \mathrm{Re}(q) - 2y\mathrm{Re}(q) - 2xy\partial_y \mathrm{Im}(q) - 2x\mathrm{Im}(q)\right]$
$\partial_x \mathrm{Im}(t_6)$	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a + r_b)}\left[2xy\partial_x \mathrm{Re}(q) + 2y\mathrm{Re}(q) + (x^2-y^2)\partial_x \mathrm{Im}(q) + 2x\mathrm{Im}(q)\right]$
$\partial_y \mathrm{Im}(t_6)$	$\dfrac{[\![eE]\!]}{4k^2 n_a n_b (r_a + r_b)}\left[2xy\partial_y \mathrm{Re}(q) + 2x\mathrm{Re}(q) + (x^2-y^2)\partial_y \mathrm{Im}(q) - 2y\mathrm{Im}(q)\right]$

Table 6. Derivatives of t_7, t_8, and t_9. For derivatives of q, f, and y, see Table 8.

$\partial_x \mathrm{Re}(t_7)$	$\frac{k[\![eE]\!]}{(r_a+r_b)} \cdot \frac{(x^2+y^2)\partial_x C - 4xC}{(x^2+y^2)^3}$	$C=(x^2-y^2)\mathrm{Re}(y)+2xy\,\mathrm{Im}(y)$
$\partial_y \mathrm{Re}(t_7)$	$\frac{k[\![eE]\!]}{(r_a+r_b)} \cdot \frac{(x^2+y^2)\partial_y C - 4yC}{(x^2+y^2)^3}$	$C=(x^2-y^2)\mathrm{Re}(y)+2xy\,\mathrm{Im}(y)$
$\partial_x \mathrm{Im}(t_7)$	$\frac{k[\![eE]\!]}{(r_a+r_b)} \cdot \frac{(x^2+y^2)\partial_x D - 4xD}{(x^2+y^2)^3}$	$D=(x^2-y^2)\mathrm{Im}(y)-2xy\,\mathrm{Re}(y)$
$\partial_y \mathrm{Im}(t_7)$	$\frac{k[\![eE]\!]}{(r_a+r_b)} \cdot \frac{(x^2+y^2)\partial_y D - 4yD}{(x^2+y^2)^3}$	$D=(x^2-y^2)\mathrm{Im}(y)-2xy\,\mathrm{Re}(y)$
$\partial_x \mathrm{Re}(t_8)$	$\frac{2k[\![m]\!]}{(r_a+r_b)} \cdot \frac{(y^2-x^2)}{(x^2+y^2)^2}$	
$\partial_y \mathrm{Re}(t_8)$	$\frac{-4k[\![m]\!]}{(r_a+r_b)} \cdot \frac{xy}{(x^2+y^2)^2}$	
$\partial_x \mathrm{Im}(t_8)$	$\frac{4k[\![m]\!]}{(r_a+r_b)} \cdot \frac{xy}{(x^2+y^2)^2}$	
$\partial_y \mathrm{Im}(t_8)$	$\frac{2k[\![m]\!]}{(r_a+r_b)} \cdot \frac{(y^2-x^2)}{(x^2+y^2)^2}$	
$\partial_x \mathrm{Re}(t_9)$	$2k(n_b a_b - n_a a_a)\,\frac{(y^2-x^2)}{(x^2+y^2)^2}$	
$\partial_y \mathrm{Re}(t_9)$	$-4k(n_b a_b - n_a a_a)\,\frac{xy}{(x^2+y^2)^2}$	
$\partial_x \mathrm{Im}(t_9)$	$4k(n_b a_b - n_a a_a)\,\frac{xy}{(x^2+y^2)^2}$	
$\partial_y \mathrm{Im}(t_9)$	$2k(n_b a_b - n_a a_a)\,\frac{(y^2-x^2)}{(x^2+y^2)^2}$	

Table 7. Derivatives of t_{10}, t_{11}, and t_{12}. For derivatives of q, f, and y, see Table 8.

$\partial_x \mathrm{Re}(t_{10})$	$\left(\frac{a_b}{n_a} - \frac{a_a}{n_b}\right)\frac{x}{k}$
$\partial_y \mathrm{Re}(t_{10})$	$-\left(\frac{a_b}{n_a} - \frac{a_a}{n_b}\right)\frac{y}{k}$
$\partial_x \mathrm{Im}(t_{10})$	$\left(\frac{a_b}{n_a} - \frac{a_a}{n_b}\right)\frac{y}{k}$
$\partial_y \mathrm{Im}(t_{10})$	$\left(\frac{a_b}{n_a} - \frac{a_a}{n_b}\right)\frac{x}{k}$
$\partial_x \mathrm{Re}(t_{11})$	$\frac{[[m]]}{kn_a n_b (r_a + r_b)}x$
$\partial_y \mathrm{Re}(t_{11})$	$\frac{-[[m]]}{kn_a n_b (r_a + r_b)}y$
$\partial_x \mathrm{Im}(t_{11})$	$\frac{[[m]]}{kn_a n_b (r_a + r_b)}y$
$\partial_y \mathrm{Im}(t_{11})$	$\frac{[[m]]}{kn_a n_b (r_a + r_b)}x$
$\partial_x \mathrm{Re}(t_{12})$	$\frac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a + r_b)}\left[(x^2 - y^2)\partial_x \mathrm{Re}(q) + 2x\,\mathrm{Re}(q) - 2xy\partial_x \mathrm{Im}(q) - 2y\,\mathrm{Im}(q)\right]$
$\partial_y \mathrm{Re}(t_{12})$	$\frac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a + r_b)}\left[(x^2 - y^2)\partial_y \mathrm{Re}(q) - 2y\,\mathrm{Re}(q) - 2xy\,\partial_y \mathrm{Im}(q) - 2x\,\mathrm{Im}(q)\right]$
$\partial_x \mathrm{Im}(t_{12})$	$\frac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a + r_b)}\left[2xy\partial_x \mathrm{Re}(q) + 2y\,\mathrm{Re}(q) + (x^2 - y^2)\partial_x \mathrm{Im}(q) + 2x\,\mathrm{Im}(q)\right]$
$\partial_y \mathrm{Im}(t_{12})$	$\frac{(e_a E_a + e_b E_b)}{4kn_a n_b (r_a + r_b)}\left[2xy\,\partial_y \mathrm{Re}(q) + 2x\,\mathrm{Re}(q) + (x^2 - y^2)\partial_y \mathrm{Im}(q) - 2y\,\mathrm{Im}(q)\right]$

Table 8. Derivatives of q, f, and y.

$\partial_x \operatorname{Re}(q)$	$\dfrac{[[eE]](e_a+e_b)-2\operatorname{Re}(q)\,(e_a+e_b)[x(e_a+e_b)+s_a+s_b]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Re}(q)$	$\dfrac{-2\operatorname{Re}(q)(e_a+e_b)^2y}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_x \operatorname{Im}(q)$	$\dfrac{-2\operatorname{Im}(q)(e_a+e_b)[x(e_a+e_b)+(s_a+s_b)]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Im}(q)$	$\dfrac{-(e_a+e_b)\left[[[eE]]-2\operatorname{Im}(q)(e_a+e_b)^2y\right]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_x \operatorname{Re}(f)$	$\dfrac{(e_a+e_b)\{e_bE_bs_a-e_aE_as_b-2e_ae_b[[E]]x-2[(e_a+e_b)x+s_a+s_b]\operatorname{Re}(q)\}-e_ae_b[[E]](s_a+s_b)}{[x(e_a+e_b)+s_a+s]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Re}(f)$	$\dfrac{-2y(e_a+e_b)[e_ae_b[[E]]+\operatorname{Re}(f)(e_a+e_b)]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_x \operatorname{Im}(f)$	$\dfrac{-2\operatorname{Im}(f)(e_a+e_b)[x(e_a+e_b)+s_a+s_b]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Im}(f)$	$\dfrac{(e_a+e_b)(e_aE_as_b-e_bE_bs_a)-e_ae_b[[E]](s_a+s_b)\;-2\operatorname{Im}(f)(e_a+e_b)^2y}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_x \operatorname{Re}(y)$	$\dfrac{(e_a+e_b)\{E_as_a+E_bs_b+2x(e_aE_a+e_bE_b)-2\operatorname{Re}(y)[x(e_a+e_b)+s_a+s_b]\}+(s_a+s_b)(e_aE_a+e_bE_b)}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Re}(y)$	$\dfrac{2y(e_a+e_b)[e_aE_a+e_bE_b-\operatorname{Re}(y)(e_a+e_b)]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_x \operatorname{Im}(y)$	$\dfrac{-2\operatorname{Im}(y)(e_a+e_b)[x(e_a+e_b)+s_a+s_b]}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$
$\partial_y \operatorname{Im}(y)$	$\dfrac{[[E]](s_be_a-e_bs_a)-2\operatorname{Im}(y)(e_a+e_b)^2y}{[x(e_a+e_b)+s_a+s_b]^2+(e_a+e_b)^2y^2}$