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On Some General Properties of Parabolic Conservation Equations

Lawrence Dresner

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On Some General Properties of Parabolic Conservation Equations

Lawrence Dresner

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To my grandchildren

..to explore, to guess, to test, to estimate, and to prove.

Lynn Arthur Steen

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Abstract

This report deals with certain general properties of partial differential equations of the form $S(c)c_t + q_z = Q(c)$, where t may be thought of as time, z as distance, c as an intensive quantity (e.g., temperature), and q its flux (e.g., heat flux), and where q depends on both c and c_z. Six topics are studied, namely:

§ maximum and minimum principles,

§ ordering of solutions,

§ invariance to stretching (affine) groups,

§ stability of steady states,

§ comparability of solutions, and

§ traveling wave solutions.

Illustrative examples are given from the field of nonlinear diffusion, applied superconductivity, and helium cryogenics.

1. Introduction

Conservation equations are partial differential equations of the form

$$c_t + q_z = 0 \tag{1}$$

where t may be thought of as time, z distance, c as an intensive quantity (e.g. temperature) and q its flux (e.g., heat flux). These identifications are not essential to what follows, of course, but serve the twofold purpose of establishing a useful nomenclature and suggesting an interpretation.

To treat conservation equations, one must postulate some connection between q and c. Lax[1] has studied the hyperbolic equations that arise when q is purely a function of c. In this paper, we study the parabolic equations that arise when q also depends on the space derivative of c, namely c_z. Typical of the equations that we have in mind are

(i) $c_t = c_{zz}$, the ordinary diffusion equation ($q = -c_z$);

(ii) $c_t = (c_z^{1/3})_z$, the so-called Gorter-Mellink diffusion equation, which describes heat transport in turbulent superfluid helium ($q = -c_z^{1/3}$);

(iii) $c_t = (c^n)_{zz}$, which occurs in hydrology, plasma physics, and applied superconductivity and is often called the porous medium equation ($q = -nc^{n-1}c_z$).

The thrust in this paper is towards certain common properties of these equations that enable us to tell useful things about their solutions without actually solving them explicitly. The properties we shall be discussing are

(i) **Maximum and minimum principles.** A partial differential equation is said to obey a maximum (minimum) principle if the largest (smallest) value of its solutions in any region R of the (z,t)-plane lies on the boundary of R. In this paper, we shall include in the definition of maximum (minimum) principles the following addition: if we choose the region R to be a rectangle in the (z,t)-plane whose sides are parallel to the axes, then the largest (smallest) value of c

cannot occur on the upper side parallel to the z-axis (side AB in Fig. 2). Thus the largest (smallest) value of c is either an initial value or a boundary value.

As we shall see later, as a consequence of the second law of thermodynamics, equations of the form (1) that can be interpreted as representing real processes necessarily obey maximum and minimum principles in this extended sense.

(ii) **Ordering of solutions.** By *ordering* I mean that the difference of two solutions also obeys a maximum and a minimum principle. Such ordering has the useful consequence that if two solutions $c_1(z,t)$ and $c_2(z,t)$ obey the same boundary conditions on an interval $a \leq z \leq b$, i.e., if $c_1(a,t) = c_2(a,t)$ and $c_1(b,t) = c_2(b,t)$, but if $c_1(z,0) \geq c_2(z,0)$, then $c_1(z,t) \geq c_2(z,t)$ for all $t > 0$. We describe this latter situation by saying the solutions are *ordered according to their initial values.*

For linear equations, such as the ordinary diffusion equation, ordering is a simple consequence of the existence of maximum and minimum principles because the difference of two solutions obeys the same equation as the solutions themselves. For nonlinear equations, a separate investigation is necessary.

(iii) **Group invariance**. Some of the equations given as examples above are invariant to a one-parameter family of stretching (affine) groups of the form

$$c' = \lambda^{\alpha} c \tag{2a}$$

$$t' = \lambda^{\beta} t \qquad 0 < \lambda < \infty \tag{2b}$$

$$z' = \lambda z \tag{2c}$$

where λ labels transformations of a group and the parameters α and β, which obey a linear constraint

$$M\alpha + N\beta = L \tag{3}$$

label groups of the family. The coefficients M, N, and L are determined by the partial differential equations.

A transformation of the type (2), which leaves the partial differential equation unchanged (invariant), in general carries one solution into another. Among the solutions, there may be some that are carried into themselves by all the transformations of *one* group of the family, say, that labeled by particular values of the parameters α and β. It can be shown that such solutions must have the form

$$c = t^{\alpha/\beta}\, y(z/t^{1/\beta}) \tag{4}$$

where $y(x)$ is a function of $x = z/t^{1/\beta}$ still to be determined. Invariant solutions like that of Eq. (4) are called similarity solutions. The importance of similarity solutions stems from the fact that $y(x)$ is a function of one variable only. If we substitute Eq. (4) into the partial differential equation, we obtain an *ordinary* differential equation for $y(x)$.

I have shown[2] that this ordinary differential equation is invariant to the associated group

$$y' = \mu^{L/M} y \tag{5a}$$

$$0 < \mu < \infty$$

$$x' = \mu x \tag{5b}$$

Again, a transformation of the group (5) carries a solution $y(x)$ of the ordinary differential equation into another solution $y'(x')$. Among the solutions $y(x)$ there may be some that are carried into themselves by the transformations of the group (5). Such solutions have the form $y = Ax^{L/M}$, where A is a constant determined by the partial differential equation. If we substitute the values of y and x from Eq. (4) into this last expression we find

$$c/t^{\alpha/\beta} = A(z/t^{1/\beta})^{L/M} \tag{6a}$$

or in view of the constraint (3)

$$c = Az^{L/M} t^{-N/M} \tag{6b}$$

Since α and β no longer appear in Eq. (6b), this solution is invariant to *all* groups of the family. This can be proved directly using Eq. (2). The permissible values of A can be found by substituting Eq. (6) into the partial differential equation.

The importance of the totally invariant solution (6b) is that under certain broad circumstances, it gives the asymptotic form of the solutions for large z or small t.

Certain generalizations of Eq. (1) occur in practice. For example, when Eq. (1) describes heat diffusion (so that c really is the temperature), the first term is multiplied by $S(c)$, the heat capacity per unit volume. Another conservation equation in which a function of c multiplies the term c_t is (iv) $cc_t = c_{zz}$, which figures in the theory of thermal expulsion of a compressible fluid from a long, slender tube. Equations of this generalized type often share some or all of the three properties discussed above.

(iv) Stability of steady states. Another generalization of Eq. (1) arises when we add a source term $Q(c)$ to the right-hand side. When a source term is present, the partial differential equation sometimes has one or more steady states. Furthermore, as external parameters change, steady states may appear and disappear. Only steady states that are stable against perturbation can occur in the laboratory. Thus the stability of steady states becomes a fourth general property of interest.

As we shall see later on, under very broad conditions, equations like Eq. (1) possess a functional whose value for any solution either decreases with time or remains stationary. Such a functional plays a role much like that played by the gravitational potential of a ball rolling on a curved surface. Its critical points correspond to steady states: minima correspond to stable steady states, and maxima and saddle points correspond to unstable steady states. By studying the topological properties of this functional, we shall be able to draw some general conclusions about the stability of steady states.

(v) Comparability of solutions. It is possible to compare solutions of specimens of Eq.(1) [or its generalization Eq. (7)] that have different source terms $Q(c)$. (Here we use the word *compare* in the sense of Hardy, Littlewood, and Polya[3]: two functions are comparable over a domain if one is greater than or equal to the other everywhere in the domain.) We shall show later that if $c_1(z,t)$ and $c_2(z,t)$ obey the same initial conditions and the same boundary conditions on an interval $a \le z \le b$, i.e., if $c_1(z,0) = c_2(z,0)$, $c_1(a,t) = c_2(a,t)$ and $c_1(b,t) = c_2(b,t)$, and if $Q_1(c) \ge Q_2(c)$, then $c_1(z,t) \ge c_2(z,t)$ for all $t > 0$. Comparability of solutions is then a fifth general property of interest.

(vi) Traveling wave solutions. Occasionally Eq. (1) is invariant to the translation groups $z' = z + \lambda$ and $t' = t + \mu$, $-\infty < \lambda, \mu < \infty$. Solutions $c(z,t)$ invariant to both groups are simply constants, but we may look for solutions invariant to the one-parameter subgroup $z' = z - v\lambda$ and $t' = t + \lambda$, $-\infty < \lambda < \infty$, where v is a fixed constant. Such solutions have the form $c(z+vt)$ and are called traveling wave solutions. The form $c(z+vt)$ represents a wave traveling from right to left without change of shape when v is positive. The importance of traveling wave solutions is that they, like similarity solutions, may be calculated by solving an *ordinary* differential equation. The existence and nature of such solutions is a sixth area of general interest.

2. Maximum Principles

If we introduce the auxiliary variable $h = \int_0^c S(c')\,dc'$, we can deal directly with the first generalization of Eq. (1), which now takes the form

$$h_t + q_z = 0 \tag{7}$$

Let us now consider a region R of the (z,t)-plane bounded by a curve B and apply Green's theorem:

$$0 = \int\int_R (h_t + q_z)\,dz\,dt = \int_B (-h\,dz + q\,dt) \tag{8}$$

If the solution c has a maximum at a point P in the interior of R, P is surrounded by closed level curves of c, that is, closed curves on which c = constant. On any of these level curves C, c and h are constant. Hence the first integral on the extreme right-hand side of Eq. (8) is zero over C and

$$\int_C q\,dt = 0 \tag{9a}$$

Now since C is a level curve of c in the (z,t)-plane, the normal vector to C, (c_z, c_t), points inward towards P (see Fig. 1). Thus $c_z\,dt < 0$ at every point of C (except at the top and bottom, where $c_z\,dt = 0$ momentarily). Therefore, if q and c_z always have opposite algebraic signs,

$$\int_C q\,dt > 0 \tag{9b}$$

which contradicts Eq. (9a). Thus the assumption that c has a maximum at an interior point P in R cannot be true, and the largest value of c in R must lie on the boundary B.

Comment 1: In processes that obey the second law of thermodynamics, q and c_z always have opposite signs.

Comment 2: The conclusion is also true if q and c_z always have the same sign.

Comment 3: A similar proof shows that c obeys a minimum principle, that is, that the smallest value of c in R lies on the boundary B.

3. Maximum Principles (part 2)

Protter and Weinberger[4] have shown the following for the ordinary diffusion equation: If we choose the region R to be a rectangle OABC in the (z,t)-plane whose sides are parallel to the axes (see Fig. 2), then the largest value of c cannot occur on the interior of segment AB. In other words, the largest value of c is either an initial or a boundary value. The same is true of the conservation equations (1) and (7) if $S(c) > 0$.

Suppose the largest value of c in R did occur at a point Q in the interior of AB. In a sufficiently small neighborhood of Q, ST, c is increasing on the segment SQ and decreasing on the segment QT. Let the value of c be the same at S and T, and let C' be the level curve in R around Q for which c has the common value $c_S = c_T$. Now since c is greater on the segment SQT than it is on the curve C',

$$\int h\,dz < 0 \tag{10a}$$

where the integral is taken over the entire closed path C'-TQS. But then it follows from Eq. (8) that

$$\int q\,dt < 0 \tag{10b}$$

over the same closed path. Now if q and c_z have opposite signs, it again follows as it did in Fig. 1 that

$$\int q\,dt > 0 \tag{10c}$$

over the path C'-TQS, which contradicts Eq. (10b). Thus the assumption that the largest value lay at Q in the interior of AB must be false.

Comment 1: This proof only holds if q and c_z have opposite signs and if $S(c) > 0$.

Comment 2: A similar proof shows that the smallest value of c cannot occur at a point Q in the interior of AB.

In all four of the examples given in the introduction, q and c_z have opposite algebraic signs. In the fourth example, the thermal expulsion equation $cc_t = c_{zz}$, the proofs given apply only when $c > 0$ since $S(c) = c$. The same restriction can be applied to example (iii), where taking $c > 0$ is sufficient to ensure that q and c_z have opposite algebraic signs. In all of these cases, then, the largest and smallest values of c are initial or boundary values.

4. Ordering of Solutions

If we consider Eq. (7) written for two different solutions c_1 and c_2 we obtain

$$(h_1 - h_2)_t + (q_1 - q_2)_z = 0 \tag{11}$$

If

$$(h_1 - h_2)_z (q_1 - q_2) < 0 \tag{12}$$

then the two theorems just proved hold for the difference $h_1 - h_2$. Now if, for example, $c_1 = c_2$ on OA and BC in Fig. 2 but $c_1 \geq c_2$ on OC, then $h_1 = h_2$ on OA and BC, but $h_1 \geq h_2$ on OC. Then the smallest initial or boundary value of $h_1 - h_2$ is zero so that $h_1 - h_2 \geq 0$ everywhere in R. Then if $S(c) > 0$, $c_1 \geq c_2$ everywhere in R. So if Eq. (12) is fulfilled, the solutions are ordered according to their initial values.

In the case of the ordinary diffusion equation, $S = 1$, $h = c$, and $q = -c_z$. Therefore

$$(h_1 - h_2)_z (q_1 - q_2) = -[(c_1 - c_2)_z]^2 < 0 \tag{13}$$

so that solutions of the ordinary diffusion equation are ordered according to their initial values.

In the case of the Gorter-Mellink diffusion equation, $S = 1$, $h = c$, and $q = -c_z^{1/3}$ so that

$$\begin{aligned}(h_1 - h_2)_z (q_1 - q_2) &= -(q_1^3 - q_2^3)(q_1 - q_2) \\ &= -(q_1^2 + q_1 q_2 + q_2^2)(q_1 - q_2)^2 \\ &< 0\end{aligned} \tag{14}$$

since both factors on the right-hand side are always positive. Therefore the solutions of the Gorter-Mellink diffusion equation are also ordered according to their initial values.

In the case of example (iv), the thermal expulsion equation, $S = c$, $h = c^2/2$, and $q = -c_z$ so that

$$\begin{aligned}(h_1 - h_2)_z (q_1 - q_2) &= -(c_1q_1 - c_2q_2)(q_1 - q_2) \\ &= -(c_1q_1^2 - (c_1+c_2)q_1q_2 + c_2q_2^2) \qquad (15)\end{aligned}$$

Now, unfortunately, the quadratic form on the right-hand side of Eq. (15) has a positive discriminant, namely, $(c_1 - c_2)^2$, so that it is not always of one sign for all values of the ratio q_1/q_2. The roots of the quadratic form are $q_1/q_2 = 1$ and $q_1/q_2 = c_2/c_1$. Between these roots, the right-hand side of Eq. (15) is positive; beyond them it is negative (we restrict our considerations to solutions c that are > 0). So if we are to prove the ordering of the solutions of the partial differential equation, we must refine our analysis.

5. Ordering of Solutions (part 2)

To show the ordering of two solutions, it is necessary only to consider the difference of two infinitesimally close solutions c and $c + u$, where $u << c$. (No generality is lost by this restriction--see the note at the end of part 3.) To first order, the equation $cc_t = c_{zz}$ yields

$$uc_t + cu_t = u_{zz} \qquad (16)$$

To show, for example, that the solutions c are ordered according to their initial conditions, we must show that $u \geq 0$ in the rectangle OABC of Fig. 2 when $u = 0$ on OA and BC and $u \geq 0$ on OC. We begin by excluding the possibility that u has a negative minimum at a point P in the interior of the rectangle. Then we would have $u_{zz}(P) \geq 0$, $u_t(P) = 0$, and $u(P) < 0$. The idea is to show that these conditions are inconsistent with Eq. (16), but because we cannot be sure of the sign of c_t, this argument is not conclusive.

There is a standard trick (see Ref. 4) for overcoming this difficulty. Set $v = ue^{-\lambda t}$. Then v, too, is zero on OA and BC and ≥ 0 on OC. Now v obeys the equation

$$v(c_t + \lambda c) + cv_t = v_{zz} \qquad (17)$$

If we restrict consideration to positive solutions c, we can choose λ large enough to make $c_t + \lambda c$ positive everywhere in OABC. Then the conditions $v(P) < 0$, $v_t(P) = 0$, $v_{zz}(P) \geq 0$ for a negative minimum at P are inconsistent with Eq. (17).

Similarly, v cannot have a negative minimum at a point Q on the interior of segment AB. For at Q we would have $v(Q) < 0$ and $v_{zz}(Q) \geq 0$, which would require $v_t(Q) > 0$. But then there would be even smaller values of v in OABC just below Q, contrary to the hypothesis that its (negative) smallest value lay at Q.

Clearly this state of affairs requires that the smallest value of v in the rectangle OABC or on its boundary be zero, so that $v \geq 0$ and therefore $u \geq 0$ in OABC. This means then that positive solutions of $cc_t = c_{zz}$ are ordered according to their initial conditions.

The situation with respect to example (iii), the porous medium equation $c_t = (c^n)_{zz}$, is the same that as with respect to example (iv) just discussed: the quantity

$$(h_1 - h_2)_z (q_1 - q_2) = - n [c_{1z}^2 c_1^{n-1} - (c_1^{n-1} + c_2^{n-1}) c_{1z} c_{2z} + c_{2z}^2 c_2^{n-1}] \quad (18)$$

is again a quadratic form with a positive discriminant and thus may change sign. But a straightforward repetition of the argument given earlier in this section shows that the positive solutions of the porous medium equation are also ordered according to their initial conditions.

Finally, it should be clear now that the positive solutions in all four examples are also ordered according to their boundary conditions, i.e., that if $c_1(z,0) = c_2(z,0)$, $c_1(a,t) \geq c_2(a,t)$, and $c_1(b,t) \geq c_2(b,t)$, then $c_1(z,t) \geq c_2(z,t)$ in rectangle OABC.

6. Ordering of Solutions (part 3)

The argument used in part 2 can be applied directly to Eq. (7) to find simple sufficient conditions for the solutions to be ordered. We begin again by considering the difference u of two infinitesimally close solutions c and $c + u$. To first order then

$$Su_t = -(\partial q/\partial c_z) u_{zz} - [\partial/\partial z(\partial q/\partial c_z) + \partial q/\partial c] u_z$$

$$-[(dS/dc)c_t + \partial/\partial z(\partial q/\partial c)]u \quad (19)$$

where S, q, and the derivatives of q are evaluated at the solution c. We can now follow the argument given in part 2 without change. The value of the coefficient of u_z does not matter, for $u_z = 0$ at the points P and Q. The value of the coefficient of u also does not matter because it can be dominated by $-\lambda S$ as long as $S(c) > 0$. And finally, the argument only goes through if $\partial q/\partial c_z \leq 0$. **So if $S(c) > 0$ and $\partial q/\partial c_z \leq 0$, the solutions of Eq. (7) are ordered according to their boundary conditions or their initial conditions.** This conclusion is unaffected by the presence of a source term $Q(c)$, which merely adds a term $(dQ/dc)u$ to the right-hand side of Eq. (19).

Note: To see that no generality is lost by considering the difference between infinitesimally close solutions, subtract Eq. (7) written for c from Eq. (7) written for $c + u$, where u is no longer treated as small. Using the law of the mean, we find Eq. (19) for u, except that now S, q, and their derivatives are evaluated at values intermediate between c and $c + u$. But this does not affect the foregoing proof of the conclusion given above in bold type.

7. Comparability of Solutions

Now we show that if $c_1(z,t)$ and $c_2(z,t)$ obey the same initial conditions and the same boundary conditions on an interval $a \leq z \leq b$, i.e., if $c_1(z,0) = c_2(z,0)$, $c_1(a,t) = c_2(a,t)$ and $c_1(b,t) = c_2(b,t)$, and if $Q_1(c) \geq Q_2(c)$, then $c_1(z,t) \geq c_2(z,t)$ for all $t > 0$. Let c_1 satisfy Eq. (7) with the source term Q_1 and let $c_2 = c_1 + u$ satisfy Eq. (7) with the source term Q_2:

$$h_t(c_1) + q_z(c_1, c_{1z}) = Q_1(c_1) \tag{20a}$$

$$h_t(c_2) + q_z(c_2, c_{2z}) = Q_2(c_2) \tag{20b}$$

We subtract Eq. (20b) from Eq. (20a) and, using the law of the mean, we obtain Eq. (19) with the additional term $Q_1(c_1) - Q_2(c_2)$ on the right-hand side. Now,

$$Q_1(c_1) - Q_2(c_2) = Q_1(c_1) - Q_2(c_1) + Q_2(c_1) - Q_2(c_2) \tag{21a}$$

$$= Q_1(c_1) - Q_2(c_1) + (dQ_2/dc)u \tag{21b}$$

so that now

$$Su_t = -(\partial q/\partial c_z)u_{zz} - [\partial/\partial z(\partial q/\partial c_z) + \partial q/\partial c]u_z$$

$$- [(dS/dc)c_t + \partial/\partial z(\partial q/\partial c) - dQ_2/dc]u + Q_1(c_1) - Q_2(c_1) \tag{22}$$

Since $Q_1(c_1) - Q_2(c_1) \geq 0$, the proof given above in part 2 that $u \geq 0$, i.e., that $c_1 \geq c_2$, in the rectangle OABC and on its boundary goes through without change.

I have used the foregoing comparability theorem to study the following problem of heat diffusion that arises in applied superconductivity. On the infinite interval $-\infty < z < \infty$, let c obey the ordinary diffusion equation with the source term $Q(c)$ shown in Fig. 3:

$$c_t = c_{zz} + Q(c) \tag{23}$$

Let c initially be given by an instantaneous pulse occurring at time $t = 0$ at the origin $z = 0$, i.e., let

$$c = H \exp(-z^2/4t) / (4\pi t)^{1/2} \tag{24}$$

If H, the pulse strength, is small, the solution of Eq. (23) will ultimately tend to zero. If H is large, the solution will ultimately grow without bound. What value of H divides these two regions of behavior?

Equation (23) with Q given by Fig. 3 is not solvable in simple terms. But if we replace $Q(c)$ by the upper limit given by the dashed line, namely $(c/c_f)Q_f$, the resulting equation can easily be solved to give

$$c_1 = H \exp(Q_f t/c_f) \exp(-z^2/4t) / (4\pi t)^{1/2} \tag{25}$$

The peak of c occurs at $z = 0$, and it reaches its smallest value at $t = c_f/2Q_f$. This smallest value is

$$c_{1\min} = H\,(2\pi c_f/eQ_f)^{-1/2} \tag{26}$$

Now if $c_{1\min} = c_f$, then at the instant that c_1 achieves its smallest peak value, the solution c, which is $\leq c_1$, will be $\leq c_f$ everywhere. But then $Q(c)$ given in Fig. 3 will be zero everywhere, and from that time on c will decay to zero. Clearly then if we set $c_f = c_{1\min}$ in Eq. (26), we get a <u>lower</u> bound to H, namely,

$$H > (2\pi/e)^{1/2}\, c_f^{3/2}\, Q_f^{-1/2} \tag{27}$$

Solutions belonging to values of H smaller than the right-hand side of Eq. (27) will surely tend to zero as t tends to infinity.

8. Asymptotic Behavior of Solutions

I have used the ordering property in a variety of ways, but none more important than determining the asymptotic behavior of certain solutions of conservation equations that are invariant to a family of stretching groups. We begin by considering the similarity solutions, whose form is given in Eq. (4). At the origin they necessarily obey boundary conditions of the form

$$c(0,t) = Bt^{\alpha/\beta} \tag{28a}$$

We are interested here in solutions that vanish at $z = \infty$, i.e., solutions that obey the boundary condition

$$c(\infty,t) = 0 \qquad (t > 0) \tag{28b}$$

Because the similarity solutions have the form (4), they must also obey the initial condition

$$c(z,0) = 0 \qquad (z > 0) \tag{28c}$$

Finally, we begin by restricting our attention to equations for which L/M and N/M are both < 0 and $A > 0$. The Gorter-Mellink equation $c_t = (c_z^{1/3})_z$ and the thermal expulsion equation $cc_t = c_{zz}$ are of this type.

When $L/M < 0$, the similarity solution (4) evaluated at $z = 0$, $t > 0$, i.e., on the open segment OA in Fig. 2, is less than the totally invariant solution (6b), since the latter is infinite there. Both solutions obey conditions (28b) and (28c). Since the solutions are assumed to be ordered according to their boundary conditions at $z = 0$, $t > 0$, we see that

$$c(z,t) < Az^{L/M}\, t^{-N/M} \tag{29a}$$

which is equivalent to

$$y(x) < Ax^{L/M} \tag{29b}$$

in light of Eqs. (3) and (4). The function $y(x)$ obeys the boundary conditions $y(0)= B$ [Eq. (28a)] and $y(\infty) = 0$ [Eqs. (28b) and (28c)]. As noted earlier, when the function $y(x)$ is transformed by the transformations of the group (5), its

images are also solutions of the ordinary differential equation for $y(x)$. Each image has $y'(\infty) = 0$ and $y'(0) = \mu^{L/M}B$. Thus any solution and its one-parameter family of images under group (5) form the totality of solutions of the ordinary differential equation having $y(\infty) = 0$ and $y(0) = B$, a positive constant. Moreover, owing to the ordering of the boundary value $c(0,t) = Bt^{\alpha/\beta}$ with respect to B, the functions $y(x)$ are also ordered with respect to B.

Since the ordered functions $y(x)$ are bounded from above by the totally invariant solution [see Eq. (29b)], they must have a limit as $B \to \infty$. This limit, being arbitrarily close to solutions of the family $y(x)$, must also be a solution. It is, furthermore, invariant to the associated group (5) since that group carries the family into itself. It must therefore be the solution $y = A\,x^{L/M}$. Thus

$$\lim_{B\to\infty} y(x)/x^{L/M} = A \tag{30}$$

Now consider a particular integral curve belonging to a fixed value of B. Let us focus our attention on a set of images (x',y') of the points (x,y) on the curve just defined by setting $\mu = a/x$, where a is some fixed number. Then the images are

$$x' = a \tag{31a}$$

$$y' = (a/x)^{L/M}\,y \tag{31b}$$

The point (x',y') lies on the integral curve y(x) belonging to the value

$$B' = (a/x)^{L/M}B \tag{31c}$$

Thus as $x \to \infty$, so does B' (remember that $L/M < 0$). Finally, then,

$$\lim_{x\to\infty} y(x)/x^{L/M} = \lim_{B'\to\infty} y'(x')/x'^{L/M} = A \tag{32}$$

which means

$$\lim_{z\to\infty} c(z,t)/Az^{L/M}t^{-N/M} = 1 \tag{33a}$$

or

$$\lim_{t\to 0} c(z,t)/Az^{L/M}t^{-N/M} = 1 \tag{33b}$$

for the similarity solutions $c(z,t)$ obeying the boundary and initial conditions (28).

Now all solutions obeying Eqs. (28b) and (28c) and having finite $c(0,t)$ must obey Eq. (29a). Since the solutions are ordered according to their boundary condition $c(0,t)$, Eqs. (33a) and (33b) thus hold for any solution for which $c(0,t)$ can be bounded from below by an expression of the form (28a).

9. Asymptotic Behavior: Example

The asymptotic behavior (32) will fall out of any explicit representation of the function of $y(x)$ obtained by solving the ordinary differential equation $y(x)$ obeys. But even when no explicit representation can be obtained, the asymptotic behavior (32) can be deduced from a study of the ordinary differential equation for $y(x)$. The following example makes clear how this happens.

The Gorter-Mellink diffusion equation $c_t = (c_z^{1/3})_z$ is invariant to the family of groups (2) when the coefficients in the linear constraint (3) are $M = 2$, $N = -3$, and $L = -4$. If we substitute the right-hand side of Eq. (4) for c, we find the following ordinary differential equation for $y(x)$:

$$\beta(y'^{1/3})' + xy' - \alpha y = 0 \tag{34}$$

As expected, this ordinary differential equation--called the principal differential equation in reference 2--is invariant to the group (5) (N.B.: now $L/M = -2$ and $N/M = -3/2$).

The method of treating Eq. (34) outlined in reference 2 is based on a theorem of Lie's,[5] which tells us that if we choose as new variables an invariant u and a first differential invariant v of the group (5), the differential equation relating v to u will be of first order. The invariants that we use are $u = xy^{1/2}$ and $v = xy'^{1/3}$. Then

$$x(dv/dx) = v + x^2 d(y'^{1/3})/dx = v + (\alpha/\beta)u^2 - v^3/\beta \equiv G(u,v) \tag{35a}$$

$$x(du/dx) = u + v^3/2u \equiv F(u,v) \tag{35b}$$

so that

$$dv/du = u(2\beta v - 2v^3 + 2\alpha u^2)/\beta(2u^2 + v^3) \tag{35c}$$

Now the totally invariant solution (6b), being equivalent to $y(x) = Ax^{L/M} = Ax^{-2}$, corresponds to <u>fixed</u> values of u and v, namely $u = \pm A^{1/2}$ and $v = (-2A)^{1/3}$.

These fixed values of u and v thus represent solutions of the differential equation (35). For these solutions, $du = dv = 0$, which means that the points $P_\pm$: $(\pm A^{1/2},(-2A)^{1/3})$ are *critical points* of Eq. (35). Substitution of Eq. (6b) into the Gorter-Mellink diffusion equation yields $4(\sqrt{3})/9$ as the value of A, so that the coordinates of the critical points $P_\pm$ are $(\pm 2(3^{1/4})/3,-2(\sqrt{3})/3)$.

We can verify these coordinates directly from Eq. (35), for the critical points are those points at which the numerator and denominator of Eq. (35) vanish simultaneously. It follows at once that the only critical points are O: (0,0) and $(\pm[2^{1/3}\beta/(\alpha+2)]^{3/4},-[2\beta/(\alpha+2)]^{1/2})$. Now from the linear constraint (3) it follows that $\beta/(\alpha+2) = 2/3$, so that the coordinates of the latter critical points are exactly those given above for the points $P_\pm$.

Now because the coordinates of the points $P_\pm$ do not depend on α and β, $P_\pm$ are critical points of Eq. (35) for all α and β. Thus the constant values of $u = \pm A^{1/2}$ and $v = (-2A)^{1/3}$ that are their coordinates represent solutions of Eq. (35) for all α and β. But this means that $y(x) = Ax^{-2}$ is a solution of Eq. (34) for all α and β and thus corresponds to a totally invariant solution of the Gorter-Mellink diffusion equation. So by studying the critical points of the first-order differential equation (35)--called the associated differential equation in reference 2--we could have been led to the totally invariant solution.

But how can we see from Eq. (35) that the totally invariant solution gives the asymptotic behavior of the function $y(x)$? To answer this question, we must study the phase diagram of Eq. (35). In the applications of the Gorter-Mellink diffusion equation, interest focuses on solutions for which $y > 0$ and $y' < 0$. Accordingly, we shall need only the fourth quadrant of the phase diagram. It is sketched in Fig. 4 for the case $\alpha > 0$, $\beta > 0$. Shown are the loci of zero slope $u = 0$ and C_0: $2\beta v - 2v^3 + 2\alpha u^2 = 0$, and the locus of infinite slope C_∞: $2u^2 + v^3 = 0$. These loci, which intersect in the two critical points O and P_+, divide the quadrant into regions in each of which the slope dv/du has one sign. From the diagram we see that O is a node and P_+ is a saddle point.

For solutions $y(x)$ that have finite $y(0)$ and $y'(0)$, the value $x = 0$ corresponds to $u = v = 0$, i.e., to the origin O. So the solution of Eq. (35c) that we want must pass through the origin. The separatrix S passes through the point P_+, and now we show that as we pass from O to P_+ along S, x tends to ∞.

If we approach the singular point P_+ along an integral curve S, then in the immediate vicinity of P_+, Eq. (35b) can be written

$$x(du/dx) = (F_u + mF_v)(u - u_{P+}) \tag{36}$$

where m is the slope of the integral curve S at P_+ and the partial derivatives F_u and F_v are evaluated at P_+. Equation (36) means that near P_+,

$$x = \text{const}\,(u - u_{P+})^{1/(F_u + mF_v)} \tag{37}$$

Thus as $u \to u_{P+}$, x approaches 0 or ∞ according to whether $F_u + mF_v$ is positive or negative. Now $F_u + mF_v$ is the directional derivative of F along the curve S in the direction of increasing u. Since S and C_∞ ($F = 0$) intersect at P_+, $F_u + mF_v$ is positive if as we move in the direction of increasing u, S crosses C_∞ in the direction of increasing F and negative if S crosses C_∞ in the direction of decreasing F. We summarize this result as follows: **At a singular point P, if S crosses C_∞ in the direction of increasing u, and F increases (decreases), then $x \to 0$ ($\pm\infty$) as we approach P along S.** The same rule holds if we replace u by v, C_∞ by C_0, and F by G. Thus as we approach P_+ along the separatrix S, x becomes infinite. Then, for large x, $u \approx u_{P+}$ and $y \sim u_{P+}{}^2x^{-2} = Ax^{-2}$, as we wished to show.

For any fixed values of α and β, the separatrix S corresponds to an entire family of solutions $y(x)$ that are images of one another under the transformations of the associated group (5): for, two points (x,y) and (x',y') related as in Eq. (5) correspond to the same values of u and v. Every member of the family has the asymptotic form $y \sim Ax^{-2}$ but the shape of the curves $y(x)$ for small x depends on the particular values of α and β. The values of α and β in turn depend on the boundary condition (28a) imposed on $c(z,t)$. For example, when $\alpha = 1$ and $\beta = 2$, $c(0,t) = Bt^{1/2} = y(0)t^{1/2}$ and $c_z(0,t) = y'(0)$, a constant. This boundary condition can be used to solve the problem of heat flow in a long tube filled with superfluid helium initially at a uniform temperature, which, beginning at $t = 0$, is subjected to a constant flux of heat $-[c_z(0,t)]^{1/3}$ through the surface $z = 0$. This problem, called the clamped-flux problem, has been dealt with in detail in reference 2.

10. Asymptotic Behavior : Example (part 2)

A similar analysis holds for the case in which $\alpha < 0$ and $\beta > 0$ (β must be > 0 in order that the similarity solution (4) spread out with time rather than contract). But whereas the existence of the lenticular region between C_0 and C_∞ in Fig. 4 makes it easy to see that when $\alpha > 0$ one of the separatrices through P_+ also passes through O, it is somewhat more difficult to see when $\alpha < 0$. For the sake of completeness, the analysis of the phase diagram when $\alpha < 0$ is given here, but the reader not interested in the details may pass on now to the next section.

Shown in Fig. 5 are the first and fourth quadrants of the phase diagram of Eq. (35) when $\alpha < 0$ and $\beta > 0$; the second and third quadrants are mirror images of the first and fourth, respectively, in the v-axis. The curves C_0 and C_∞ are shown as heavy lines; typical integral curves are shown as light lines. Note that the curve C_0 has two parts.

The point P_+ is a saddle point through which two separatrices pass. This is more or less clear from the diagram but can be proved indubitably by applying L'Hospital's rule to the right-hand side of Eq. (35c) to find possible values of the slope m of the integral curves at the point P_+. We then obtain the quadratic equation

$$m = u_{P+}(2\beta m - 6v_{P+}^2 m + 4\alpha u_{P+})/\beta(4u_{P+} + 3v_{P+}^2 m) \tag{38a}$$

or

$$4\beta m^2 + u_{P+}(2\beta + 8)m - 4\alpha u_{P+}^2 = 0 \tag{38b}$$

by noting that $3v_{P+}^2 = 4$. The discriminant of Eq. (38b), calculated with the help of Eq. (3), is

$$u_{P+}^2(100\beta^2 - 96\beta + 64) \tag{38c}$$

which is always positive. Thus Eq. (38b) always has two real roots; these are the slopes of the separatrices. When $\alpha < 0$, both must be negative, as shown in Fig. 5.

Using the rule given at the end of the last section, we see that as we approach P_+ along the separatrix S in the direction of increasing u, F decreases so that $x \to \infty$. So it would appear that S is the integral curve of Eq. (35c) that we want. It remains to show that S also passes through the origin O.

Now in the neighborhood of the origin, integral curves exist for all values of m that behave as $v = mu$ does. This we can see by substituting $v = mu$ into Eq. (35c) and keeping only the lowest order terms in the numerator and the denominator. Thus the origin is a node. But in addition to these integral curves, there is an exceptional integral curve, S': $v = u^{2/3}$, which also satisfies Eq. (35c) to lowest order. The configuration of integral curves in the neighborhood of the origin is thus as shown in Fig. 5. Since integral curves do not intersect, except possibly at critical points, integral curves that lie below S' must pass through O as we wished to prove.

We can check these results in the case $\alpha = -1$, for then Eq. (34) is explicitly solvable[6]; the solution for which $y(\infty) = 0$ is

$$y = (4(\sqrt{3})/9)(x^4 + a^4)^{-1/2} \tag{39}$$

where a is a constant of integration. By direct calculation we find that the one-parameter family (39) corresponds to the curve $v = -3u^2/2$ in the (u,v)-plane. This curve passes through both singularities O and P_+ and is thus the separatrix S. As expected, all curves of the family (39) have the asymptotic behavior $y \sim (4(\sqrt{3})/9)x^{-2} = Ax^{-2}$.

The value $\alpha = -1$ corresponds to the condition

$$\int_{-\infty}^{\infty} c\, dz = \int_{-\infty}^{\infty} t^{-1/\beta} y(z/t^{1/\beta})\, dz = \int_{-\infty}^{\infty} y(x)\, dx \tag{40}$$

if we note that the solution (39) is even in x. The condition (40) says that the integral of c over all space is *constant in time*. This means that there is global conservation of the quantity c. The function of z and t that Eq. (39) represents is more and more peaked around $z = 0$, the smaller t is. For this reason, the solution (39) represents the development of an instantaneous pulse of the quantity c in the plane $z = 0$ at the time $t = 0$. In reference 6, the problem $\alpha = -1$ is called the *pulsed-source problem*. Its importance, aside from its practical uses, is that it is analytically solvable for all equations of the form (1) with $q = q(c,c_z)$ that are invariant to the family of groups (2).

Another problem analytically solvable for invariant differential equations is that for which $\alpha = 0$ when $q = q(c_z)$. In the case at hand, the solution when $\alpha = 0$ is

$$y = (8(\sqrt{3})/9a^2)[1 - x/(x^2 + a^2)^{1/2}] \tag{41}$$

where again a is a constant of integration. As expected, $y \sim (4(\sqrt{3})/9)x^{-2} = Ax^{-2}$.

11. Asymptotic Behavior of Solutions (part 2)

The proof given in section 8 that the totally invariant solution $Az^{L/M}t^{-N/M}$ gives the asymptotic behavior of solutions that obey the conditions (28b,c) depends on the explicit assumptions that $L/M < 0$, $N/M < 0$, and $A > 0$. To explore what happens if these conditions are not met, we study the analytic solutions to the pulsed-source problem ($\alpha = -1$) for the generalized diffusion equation*

$$c_t = (c^m c_z^n)_z \tag{42}$$

*As it happens, Eq. (42) is the most general form of Eq. (1) with $q = q(c,c_x)$ invariant to the family of groups (2).

For Eq. (42), $M = m + n - 1$, $N = 1$, and $L = n + 1$. The similarity solutions (4), when substituted into Eq. (42), lead to the following principal ordinary differential equation for $y(x)$:

$$\beta(y^m y'^n)' + xy' - \alpha y = 0 \tag{43a}$$

When $\alpha = -1$, Eq. (43a) becomes

$$\beta(y^m y'^n)' + (xy)' = 0 \tag{43b}$$

where now $\beta = 2n + m$. Equation (43b) can be integrated once to yield

$$\beta y^m y'^n + xy = 0 \tag{43c}$$

Here we have set the constant of integration equal to zero since we are interested in even solutions, for which $y'(0) = 0$. Solutions for which $y(0)$ is finite and $y'(0) = 0$ can only occur when $n > 0$.

Now since we are interested in solutions that are everywhere positive and since both x and β are positive, it is clear that y'^n must be negative when $x > 0$. This puts another restriction of the value of n, namely, that it must be an odd integer, the reciprocal of an odd integer, or the ratio of two odd integers. When these restrictions are met, Eq. (43c) can be integrated to give

$$\beta^{1/n}[(n+1)/(m+n-1)]y^{(m+n-1)/n} = -x^{(n+1)/n} + \text{constant} \tag{44}$$

If $m + n < 1$, the left-hand side of Eq. (44) is < 0, even when $x = 0$. Therefore, the constant in Eq. (44) is also < 0. We can therefore write Eq. (44) as

$$y = [(1-m-n)(a^{(n+1)/n} + x^{(n+1)/n})/\beta^{1/n}(n+1)]^{n/(m+n-1)} \tag{45}$$

where we have written $-a^{(n+1)/n}$ for the constant in Eq. (44). Because $n > 0$, when x is large enough, $x^{(n+1)/n} >> a^{(n+1)/n}$. Then for large enough x,

$$y \sim x^{(n+1)/(m+n-1)} = x^{L/M} \tag{46}$$

and furthermore $L/M = (n+1)/(m+n-1) < 0$ and $N/M = 1/(m+n-1) < 0$. This is just the asymptotic behavior we found before.

If $m + n > 1$, the left-hand side of Eq. (44) is > 0, even when $x = 0$. Therefore, the constant in Eq. (44) is also > 0. We can therefore write Eq. (44) as

$$y = [(m+n-1)(a^{(n+1)/n} - x^{(n+1)/n})/\beta^{1/n}(n+1)]^{n/(m+n-1)} \tag{47}$$

Furthermore, now $L/M > 0$ and $N/M > 0$. It is clear from the form of Eq. (47) that the solution $y(x)$ crosses the x-axis at $x = a$. If Eq. (47) is used when $y > 0$ and if y is set equal to zero beyond the first root, a solution $c(z,t)$ is thereby created that satisfies Eqs. (28b,c) by reaching zero at a finite value of z. Such solutions were first displayed by Pattle[7]. For the sake of a definite nomenclature, I call such solutions *segmented.*

12. Asymptotic Behavior of Solutions (part 3)

The question now arises whether when L/M > 0 and N/M > 0 *all* similarity solutions are segmented rather than just those for which $\alpha = -1$. To examine this question we study the direction field of the first-order differential equation associated with Eq. (43a). For ease of calculation we choose as the invariant u and the first differential invariant v of the associated group (5) the following:

$$v = y^m y'^n x^{-(m+2n)/(m+n-1)} \tag{48a}$$

$$u = y x^{-(n+1)/(m+n-1)} \tag{48b}$$

After a short computation following the outline provided by Eqs. (35), we find

$$x(dv/dx) = -[(m+2n)/(m+n-1)]v + \alpha u/\beta - (v/u^m)^{1/n}/\beta \equiv G(u,v) \tag{49a}$$

$$x(du/dx) = -[(n+1)/(m+n-1)]u + (v/u^m)^{1/n} \equiv F(u,v) \tag{49b}$$

$$dv/du = G(u,v)/F(u,v) \tag{49c}$$

When $m+n-1 > 0$ and $n > 0$, Eq. (49) has an integral curve S through the origin which behaves like $v = -u/\beta$ for small u and v. For when $v = -u/\beta$, the term $(v/u^m)^{1/n}$ dominates the other terms in both F and G near the origin O. Thus as we approach O along S, we have

$$\lim (y^{m-1} y'^n/x) = \lim (v/u) = -1/\beta \tag{50a}$$

Now by integrating Eq.(49b) as (u,v) approaches O along S, we see that x can only approach some finite value; call it a. Then from Eq. (48b) we see that $y(a)$

= 0. When $x \to 0$, u and $v \to \infty$; thus the integral curve S represents a segmented solution with its foot at $x = a$, where

$$\lim_{x \to a} (y^{m-1} y'^{n}) = -a/\beta \tag{50b}$$

Eq. (50b) agrees with detailed calculations done in reference 2 for the cases $m = 1$, $n = 1$, $\alpha = -1$ and $m = 1$, $n = 1$, $\alpha = 1/2$. Eq. (50b) can be integrated to give the following behavior of $y(x)$ in the neighborhood of the foot $x = a$:

$$y = (a/\beta)^{1/(m+n-1)}[(m+n-1)(a-x)/n]^{n/(m+n-1)} \tag{50c}$$

Eq. (50c) agrees with the exact result (47) in the case $\alpha = -1$. In reference 2, Eq. (50b) was used to find initial conditions for numerical integration of Eq. (43a) from a to zero. Eq. (50c) can be used in the same way. Solutions for various values of a can be obtained from one another by transformation with the associated group (5).

13. Recapitulation of Sections 8-12: Asymptotic Behavior

In section 8, we showed that if $L/M < 0$, $N/M < 0$, and $A > 0$, then solutions of Eq. (1) obeying the boundary conditions (28b): $c(\infty,t) = 0$ ($t > 0$) and (28c): $c(z,0) = 0$ ($z > 0$) have the asymptotic behavior $c(z,t) \sim Az^{L/M} t^{-N/M}$ for small t or large z. In section 8, we begged the question of the existence of solutions obeying Eqs. (28b,c).

In section 9, we demonstrated the existence of such solutions by direct calculation for the Gorter-Mellink equation $c_t = (c_z^{1/3})_z$ when $\alpha > 0$. In section 10, we extended these calculations to the case $\alpha < 0$.

In section 11, we considered the pulsed-source problem ($\alpha = -1$) for the generalized diffusion equation (42): $c_t = (c^m c_z^n)_z$ in order to see what would happen if $L/M > 0$, $N/M > 0$, or $A < 0$. The pulsed-source problem was chosen because it is analytically solvable for Eq. (42). We found that similarity solutions could only exist when $n > 0$. (The exponent n must also be > 0 if we wish to apply the ordering theorem to Eq. (42), for that theorem requires that $\partial q/\partial c_z \leq 0$.) When $L/M = (n+1)/(m+n-1) < 0$, we found by direct calculation the asymptotic behavior predicted in section 8, but when $L/M = (n+1)/(m+n-1) > 0$, we found that the similarity solutions of the pulsed-source problem are segmented, i.e., vanish at and beyond a finite value of z.

In section 12, we found that when $L/M > 0$, the similarity solutions of Eq. (42) are segmented for all values of α and β ($\beta > 0$).

14. Steady States

As mentioned in item (iv) of the introduction, when a source term $Q(c)$ is added to the right-hand side of Eq. (1), the resulting partial differential equation may have more than one steady state. Only steady states that are stable against perturbation can occur in the laboratory (owing to the ubiquitous presence of thermal fluctuations).

The key to determining the stability of steady states is the Lagrangean. Let us suppose that the steady-state version of Eq. (1) with a source term is the Euler-Lagrange equation of a Lagrangean $L[z,c,c_z]$,* i.e., let us suppose we can write Eq. (1) with a source term as

$$S(c)c_t = Q(c) - q_z(c,c_z) = \partial/\partial z(\partial L/\partial c_z) - \partial L/\partial c \qquad (51)$$

where we have generalized Eq. (1) by the insertion of a coefficient $S(c) > 0$ on

*Bolza[8] shows that such a Lagrangean always exists.

the left-hand side. If the steady states are defined by the boundary conditions $c = c_a$ at $z = a$ and $c = c_b$ at $z = b$, then **the action**

$$A = \int_a^b L[z,c(z,t), c_z(z,t)]\, dz \qquad (52)$$

decreases monotonically with time for solutions of Eq. (51) that obey the boundary conditions $c(a,t) = c_a$ and $c(b,t) = c_b$. **Proof:**

$$dA/dt = \int_a^b [(\partial L/\partial c_z)c_{zt} + (\partial L/\partial c)c_t]\, dz \qquad (53a)$$

$$= c_t(\partial L/\partial c_z)\Big|_a^b - \int_a^b c_t(\partial/\partial z(\partial L/\partial c_z) - \partial L/\partial c)\, dz \qquad (53b)$$

$$= - \int_a^b S(c)c_t^2\, dz < 0 \qquad (53c)$$

The integrated term vanished because $c_t(a,t) = c_t(b,t) = 0$ as a consequence of the boundary conditions imposed on c. It follows at once from Eq. (53) that $dA/dt = 0$ if and only if $c_t = 0$, i.e., that the stationary values of A correspond to steady states and conversely.

As mentioned in the introduction, the action A plays a role much like that played by the gravitational potential of a ball rolling on a curved surface. Imagine the action plotted as an infinite-dimensional surface whose base

coordinates specify any function of z on the interval $a \leq z \leq b$. The base coordinates might be, for example, the Fourier coefficients of the function. At any instant, the solution of Eq. (51) corresponds to a point on the action surface, and as time advances the point traces out a curve on the surface. Such curves always progress in the direction of decreasing action. So, in a manner of speaking, the solution point rolls down the action surface. The stationary (or critical) points of the action surface correspond to steady states, so the analogy with the potential energy of a ball rolling on a surface is complete.

Local minima on the action surface correspond to states that are stable against small perturbations, while local maxima, saddle points, and points of inflection correspond to unstable states. If we imagine the action surface to be an inclined rubber sheet and make a dimple in it downwards with a finger, we create a local minimum at the center and a saddle on the rim. Thus we create two steady states, one stable and the other unstable. If we make the dimple upwards, we create a local maximum and a saddle. It would appear, then, that new steady states may appear in pairs, at least one of which (the one with the larger action) is always unstable. The state with the smaller action may be either stable or unstable.

By imagining (or experimentally determining) the different shapes a rubber membrane can take, we can guess what different combinations of stable and unstable steady states are possible. The analogy between the infinite-dimensional action surface and a rubber membrane is thus of great heuristic value, but, as it stands, it does not constitute a proof of any of the above statements.

15. Steady States (part 2)

It is possible to prove convincingly the following: If there are just two steady states, the one with the larger action is unstable. **Proof:** Let $c_1(z)$ and $c_2(z)$ denote the two steady states and let their respective actions be $A_1 > A_2$. Consider now the one-parameter family of initial conditions

$$c(z,0) = \lambda c_1(z) + (1-\lambda)c_2(z), \qquad 0 \leq \lambda \leq 1 \qquad (54)$$

Sketched in Fig. 6 as a function of λ is the initial action A that corresponds to $c(z,0)$. When $\lambda = 0$, $A = A_2$, a stationary value, and when $\lambda = 1$, $A = A_1$, also a stationary value. Joining A_1 and A_2 is a continuous curve, which intersects the intermediate value A_* of the action at the point P. As time goes on, all the points on this curve move down ($dA/dt < 0$) except the fixed end points A_1 and A_2, which correspond to the steady states c_1 and c_2. The point P can therefore only

move to the right. Since P is bounded on the right by the line $\lambda = 1$, it must approach a limiting value $\lambda \leq 1$; call this limiting value λ_*.

If $\lambda_* < 1$, then the action of the descendant of the initial condition (54) having the λ-value λ_* approaches A_* in the limit of long times and the descendant itself therefore approaches a steady state different from either c_1 or c_2. This is contrary to hypothesis; therefore $\lambda_* = 1$. Then eventually the point P will lie in any neighborhood of $\lambda_* = 1$, no matter how small. This means that there are initial conditions in every neighborhood of c_1, no matter how small, whose descendants do not approach c_1. Thus c_1 is unstable, as was to be proved.

When there are more than two steady states, we can use the above proof to show that the one with the largest action (call it c_1) is unstable by letting c_2 represent the steady state with the next largest action. But this method of proof fails for any other steady state, so that we cannot use it to draw conclusions about the stability of these other states. Figure 7 illustrates why the proof fails except for the steady state with the largest action.

Figure 7 shows the action of the initial conditions (54) plotted against λ just as in Fig. 6, but in addition, Fig. 7 shows a horizontal line $A = A_3$ marking the higher action of another steady state c_3. It is possible that the action of some initial states (54) exceeds A_3 and it is further possible that as least one of them, say that of point R, approaches the steady state c_3. Thus the point P is bounded on the right by λ_R. It is then no longer possible to conclude that the state c_1 is unstable. There is no way to exclude this possibility except when no such state c_3 can exist. Therefore, all we can conclude in general is that the state with the highest action is unstable.

16. Steady States (part 3)

The discussion of the last section outlines what we can conclude from purely topological arguments. To say more, we must add more hypotheses, i.e., we must restrict our considerations to more specialized situations. One special case I have dealt with[9] is that of the ordinary diffusion equation in the presence of a positive source $Q(c)$, i.e., the case in which $q = -c_z$. Then

$$L = c_z^2/2 - \int_0^c Q(c')\, dc' \tag{55a}$$

for which the Euler-Lagrange equation (51) becomes

$$S(c)c_t = c_{zz} + Q(c) \tag{55b}$$

Since Eq. (55b) is invariant to translation in the z-coordinate, we may relocate the origin midway between the points $z = a$ and $z = b$; henceforth, therefore, we shall be considering solutions of Eq. (55b) on the interval $-a \leq z \leq a$ for which $c(\pm a,t) = 0$.

The steady states of Eq. (55b) are defined by the ordinary differential equation and boundary conditions

$$d^2c/dz^2 + Q(c) = 0; \quad c(\pm a) = 0. \tag{56}$$

The steady states we seek are even on the interval $-a \leq z \leq a$. It is perhaps worth pointing out that for the time being we are again begging the question of the existence of such solutions.

Let us suppose that exactly two steady states exist, c_1 and c_2. These states are ordered according to their maximum values, which occur at $z = 0$, i.e., if $c_1(0) > c_2(0)$, then $c_1(z) > c_2(z)$ for $-a \leq z \leq a$. To see this consider the solutions that grow from the initial conditions $c_1(z,0) = c_1(0)(1 - z/a)$ and $c_2(z,0) = c_2(0)(1 - z/a)$ with the following boundary conditions on the interval $0 \leq z \leq a$: $c_1(0,t) = c_1(0)$, $c_2(0,t) = c_2(0)$, $c_1(a,t) = 0$, $c_2(a,t) = 0$. The first solution matures into the steady state $c_1(z)$, the second into the steady state $c_2(z)$. Since $c_1(z,0) > c_2(z,0)$, $c_1(z,t) > c_2(z,t)$ by the ordering theorem of section 6 ($\partial q/\partial c_z = -1$!). Thus letting t become infinite, we find $c_1(z) > c_2(z)$.

If $Q(c)$ is concave upwards, then $A_1 > A_2$. **Proof:** Let us consider the action $A(t,\lambda)$ of the descendant $c(z,t;\lambda)$ of an initial condition of the family (54). Since $c(z,t;\lambda)$ is even

$$A(t,\lambda) = \int_0^a [c_z^2 - 2\int_0^c Q(c')\,dc']\,dz \tag{57}$$

Then,

$$dA/d\lambda = \int_0^a [2c_z c_{z\lambda} - 2Q(c)c_\lambda]\,dz \tag{58a}$$

$$= 2c_z c_{z\lambda}\Big|_0^a - 2\int_0^a c_\lambda[c_{zz} + Q(c)]\,dz \tag{58b}$$

$$= -2\int_0^a S(c)c_t c_\lambda\,dz \tag{58c}$$

since $c_z(0,t) = c_\lambda(a,t) = 0$. When $t = 0$, $c_\lambda = c_1(z) - c_2(z)$. Therefore,

$$(dA/d\lambda)_{t=0} = -2\int_0^a [S(c)c_t]_{t=0}\,(c_1 - c_2)\,dz \tag{58d}$$

Now

$$[S(c)c_t]_{t=0} = c_{zz}(z,0) + Q[c(z,0)] \tag{59a}$$

$$= \lambda c_{1zz} + (1-\lambda)c_{2zz} + Q[\lambda c_1+(1-\lambda)c_2] \tag{59b}$$

$$= -\lambda Q(c_1) - (1-\lambda)Q(c_2) + Q[\lambda c_1+(1-\lambda)c_2] \tag{59c}$$

so that

$$(dA/d\lambda)_{t=0} = 2 \int_0^a (c_1 - c_2) \{\lambda Q(c_1) + (1-\lambda)Q(c_2) - Q[\lambda c_1+(1-\lambda)c_2]\} dz \tag{59d}$$

If $Q(c)$ is concave upwards, $(dA/d\lambda)_{t=0} > 0$ for $0 < \lambda < 1$, $(dA/d\lambda)_{t=0} < 0$ for $\lambda < 0$ or $\lambda > 1$, and $(dA/d\lambda)_{t=0} = 0$ for $\lambda = 0$ or $\lambda = 1$. Thus $A(t=0)$ must look like Fig. 8. Therefore, $A_1 > A_2$, as was to be shown. Furthermore, it is obvious that c_1 is unstable since in any neighborhood of it, no matter how small, there are initial conditions with actions $< A_1$. Because the action always decreases, these initial conditions can never return to c_1.

Now we are in a position to show that c_2 is stable. We argue by *reductio ad absurdum.* Suppose c_2 were unstable. Then in every neighborhood of c_2 there is an initial condition $c'(z,0)$ that diverges from c_2. Let us choose as the the neighborhood bounded by the initial conditions (54) with $\lambda = \varepsilon$ and $\lambda = -\varepsilon$, $\varepsilon << 1$. Now because $A(-\varepsilon) < A_1$ and $A(\varepsilon) < A_1$, $c(z,t;-\varepsilon)$ and $c(z,t;\varepsilon)$ can only tend to c_2.* But by the ordering theorem, $c(z,t;-\varepsilon) < c'(z,t) < c(z,t;\varepsilon)$. Thus $c'(z,t)$ must tend to c_2, contrary to hypothesis. Therefore c_2 is stable.

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* A subtlety has been glossed over here, namely, that if $c(z,t;-\varepsilon)$ and $c(z,t;\varepsilon)$ cannot approach c_1 they must approach c_2. The only way for $c(z,t;-\varepsilon)$ and $c(z,t;\varepsilon)$ to avoid this if for $A(t,-\varepsilon)$ and $A(t,\varepsilon)$ to decrease without bound, for the only stationary values the action has are A_1 and A_2. From Eq. (57) we see that $A > -2 \int_0^a \int_0^c Q(c')\, dc'] dz$, so the only way $A(t,-\varepsilon)$ and $A(t,\varepsilon)$ can decrease without bound is for $c(z,t;-\varepsilon)$ and $c(z,t;\varepsilon)$ to increase without bound (since $Q(c) > 0$). But since $c(z,t;-\varepsilon)$ and $c(z,t;\varepsilon)$ are bounded from above by c_1 (ordering theorem), this is not possible.

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17. Steady States: Example

An example of the situation just discussed in which detailed calculation of the steady states is possible is the case $Q(c) = B\exp(\gamma c)$. In strict point of fact, Eq. (56) can always be solved in terms of quadratures, but the functional dependence of c on z, while given explicitly in the form of integrals, is far from transparent. Only in rather special cases can it be clarified, and the case $Q(c) = B\exp(\gamma c)$ is one of them.

If we multiply Eq. (56) by dc/dz and integrate from 0 to z, we find

$$(dc/dz)^2 + 2\int_0^z Q(c)(dc/dz)\,dz = 0 \tag{60a}$$

since $(dc/dz)_{z=0} = 0$. Since $(dc/dz)\,dz = dc$, Eq. (60a) can be written

$$(dc/dz)^2 = 2\int_c^{c_m} Q(c')\,dc' \tag{60b}$$

so that

$$z = \int_c^{c_m} \left(2\int_{c'}^{c_m} Q(c'')\,dc''\right)^{-1/2} dc' \tag{60c}$$

Here c_m is the maximum value of c that occurs at $z = 0$. If we set $z = a$, in Eq. (60c), we find

$$a = \int_0^{c_m} \left(2\int_c^{c_m} Q(c')\,dc'\right)^{-1/2} dc \tag{60d}$$

When $Q(c) = B\exp(\gamma c)$, which is both positive and concave upwards, Eq. (60d) can be integrated to yield

$$(\gamma B/2)^{1/2}a = \ln[y + (y^2 - 1)^{1/2}]/y; \quad y = \exp(\gamma c_m/2) \tag{61}$$

Figure 9 shows a sketch of y plotted versus $(\gamma B/2)^{1/2}a$. We see at once that depending on the value of $(\gamma B/2)^{1/2}a$ there can be no steady state, one steady state, or two steady states.

When there are two steady states (small enough a), $dc_m/da < 0$ for the one with the larger peak temperature c_m, and $dc_m/da > 0$ for the one with the

smaller c_m. Now Eq. (56) can serve as a model for an uncooled, current-carrying wire heat sunk at its ends and having a resistivity that increases with temperature c[9]. Presumably, steady states whose peak temperature *decreases* as the cooled length *increases* do not occur in the laboratory. So we suspect that the upper steady state (the one with the larger c_m) is the unstable one. But how to prove this conjecture?

We begin by showing that **for the steady solutions of Eq. (56), $dA/da < 0$. Proof:**

$$dA/da = [c_z^2 - 2 \int_0^c Q(c')\, dc']_{z=a} + 2 \int_0^a [c_z c_{za} - Q(c) c_a]\, dz \tag{62a}$$

$$= [c_z(a)]^2 + 2c_z(a)c_a(a) - 2 \int_0^a [c_{zz} + Q(c)]c_a\, dz \tag{62b}$$

$$= [c_z(a)]^2 + 2c_z(a)c_a(a) \tag{62c}$$

The integral in the first term of Eq. (62a) vanishes because $c(a) = 0$. The integral in Eq. (62b) vanishes because c satisfies Eq. (56). Now, finally, we note that $c_a(a) = -c_z(a)$; see Fig. 10. Therefore,

$$dA/da = -[c_z(a)]^2 < 0 \tag{62d}$$

Furthermore, from Eq. (60b) we see that

$$[c_z(a)]^2 = 2 \int_0^{c_m} Q(c)\, dc \tag{62e}$$

Thus A decreases with increasing a and does so faster on the upper branch than on the lower. If we go backwards from the single state of largest a, the action A increases on both branches but it increases faster on the upper branch than on the lower. Thus for a given a, A is larger on the upper branch than on the lower. From the results of sections 14, 15, and 16, we see then that the upper state is unstable and the lower state stable.

18. Steady States: Second Example

A related example arises in the case

$$Q(c) = 0, \qquad c < c_s \tag{63a}$$

$$Q(c) = B \exp[\gamma(c - c_s)], \qquad c > c_s \tag{63b}$$

Again Eq. (60d) can be integrated and yields

$$(\gamma B/2)^{1/2}a = \ln[y + (y^2 - 1)^{1/2}]/y + (\gamma c_s/2)(y^2 - 1)^{-1/2}; \quad y = \exp[\gamma(c_m - c_s)/2] \quad (64)$$

For small values of $\gamma c_s/2$ (roughly less than 0.2), the curve of y versus $(\gamma B/2)^{1/2}a$ looks like the sketch in Fig. 11. The most interesting region is that between the turning points P and Q, where there are four possible steady states (including the state $c = 0$). Again $dA/da < 0$, so that $A_3 > A_2$ and $A_1 > A_2$. These relations encourage the conjecture that states 3 and 1 are unstable, while states 0 and 2 are stable.

To prove this conjecture, we take advantage of the ordering of the solutions with respect to their initial conditions. Consider, for example, steady states 2 and 3 and the initial conditions

$$c(z,0) = \lambda c_3(z) + (1-\lambda)c_2(z), \qquad 0 \le \lambda \le 1 \qquad (65a)$$

all of which lie between c_2 *and* c_3. The difficulty alluded to in Fig. 7 cannot now arise. For, if the action of a point such as R were to approach a limit other than A_2, it would correspond to a steady state that, perforce, would have to lie between c_2 and c_3. There being no such steady state, we see at once that all of the states (65a) tend to c_2 except that for which $\lambda = 1$. Thus c_3 is unstable. By a similar argument regarding c_1 and c_2, we see that c_1 is unstable, and all of the initial states

$$c(z,0) = \lambda c_1(z) + (1-\lambda)c_2(z), \qquad 0 \le \lambda \le 1 \qquad (65b)$$

tend to c_2 except that for which $\lambda = 1$.

Now we can show that c_2 is stable by slightly extending the argument by *reductio ad absurdum* of section 16: we merely choose one of the curves bounding the neighborhood of c_2 from the family (65a) and the other from the family (65b).

19. Traveling Wave Solutions

If the Lagrangean has no explicit dependence on z, then Eq. (51) is invariant to the translation groups $z' = z + \lambda$ and $t' = t + \mu$, $-\infty < \lambda, \mu < \infty$. Solutions $c(z,t)$ invariant to both groups are simply constants, but that is because of the very high algebraic symmetry we have imposed. Suppose we look for solutions invariant to the one-parameter subgroup $z' = z - v\lambda$ and $t' = t + \lambda$, $-\infty < \lambda < \infty$, where v is a fixed constant. Such solutions must have the form

$c(z+vt)$ and are called traveling wave solutions. The form $c(z+vt)$ represents a wave traveling from right to left without change of shape when v is positive.

In the field of applied superconductivity, much time and effort have been spent calculating the traveling wave solutions of Eq. (55b). When we substitute the form $c(x) = c(z+vt)$ into Eq. (55b), we find the ordinary differential equation

$$c'' - vS(c)c' + Q(c) = 0 \tag{66}$$

At least two different kinds of source terms $Q(c)$ are of interest: (1) those like that of Eq. (63) that are zero up to and positive beyond a certain threshold value of c, and (2) those of the form shown in Fig. 12.

Suppose we seek solutions of Eq. (66) that look as shown in Fig. 13. The flat asymptotes of the solution at $x = -\infty$ and $x = \infty$ must be roots of the function $Q(c)$. In both cases of interest, $Q(0) = 0$, so the left-hand asymptote is admissible. In case (1), $Q(c)$ does not have another root, so it cannot admit traveling wave solutions of the form shown in Fig. 13. Case (2), however, can. So far it would appear that either $c(P)$ or $c(Q)$ in Fig. 12 can serve as c_m.

In fact, only $c(Q)$ can serve as c_m. For as we shall now see, the flat solution $c = c(P)$ is unstable against small perturbation, whereas the flat solutions $c = 0$ and $c = c(Q)$ are stable. We take a direct approach: consider the difference u between $c = c(P)$ or $c = c(Q)$ and an infinitesimally close solution $c + u$. If we subtract Eq. (55b) written for c from Eq. (55b) written for $c + u$, we find ($c_t = 0$!)

$$S(c)u_t = u_{zz} + Q'(c)u \tag{67a}$$

Note now that since c is a flat solution, $S(c)$ and $Q'(c)$ are constants. If we now consider a separable solution of Eq. (67a) of the form $u = \exp(ikx + \omega t)$, we find the dispersion relation

$$S\omega = -k^2 + Q' \tag{67b}$$

If $Q'(c) < 0$, as it is for $c = 0$ and $c = c(Q)$, $\omega < 0$ for all k, and thus the states $c = 0$ and $c = c(Q)$ are stable. If $Q'(c) > 0$, $\omega > 0$ for some k, and thus the state $c = c(P)$ is unstable.

Finally, if we multiply Eq. (66) by c' and integrate from $-\infty$ to ∞, we find

$$v \int_{-\infty}^{\infty} S(c)\, c'^2\, dc = \int_0^{c(Q)} Q(c)\, dc \tag{67c}$$

Since the left-hand side of Eq. (67c) is > 0, traveling wave solutions can only exist if the right-hand side of Eq. (67c) is also > 0, i.e., if the area of lobe 2 in Fig. 12 is greater than the area of lobe 1[10].

20. Traveling Wave Solutions: Example

In general, Eq. (66) has solutions of the type shown in Fig. 13 only for certain values of the velocity v, as the following example shows. Take $Q(c)$ to have the particular form:

$$Q(c) = -hc, \qquad 0 < c < c_s \tag{68a}$$

$$Q(c) = h(c_m - c), \qquad c_s < c < c_m \tag{68b}$$

which is of the type shown in Fig.12, and take $S(c) = S$, a constant. Now Eq. (66) is invariant to the translation group $c^1 = c$, $x^1 = x + \lambda$, $-\infty < \lambda < \infty$. We can use Lie's theorem[5] to reduce its order by introducing as new dependent and independent variables a first differential invariant $q \equiv c'$ and an invariant c, respectively. Then since $c'' \equiv d^2c/dx^2 = d(dc/dx)/dx = [d(dc/dx)/dc](dc/dx) = q(dq/dc)$, we find

$$q(dq/dc) - vSq + Q(c) = 0 \tag{69a}$$

and the boundary conditions

$$q(0) = 0, \qquad q(c_m) = 0 \tag{69b}$$

The boundary conditions, being two in number, generally overdetermine the solution of Eq. (69a), a first-order differential equation, and that is the reason why solutions exist only for particular values of v.

When $0 < c < c_s$, Eq. (69a) has the solution $q = mc$, where m is a root of the quadratic equation

$$m^2 - vSm - h = 0 \tag{70}$$

Since we are looking for solutions like that shown in Fig. 13 for which $q > 0$, we take the positive root. Thus

$$q = c[vS + (v^2S^2 + 4h)^{1/2}]/2, \qquad 0 < c < c_s \tag{71a}$$

When $c_s < c < c_m$, q also varies linearly with c, now with the negative slope given by Eq. (70):

$$q = (c - c_m)[vS - (v^2S^2 + 4h)^{1/2}]/2, \qquad c_s < c < c_m \tag{71b}$$

Equations (71a) and (71b) must give the same value of q at $c = c_s$; equating their right-hand sides yields

$$v = (h^{1/2}/S)(c_m - 2c_s)/[c_s(c_m - c_s)]^{1/2} \tag{71c}$$

The difference of the areas of lobes 1 and 2 is $(hc_m/2)(c_m - 2c_s)$ so that the quantity $c_m - 2c_s > 0$ when the area of lobe 2 > the area of lobe 1.

21. Traveling Wave Solutions (part 2)

The manner of solving Eq. (66) in the foregoing example can be applied to Eq. (66) in the general case—we therefore focus our attention on Eqs (69). Fig. 14 shows the direction field of Eq. (69a) for a function $Q(c)$ of the type sketched in Fig. 12. The singularities at O and Q are both saddle points, and the integral curve we want must coincide with the separatrix of positive slope at O and that of negative slope at Q. From the foregoing example, we expect that the two separatrices will coincide only for particular values of v.

It is clear from Fig. 14 that $c' = q > 0$, i.e., that $c(x)$ is monotone increasing, so that the traveling wave propagating from right to left does in fact look like the curve sketched in Fig. 13.

The form of Eq. (69a) suggests that if $S_1(c) > S_2(c)$, then $v_1 \leq v_2$ for the same $Q(c)$. The special case in which $S(c)$ is a constant confirms this supposition [cf. Eq. (71c)]. A proof for general $S(c) > 0$ is as follows:

$$h_{1t}(c_1) + q_z(c_1, c_{1z}) = Q(c_1) \tag{72a}$$

$$h_{2t}(c_2) + q_z(c_2, c_{2z}) = Q(c_2) \tag{72b}$$

If we subtract Eq. (72b) from Eq. (72a) and proceed as we did in section 7, we find

$$S_1 u_t = -(\partial q/\partial c_z)u_{zz} - [\partial/\partial z(\partial q/\partial c_z) + \partial q/\partial c]u_z$$

$$- [(dS_1/dc)c_t + \partial/\partial z(\partial q/\partial c) - dQ/dc]u$$

$$- [S_1(c_2) - S_2(c_2)]c_{2t} \tag{73}$$

Now if we take the initial condition $c_2(z,0)$ to be the traveling wave solution belonging to $S_2(c)$ at $t = 0$, then $c_2(z,t)$ is the same traveling wave solution at any t. If we take $c_1(z,0) = c_2(z,0)$, then $c_1(z,0)$ eventually matures into the traveling wave solution belonging to $S_1(c)$. Now the last term in Eq. (73) is negative since $S_1(c_2) > S_2(c_2)$ by hypothesis and $c_{2t} > 0$ (since for the traveling wave solution $c' > 0$). Thus, by reasoning now familiar, $c_1(z,t) \leq c_2(z,t)$, which means $v_1 \leq v_2$. Note that the above proof applies no matter what the form of $Q(c)$.

It also follows similarly from the comparison theorem of section 7, that if $Q_1(c) < Q_2(c)$, then $v_1 \leq v_2$ for the same $S(c)$.

These comparison theorems are useful for bounding the velocity v by using results like that of Eq. (71) in cases where Eq. (69a) is difficult to solve.

22. Traveling Wave Solutions (part 3)

When $Q(c)$ is non-negative (case (1) of section 19), the traveling wave solutions cannot have the shape shown in Fig. 13, for in this figure the values $c = 0$ and $c = c_m$ are both roots of $Q(c)$. What, then, is the asymptotic behavior at large x of the traveling wave solution that progresses from right to left?

The best way to answer this question is to start by studying the simple case in which S = constant and $Q = 0$ for $c < c_s$ and Q = constant for $c > c_s$. In this case, the solution $q = vSc$ is the unique solution for which $q(0) = 0$ when $c < c_s$. The direction field of Eq. (69a) for $c > c_s$ is shown in Fig. 15.

Suppose we now consider traveling wave solutions that grow from a localized disturbance (see Fig. 16). The left-hand halves of the c-profiles in Fig. 16 eventually mature into a traveling wave moving to the left. When plotted as curves of $q = c'$ versus c, these c-profiles look like the lower family of curves in Fig. 15, i.e., the family for which $q = 0$ at the maximum value of c [cf. Fig. 16(ii)]. At time goes on and the central value of c increases, the q-c profile approaches the dotted curve in Fig. 15 near $c = c_s$. So eventually $q = Q/vS$ at $c = c_s$. If we join this solution to the solution $q = vSc$ that obtains for $c < c_s$, we find

$$v = (Q/c_s)^{1/2}/S \tag{74}$$

The separatrix $q = Q/vS$ corresponds to the asymptotic behavior $c \sim Qx/vS$ of the traveling wave solution, which replaces the condition $q(c_m) = 0$.

Such a separatrix exists in many cases, but it does not always exist as we show next.

23. Traveling Wave Solution: Counter-example

Consider now the case S = constant, $Q = 0$ for $c < c_s$ and $Q = Q_* c$ for $c > c_s$. In this case the direction field of Eq. (69a) for $c > c_s$ is shown in Fig. 17. The line L_0: $q = Q_* c/vS$ represents the locus of zero slope ($dq/dc = 0$). The lines L_+ and L_- are the integral curves of Eq. (69a) $q = m_+ c$ and $q = m_- c$, respectively, where $m_\pm$ are the roots of the quadratic equation

$$m^2 - vSm + Q_* = 0 \tag{75}$$

It follows from Eq. (75) that $m_+ + m_- = vS$ and $m_+ m_- = Q_*$, so that the slope $m_0 = Q_*/vS$ equals $m_+ m_-/(m_+ + m_-)$. Since both m_+ and m_- are positive, $m_0 < m_- < m_+$. Thus the representation given in Fig. 17 is correct. It is clear, then, that the family of integral curves that resemble the c-profiles of Fig. 16(ii) is asymptotic to the integral curve L-. The velocity is then given by the condition

$$vS = m_- \tag{76a}$$

which is the same as

$$vS = -[(vS)^2 - 4Q_*]^{1/2} \tag{76b}$$

No solution for v exists for Eqs. (76b).

The reasoning presented above is based on the assumption that Eq. (75) has real roots, i.e., that the discriminant $(vS)^2 - 4Q_*$ is ≥ 0. If it is not, the integral curves represented by the lines $L_\pm$ do not exist. Then the transient q-c-profiles continue to increase without any asymptotic limit so that no traveling wave solution exists.

The conclusion that there is no traveling wave solution leaves moot the question of what really happens. Now since the foregoing argument does not depend on c_s having any special value, it holds as well when $c_s = 0$. In this case, the time-dependent Eq. (55b) can be can be solved just as we solved Eq. (23) in section 7:

$$c = \exp(-z^2 S/4t + Q_* t/S)/(4\pi t/S)^{1/2} \tag{77}$$

The locus of a constant value of c in the (z,t)-plane is given by

$$z = [4Q_*t^2/S^2 - (2t/S)\ln(4\pi t/S) - (4t/S)\ln c]^{1/2} \tag{78a}$$

$$= 2Q_*^{1/2}t/S - (1/[2Q_*^{1/2}])\ln(4\pi t/S) - \ln c/Q_*^{1/2} + \ldots \tag{78b}$$

so that

$$dz/dt = 2Q_*^{1/2}/S - 1/[2Q_*^{1/2}t] + \ldots \tag{78c}$$

Thus the locus always accelerates, but at an ever decreasing rate so that for very large t, it is essentially moving with a "velocity" $v = 2Q_*^{1/2}/S$.

24. Traveling Wave Solutions: Geometric Considerations

From the analysis presented so far it is clear that the study of the direction field of Eq. (69a) sheds light on the questions of the existence and nature of the traveling wave solutions. If the locus of zero slope L_0: $q = Q(c)/vS(c)$ is asymptotically monotone decreasing (see Fig. 18, for example) then a separatrix S must exist that is the asymptotic limit of the lower family of curves. In such a case, we may expect a traveling wave solution to exist. Furthermore, if L_0 is monotone decreasing for all $c > c_s$, then

$$v \int_0^{c_s} S(c)\, dc < Q(c_s+)/vS(c_s) \tag{79a}$$

so that

$$v < \{Q(c_s+)/[S(c_s) \int_0^{c_s} S(c)\, dc]\}^{1/2} \tag{79b}$$

The bound in Eq. (79b) resembles an estimate of Whetstone and Roos[11].

If L_0 is monotone increasing for $c > c_s$, it appears difficult to decide the question of the existence of traveling wave solutions. For, in the case of the counter-example of section 23, it was not possible to rule out traveling wave solutions solely with a geometric argument based on Fig. 17. Instead, a contradiction had to be obtained by solving Eq. (69a) explicitly.

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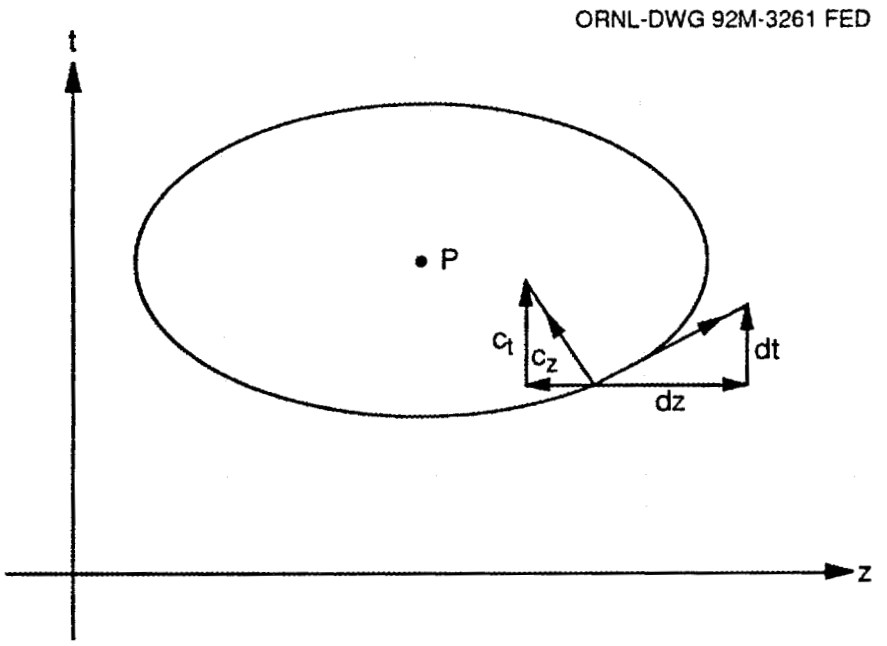

Fig. 1. A level curve of c surrounding the maximum at P in the (z,t)-plane.

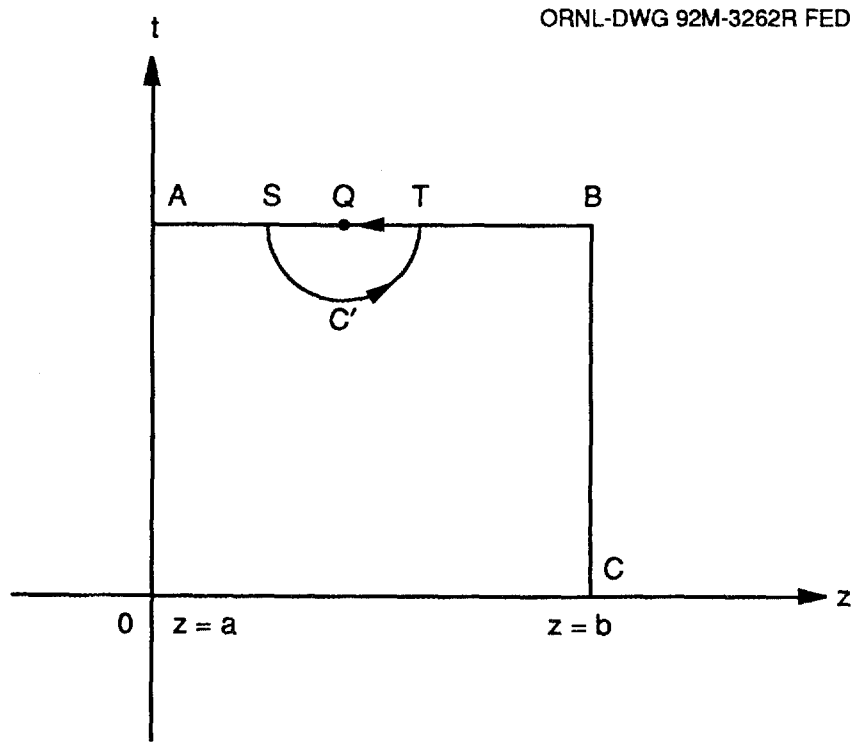

Fig. 2. The rectangular region OABC in the (z,t)-plane whose sides are parallel to the axes. The largest value of c is assumed to occur at point Q.

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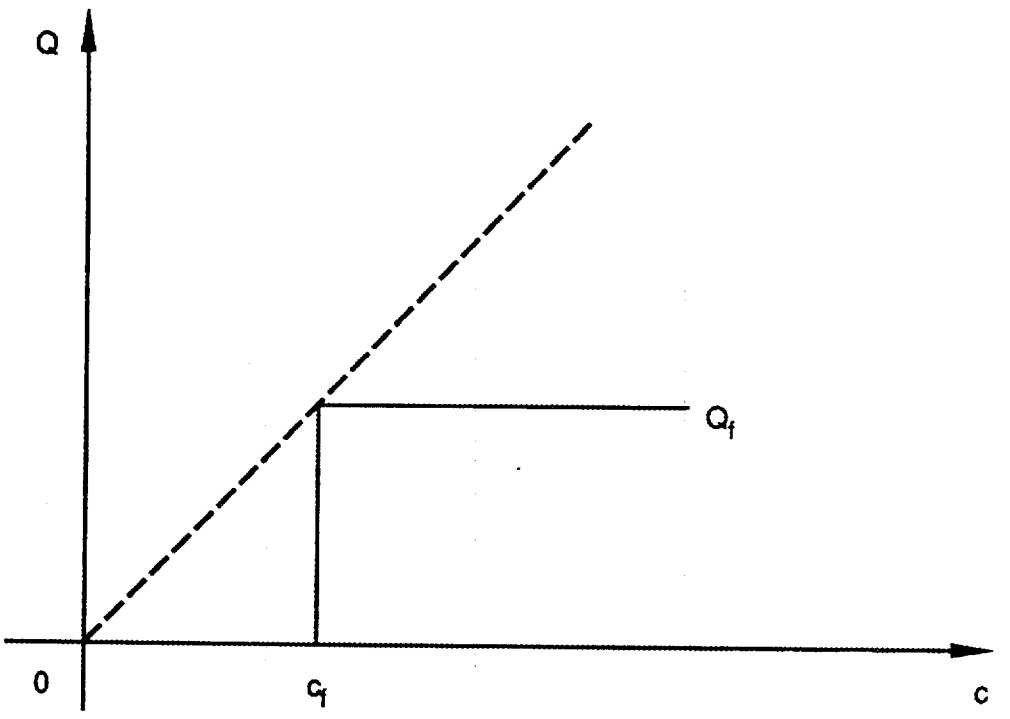

Fig. 3. The source term $Q(c)$ in Eq. (23) is a step function.

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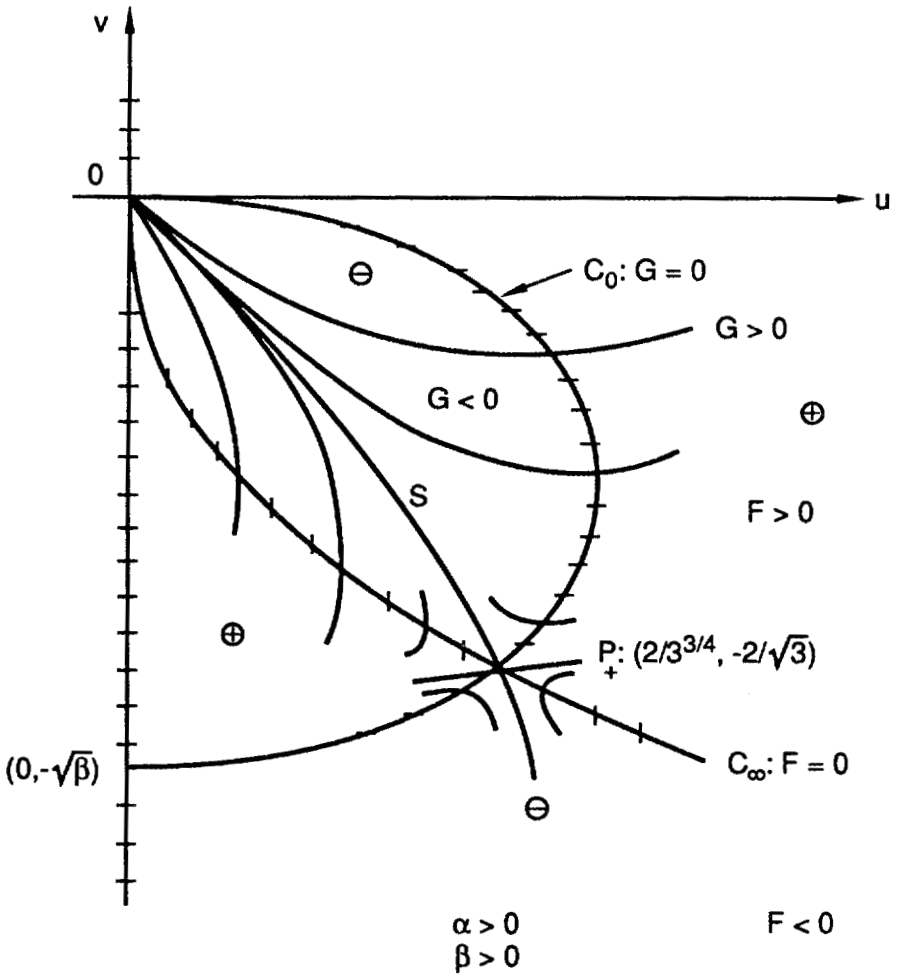

Fig. 4. The fourth quadrant of the phase diagram of Eq. (35) when $\alpha > 0$, $\beta > 0$. The heavy curves C_0 and C_∞ divide the quadrant into regions in each of which the slope dv/du has one sign only (shown encircled).

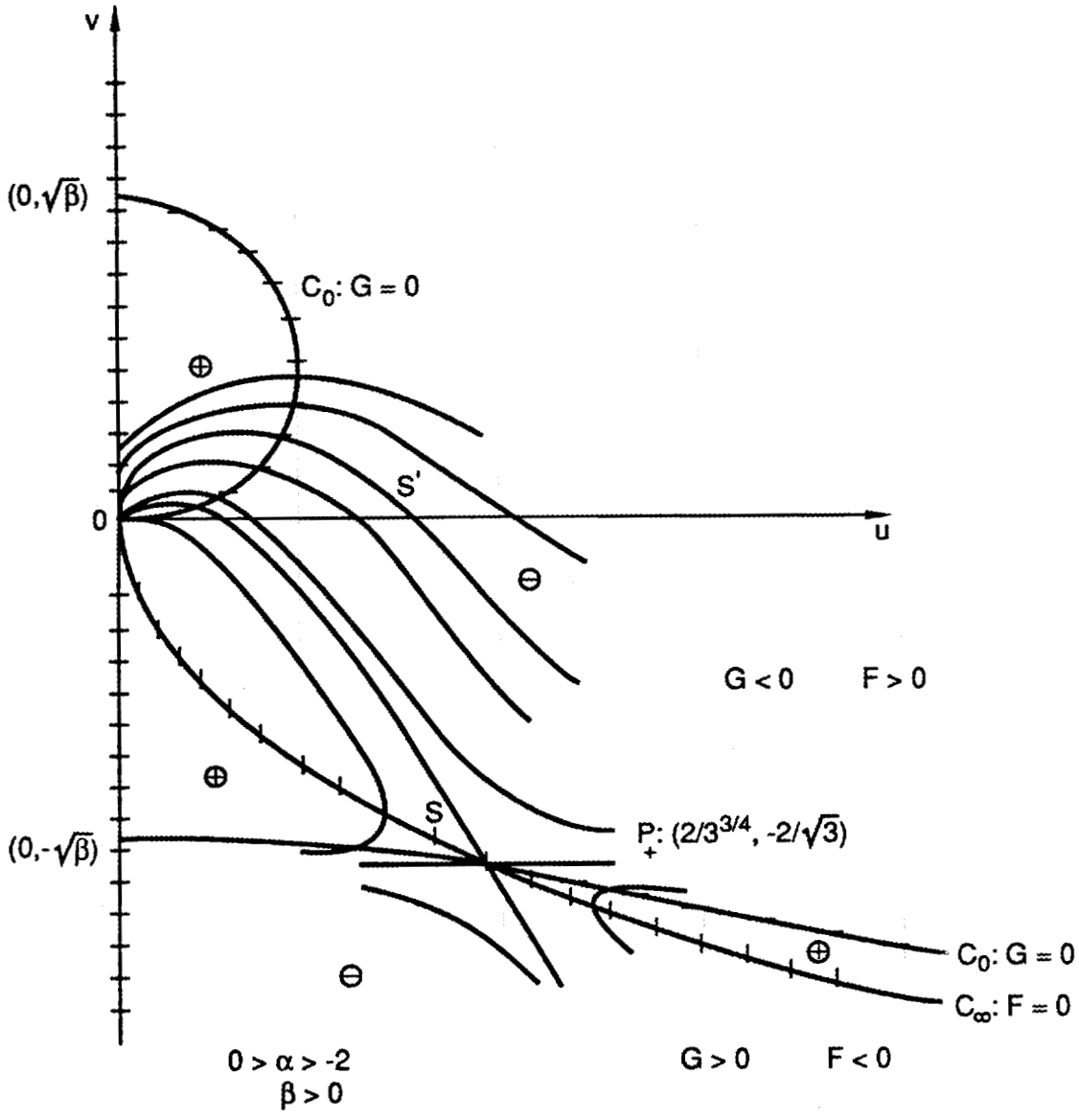

Fig. 5. The first and fourth quadrants of the phase diagram of Eq. (35) when $-2 < \alpha < 0$, $\beta > 0$. The heavy curves C_0 and C_∞ divide the quadrants into regions in each of which the slope dv/du has one sign only (shown encircled).

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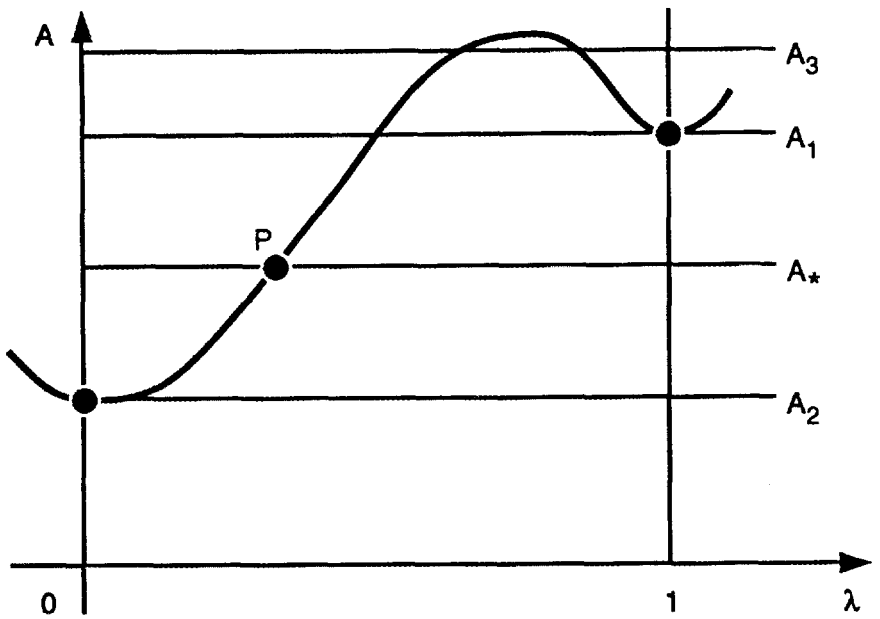

Fig. 6. A sketch of the action (52) for the family of initial conditions (54) when there are just two steady states.

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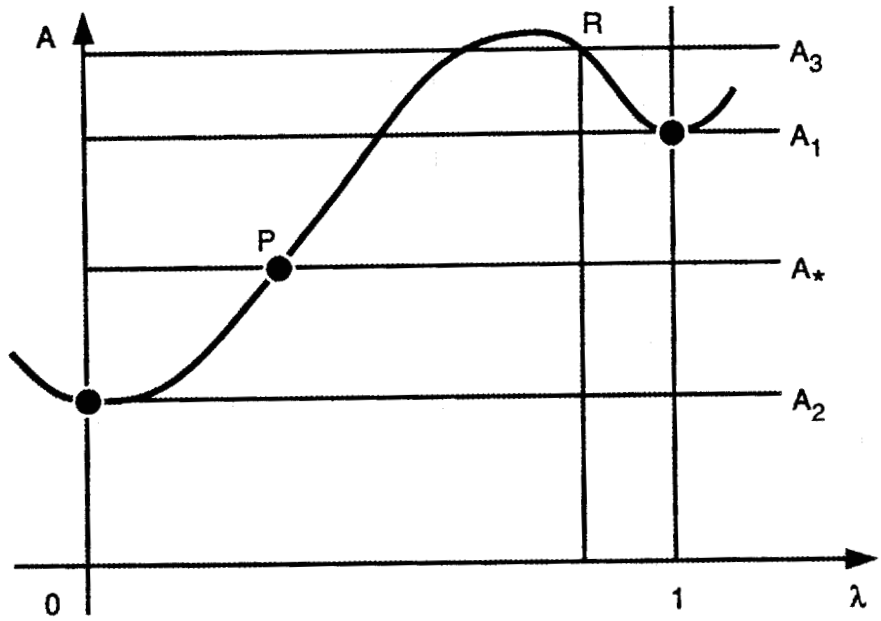

Fig. 7. A sketch of possible behavior of the action (52) for the family of initial conditions (54) when there are more than two steady states.

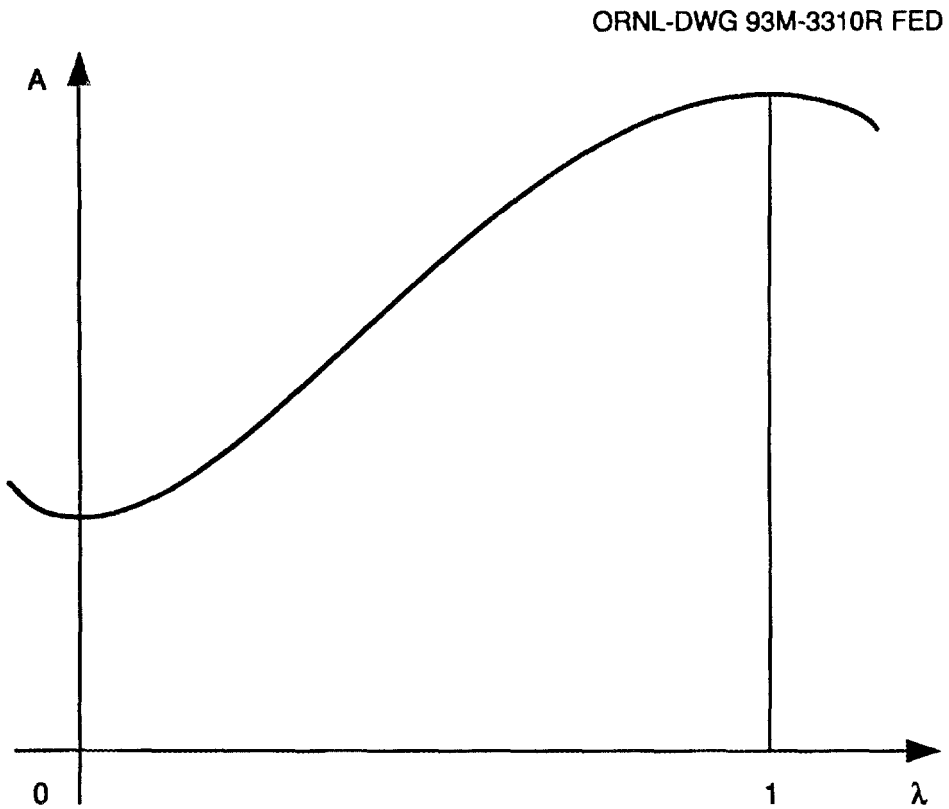

Fig. 8. A sketch of the action for the family of initial conditions (54) when $Q(c)$ is concave upwards.

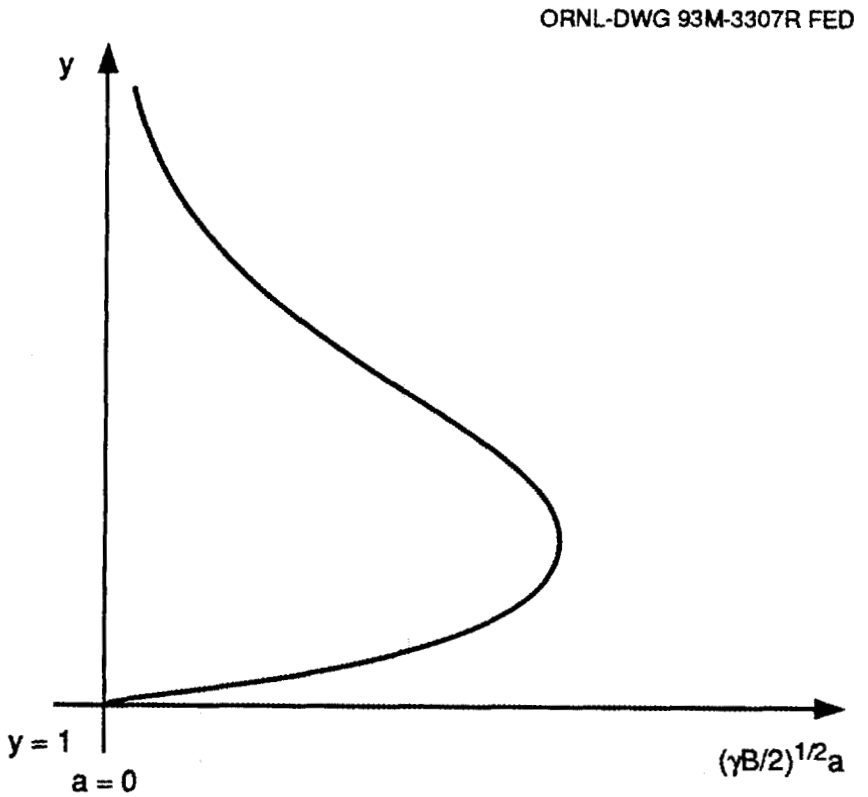

Fig. 9. A sketch of Eq. (61) showing y plotted as ordinate and $(\gamma B/2)^{1/2}a$ as abscissa.

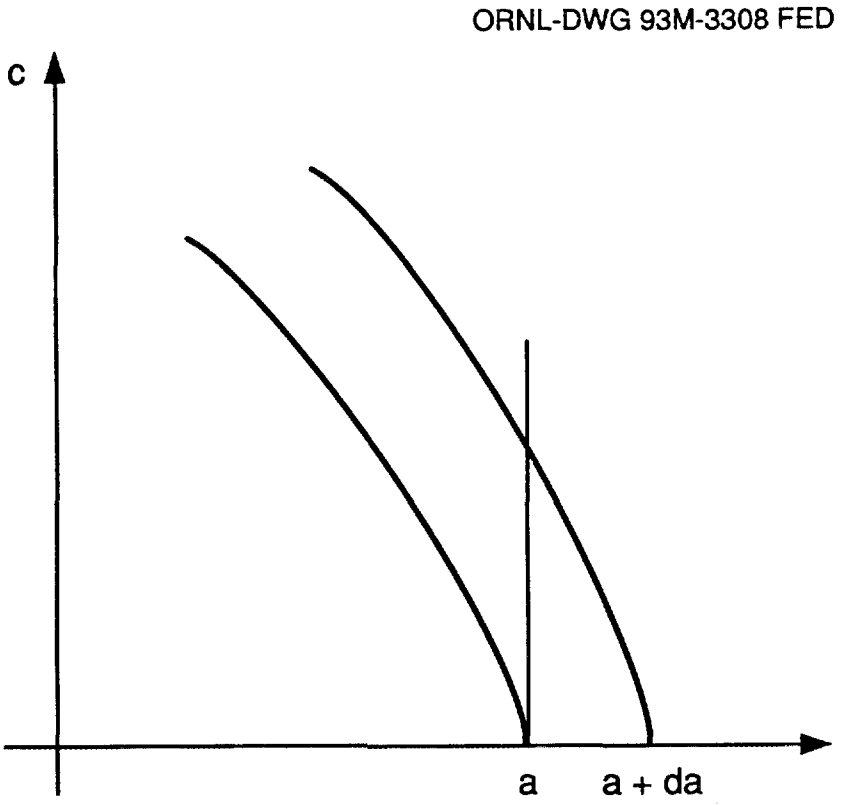

Fig. 10. Geometric relations near the foot of the solution curve.

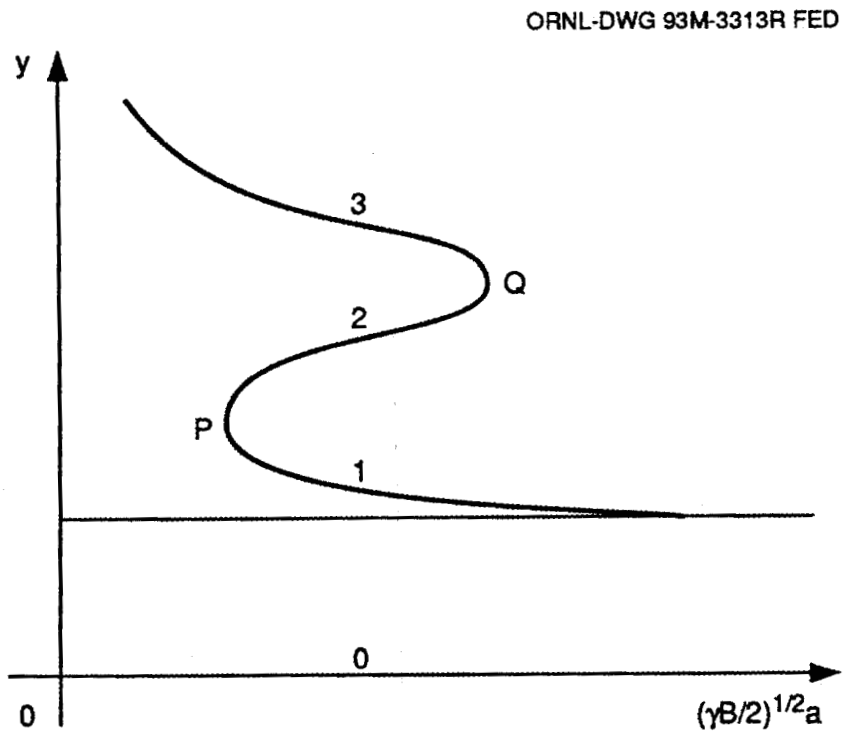

Fig. 11. A sketch of Eq. (64) showing y plotted as ordinate and $(\gamma B/2)^{1/2}a$ as abscissa when $\gamma c_s/2 < 0.2$.

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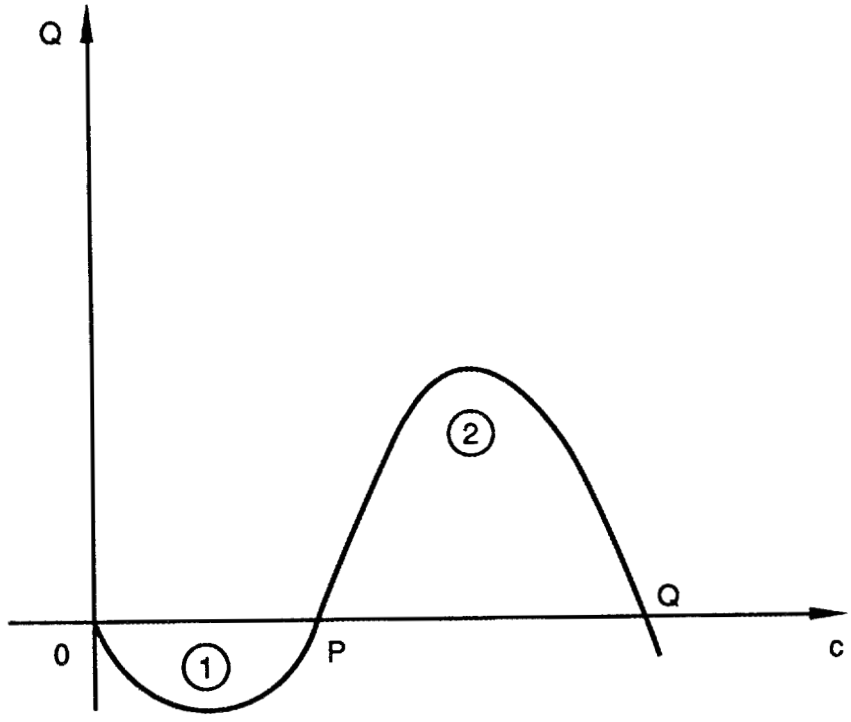

Fig. 12. One kind of source term $Q(c)$ of interest in applied superconductivity.

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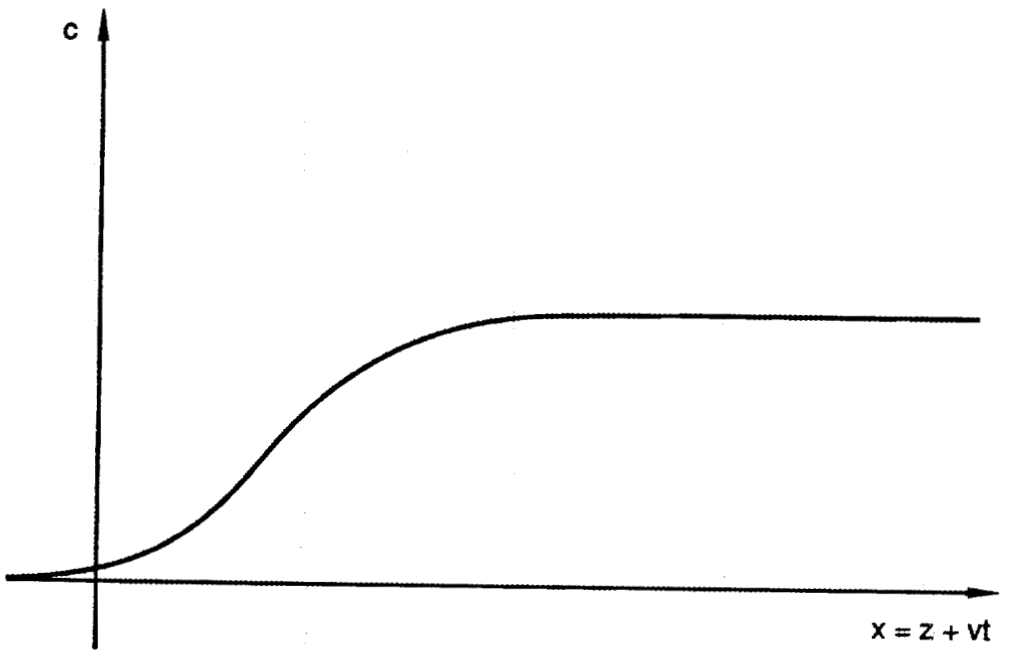

Fig. 13. The instantaneous profile of a traveling wave solution advancing from right to left.

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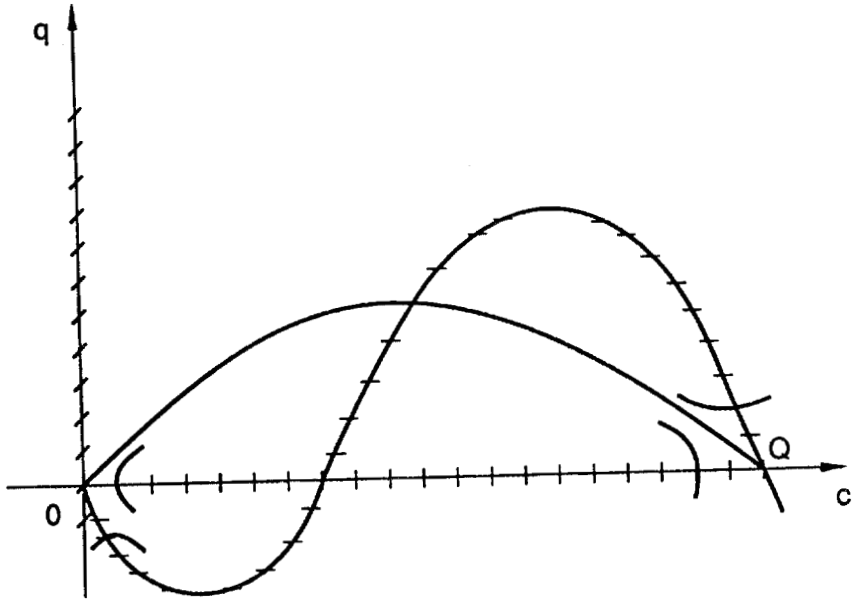

Fig. 14. A sketch of the direction field of Eq. (69a) for a source term of the type shown in Fig. 12.

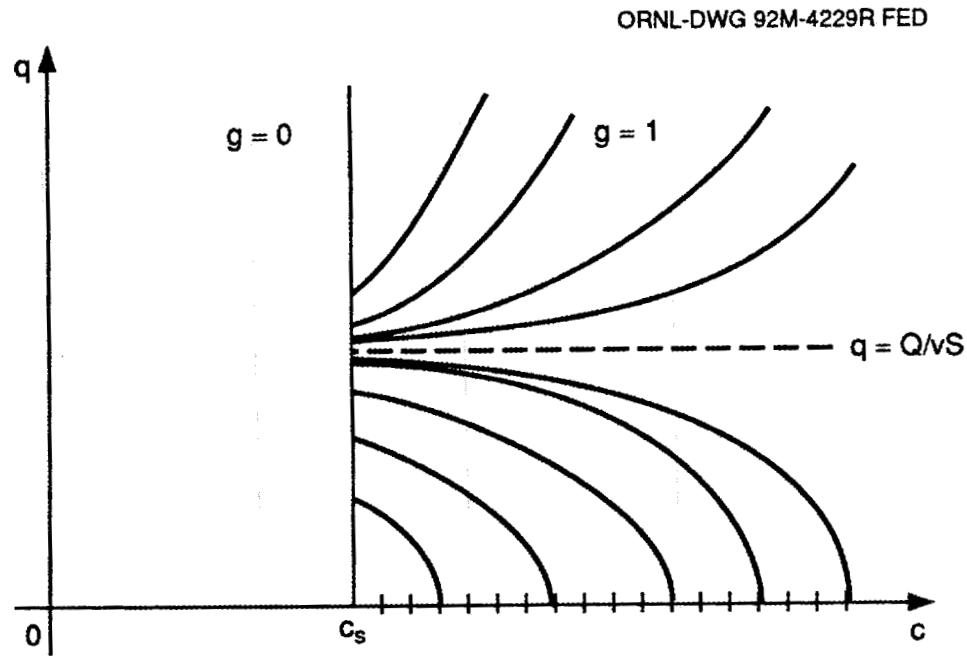

Fig. 15. A sketch of the direction field of Eq. (69a) for $c > c_s$ in the simple case in which S = constant and $Q = 0$ for $c < c_s$ and Q = constant for $c > c_s$.

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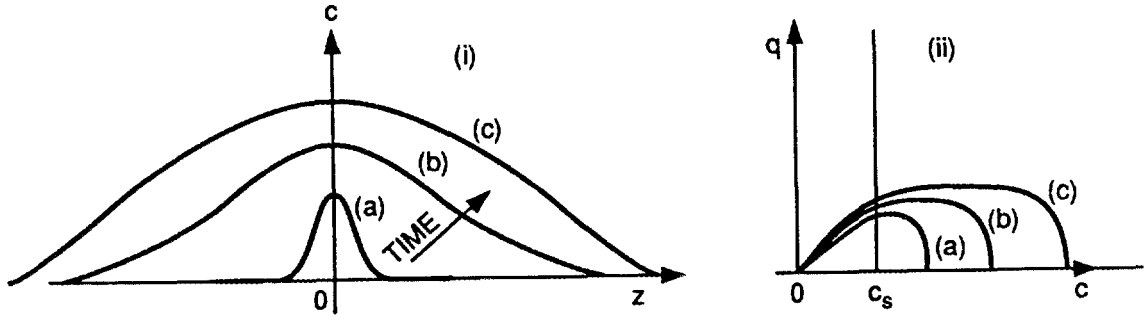

Fig. 16. Part (i): sketches showing the development of a localized distrubance. Part (ii): sketches showing the associated profiles of *q* versus *c*.

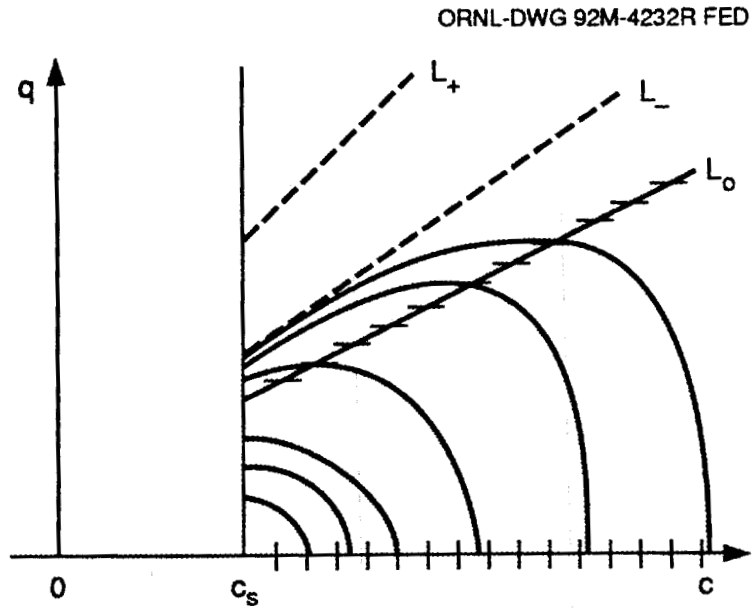

Fig. 17. A sketch of the direction field for $c > c_s$ in the case in which S = constant and $Q = 0$ for $c < c_s$ and $Q = Q_0 c$ for $c > c_s$.

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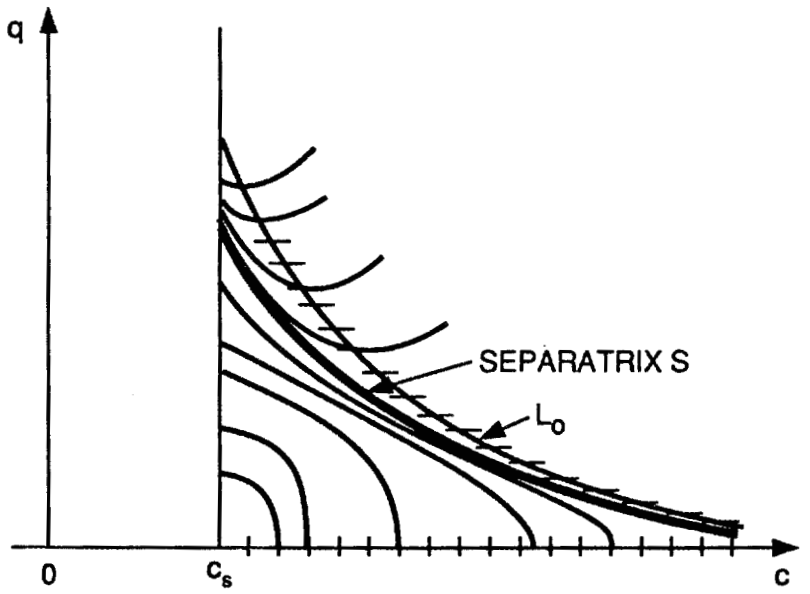

Fig. 18. A sketch of the direction field for $c > c_s$ when the locus L_0 of zero slope is monotone decreasing.

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