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A Rigid Body Model and Decoupled Control Architecture for Two Manipulators Holding a Complex Object

M. A. Unseren

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ORNL/TM-11752
CESAR-91/01

Engineering Physics and Mathematics Division

A RIGID BODY MODEL AND DECOUPLED CONTROL ARCHITECTURE FOR TWO MANIPULATORS HOLDING A COMPLEX OBJECT

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DATE PUBLISHED — January 1991

Prepared for the
Engineering Research Program
Office of Basic Energy Sciences

Prepared by the
OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37831
managed by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
for the
U.S. DEPARTMENT OF ENERGY
under contract DE-AC05-84OR21400

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CONTENTS

ABSTRACT v

1. INTRODUCTION 1

2. PROBLEM STATEMENT AND SYSTEM DESCRIPTION 5
   2.1. SYSTEM VARIABLES AND COORDINATE FRAMES 5
   2.2. MANIPULATOR MODELS 8

3. DYNAMIC COUPLING 9

4. KINEMATIC COUPLING 13

5. CLOSED-CHAIN MODEL IN JOINT SPACE 15
   5.1. SOLVING FOR FORWARD DYNAMICS 18
   5.2. SOLVING FOR INVERSE DYNAMICS 18

6. REDUCED ORDER MODEL AND CONTACT FORCE DETERMINATION 21
   6.1. SOLVING FOR FORWARD DYNAMICS 22
   6.2. SOLVING FOR INVERSE DYNAMICS 24

7. CONTROL ARCHITECTURE 25

8. CONCLUSION 29

9. ACKNOWLEDGEMENT 31

REFERENCES 33

APPENDIX A 37

APPENDIX B 41

APPENDIX C 43

Abstract

A rigid body dynamical model and control architecture are developed for the closed-chain motion of two manipulators holding an object containing a spherical joint in a three-dimensional workspace. The manipulators may have an equal or unequal number of joints. Dynamic and kinematic constraints are determined and combined with the equations of motion of the manipulators to obtain a dynamical model of the entire system in the joint space. The problem of solving the joint space model for the unknown variables is discussed. This includes a new proof that the problem of solving the model for the generalized input forces applied to the joint actuators (i.e., computing the joint torques) is underspecified in nature. The system is transformed to reduce the dimensionality of the model and eliminate the generalized contact forces, which are calculated separately. The problem of solving the reduced order model for the unknown variables is also addressed. A new control architecture consisting of the sum of the outputs of a primary and secondary controller is suggested which, according to the model, decouples the force- and position-controlled degrees of freedom during motion of the system. The proposed composite controller enables the designer to develop independent, non-interacting control laws for the force- and position-control of the complex closed-chain system.

1 INTRODUCTION

The lifting and transport of a jointed object, i.e., an object consisting of two rigid bodies connected by a joint, invariably requires two-handed cooperative manipulation. It would be difficult for a single manipulator to transport such an object in a safe and controlled manner since it could only hold one of the rigid bodies comprising the object while the other swings freely. Additionally, the mass of the object may be beyond the carrying capacity of a single manipulator. However, two cooperating manipulators could securely hold such an object by its two rigid bodies and move it in a desirable, well-behaved manner. Unfortunately, the addition of a second manipulator to accomplish the task leads to a more complicated system. Indeed, when two manipulators mutually hold an object, the three form a single closed-chain mechanism and a loss of degrees of freedom(DOF) occurs. Strong kinematic and dynamic interactions between the two manipulators due to the shared payload result in a constrained system motion. The manipulators can no longer function independently in this closed-chain configuration. The importance of understanding the problems and issues associated with controlling two cooperating manipulators is discussed in [37].

Various dynamical models have been developed and position-only as well as position/force control schemes have been proposed for two manipulators holding a rigid object[9 – 31], but the literature on the modeling and control of two manipulators holding a complex, jointed object is very scarce. Kinematic constraints and an entire system model for the closed-chain consisting of two six-axis manipulators holding a jointed object are developed in [9] by considering two types of objects: a pair of pliers and a part containing a spherical joint. The common load is viewed only as a pure kinematic structure imposing constraints on the motions of the manipulators. One manipulator is the leader (master), the other is the follower (slave). The joint velocities and accelerations of the follower are determined from the kinematic constraints when the joint positions of both manipulators and the joint velocities and accelerations of the leader are given. A minimum Euclidean norm solution for the joint torques for trajectory tracking is computed from the model using pseudoinverse techniques in [9]. However, it does not address the problem of controlling the object contact forces. Also, the proposed control law in [9] is open-loop and involves no servoing.

The significance of the motion dynamics in the modeling and control of a manipulator when the end effector is strongly constrained by environmental interactions has only recently been recognized [1 – 8]. Such environmental interactions are produced by hard, physical contact between the end effector and rigid structures in the work area. Such systems usually contain closed-chains and the manipulator motion is governed by its own dynamics, the generalized contact forces which arise due to the contact, and the kinematic constraints imposed on the motion of the manipu-

lator [1 – 8]. In particular, Kankaanranta and H.N. Koivo [1, 2] and McClamroch and Wang [3] have recently derived dynamical models and control laws for the hard contact motion of a single manipulator. The kinematic constraints governing the generalized coordinates are modeled as a constraint function and incorporated into the manipulator equations of motion to obtain a dynamical model for the entire closed-chain system. Using the constraint function, transformations are introduced which separate the model into two sets of equations. One set of equations characterizes the manipulator motion along the constraint surface and contains no generalized forces of contact. The other set of equations is used to calculate the generalized contact forces which are caused by strong interactions between the end effector and the constraint surface. Nonlinear controllers are proposed which, according to the latter form of the model, lead to the exact decoupling of the force- and position-controlled directions of motion [1, 2, 3].

The work of Kankaanranta et al. [1, 2] and McClamroch et al. [3] demonstrated that considerable insight into the design of a control architecture for the complex, constrained motion of a single manipulator can be obtained by considering the dynamical model in the analysis. Motivated by their results, the general theory discussed in [1, 2, 3] has recently been extended to the modeling and control of a pair of N-joint manipulators holding a rigid object in a three-dimensional workspace [38, 39]. In [1, 2, 3], the coupling effects between the manipulator and the environment, e.g., a manipulator turning a crank, are taken into account by regarding the environment, i.e., the crank, only as a pure kinematic structure imposing constraints on the motion of the manipulator. In our recent work [38, 39], the rigid object held by two manipulators is considered to be large and heavy; thus its dynamics can be significant and have been accounted for in the analysis. The control architecture proposed in [38, 39] is such that the position variables and an independent subset of the generalized contact forces are decoupled, i.e., they can be controlled independently.

When the payload held by two manipulators possesses a joint, the overall system is more complicated and has more DOF when compared to the rigid payload case. Furthermore, to accomplish successfully the task of lifting and transporting such an object using two cooperating manipulators requires force control in addition to position control. Indeed, the generalized contact forces imparted to the object by the manipulators must be controlled in addition to controlling the position of the system. In order to better understand the problems associated with two strongly interacting manipulators, a model describing the dynamic behavior of the system is required. In this report a rigid body dynamical model for two manipulators holding an object containing a spherical joint in a three-dimensional workspace is first developed in the joint space. The system is transformed to obtain a reduced order model not explicit in the generalized contact forces, which are calculated separately. Additionally, a control architecture, which according to the latter form of the model decouples the

force- and position-controlled DOF during motion of the system, is suggested. For the case of two manipulators holding a jointed object, this is a new result that enables the designer to develop independent, non-interacting control laws for the force- and position-control of the closed-chain system.

This report is a first step towards analyzing the increased dynamic complexity which occurs when the mutually held payload possesses a joint which is structurally distinct from the single-DOF joints contained in the two manipulators. It generalizes the approach given in [38, 39] and the manipulators are no longer restricted to have the same number of joints. The dynamics of all components of the closed-chain including those of the complex payload are accounted for in the rigid body model and control architecture presented here.

The problem of determining the solution of the model for the forward dynamics(i.e., to determine the output response to given inputs) and inverse dynamics(i.e., to determine the required inputs when the desired output response of the system is specified) is not addressed in the previous work on the modeling and control of a single manipulator [1, 2, 3]. For the case of two manipulators holding a rigid object, the solution for the forward dynamics is well-specified and for the inverse dynamics underspecified [38, 39]. The problem of determining the nature of the solution to the closed-chain model for two manipulators holding a spherically jointed object is investigated here.

The rigid body model and control architecture developed in this report can form the basis for an analysis of the closed-loop system properties and behavior which arise from the use of various force- and position-control schemes proposed in the literature [32, 33]. The position controlled variables explicit in the model are general "pseudovariables" to be selected by the designer for a specific application. It should be mentioned that the control architecture derived here assumes perfect knowledge of the dynamical terms in the entire system model. Also, the computational requirements for the suggested controller are not addressed in this report.

The report is organized as follows. To model the closed-chain mechanism, the dynamical and kinematic couplings between the manipulators due to the common load are first determined. The equations of motion of the manipulators are combined with the dynamic and kinematic constraints to obtain a closed-chain system model in the joint space. A reduced order model is derived that is not a function of the generalized contact forces, which are calculated separately. The problem of solving the joint space and reduced order models for the forward and inverse dynamics is discussed. Finally, a decoupled control architecture is developed based on the reduced order model and the generalized contact force equations. A condensed version of this report can be found in [40].

2 PROBLEM STATEMENT AND SYSTEM DESCRIPTION

The problem is to develop a dynamical model and control architecture for two serial-link manipulators holding an object in a three-dimensional workspace. Manipulator i(i=1,2) has a stationary base and contains N_i single-DOF joints. Thus the two manipulators can have an equal($N_1 = N_2$) or unequal($N_1 \neq N_2$) number of joints. The common load securely held by both manipulators consists of two rigid bodies connected by a three-DOF spherical joint. There is no relative motion between the end effector of manipulator i and rigid body i(i=1,2) of the object, i.e., no slipping. It is assumed that the manipulators are initially holding the object. The problem of modeling and controlling the manipulators when they initially acquire (grasp) the object which results in collisions between the end effectors and shared load is not considered in this report. The dynamical models of both manipulators are assumed to be known. The configuration of the system is shown in Fig. 1.

2.1 System Variables and Coordinate Frames

The joint positions of the two manipulators are the generalized coordinates describing the configuration of the system. The system variables include the generalized coordinates, velocities, and accelerations, the generalized contact forces exerted by the end effectors on the common object, the contact forces transmitted across the spherical joint of the shared load, and the generalized input forces(i.e., the joint torques) applied to the joint actuators.

A stationary world coordinate frame (X_w, Y_w, Z_w) shown in Fig. 1 serves as a reference frame. The location of this coordinate system is based upon the task geometry. As shown in Fig. 1, the coordinate frame $(X_k^{(i)}, Y_k^{(i)}, Z_k^{(i)})$ is assigned to the k_{th} link of manipulator i(=1,2), where $k = 0, 1, \ldots, N_i$.

An orthogonal (3×3) rotation matrix ${}^{i}R_w^{N_i}(q_i)$ describes the orientation of the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate frame which has its origin at the centerpoint of the end effector of manipulator i in the world coordinates. The tips of the (3×1) vectors ${}^{i}r$ and ${}^{iw}r$ emanating from the centerpoint of the end effector of manipulator i coincide with point CM_{oi}, the center of mass of the rigid body i of the object, as shown in Fig. 2. Vectors ${}^{i}r$ and ${}^{iw}r$ are expressed in the local and world coordinate systems, respectively. They are related by:

$$ {}^{iw}r(q_i) = {}^{i}R_w^{N_i}(q_i)\,{}^{i}r\,. \tag{1} $$

The tips of the (3×1) vectors ${}^{il}r$ and ${}^{ilw}r$ emanating from the centerpoint of the end effector of manipulator i coincide with point Q_{cs}, the center of the spherical joint, as shown in Fig. 2. Vectors ${}^{il}r$ and ${}^{ilw}r$ are expressed in the local and world coordinate systems, respectively. They are related by:

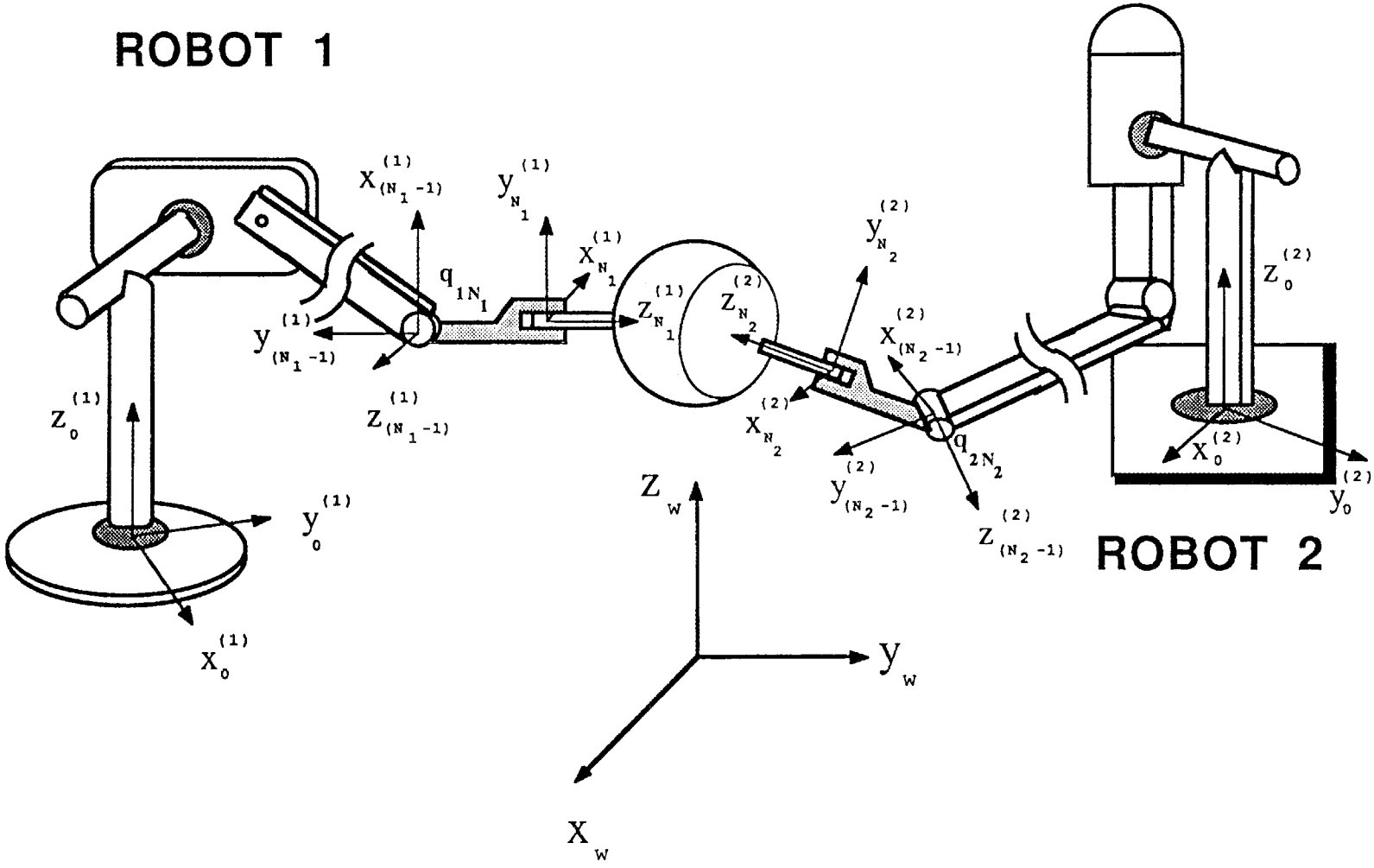

Figure 1. System Configuration and Coordinate System Assignment

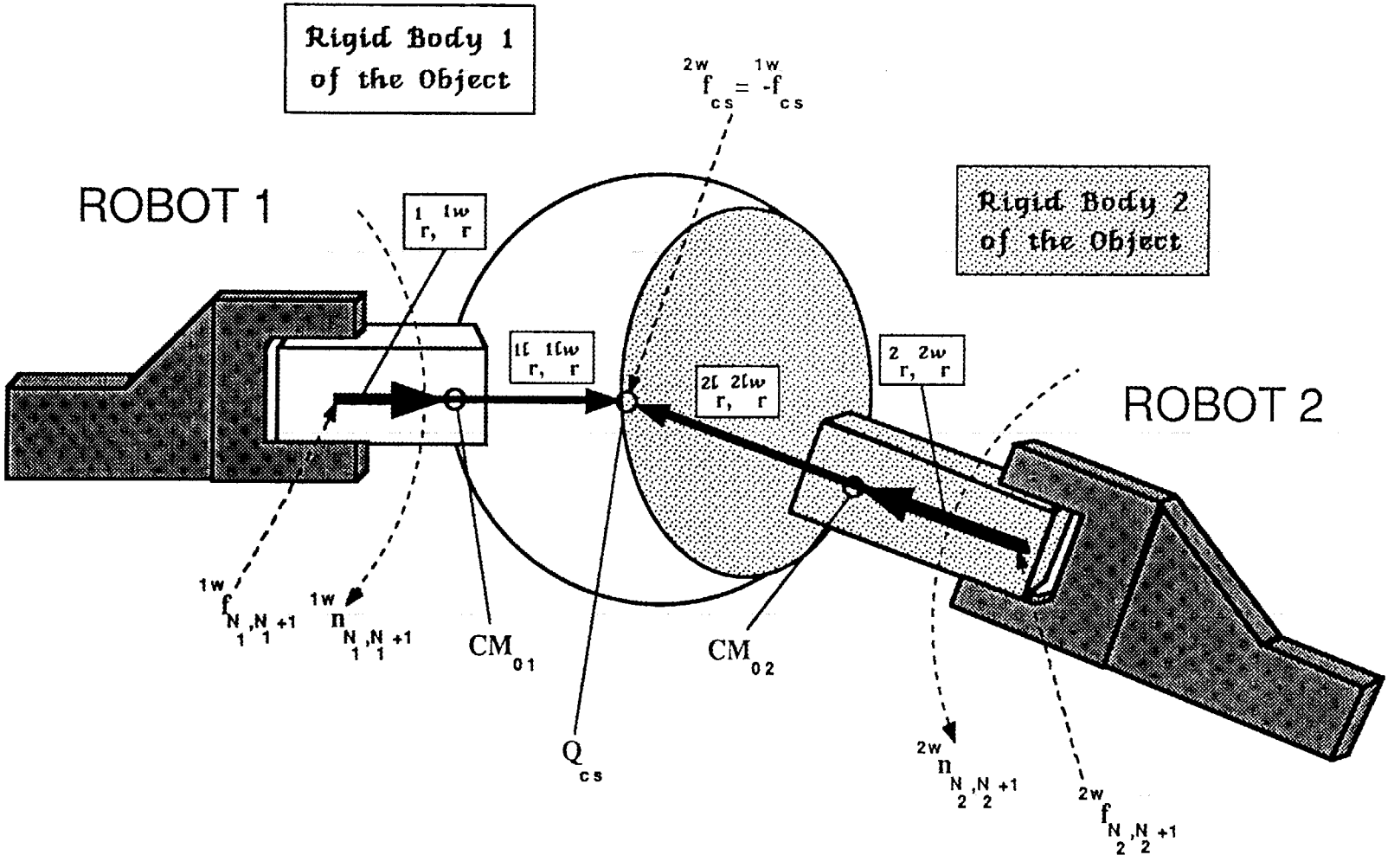

Figure 2. Freebody Diagram for the Object with Spherical Joint

$$^{ilw}r(q_i) = {}^{i}R_w^{N_i}(q_i)\,{}^{il}r\,. \tag{2}$$

2.2 Manipulator Models

The dynamical model for an individual serial-link manipulator i (=1,2) shown in Fig. 1 whose end effector is exerting a generalized contact force f_{ci} on the environment, i.e., on the spherically jointed object, is given in Lagrange's formulation by:

$$^{i}\tau \;=\; D_i(q_i)\,\ddot{q}_i \;+\; C_i(q_i\,,\,\dot{q}_i) \;+\; J_{iw}^T(q_i)\,f_{ci} \tag{3}$$

where superscript T denotes a transposition. The joint positions of manipulator i are represented by vector $q_i = [q_{i1}, q_{i2}, \ldots, q_{iN_i}]^T$ and the joint torques applied to the joint actuators by the vector $^{i}\tau = [^{i}\tau_1, {}^{i}\tau_2, \ldots, {}^{i}\tau_{N_i}]^T$. The $(N_i \times N_i)$ symmetric and positive definite inertia matrix is $D_i(q_i)$, and the Coriolis, centripetal, and gravity forces for manipulator i are described by the $(N_i \times 1)$ vector $C_i(q_i, \dot{q}_i)$. Manipulator i imparts contact forces and contact torques to the object at the contact surfaces between its end effector and the load. The (6×1) generalized contact force vector f_{ci} in Eq. (3) represents the equivalent contact force and torque acting at the origin of the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate system, i.e., at the centerpoint of the end effector of manipulator i. The equivalent generalized force at the centerpoint produces the same effect on the object as the applied forces and torques at the contact surfaces. f_{ci} is expressed in the world coordinates. It is comprised of a (3×1) force vector $^{iw}f_{N_i,N_i+1}$ and a (3×1) torque vector $^{iw}n_{N_i,N_i+1}$ shown in Fig. 2:

$$f_{ci} \;=\; \left[{}^{iw}f_{N_i,N_i+1}^T,\; {}^{iw}n_{N_i,N_i+1}^T\right]^T\,. \tag{4}$$

In Eq. (3), the $(N_i\times6)$ transposed Jacobian matrix $J_{iw}^T(q_i)$ of manipulator i transforms the generalized contact force f_{ci} to the joint space.

The equations of motion for manipulators 1 and 2 defined in conjunction with Eq. (3) with i=1,2 will be combined along with the equations describing the dynamic and kinematic couplings to obtain the model of the entire system. The dynamic coupling between the manipulators is discussed next.

3 DYNAMIC COUPLING

The motions of the manipulators are dynamically coupled because the generalized contact forces are interrelated through the common object. The coupling effects can be quantified by the dynamics of the load. The object is decomposed into two rigid freebodies connected by a spherical joint. Newton's and Euler's equations for rigid freebody i(i=1,2) of the object shown in Fig. 2 are:

$$m_{ci}\,\vec{\dot{v}}_{ci} = m_{ci}\,\vec{g} + {}^{iw}\vec{f}_{N_i,N_i+1} + {}^{iw}\vec{f}_{cs}\,, \tag{5}$$

$$(D_{ci}\vec{\dot{\omega}}_{ci}) + \vec{\omega}_{ci} \times (D_{ci}\vec{\omega}_{ci}) = {}^{iw}\vec{n}_{N_i,N_i+1} + \left(-{}^{iw}\vec{r} \times {}^{iw}\vec{f}_{N_i,N_i+1}\right) + \left({}^{ilw}\vec{r} - {}^{iw}\vec{r}\right) \times {}^{iw}\vec{f}_{cs} \tag{6}$$

where superscript $\vec{}$ denotes that the quantity is an explicit function of the unit vectors directed along the X_w, Y_w and Z_w axes of the world coordinate frame. In Eqs. (5) and (6), m_{ci} is the mass of the rigid body i, and D_{ci} is the (3×3) symmetric inertia matrix of the rigid body i about its center of mass at point CM_{oi}. The quantities $(D_{ci}\vec{\dot{\omega}}_{ci})$ and $(D_{ci}\vec{\omega}_{ci})$ on the left hand side of eq. (6) denote that the expressions $(D_{ci}\dot{\omega}_{ci})$ and $(D_{ci}\omega_{ci})$ are considered to be single vector quantities, respectively. The vector $\vec{g}$ represents the gravitational acceleration of the rigid freebody. The vectors $[\vec{v}_{ci}\,,\,\vec{\omega}_{ci}]$ and $[\vec{\dot{v}}_{ci}\,,\,\vec{\dot{\omega}}_{ci}]$ denote the Cartesian velocity and acceleration of the center of mass of rigid freebody i, respectively, with $(\vec{v}_{ci}\,,\,\vec{\dot{v}}_{ci})$ being the translational and $(\vec{\omega}_{ci}\,,\,\vec{\dot{\omega}}_{ci})$ the rotational components. In Eqs. (5) and (6), ${}^{iw}\vec{f}_{cs}$ is a vector denoting the contact force the other rigid freebody of the object imparts to rigid body i at point Q_{cs}, as shown in Fig. 2. It is evident that:

$${}^{1w}\vec{f}_{cs} = -{}^{2w}\vec{f}_{cs}\,. \tag{7}$$

It should be noted that there are no contact torques transmitted across the spherical joint, i.e., rotational motion about the center of the spherical joint is allowed. The quantities $-{}^{iw}\vec{r}$ and $({}^{ilw}\vec{r} - {}^{iw}\vec{r})$ are moment arms from point CM_{oi} to the points where ${}^{iw}\vec{f}_{N_i,N_{i+1}}$ and ${}^{iw}\vec{f}_{cs}$ act on rigid body i of the object, respectively.

Since all quantities are to be expressed in matrix/column-vector notation throughout this report, Eqs. (5), (6), and (7) are rewritten in a concise form:

$$\begin{bmatrix} m_{ci}\,I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & D_{ci} \end{bmatrix} \begin{bmatrix} \dot{v}_{ci} \\ \dot{\omega}_{ci} \end{bmatrix} + \begin{bmatrix} -m_{ci}\,g \\ \Omega_{ci}\,D_{ci}\,\omega_{ci} \end{bmatrix} = [L_{mi}(q_i)\,;\,L_{si}(q_i)] \begin{bmatrix} f_{ci} \\ {}^{iw}f_{cs} \end{bmatrix}, \tag{8}$$

$${}^{1w}f_{cs} = -{}^{2w}f_{cs} \tag{9}$$

where all vectors are expressed in the world coordinate system. $I_{3\times3}$ and $0_{3\times3}$ denote (3×3) identity and zero matrices, respectively. f_{ci} is defined in Eq. (4) and ${}^{iw}f_{cs}$

is a (3×1) vector. In Eq. (8), $L_{mi}(q_i)$ is a (6×6) force/torque transmission matrix which transforms the generalized contact force of manipulator i to its equivalent at the center of mass of the rigid body i of the object at point CM_{oi}. The equivalent generalized force at point CM_{oi} produces the same effect on the rigid freebody as the applied generalized force at the point of contact [39]. Matrix $L_{mi}(q_i)$ is defined by:

$$L_{mi}(q_i) = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ -E_i(q_i) & I_{3\times3} \end{bmatrix}. \tag{10}$$

The (3×3) matrix $E_i(q_i)$ in Eq. (10) arises from representing the cross product expression $(-{}^{iw}\vec{r} \times {}^{iw}\vec{f}_{N_i,N_i+1})$ in Eq. (6) in a matrix/column-vector notation. $E_i(q_i)$ is defined as:

$$E_i(q_i) = \begin{bmatrix} 0 & -{}^{iw}r_z & {}^{iw}r_y \\ {}^{iw}r_z & 0 & -{}^{iw}r_x \\ -{}^{iw}r_y & {}^{iw}r_x & 0 \end{bmatrix} \tag{11}$$

where $[{}^{iw}r_x, {}^{iw}r_y, {}^{iw}r_z]^T = {}^{iw}r$ is defined in Eq. (1). Matrix $L_{mi}(q_i)$ in Eq. (10) is nonsingular with determinant $\{L_{mi}(q_i)\} = 1$.

In Eq. (8), $L_{si}(q_i)$ is a (6×3) force transmission matrix that transforms the contact force ${}^{iw}f_{cs}$ acting at point Q_{cs} to its equivalent at point CM_{oi}. Matrix $L_{si}(q_i)$ is defined as:

$$L_{si}(q_i) = \begin{bmatrix} I_{3\times3} \\ E_{il}(q_i) - E_i(q_i) \end{bmatrix}. \tag{12}$$

The matrix term $(E_{il}(q_i) - E_i(q_i))$ in Eq. (12) arises from the cross product term $[({}^{ilw}\vec{r} - {}^{iw}\vec{r}) \times {}^{iw}\vec{f}_{cs}]$ in Eq. (6). $E_i(q_i)$ is given in Eq. (11) and $E_{il}(q_i)$ is a (3×3) matrix defined as:

$$E_{il}(q_i) = \begin{bmatrix} 0 & -{}^{ilw}r_z & {}^{ilw}r_y \\ {}^{ilw}r_z & 0 & -{}^{ilw}r_x \\ -{}^{ilw}r_y & {}^{ilw}r_x & 0 \end{bmatrix} \tag{13}$$

where $[{}^{ilw}r_x, {}^{ilw}r_y, {}^{ilw}r_z]^T = {}^{ilw}r$ is defined in Eq. (2).

In Eq. (8), $(\Omega_{ci}\, D_{ci}\, \omega_{ci})$ is a (3×1) vector arising from the cross product expression $(\vec{\omega}_{ci} \times (D_{ci}\, \omega_{ci}))$ in Eq. (6), where Ω_{ci} is a (3×3) matrix defined as:

$$\Omega_{ci} = \begin{bmatrix} 0 & -\omega_{ciz} & \omega_{ciy} \\ \omega_{ciz} & 0 & -\omega_{cix} \\ -\omega_{ciy} & \omega_{cix} & 0 \end{bmatrix} \tag{14}$$

where $[\omega_{cix}, \omega_{ciy}, \omega_{ciz}]^T = \omega_{ci}$. The object dynamics in Eqs. (8) with i=1,2 and (9) quantify the dynamic coupling between the manipulators. Indeed, they represent fifteen scalar dynamic constraints which characterize the dependence among the eighteen scalar components of the generalized contact force $[f_{c1}^T, {}^{1w}f_{cs}^T, f_{c2}^T, {}^{2w}f_{cs}^T]^T$.

They will be incorporated in the closed-chain dynamical equations. In addition to the dynamical constraints, the kinematic coupling between the manipulators also constrains the motion of the closed-chain mechanism. It is presented next.

4 KINEMATIC COUPLING

The manipulators are kinematically coupled because the $(N_1 + N_2)$ generalized coordinates $(q_1^T, q_2^T)^T$ are interrelated. The kinematic coupling effects can be quantified by equations showing the dependence among the generalized coordinates and their derivatives. These kinematic constraints can be determined by expressing the Cartesian translational velocity of the center of the spherical joint contained in the object in terms of the joint velocities of manipulator i(=1,2), and then obtaining a relationship among the generalized velocities.

The Cartesian translational velocity of the object at point Q_{cs} and the object velocity at point CM_{oi} are related through the force transmission matrix $L_{si}(q_i)$. The relation among these velocities is:

$$v_{cs} = L_{si}^T(q_i) \begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} \tag{15}$$

where the (3×1) vector v_{cs} is the translational velocity of the center of the spherical joint in the world coordinates. Furthermore, the Cartesian velocities of rigid body i of the object at point CM_{oi} and at the point of contact of the external forces imparted by manipulator i are related by:

$$\begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} = (L_{mi}^T(q_i))^{-1} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \tag{16}$$

where the (3×1) vectors v_i and ω_i(i=1,2) represent the translational and rotational velocities, respectively, of the end effector of manipulator i at its centerpoint in the world coordinates. A detailed derivation to obtain the result of Eqs. (15) and (16) based only on the geometry and kinematics of the closed-chain is provided in Appendix A. It is straightforward to verify by Eqs. (10) and (16) that $\omega_i = \omega_{ci}$, since every point on the rigid end effector of manipulator i and the rigid body i of the complex object has the same Cartesian angular velocity in the world coordinate system. However, it should be noticed that in general $\omega_{c1} \neq \omega_{c2}$ due to the spherical joint.

The velocities of the end effector of manipulator i in the Cartesian world coordinate frame and the joint space are related through the $(6 \times N_i)$ Jacobian matrix $J_{iw}(q_i)$, that is:

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = J_{iw}(q_i)\, \dot{q}_i \,. \tag{17}$$

Eliminating $[v_{ci}^T, \omega_{ci}^T]^T$ from Eq. (15) using Eq. (16), and equating the right hand sides of the two sets of resulting equations obtained with i=1,2 give a relationship:

$$\left[L_{s1}^T(q_1)\,(L_{m1}^T(q_1))^{-1}\, J_{1w}(q_1)\,;\, -L_{s2}^T(q_2)\,(L_{m2}^T(q_2))^{-1}\, J_{2w}(q_2)\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0_{3\times1} \tag{18}$$

in which Eq. (17) has been applied. Equation (18) comprises three scalar constraint equations characterizing the kinematic dependence among the generalized velocities when the manipulators operate in the closed-chain configuration. Each independent scalar constraint contained in Eq. (18) causes the loss of one DOF in the closed-chain [34]. This is significant since the system DOF specifies the number of independent ways the multi-manipulator closed-chain can move without violating the constraints given by Eq. (18) [34]. The number of DOF for a three-dimensional kinematic chain mechanism consisting of rigid links serially connected by j_1 single-DOF revolute joints and j_2 three-DOF spherical joints is determined by:

$$DOF = 6\,(k_{tot} - 1) - 5\,j_1 - 3\,j_2 \tag{19}$$

where k_{tot} is the total number of links(including a stationary, zeroth reference link the mechanism is mounted on). Equation (19) reflects the fact that each moving link has six DOF whereas the zeroth link has none, and each single-DOF joint causes a loss of five DOF for a moving link, while each three-DOF joint causes a loss of three DOF for a moving link. For the multi-manipulator configuration shown in Fig. 1, $k_{tot} = (N_1 + N_2 + 1)$, $j_1 = (N_1 + N_2)$, $j_2 = 1$, and the total sytem DOF is $(N_1 + N_2 - 3)$ by Eq. (19). Indeed, the DOF for the closed-chain is equal to the number of generalized coordinates $(q_1^T, q_2^T)^T$ minus the number of independent scalar equations of constraint [34].

A dynamical model for the multiple manipulator system is presented next.

5 CLOSED-CHAIN MODEL IN JOINT SPACE

The dynamic constraints represented by Eqs. (8) with i=1,2 and (9) as well as the kinematic constraints in Eq. (18) must be satisfied during the closed-chain motion. The constraints may be combined with the equations of motion of the manipulators to obtain a joint space dynamical model of the entire system. The model is formed by first combining the manipulator models and the dynamic constraints to obtain dynamical equations which contain explicitly an independent subset of the generalized contact forces. The dynamical equations are incorporated into the kinematic constraints to obtain a functional relation for the generalized contact forces. These contact forces can then be eliminated from the equations of motion to obtain a closed-chain model for the entire system.

The dynamics of rigid body i of the object in Eq. (8) can be incorporated in the manipulator i model in Eq. (3) by the following two step procedure: (a) Eq. (8) is solved for f_{ci} and (b) f_{ci} is eliminated from Eq. (3) using the solution obtained in step (a). This procedure gives the composite dynamics for manipulator i and rigid body i of the object:

$$\begin{aligned} {}^{i}\tau &= D_i(q_i)\,\ddot{q}_i + C_i(q_i,\dot{q}_i) + J_{iw}^T(q_i)\,L_{mi}^{-1}(q_i)\,\Lambda_i \begin{bmatrix} \dot{v}_{ci} \\ \dot{\omega}_{ci} \end{bmatrix} + \\ & J_{iw}^T(q_i)\,L_{mi}^{-1}(q_i) \left\{ \begin{bmatrix} -m_{ci}\,g \\ \Omega_{ci}\,D_{ci}\,\omega_{ci} \end{bmatrix} - L_{si}(q_i)\,{}^{iw}f_{cs} \right\} \end{aligned} \tag{20}$$

where Λ_i is a (6×6) matrix:

$$\Lambda_i = \begin{bmatrix} m_{ci}\,I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & D_{ci} \end{bmatrix}. \tag{21}$$

It should be noted that Eq. (20) is an explicit function of the spherical joint contact force vector ${}^{iw}f_{cs}$ which is defined in conjunction with Eqs. (5) – (7) and (9).

Since the closed-chain model is to be expressed in the joint space, the Cartesian variables ω_{ci}, $\dot{\omega}_{ci}$ and $\dot{v}_{ci}$ of the object are eliminated from Eq. (20). The Cartesian acceleration of the object at point CM_{oi} can be expressed in terms of the joint accelerations of manipulator i by differentiating Eq. (16) and applying Eq. (17):

$$\begin{aligned} \begin{bmatrix} \dot{v}_{ci} \\ \dot{\omega}_{ci} \end{bmatrix} &= (\dot{L}_{mi}^T)^{-1}\,J_{iw}(q_i)\,\dot{q}_i + \\ & (L_{mi}^T(q_i))^{-1}\left[\dot{J}_{iw}\,\dot{q}_i + J_{iw}(q_i)\,\ddot{q}_i\right] \end{aligned} \tag{22}$$

where the (6 × 6) matrix $(\dot{L}_{mi}^T)^{-1}[= (\partial((L_{mi}^T)^{-1})/\partial q_i)\dot{q}_i]$ and the (6 × N_i) matrix $\dot{J}_{iw}[= (\partial J_{iw}/\partial q_i)\dot{q}_i]$ are functions of the joint positions and velocities of manipulator i.

The closed-chain dynamics for the entire system are derived in two steps: (a) Incorporate Eqs. (16), (17) and (22) into Eq. (20) to obtain the composite manipulator i - rigid body i(of object) dynamics in the joint space; (b) The two sets of equations obtained from (a) with i=1,2 are combined and expressed in a concise form:

$$\tau = D(q)\,\ddot{q} + C(q,\,\dot{q}) + H_A(q,\,\dot{q}) + H_B(q,\,\dot{q})\,\dot{q} + A^T(q)\,{}^{2w}f_{cs} \tag{23}$$

in which Eq. (9) has been used to eliminate ${}^{1w}f_{cs}$ and where $\tau = [{}^1\tau^T,\,{}^2\tau^T]^T$ and $q = [q_1^T,\,q_2^T]^T$ are the joint torques and generalized coordinates for the entire system, respectively. The $((N_1+N_2)\times 1)$ vector $C(q,\,\dot{q})$ is defined as:

$$C(q,\dot{q}) = \begin{bmatrix} C_1(q_1,\,\dot{q}_1) \\ C_2(q_2,\,\dot{q}_2) \end{bmatrix}. \tag{24}$$

The $((N_1+N_2)\times(N_1+N_2))$ matrix $D(q)$ is defined by:

$$D(q) = \begin{bmatrix} D_1 + (J_{1w}^T\,L_{m1}^{-1})\,\Lambda_1\,(J_{1w}^T\,L_{m1}^{-1})^T & 0_{N_1\times N_2} \\ 0_{N_2\times N_1} & D_2 + (J_{2w}^T\,L_{m2}^{-1})\,\Lambda_2\,(J_{2w}^T\,L_{m2}^{-1})^T \end{bmatrix} \tag{25}$$

where matrix Λ_i (i=1,2) in Eq. (25) is defined by Eq. (21). $D(q)$ is the inertia matrix for the entire closed-chain system. It is symmetric, positive definite, and therefore nonsingular.

The $((N_1+N_2)\times 1)$ vector $H_A(q,\,\dot{q})$ and $((N_1+N_2)\times(N_1+N_2))$ matrix $H_B(q,\,\dot{q})$ in Eq. (23) are defined as:

$$H_A(q,\,\dot{q}) = \begin{bmatrix} J_{1w}^T(q_1)\,L_{m1}^{-1}(q_1)\begin{bmatrix} -\,m_{c1}\,g \\ \Omega_{c1}\,D_{c1}\,[0_{3\times 3}\,;\,I_{3\times 3}]\,J_{1w}(q_1)\dot{q}_1 \end{bmatrix} \\ \\ J_{2w}^T(q_2)\,L_{m2}^{-1}(q_2)\begin{bmatrix} -\,m_{c2}\,g \\ \Omega_{c2}\,D_{c2}\,[0_{3\times 3}\,;\,I_{3\times 3}]\,J_{2w}(q_2)\dot{q}_2 \end{bmatrix} \end{bmatrix}, \tag{26}$$

$$H_B(q,\,\dot{q}) = \begin{bmatrix} J_{1w}^T\,L_{m1}^{-1}\,\Lambda_1\,\frac{d}{dt}\left\{(J_{1w}^T\,L_{m1}^{-1})^T\right\} & 0_{N_1\times N_2} \\ 0_{N_2\times N_1} & J_{2w}^T\,L_{m2}^{-1}\,\Lambda_2\,\frac{d}{dt}\left\{(J_{2w}^T\,L_{m2}^{-1})^T\right\} \end{bmatrix} \tag{27}$$

where $\frac{d}{dt}\{(J_{iw}^T\,L_{mi}^{-1})^T\} = [(\dot{L}_{mi}^T)^{-1}\,J_{iw}(q_i) + (L_{mi}^T(q_i))^{-1}\,\dot{J}_{iw}]$ with i=1,2 in Eq. (27).

The $((N_1+N_2)\times 3)$ matrix $A^T(q)$ in Eq. (23) is termed the contact force coefficient matrix. It is defined by:

$$A^T(q) = \begin{bmatrix} J_{1w}^T(q_1)L_{m1}^{-1}(q_1)\,L_{s1}(q_1) \\ -\,J_{2w}^T(q_2)L_{m2}^{-1}(q_2)\,L_{s2}(q_2) \end{bmatrix} \tag{28}$$

and is assumed to have full rank three $(3 < (N_1 + N_2))$.

The dynamics of both manipulators and mutually held object are represented in Eq. (23) as $(N_1 + N_2)$ second order differential equations. Equation (23) contains explicitly an independent subset of the generalized contact forces $[f_{c1}^T, {}^{1w}f_{cs}^T, f_{c2}^T, {}^{2w}f_{cs}^T]^T$ acting on the object, namely ${}^{2w}f_{cs}$. While Eq. (23) describes the dynamics of all components of the closed-chain, it does not account for the kinematic constraints between the manipulators due to the common jointed load. To combine the kinematic constraints given by Eq. (18) and the closed-chain dynamics to obtain a model for the entire system, Eq. (18) is first expressed in a simplified, concise form in terms of the contact force coefficient matrix:

$$A(q)\,\dot{q} \;=\; 0_{3\times 1} \tag{29}$$

in which Eq. (28) has been applied. The corresponding acceleration constraints result from differentiating Eq. (29):

$$A\,\ddot{q} \;+\; \dot{A}\,\dot{q} \;=\; 0_{3\times 1}\,. \tag{30}$$

The $(3\times(N_1+N_2))$ matrix $\dot{A}[=(\partial A/\partial q)\dot{q}]$ in Eq. (30) is a function of the generalized coordinates and velocities.

Equations (23) and (29) are derived independently based on the dynamics and kinematic constraints for the closed-chain, respectively. Matrix $A(q)$ quantifies the coupling among the generalized velocities in Eq. (29). Interestingly, Eq. (23) reveals that $A^T(q)$ maps the three components of contact force acting at the center of the spherical joint in the object (${}^{2w}f_{cs}$) to the joint space. A functional relation for the calculation of the object contact forces is obtained by solving Eq. (23) for the generalized accelerations and substituting the result in Eq. (30):

$$A\,D^{-1}\,A^T\,{}^{2w}f_{cs} \;=\; \dot{A}\,\dot{q} \;+\; A\,D^{-1}\,\{\tau \;-\; C \;-\; H_A \;-\; H_B\,\dot{q}\}\;. \tag{31}$$

Since A has rank three and D^{-1} is positive definite and symmetric, then the (3×3) coefficient matrix $(A\,D^{-1}\,A^T)$ on the left hand side of Eq. (31) is positive definite, symmetric, and therefore nonsingular. Thus, the determination of ${}^{2w}f_{cs}$ using Eq. (31) is a well-specified problem. Interestingly, the object contact forces transmitted across the center of the spherical joint are functions of the generalized coordinates q, the generalized velocities $\dot{q}$, and the joint torques τ using Eq. (31).

The dynamical model of the closed-chain system is determined by eliminating the vector ${}^{2w}f_{cs}$ in Eq. (23) using Eq. (31). After rearranging, the resulting equations are:

$$\Delta(q)\;\{\tau \;-\; C \;-\; H_A \;-\; H_B\,\dot{q}\} \;=\; D\,\ddot{q} \;+\; A^T\left(A\,D^{-1}\,A^T\right)^{-1}\dot{A}\,\dot{q} \tag{32}$$

where $\Delta(q)$ is a $((N_1 + N_2) \times (N_1 + N_2))$ matrix defined as:

$$\Delta(q) \;=\; I_{(N_1+N_2)\times(N_1+N_2)} \;-\; A^T\left(A\,D^{-1}\,A^T\right)^{-1} A\,D^{-1}\,. \tag{33}$$

Interestingly, matrix $\Delta(q)$ is idempotent, i.e., $\Delta(q) = \Delta^2(q)$, and it is an orthogonal complement of the contact force coefficient matrix, i.e., $\Delta(q)\, A^T(q) = 0_{(N_1+N_2)\times 3}$.

Equation (32) represents a joint space rigid body dynamical model for the entire system in the Lagrange's formulation. It incorporates all constraint effects and contains $(N_1 + N_2)$ second order differential equations. The solution of Eq. (32) for the forward and inverse dynamics is discussed next.

5.1 Solving for Forward Dynamics

The joint torques $\tau = ({}^1\tau^T, {}^2\tau^T)^T$ over the specified time interval $(t_0 \leq t \leq t_f)$ are assumed to be given. The variables to be determined over the specified time interval are the $(N_1 + N_2)$ generalized coordinates $q = (q_1^T, q_2^T)^T$ and the $(N_1 + N_2)$ generalized velocities $\dot{q} = (\dot{q}_1^T, \dot{q}_2^T)^T$, which are assumed to be given and to satisfy Eq. (29) at time t_0.

The closed-chain model in Eq. (32) contains $(N_1 + N_2)$ second order differential equations and $(N_1 + N_2)$ generalized accelerations. Since matrix $D(q)$ is nonsingular, Eq. (32) can be solved for the highest order derivative $\ddot{q}$ and expressed as $2(N_1 + N_2)$ first order differential equations containing $2(N_1 + N_2)$ unknown variables $(q, \dot{q})$. Since the number of first order differential equations is equal to the number of unknown scalar variables, the solution for the forward dynamics is well-specified when input τ is known over the given time interval. Therefore the model given in Eq. (32) can be numerically integrated to provide dynamical simulation of the closed-chain motion.

5.2 Solving for Inverse Dynamics

The desired trajectories of the generalized coordinates and their derivatives: q, $\dot{q}$, $\ddot{q}$ over the specified time interval $(t_0 \leq t \leq t_f)$ are assumed to be given which satisfy Eqs. (29) and (30). The variables to be determined over the specified time interval are the $(N_1 + N_2)$ joint torques $\tau = ({}^1\tau^T, {}^2\tau^T)^T$.

The closed-chain model in Eq. (32) involves $(N_1 + N_2)$ second order differential equations containing $(N_1 + N_2)$ joint torques. The rank of the $((N_1 + N_2) \times (N_1 + N_2))$ coefficient matrix $\Delta(q)$ defined in Eq. (33) which premultiplies vector τ in the model determines the nature of the solution to the inverse dynamics. It is proven in Appendix B by analytical techniques that $\Delta(q)$ is a singular matrix and that its rank does not exceed the DOF of the closed-chain system, namely $(N_1 + N_2 - 3)$. It should be mentioned that the proof in Appendix B requires knowledge of particular matrix- and vector-variables which are introduced in the next section(6). The proof that matrix $\Delta(q)$ is singular and that the $rank(\Delta(q)) \leq (N_1 + N_2 - 3)$ is a new result for the case of two manipulators holding a spherically jointed payload. Therefore the problem of solving for the inverse dynamics is underspecified in nature. Equation (32)

has infinitely many solutions for the joint torques when the generalized coordinates and their derivatives are known over the specified time interval.

The structure of the closed-chain joint space model given by Eq. (32) is not particularly suitable for the development of a control architecture, i.e., the joint torques cannot be solved by inverting matrix $\Delta(q)$, since it is singular. Furthermore, the number of scalar second order differential equations contained in Eq. (32) exceeds the DOF of the system. However, as shown in the next section, the joint space model is amenable to the realization of a reduced order model which is useful for controller design purposes.

6 REDUCED ORDER MODEL AND CONTACT FORCE DETERMINATION

The closed-chain system consisting of two serial-link manipulators containing N_1 and N_2 joints, respectively, mutually holding an object containing a spherical joint described by $(N_1 + N_2)$ second order differential equations in Eq. (32) has $(N_1 + N_2 - 3)$ DOF. The number of second order differential equations of motion is reduced from $(N_1 + N_2)$ to $(N_1 + N_2 - 3)$ in this section. The generalized contact forces are eliminated from the reduced order model, but can be calculated separately. The solution of the model for the unknown variables is discussed.

To determine a reduced order model, a new vector variable $\nu = [\nu_1, \nu_2, \ldots, \nu_{N_1+N_2-3}]^T$ referred to as the pseudovelocity [1, 34] is introduced. The number of scalar components of ν is equal to the DOF of the system. The pseudovelocity is defined by:

$$\nu = B(q)\,\dot{q}. \tag{34}$$

The $((N_1 + N_2 - 3) \times (N_1 + N_2))$ matrix $B(q)$ in Eq. (34) is selected so that the composite $((N_1 + N_2) \times (N_1 + N_2))$ matrix $(A^T(q);\, B^T(q))^T$ is nonsingular. It is convenient to partition the inverse of $(A^T(q);\, B^T(q))^T$ into two matrices:

$$\begin{bmatrix} A(q) \\ B(q) \end{bmatrix}^{-1} = [\Pi(q);\, \Sigma(q)] \tag{35}$$

where $\Pi(q)$ is a $((N_1+N_2)\times 3)$ matrix and $\Sigma(q)$ a $((N_1+N_2)\times(N_1+N_2-3))$ matrix. Eq. (35) implies that $A\,\Pi = I_{3\times 3}$, $A\,\Sigma = 0_{(3\times(N_1+N_2-3))}$, $B\,\Pi = 0_{((N_1+N_2-3)\times 3)}$, $B\,\Sigma = I_{(N_1+N_2-3)\times(N_1+N_2-3)}$ and $(\Pi\,A + \Sigma\,B) = I_{(N_1+N_2)\times(N_1+N_2)}$. The terminology pseudovelocity emphasizes the fact that the linear combinations of the generalized velocities on the right hand side of Eq. (34) are not necessarily the time derivatives of any physical coordinates [1, 34]. Indeed, the choice of matrix $B(q)$ by the designer is somewhat arbitrary. Two examples of the selection of matrix $B(q)$ given specific choices for the pseudovelocity vector ν by the designer are provided in Appendix C. Furthermore, the examples show that matrices $\Pi(q)$ and $\Sigma(q)$ may be defined analytically thus avoiding the numerical inversion of matrix $(A^T(q); B^T(q))^T$ for calculating their values.

Differentiating Eq. (34) establishes a relationship between the pseudo- and generalized accelerations:

$$\dot{\nu} = B\,\ddot{q} + \dot{B}\,\dot{q}. \tag{36}$$

The $((N_1 + N_2 - 3) \times (N_1 + N_2))$ matrix $\dot{B}[= (\partial B/\partial q)\dot{q}]$ in Eq. (36) is a function of the generalized coordinates and velocities.

Equations (29) and (34) can be solved for the generalized velocities:

$$\dot{q} = \Sigma\,\nu \tag{37}$$

in which Eq. (35) has been applied. Likewise, Eqs. (30) and (36) can be solved for the generalized accelerations:

$$\ddot{q} = \Sigma\,\dot{\nu} - [\Pi\,\dot{A} + \Sigma\,\dot{B}]\,\Sigma\,\nu\,. \tag{38}$$

Substituting for $\dot{q}$ in Eq. (29) using Eq. (37) yields the kinematic velocity constraint equation $A\,\Sigma\,\nu = 0_{3\times1}$, which is identically true because $A\,\Sigma = 0_{(3\times(N_1+N_2-3))}$ in accordance with Eq. (35). Thus the kinematic constraints are satisfied regardless of the values of the pseudovelocities; therefore, the closed-chain dynamics in Eq. (23) expressed in the pseudospace implicitly satisfy the kinematic constraints.

Premultiplying Eq. (23) by the matrix Σ^T and utilizing the properties of Eq. (35) obtain:

$$\Sigma^T\,D\,\ddot{q} = \Sigma^T\,\{\tau - C - H_A - H_B\,\dot{q}\}\,. \tag{39}$$

Substituting Eqs. (37) and (38) into Eq. (39) yields the reduced order equations of motion in the pseudospace:

$$\Sigma^T D\,\Sigma\,\dot{\nu} = \Sigma^T\left\{\tau + \left\{D\left[\Pi\,\dot{A} + \Sigma\,\dot{B}\right] - H_B\right\}\Sigma\,\nu - C - H_A\right\}. \tag{40}$$

Eq. (40) is comprised of $(N_1 + N_2 - 3)$ second order differential equations, which is equal to the DOF of the entire system. It should be noted that the term $(A^{T\,2w}f_{cs})$ in Eq. (23) is implicitly cancelled in the reduced order model because $\Sigma^T\,A^T = 0_{((N_1+N_2-3)\times3)}$ in accordance with Eq. (35). Thus, Eqs. (39) and (40) are not functions of the generalized contact forces which may be calculated in terms of the pseudovelocities using Eqs. (31) and (37):

$$A\,D^{-1}A^{T\,2w}f_{cs} = \dot{A}\,\Sigma\,\nu + A\,D^{-1}\,\{\tau - C - H_A - H_B\,\Sigma\,\nu\}\,. \tag{41}$$

It should be mentioned that ${}^{2w}f_{cs}$ can be calculated as a function of $\dot{\nu}$ through premultiplying Eq. (23) by the matrix Π^T and incorporating Eqs. (35), (37), and (38).

It will be shown in the next section that the separated form of the model for the entire closed-chain system given by Eqs. (40) and (41) is useful for controller design purposes.

The solution of the reduced order model for the forward and inverse dynamics is now discussed.

6.1 Solving for Forward Dynamics

The joint torques $\tau = ({}^1\tau^T, {}^2\tau^T)^T$ over the specified time interval $(t_0 \le t \le t_f)$ are assumed to be given. The variables to be determined over the specified time

interval are the $(N_1 + N_2 - 3)$ pseudocoordinates p $\{= [p_1, p_2, \ldots, p_{N_1+N_2-3}]^T\}$ and the $(N_1 + N_2 - 3)$ pseudovelocities ν, where $\nu = \dot{p}$. It is assumed that q and ν are given at time t_0 which satisfy Eq. (37).

The reduced order model in Eq. (40) involves $(N_1 + N_2 - 3)$ second order differential equations containing $(N_1 + N_2 - 3)$ pseudoaccelerations. Since matrix Σ^T has full rank $(N_1 + N_2 - 3)$ and matrix $D(q)$ is positive definite and symmetric, then the square coefficient matrix $(\Sigma^T D\, \Sigma)$ in Eq. (40) is positive definite, symmetric, and therefore nonsingular. Thus Eq. (40) can be solved for the pseudoaccelerations and expressed as $2(N_1 + N_2 - 3)$ first order differential equations containing $2(N_1 + N_2 - 3)$ unknown variables $(p\,,\, \nu)$. Since the number of first order differential equations is equal to the number of unknown scalar variables, the solution for the forward dynamics is well-specified when the input τ is known over the given time interval.

The reduced order model in Eq. (40) leads to a computationally efficient dynamical simulation of the closed-chain motion when compared to the joint space model given by Eq. (32). For example, with two six-joint manipulators $(N_1 = N_2 = 6)$ holding a spherically jointed object, Eqs. (40) and (32) contain nine and twelve second order differential equations of motion, respectively. However, since the problem of analytically expressing the generalized coordinates $q = (q_1^T\,,\, q_2^T)^T$ in a reduced manner remains unsolved [1, 2, 3, 7], Eqs. (40) and (41) are explicit functions of q and not of the pseudocoordinate vector p. Thus it is nontrivial to numerically integrate the reduced order model. A method for circumventing this problem in discrete time which solves for the pseudocoordinates numerically is now discussed.

Let Γ denote the sampling period and assume that the joint torques $\tau(k\,\Gamma)$ are given over the specified time interval $k = k_0,\, k_0 + 1,\, \ldots,\, k_f$, where time $t = k\,\Gamma$ and $\{k_0,\, k_f\}$ are non-negative integers. The variables q and ν are assumed to be given and to satisfy Eq. (37) at $t = k_0\,\Gamma$. The nonsingular coefficient matrix $(\Sigma^T\, D\, \Sigma)$ on the left hand side of Eq. (40) is a function of q. The right hand side of Eq. (40) is a function of the variables $(q,\, \nu,\, \tau)$.

At the initial time Eq. (40) can be solved for the highest order derivative $(\dot{\nu}(k_0\,\Gamma))$ and numerically integrated over the sampling period from $t = k_0\,\Gamma$ to $t = (k_0 + 1)\,\Gamma$ to obtain $p(t)$ and $\nu(t)$ at $t = (k_0 + 1)\,\Gamma$. The drawback of the reduced order model being a function of the generalized coordinates is now evident. Indeed, $q((k_0 + 1)\,\Gamma)$ must be determined before Eq. (40) can be solved for the highest order derivative $(\dot{\nu}((k_0 + 1)\,\Gamma))$ and numerically integrated from time $t = (k_0 + 1)\,\Gamma$ to $t = (k_0 + 2)\,\Gamma$. To solve this problem, Eq. (37) could be numerically integrated over the sampling period from $t = k_0\,\Gamma$ to $t = (k_0 + 1)\,\Gamma$ to obtain $q((k_0 + 1)\,\Gamma)$. This approach involves numerically integrating $(N_1 + N_2)$ first order differential equations and is purely kinematical in nature.

Given $(q,\, \nu,\, \tau)$ at $t = (k_0 + 1)\,\Gamma$, the reduced order model can be solved for

$(\dot{\nu}((k_0 + 1)\Gamma))$ and numerically integrated to obtain $p(t)$ and $\nu(t)$ at $t = (k_0 + 2)\Gamma$. $q(t)$ at $t = (k_0 + 2)\Gamma$ can be obtained by integrating Eq. (37) from time $t = (k_0 + 1)\Gamma$ to $t = (k_0 + 2)\Gamma$.

Continuing in the aforementioned manner, the dynamical, discrete time simulation of Eq. (40) can be accomplished.

6.2 Solving for Inverse Dynamics

The desired trajectories of the generalized coordinates and their derivatives: q, $\dot{q}$, $\ddot{q}$ over the specified time interval $(t_0 \leq t \leq t_f)$ are assumed to be given which satisfy Eqs. (29) and (30). The (N_1+N_2) scalar variables to be determined over the specified time interval are the joint torques $\tau = ({}^1\tau^T, {}^2\tau^T)^T$.

The desired pseudospace trajectory $(\nu, \dot{\nu})$ is first obtained by Eqs. (34) and (36) since the desired joint space variables and their derivatives are known. The reduced order model in Eq. (40) contains $(N_1 + N_2 - 3)$ second order differential equations with $(N_1 + N_2)$ unknown joint torques. Moreover, the coefficient matrix Σ^T which premultiplies τ in Eq. (40) is rectangular with fewer rows than columns. Therefore, the solution to the inverse dynamics is underspecified, since the number of unknown joint torques exceeds the number of equations. Equation (40) has infinitely many solutions for τ when the desired trajectories of the generalized coordinates and their derivatives are given over the specified time interval. A solution for the joint torques obtained from the reduced order model would not consider the spherical joint contact forces, which are governed by Eq. (41). In view of this a control architecture for the joint torques based on Eqs. (40) and (41) is derived in the next section.

7 CONTROL ARCHITECTURE

The problem considered is to derive a control law for the $(N_1 + N_2)$ joint torques $\tau = ({}^1\tau^T, {}^2\tau^T)^T$ so that the force and position of the closed-chain mechanism behave in desirable manners. The controller is designed on the basis of the separated form of the closed-chain model given by Eqs. (40) and (41). Specifically, a control input τ is determined so that the pseudovariables $\{\nu, \dot{\nu}\}$ and the object contact forces ${}^{2w}f_{cs}$ acting at the center of the spherical joint will be controlled separately. Finally, the modeling and control methodology presented here are compared to another model and control scheme recently proposed in the literature.

The controller consists of the sum of the outputs of a $((N_1 + N_2) \times 1)$ primary controller (τ^p) that is designed for cancellation of nonlinear terms in the model and a $((N_1 + N_2) \times 1)$ secondary controller (τ^s) that performs closed-loop servoing. The composite control (τ) is specified as $\tau = \tau^p + \tau^s$. τ^p is realized by introducing nonlinear feedback and/or feedforward loops into the closed-chain system such that the nonlinear terms in Eqs. (40) and (41) that are not explicit in the variables $({}^{2w}f_{cs}, \dot{\nu})$ are cancelled. The primary controller is chosen as:

$$\tau^p = -\left\{\hat{D}\left[\hat{\Pi}\,\dot{\hat{A}} + \hat{\Sigma}\,\dot{\hat{B}}\right] - \hat{H}_B\right\}\hat{\Sigma}\,\dot{\nu} + \hat{C} + \hat{H}_A \tag{42}$$

where the superscript $\hat{}$ denotes that the quantity is estimated as a function of the desired(reference) feedforward trajectory (q^{ref}, ν^{ref}) and/or the feedback variables (q, ν). The composite control $(\tau = \tau^p + \tau^s)$ is substituted into Eqs. (40) and (41). The resulting equations, under the assumption that the following relations hold:

$$\hat{D} = D \quad \hat{C} = C \quad \hat{A} = A \quad \hat{B} = B \quad \dot{\hat{A}} = \dot{A} \quad \dot{\hat{B}} = \dot{B}$$

$$\hat{H}_A = H_A \quad \hat{H}_B = H_B \quad \left[\hat{\Pi}\,;\,\hat{\Sigma}\right] = \left\{\left[(\hat{A})^T\,;\,(\hat{B})^T\right]^T\right\}^{-1} \tag{43}$$

can be expressed as:

$$\dot{\nu} = \left(\Sigma^T D\,\Sigma\right)^{-1}\Sigma^T\tau^s, \tag{44}$$

$${}^{2w}f_{cs} = \left(A\,D^{-1}A^T\right)^{-1}A\,D^{-1}\tau^s \tag{45}$$

in which Eqs. (35) and (42) have been applied.

The design of the secondary controller (τ^s) is based on the structure of Eqs. (44) and (45) so as to obtain a closed-loop control system where the pseudovariables and object contact forces are servoed separately. The secondary controller is selected as:

$$\tau^s = \hat{A}^T\tau_1^s + \hat{D}\,\hat{\Sigma}\,\tau_2^s \tag{46}$$

where τ_1^s and τ_2^s are (3×1) and $((N_1+N_2-3)\times1)$ vectors, respectively, representing control variables to be determined. Substituting Eq. (46) into Eqs. (44) and (45), utilizing the matrix relations in Eq. (35), and rearranging give:

$$\dot{\nu} = \tau_2^s, \tag{47}$$

$$^{2w}f_{cs} = \tau_1^s \tag{48}$$

where the assumption of perfect knowledge of the nonlinear terms in Eq. (43) has been applied.

Eqs. (47) and (48) represent the reduced order model equation (40) and the spherical joint contact force equation (41) in the closed-loop system, respectively. Suppose the secondary controller components τ_2^s and τ_1^s are feedback and/or feed-forward control laws selected by the designer to servo the pseudovariable error and the object contact force error, respectively. Since Eqs. (47) and (48) are completely decoupled, τ_1^s and τ_2^s are independent, non-interacting controllers for the force- and position-control of the constrained multi-manipulator system, respectively. Thus, the force- and position-controlled DOF are decoupled when the composite control law ($\tau = \tau^p + \tau^s$) defined by Eqs. (42) and (46) is used in conjunction with Eqs. (40) and (41). This is a new result for the case of two manipulators holding a jointed object.

Interestingly, the structure of the composite controller ($\tau = \tau^p + \tau^s$) reveals that it is a hybrid control strategy [32, 33]. In the secondary controller defining Equation (46), the $((N_1+N_2)\times3)$ matrix $\hat{A}^T$ transforms the contact force controller τ_1^s to the joint space. Likewise, the $((N_1+N_2)\times(N_1+N_2-3))$ matrix $(\hat{D}\,\hat{\Sigma})$ in Eq. (46) transforms the pseudovariable controller τ_2^s to the joint space. Therefore all (N_1+N_2) actuated manipulator joints contribute to the force- and position-control of the entire closed-chain system.

In [9], the "orthogonal complement" Σ of the matrix A is defined by the equation $A\,\Sigma = 0_{3\times9}$ for a pair of six-axis ($N_1 = N_2 = 6$) manipulators holding an object containing a spherical joint. A joint space reduced order model in the form of Eq. (39) is obtained by premultiplying the combined manipulator equations of motion by Σ^T, which eliminates the generalized contact forces and reduces the number of second order differential equations of motion from twelve to nine [9]. Eq. (39), however, contains (N_1+N_2-3) second order differential equations of motion and (N_1+N_2) generalized accelerations $\ddot{q}$. Thus the problem of solving Eq. (39) for $\ddot{q}$ is underspecified and the reduced order model in the joint space is impractical for dynamical simulation. Eq. (39) also contains (N_1+N_2) joint torques τ. An underspecified solution for the joint torques for position tracking is computed from the model in [9]. The approach used here additionally derives functional relations in Eqs. (31) and (41) for the generalized contact forces, which is not considered in the earlier work [9].

It is difficult to develop a control input τ so as to decouple the joint space form of the reduced order model and contact force equation given by Eqs. (39) and (31), respectively. The advantage of introducing the pseudovariables $\{\nu, \dot{\nu}\}$ and determining matrices Π and Σ by Eq. (35) is that it becomes straightforward to derive a control architecture to completely decouple the reduced order model and contact force equation when they are expressed in the pseudospace by Eqs. (40) and (41), respectively.

8 CONCLUSION

The closed-chain motion of two manipulators holding an object containing a multiple-DOF spherical joint has been analyzed. A rigid body model was first developed in the joint space. It includes the dynamics of all components of the closed-chain as well as the kinematic constraints. The system was transformed to obtain a reduced order model which also includes the dynamics of both manipulators and the object, but implicitly satisfies the kinematic constraints. Either form of the model more closely represents the physical behavior of the system than the model presented in [9], which includes the manipulator dynamics but does not include the dynamics of the spherically jointed object. Furthermore, it has been shown that the solutions of the joint space and reduced order models for the forward dynamics are well-specified in nature, whereas their solutions for the inverse dynamics are underspecified.

In [9], the controller design was based only on the joint space reduced order model which contains no generalized contact forces. By considering both the reduced order model and separate contact force equation expressed in the pseudospace for the controller design, it has been shown here that the force- and position-controlled DOF can be completely decoupled. That is, the pseudovariables $\{\nu, \dot{\nu}\}$ and an independent subset of the generalized contact forces can be controlled independently using the proposed composite control architecture ($\tau = \tau^p + \tau^s$).

Derivation of an analytical expression for matrix $B(q)$ which defines the pseudovelocity vector ν is one of the main tasks when applying the reduced order model and control architecture to a specific application. Two examples of the selection of matrix $B(q)$ such that matrix $(A^T(q)\,;\,B^T(q))^T$ is nonsingular given specific choices of ν by the designer have been presented. Furthermore, the feasibility of analytically deriving matrices $\Pi(q)$ and $\Sigma(q)$ to avoid numerically inverting $(A^T(q)\,;\,B^T(q))^T$ for calculating their values was investigated in the examples. In the previous work on two manipulators holding a rigid object [38, 39], matrix $B(q)$ was presented only in general terms.

Each manipulator holding a rigid object was required to have N joints in [38, 39]. The analysis in [9] was restricted to a pair of six-joint manipulators holding a spherically jointed object. The rigid body model and decoupled control architecture presented in this report are applicable to two manipulators containing N_1 and N_2 joints, respectively.

The research presented in this report has uncovered and identified a wealth of open problems and issues associated with the closed-chain motion of two manipulators holding a spherically jointed object that warrant future attention. The total DOF in the closed-chain system is $(N_1 + N_2 - 3)$ as discussed in section 4. When there are redundant DOF in the system, i.e., when the total DOF of the closed-chain system exceeds nine, the choice for and the physical interpretation of the additional

pseudovelocities $[\nu_{10}, \nu_{11}, \ldots, \nu_{N_1+N_2-3}]^T$ need to be investigated. This includes the development of criteria or analytical methods for selecting the matrix $B(q)$.

Another interesting suggested future work area involves using the rigid body model and proposed controller to investigate properties of the closed-loop system in the presence of force disturbances, or uncertainty in the knowledge of the system dynamics and/or the kinematic constraints.

Slippage and friction were not considered in this report, but are important factors in the modeling and control of two interacting manipulators. It would be helpful and desirable to determine if the results presented here could be extended to include these effects.

The approach to the problem discussed here for the spherically jointed payload case could be applied to other interacting manipulator problems, e.g., two or more manipulators holding other complex jointed payloads such as an object containing a universal joint.

9 ACKNOWLEDGEMENT

This research was supported by the Office of Engineering Research Program, Basic Energy Sciences, of the U.S. Department of Energy under Contract No. DE-AC05-84OR21400 with Martin Marietta Energy Systems, Inc.

The author wishes to express his sincere thanks to Dr. V. Protopopescu, Dr. J. Jansen, and Dr. R. Kress for reviewing this report and providing many insightful suggestions for its improvement.

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APPENDIX A

RELATIONSHIP AMONG THE CARTESIAN VELOCITIES FOR THE RIGID BODY i OF THE COMPLEX OBJECT AT THE CENTER OF MASS AND AT THE CONTACT POINTS

The Cartesian velocity relations given by Eqs. (15) and (16) are derived in this Appendix based on the geometry and kinematics of the closed-chain. The translational velocity vector v_{ci} defined in conjunction with Eq. (5) can be derived as follows:

$$v_{ci} = \frac{d}{dt}\left\{{}^{i}p_{w}^{N_i}(q_i) + {}^{iw}r(q_i)\right\} . \tag{A.1}$$

Similarly, the translational velocity vector v_{cs} defined in conjunction with Eq. (15) can be derived as follows:

$$v_{cs} = \frac{d}{dt}\left\{{}^{i}p_{w}^{N_i}(q_i) + {}^{ilw}r(q_i)\right\} \tag{A.2}$$

where ${}^{i}p_{w}^{N_i}(q_i)$ in Eqs. (A.1) and (A.2) is a (3×1) vector emanating from the origin of the (X_w, Y_w, Z_w) world coordinate frame to the origin of the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate frame. Thus ${}^{i}p_{w}^{N_i}(q_i)$ represents the Cartesian translational position of the centerpoint of the end effector for manipulator i in the world coordinate system. The translational velocity vector v_i defined in conjunction with Eq. (16) is related to vector ${}^{i}p_{w}^{N_i}(q_i)$ as follows:

$$v_i = \frac{d}{dt}\left\{{}^{i}p_{w}^{N_i}(q_i)\right\} . \tag{A.3}$$

In Eqs. (A.1) and (A.2), $\frac{d}{dt}({}^{iw}r(q_i))$ and $\frac{d}{dt}({}^{ilw}r(q_i))$ denote the time derivatives of vectors ${}^{iw}r(q_i)$ and ${}^{ilw}r(q_i)$, respectively, with reference to the world coordinate system (X_w, Y_w, Z_w). Let $\frac{d^*}{dt}({}^{iw}r(q_i))$ and $\frac{d^*}{dt}({}^{ilw}r(q_i))$ denote the time derivatives of vectors ${}^{iw}r(q_i)$ and ${}^{ilw}r(q_i)$, respectively, with reference to the end effector coordinate system $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$. Then the following relationships hold [36]:

$$\frac{d}{dt}\left\{{}^{iw}r(q_i)\right\} = \frac{d^*}{dt}\left\{{}^{iw}r(q_i)\right\} - \begin{bmatrix} 0 & -{}^{iw}r_z & {}^{iw}r_y \\ {}^{iw}r_z & 0 & -{}^{iw}r_x \\ -{}^{iw}r_y & {}^{iw}r_x & 0 \end{bmatrix} \omega_i , \tag{A.4}$$

$$\frac{d}{dt}\left\{{}^{ilw}r(q_i)\right\} = \frac{d^*}{dt}\left\{{}^{ilw}r(q_i)\right\} - \begin{bmatrix} 0 & -{}^{ilw}r_z & {}^{ilw}r_y \\ {}^{ilw}r_z & 0 & -{}^{ilw}r_x \\ -{}^{ilw}r_y & {}^{ilw}r_x & 0 \end{bmatrix} \omega_i \tag{A.5}$$

where the last term on the right hand side of Eq. (A.4) arises from representing the cross product expression $(\vec{\omega}_i \times {}^{iw}\vec{r}(q_i))$ $[= -{}^{iw}\vec{r}(q_i) \times \vec{\omega}_i]$ in a matrix/column-vector notation. Likewise, the last term on the right hand side of Eq. (A.5) arises from the cross product expression $(\vec{\omega}_i \times {}^{ilw}\vec{r}(q_i))$.

Manipulator i securely holds the rigid body i of the complex object without slippage. Moreover, the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate frame shown in Fig. 1 moves as the end effector for manipulator i moves. Points CM_{oi} and Q_{cs}, which coincide with the tails of vectors $({}^{i}r, {}^{iw}r(q_i))$ and $({}^{il}r, {}^{ilw}r(q_i))$ respectively, as shown in Fig. 2, are stationary with respect to the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate frame. Furthermore, there is no joint variable associated with the local end effector coordinate frame. That is, the outermost scalar joint variable q_{iN_i} for manipulator i contained in the vector $q_i = [q_{i1}, q_{i2}, \cdots, q_{iN_i}]^T$ is associated with the $(X_{N_i-1}^{(i)}, Y_{N_i-1}^{(i)}, Z_{N_i-1}^{(i)})$ coordinate frame. Therefore vectors $({}^{i}r, {}^{iw}r(q_i), {}^{il}r, {}^{ilw}r(q_i))$ are constant(stationary) with respect to the $(X_{N_i}^{(i)}, Y_{N_i}^{(i)}, Z_{N_i}^{(i)})$ coordinate frame. The time derivatives of vectors ${}^{iw}r(q_i)$ and ${}^{ilw}r(q_i)$ with respect to the local moving coordinate frame are given by:

$$\frac{d^*}{dt}\left\{{}^{iw}r(q_i)\right\} = 0_{3\times1}\,, \tag{A.6}$$

$$\frac{d^*}{dt}\left\{{}^{ilw}r(q_i)\right\} = 0_{3\times1}\,. \tag{A.7}$$

Incorporating Eqs. (A.3), (A.4) and (A.6) into Eq. (A.1) yields the following relation:

$$v_{ci} = v_i - \begin{bmatrix} 0 & -{}^{iw}r_z & {}^{iw}r_y \\ {}^{iw}r_z & 0 & -{}^{iw}r_x \\ -{}^{iw}r_y & {}^{iw}r_x & 0 \end{bmatrix} \omega_i\,, \tag{A.8}$$

Similarly, Eqs. (A.3), (A.5) anα (A.7) are substituted into Eq. (A.2):

$$v_{cs} = v_i - \begin{bmatrix} 0 & -{}^{ilw}r_z & {}^{ilw}r_y \\ {}^{ilw}r_z & 0 & -{}^{ilw}r_x \\ -{}^{ilw}r_y & {}^{ilw}r_x & 0 \end{bmatrix} \omega_i\,. \tag{A.9}$$

Equations (A.8) and (A.9) can be expressed in concise forms:

$$v_{ci} = [I_{3\times3}\,;\, -E_i(q_i)] \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}, \tag{A.10}$$

$$v_{cs} = [I_{3\times3}\,;\, -E_{il}(q_i)] \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \tag{A.11}$$

in which Eqs. (11) and (13) have been applied, respectively.

As discussed in Section 4, every point on the rigid body i of the shared payload and on the end effector of manipulator i has the same Cartesian angular velocity in the world coordinates. Thus the following relation holds:

$$\omega_{ci} = \omega_i \,. \tag{A.12}$$

Equations (A.10) and (A.12) can be combined and expressed in terms of the force/torque transmission matrix $L_{mi}(q_i)$:

$$\begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} = \left(L_{mi}^T(q_i)\right)^{-1} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \tag{A.13}$$

in which Eq. (10) has been applied.

Equation (A.13) is in agreement with Eq. (16).

Eliminating $[v_i^T, \omega_i^T]^T$ from Eq. (A.11) using Eq. (A.13) gives a relationship:

$$v_{cs} = [I_{3\times3}\,; -E_{il}(q_i)]\; L_{mi}^T(q_i) \begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} . \tag{A.14}$$

Substituting for $L_{mi}^T(q_i)$ in Eq. (A.14) using Eq. (10) and simplifying yield:

$$v_{cs} = [I_{3\times3}\,;\; E_i(q_i) - E_{il}(q_i)] \begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} . \tag{A.15}$$

Finally, Eq. (A.15) can be expressed in a compact form:

$$v_{cs} = L_{si}^T(q_i) \begin{bmatrix} v_{ci} \\ \omega_{ci} \end{bmatrix} \tag{A.16}$$

in which Eq. (12) has been applied.

Equation (A.16) is in agreement with Eq. (15).

APPENDIX B

PROVING SINGULARITY OF THE JOINT TORQUE COEFFICIENT MATRIX

In this Appendix it will be proven that matrix $\Delta(q)$ which premultiplies the vector of joint torques in Eq. (32) has less that full rank and thus is singular. The definition of the pseudovelocity vector ν given in Eq. (34), the matrix relations given by Eq. (35), and the definition of $\Delta(q)$ in Eq. (33) will be used in evaluating the rank of $\Delta(q)$. Additionally, the following matrix property will be used. Suppose two $(n \times n)$ real matrices X and Y are given. Matrix X is assumed to possess full rank n and thus is nonsingular. Then the following mathematical relation applies [35]:

$$rank(Y) = rank(XY) = rank(YX) = rank(X^T Y) = rank(YX^T). \quad \text{(B.1)}$$

The rank of $\Delta(q)$ may now be determined. By the property of Eq. (B.1), the following relation holds:

$$rank(\Delta) = rank\left\{ \begin{bmatrix} \Pi^T \\ \Sigma^T \end{bmatrix} \Delta \left[A^T ; B^T\right] \right\}. \quad \text{(B.2)}$$

The rank of the $((N_1 + N_2) \times (N_1 + N_2))$ matrix within the braces on the right hand side of Eq. (B.2) is now analyzed by first decomposing it into four submatrices:

$$\begin{bmatrix} \Pi^T \\ \Sigma^T \end{bmatrix} \Delta \left[A^T ; B^T\right] = \begin{bmatrix} \Pi^T \Delta A^T & \Pi^T \Delta B^T \\ \Sigma^T \Delta A^T & \Sigma^T \Delta B^T \end{bmatrix}. \quad \text{(B.3)}$$

The submatrices can be expanded by replacing Δ with the right hand side of Eq. (33) and simplified by invoking the matrix relations in Eq. (35). Indeed, it is straightforward to verify that:

$$\begin{bmatrix} \Pi^T \\ \Sigma^T \end{bmatrix} \Delta \left[A^T ; B^T\right] = \begin{bmatrix} 0_{3\times 3} & -\left(AD^{-1}A^T\right)^{-1} AD^{-1}B^T \\ 0_{(N_1+N_2-3)\times 3} & I_{(N_1+N_2-3)\times(N_1+N_2-3)} \end{bmatrix}. \quad \text{(B.4)}$$

Equation (B.4) reveals that the first three columns of the matrix $([\Pi ; \Sigma]^T \Delta [A^T ; B^T])$ contain all zeros. Thus the matrix $([\Pi ; \Sigma]^T \Delta [A^T ; B^T])$ is singular. Furthermore, the $rank([\Pi ; \Sigma]^T \Delta [A^T ; B^T]) \leq (N_1 + N_2 - 3)$.

Since matrix $([\Pi ; \Sigma]^T \Delta [A^T ; B^T])$ is singular, then matrix Δ is also singular with $rank(\Delta) \leq (N_1 + N_2 - 3)$ because the ranks of these two matrices are the same as shown in Eq. (B.2). Interestingly, this result indicates that the rank of $\Delta(q)$ does not exceed the DOF of the closed-chain configuration, namely $(N_1 + N_2 - 3)$.

APPENDIX C

SPECIFICATION OF ANALYTICAL FORMS FOR MATRICES $B(q)$, $\Pi(q)$, AND $\Sigma(q)$

In this Appendix two examples of the selection of matrix $B(q)$ which defines the pseudovelocity vector ν in accordance with Eq. (34) are provided given specific choices for ν by the designer. Additionally, the analytical derivation of matrices $\Pi(q)$ and $\Sigma(q)$, which are defined by Eq. (35), is presented. The case of two six-axis ($N_1 = N_2 = 6$) manipulators holding a spherically jointed object is considered in the examples. It is assumed that the (6×6) Jacobian matrix $J_{iw}(q_i)$ for manipulator i(=1,2) defined in conjunction with Eq. (17) possesses full rank six and thus is non-singular. In this case the closed-chain system has nine DOF and the pseudovelocity vector ν contains nine scalar components, i.e., $\nu = [\nu_1, \nu_2, \ldots, \nu_9]^T$.

The definition of matrix $A(q)$ given by Eqs. (28) and (29) is repeated here for convenience:

$$A(q) = \left[L_{s1}^T(q_1)\,(L_{m1}^T(q_1))^{-1}\,J_{1w}(q_1)\,;\; -L_{s2}^T(q_2)\,(L_{m2}^T(q_2))^{-1}\,J_{2w}(q_2)\right]. \tag{C.1}$$

Example 1 : In this example the pseudovelocities are chosen to be the Cartesian angular velocities of rigid body 2 of the complex object as well as the Cartesian velocities of the center of mass of rigid body 1(of the object) at point CM_{o1}, i.e., $\nu = [\omega_{c2}^T, v_{c1}^T, \omega_{c1}^T]^T$. Suppose the first three pseudovelocities $[\nu_1, \nu_2, \nu_3]^T$ are expressed in terms of the joint velocities of manipulator 2 while the last six pseudovelocities are in terms of the joint velocities of manipulator 1. In this case the (9×12) matrix $B(q)$ is selected as:

$$B(q) = \begin{bmatrix} 0_{3\times6} & [0_{3\times3};\, I_{3\times3}]\,J_{2w}(q_2) \\ (L_{m1}^T(q_1))^{-1}\,J_{1w}(q_1) & 0_{6\times6} \end{bmatrix} \tag{C.2}$$

in which Eqs. (10), (16), and (17) have been used.

The (12×12) composite matrix $(A^T(q)\,;\, B^T(q))^T$ defined by Eqs. (C.1) and (C.2) with ($N_1 = N_2 = 6$) is presented here in a partitioned form:

$$\begin{bmatrix} A(q) \\ B(q) \end{bmatrix} = \begin{bmatrix} \Psi(q_1)\,(L_{m1}^T(q_1))^{-1}\,J_{1w}(q_1) & \Phi(q_2)\,J_{2w}(q_2) \\ (L_{m1}^T(q_1))^{-1}\,J_{1w}(q_1) & 0_{6\times6} \end{bmatrix} \tag{C.3}$$

where the (6×6) matrices $\Psi(q_1)$ and $\Phi(q_2)$ in Eq. (C.3) are defined as:

$$\Psi(q_1) = \begin{bmatrix} L_{s1}^T(q_1) \\ 0_{3\times6} \end{bmatrix}, \tag{C.4}$$

$$\Phi(q_2) = \begin{bmatrix} -L_{s2}^T(q_2)\,(L_{m2}^T(q_2))^{-1} \\ [0_{3\times3};\, I_{3\times3}] \end{bmatrix}. \tag{C.5}$$

Matrix $(A^T(q)\,;\, B^T(q))^T$ given by Eq. (C.3) must be nonsingular in accordance with Eq. (35). The determinant of this composite matrix is given by [35]:

$$\det \begin{bmatrix} A(q) \\ B(q) \end{bmatrix} = \det \left[(L_{m1}^T(q_1))^{-1}\, J_{1w}(q_1)\right] \cdot \det\left[\Phi(q_2)\, J_{2w}(q_2)\right] \tag{C.6}$$

where det [ ] denotes the determinant of [ ]. To determine the rank of $(A^T(q)\,;\, B^T(q))^T$, the rank of matrix $\Phi(q_2)$ contained on the right hand side of Eq. (C.6) is needed. $\Phi(q_2)$ can be simplified by incorporating Eqs. (10) and (12) with i=2 into Eq. (C.5) and rearranging:

$$\Phi(q_2) = \begin{bmatrix} -I_{3\times3} & E_{2l}(q_2) \\ 0_{3\times3} & I_{3\times3} \end{bmatrix}. \tag{C.7}$$

Interestingly, Eq. (C.7) reveals that matrix $\Phi(q_2)$ is nonsingular and that it satisfies the relation: $(\Phi(q_2))^2 = I_{6\times6}$. Given that $J_{iw}(q_i)$ and $L_{mi}(q_i)$, i=1,2, are nonsingular matrices, the application of the matrix relation given by Eq. (B.1) to the matrix product terms contained on the right hand side of Eq. (C.6) reveals that matrix $(A^T(q)\,;\, B^T(q))^T$ is nonsingular.

Matrix $(A^T(q)\,;\, B^T(q))^T$ in Eq. (C.3) can be symbolically inverted using the method of partitioning [35]:

$$\begin{bmatrix} A(q) \\ B(q) \end{bmatrix}^{-1} = \begin{bmatrix} 0_{6\times6} & J_{1w}^{-1}(q_1)\, L_{m1}^T(q_1) \\ J_{2w}^{-1}(q_2)\,\Phi(q_2) & -J_{2w}^{-1}(q_2)\,\Phi(q_2)\,\Psi(q_1) \end{bmatrix}. \tag{C.8}$$

The (12×3) matrix $\Pi(q)$ and the (12×9) matrix $\Sigma(q)$ defined by Eq. (35) with $(N_1 = N_2 = 6)$ are comprised of the first three and last nine columns, respectively, of the matrix contained on the right hand side of Eq. (C.8):

$$\Pi(q) = \begin{bmatrix} 0_{6\times3} \\ J_{2w}^{-1}(q_2) \begin{bmatrix} -I_{3\times3} \\ 0_{3\times3} \end{bmatrix} \end{bmatrix}, \tag{C.9}$$

$$\Sigma(q) = \begin{bmatrix} 0_{6\times3} & J_{1w}^{-1}(q_1)\, L_{m1}^T(q_1) \\ J_{2w}^{-1}(q_2) \begin{bmatrix} E_{2l}(q_2) \\ I_{3\times3} \end{bmatrix} & -J_{2w}^{-1}(q_2)\,\Phi(q_2)\,\Psi(q_1) \end{bmatrix} \tag{C.10}$$

in which Eq. (C.7) has been applied. It is straightforward to verify that matrices $A(q)$, $B(q)$, $\Pi(q)$, and $\Sigma(q)$ defined by Eqs. (C.1), (C.2), (C.9), and (C.10) respectively, satisfy the matrix relations given in Eq. (35) with $(N_1 = N_2 = 6)$.

Example 2 : In this example the pseudovelocities are chosen to be the Cartesian angular velocities of the two rigid bodies comprising the common load as well as the Cartesian translational velocities of the center of the spherical joint at point Q_{cs}, i.e., $\nu = [\omega_{c2}^T, v_{cs}^T, \omega_{c1}^T]^T$. Suppose the first three pseudovelocities $[\nu_1, \nu_2, \nu_3]^T$ are expressed in terms of the joint velocities of manipulator 2 while the last six pseudovelocities are in terms of the joint velocities of manipulator 1. In this case the (9×12) matrix $B(q)$ is selected as:

$$B(q) = \begin{bmatrix} 0_{3\times6} & [0_{3\times3};\, I_{3\times3}]\, J_{2w}(q_2) \\ L_{s1}^T(q_1)\,(L_{m1}^T(q_1))^{-1}\, J_{1w}(q_1) & 0_{3\times6} \\ [0_{3\times3};\, I_{3\times3}]\, J_{1w}(q_1) & 0_{3\times6} \end{bmatrix} \tag{C.11}$$

in which Eqs. (10), (16), and (17) have been used.

The (12×12) composite matrix $(A^T(q)\,;\, B^T(q))^T$ defined by Eqs. (C.1) and (C.11) with $(N_1 = N_2 = 6)$ is presented here in a partitioned form:

$$\begin{bmatrix} A(q) \\ B(q) \end{bmatrix} = \begin{bmatrix} \Psi(q_1)\,(L_{m1}^T(q_1))^{-1}\, J_{1w}(q_1) & \Phi(q_2)\, J_{2w}(q_2) \\ \lambda(q_1)\, J_{1w}(q_1) & 0_{6\times6} \end{bmatrix} \tag{C.12}$$

where the (6×6) matrices $\Psi(q_1)$ and $\Phi(q_2)$ in Eq. (C.12) are specified in Eqs. (C.4) and (C.7), respectively. The (6×6) matrix $\lambda(q_1)$ in Eq. (C.12) is defined as:

$$\lambda(q_1) = \begin{bmatrix} L_{s1}^T(q_1)\,(L_{m1}^T(q_1))^{-1} \\ [0_{3\times3};\, I_{3\times3}] \end{bmatrix}. \tag{C.13}$$

Matrix $(A^T(q)\,;\, B^T(q))^T$ given by Eq. (C.12) must be nonsingular in accordance with Eq. (35). The determinant of this composite matrix is given by [35]:

$$\det \begin{bmatrix} A(q) \\ B(q) \end{bmatrix} = \det\,[\lambda(q_1)\, J_{1w}(q_1)] \cdot \det\,[\Phi(q_2)\, J_{2w}(q_2)]\,. \tag{C.14}$$

To determine the rank of $(A^T(q)\,;\, B^T(q))^T$, the rank of matrix $\lambda(q_1)$ contained on the right hand side of Eq. (C.14) is needed. $\lambda(q_1)$ can be simplified by incorporating Eqs. (10) and (12) with i=1 into Eq. (C.13) and rearranging:

$$\lambda(q_1) = \begin{bmatrix} I_{3\times3} & -E_{1l}(q_1) \\ 0_{3\times3} & I_{3\times3} \end{bmatrix}. \tag{C.15}$$

Equation (C.15) reveals that $\lambda(q_1)$ is nonsingular and $\det\,[\lambda(q_1)] = 1$. Given that $J_{iw}(q_i)$ and $L_{mi}(q_i)$, i=1,2, and $\Phi(q_2)$ are nonsingular matrices, the application of the matrix relation given by Eq. (B.1) to the matrix product terms contained on the right hand side of Eq. (C.14) reveals that matrix $(A^T(q)\,;\, B^T(q))^T$ is nonsingular.

Matrix $(A^T(q)\,;\,B^T(q))^T$ in Eq. (C.12) can be symbolically inverted using the method of partitioning [35]:

$$\begin{bmatrix} A(q) \\ B(q) \end{bmatrix}^{-1} = \begin{bmatrix} 0_{6\times 6} & J_{1w}^{-1}(q_1)\,(\lambda(q_1))^{-1} \\ J_{2w}^{-1}(q_2)\,\Phi(q_2) & -\,J_{2w}^{-1}(q_2)\,\Phi(q_2)\,\Psi(q_1)\,(L_{m1}^T(q_1))^{-1}\,(\lambda(q_1))^{-1} \end{bmatrix}. \tag{C.16}$$

The (12×3) matrix $\Pi(q)$ and the (12×9) matrix $\Sigma(q)$ defined by Eq. (35) with $(N_1 = N_2 = 6)$ are comprised of the first three and last nine columns, respectively, of the matrix contained on the right hand side of Eq. (C.16):

$$\Pi(q) = \begin{bmatrix} 0_{6\times 3} \\ J_{2w}^{-1}(q_2) \begin{bmatrix} -\,I_{3\times 3} \\ 0_{3\times 3} \end{bmatrix} \end{bmatrix}, \tag{C.17}$$

$$\Sigma(q) = \begin{bmatrix} 0_{6\times 3} & J_{1w}^{-1}(q_1)\,(\lambda(q_1))^{-1} \\ J_{2w}^{-1}(q_2) \begin{bmatrix} E_{2l}(q_2) \\ I_{3\times 3} \end{bmatrix} & -\,J_{2w}^{-1}(q_2)\,\Phi(q_2)\,\Psi(q_1)\,(L_{m1}^T(q_1))^{-1}\,(\lambda(q_1))^{-1} \end{bmatrix} \tag{C.18}$$

in which Eq. (C.7) has been used. It is straightforward to verify that matrices $A(q)$, $B(q)$, $\Pi(q)$, and $\Sigma(q)$ defined by Eqs. (C.1), (C.11), (C.17), and (C.18) respectively, satisfy the matrix relations given in Eq. (35) with $(N_1 = N_2 = 6)$.

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