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Time Optimal Trajectories for a Two Wheeled Robot

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ABSTRACT

Our objective is to move a two wheeled robot from one posture to the next in the minimum time in a planar environment without obstacles. We assume that the maximum acceleration on each wheel is bounded. We have used Pontryagin's Maximum Principle to find the optimal paths.

The optimal trajectories are bang-bang; at every point on the optimum path, the acceleration on each wheel is either at the upper limit or at the lower limit. We can use a coordinate transformation to move the initial posture to the origin. We can reach any point by a rotation followed by a translation. Adding a final rotation moves the robot to an arbitrary posture.

A switch point is a point at which the acceleration on one of the wheels changes sign. We can characterize a trajectory by the number of switch points. A path with a smaller number of switch points will have a higher average velocity and a longer distance traveled by the wheels. The path with the smallest number of switch points has two (one of each wheel). [For translation, both wheels accelerate from zero to the maximum velocity. After the switch point, both wheels decelerate to zero.] However, there are only two paths with two switch points: translation and rotation. Rotation followed by translation requires five switch points, while rotation, translation, rotation has eight switch points.

We have explored paths with three and four switch points. The paths with three switch points and initial rotation can reach any point faster than a rotation followed by a translation. Paths with three switch points and initial translation or paths with four switch points are useful if the final orientation is considerably different than the direction of travel.

1. INTRODUCTION

A robot's path can be described by a sequence of postures (a posture is a position with an orientation). At each posture, the robot is at rest and performs a task. This paper considers the problem of finding the quickest path from one posture to the next in an environment with no obstacles.

The motivation to consider this problem was provided by the need to plan paths for the HERMIES-III mobile robot. The robot has two steerable drive wheels and four casters (in this paper, we will assume that the drive wheels are not steerable). The kinematics and control of the robot are described in Jansen and Kress.[1]

Currently, the motion system for HERMIES-III consists of five processes: global path planning, path monitoring, obstacle detection and avoidance, wheel control, and wheel drive. A mission for the robot is defined by a sequence of via points. At each via point, the robot stops and performs an action (typical actions are: laser range scan, video image acquisition, or move arm). Given the mission and a world model, the global path planner defines a sequence of set points that connect the via points and avoid the obstacles in the world model. During the execution of the path, the path monitor compares the robot's position to the plan and determines the next set or via point. While the robot is moving, sensors monitor the environment. If unexpected obstacles are detected, the robot modifies its path to avoid the obstacles. Given the next point or posture on the modified path, the wheel controller continuously determines the wheel velocities and sends them to the wheel driver. The wheel driver sends currents to the wheel motors.

In this paper, we are examining the kinematics of an idealized wheel controller. The real wheel controller would allow the final posture to change during the execution of a path. Thus, the initial wheel velocities could be nonzero (positive or negative). We will assume that the initial and final velocities are zero. The real wheel controller has bounds on both velocity and acceleration. We will neglect the velocity bounds.

Our focus is on kinematics rather than dynamics. We consider position, velocity, and acceleration and neglect mass, force, motor currents, and power supplies.

The insights gained from the study of the kinematics of an idealized wheel controller will help us design the algorithm for the real wheel controller. The design and development of the real wheel controller will be described in a subsequent paper.

Kanayama and Hartman[2] have worked on a similar problem. Their objective is to find smooth paths between postures. When they minimize curvature, the paths are circular arcs and line segments. When they minimize the derivative of the curvature of the path, the trajectories are segments of cubic spirals.

We will use Pontryagin's Maximum Principle[3] to find the optimal paths. We have been unable to find any papers that have applied the Maximum Principle to mobile robots. However, we have found several papers that apply the Maximum Principle to robotic manipulators. Since the optimum solutions for manipulators are similar to the optimum solutions for mobile robots, we will discuss the results of two recent papers.

Chen and Desrochers[4] have worked on the minimum time control problem for a manipulator with a constrained path between two endpoints (they observe that the minimum time control problem for an obstacle-free and unconstrained environment is more difficult than the constrained problem). They assume that the control torques have upper and lower bounds. For the point to point control problem, at least one of the control torques is at its bound. For the constrained problem, one and only one of the control torques is always at its bound.

Yamamoto and Mohri[5] consider the unconstrained problem. Using the Maximum Principle, they can show that the optimal path is bang-bang (the torques are always at their bounds). They propose an algorithm for solving a bang-bang control problem in which the switching times for the control torques are unknown variables.

The next section will describe the equations of motion for the robot. The third section will use the Maximum Principle to derive the conditions for optimal trajectories. The fourth section will present analytical solutions for rotation and translation. The fifth section will derive optimal trajectories for combined rotation and translation. The final section will summarize the results and present our conclusions.

2. ROBOT MOTION

We consider a robot with two wheels that cannot be steered (the wheels are perpendicular to the axis joining the wheels). The motion of each wheel is described by four variables: x, y, h, and g. The coordinates x and y locate the point of contact on the plane; the angle h is the orientation of the plane of the wheel with respect to the x axis; and the angle g is the angle of rotation of the wheel about the axis. The equations of motion for each wheel are:

$$\dot{x} = r\dot{g}\cos h \tag{1}$$

$$\dot{y} = r\dot{g}\sin h \ , \tag{2}$$

where r is the radius of the wheel and the dot signifies a time derivative.

We will refer to the two wheels by the letters R and L (for right and left). We will use a dot notation for each of the four variables for each of the two wheels; for example, $R.x$ is the x coordinate of the right wheel and $L.g$ is the angle of rotation of the left wheel. The equations of motion for the two wheels are:

$$R.\dot{x} = rR.\dot{g}\cos R.h \tag{3}$$

$$R.\dot{y} = rR.\dot{g}\sin R.h \tag{4}$$

$$L.\dot{x} = rL.\dot{g}\cos L.h \tag{5}$$

$$L.\dot{y} = rL.\dot{g}\sin L.h \tag{6}$$

We can reduce the four equations of motion to three by introducing the rigid body constraint that the distance between the wheels is fixed. We define F to be the vector from the left wheel to the right wheel and C to be the vector to the midpoint between the wheels:

$$F.x = R.x - L.x \tag{7}$$

$$F.y = R.y - L.y \tag{8}$$

$$C.x = (R.x + L.x)/2 \tag{9}$$

$$C.y = (R.y + L.y)/2 \ . \tag{10}$$

If D is the distance between the wheels, the length of F is D:

$$F.x^2 + F.y^2 = D^2 \ . \tag{11}$$

Let h be the orientation of the vector F: $\cos h = F.x/D$ and $\sin h = F.y/D$. Since the wheels are perpendicular to the axle: $\cos R.h = -\sin h$ and $\sin R.h = \cos h$.

Since the distance between the wheels is fixed, the motion of the vector F is determined by h:

$$\dot{h} = r(R.\dot{g} - L.\dot{g})/D \ . \tag{12}$$

The motion of the vector C is given by:

$$C.\dot{x} = -\ [(R.\dot{g} + L.\dot{g})\sin h]/2 \tag{13}$$

$$C.\dot{y} = [(R.\dot{g} + L.\dot{g})\cos h]/2 \tag{14}$$

Equations (12) to (14) determine the position and orientation of the robot. If the two wheels have equal velocities, the robot moves without rotation. If the wheel velocities have equal magnitudes and opposite signs, the robot rotates without translation. In general, the robot simultaneously rotates and translates.

The velocities of the wheels are changed by accelerations provided by forces. Thus, the wheel velocities are continuous, while the wheel accelerations can be discontinuous. The wheel accelerations are the control variables that determine the potential trajectories.

To express our problem in the language of control theory, we will define five state variables $[x_i]$.

$$x_1 = h \tag{15}$$

$$x_2 = C.x/(D/2) \tag{16}$$

$$x_3 = C.y/(D/2) \tag{17}$$

$$x_4 = rR.\dot{g}/D \tag{18}$$

$$x_5 = rL.\dot{g}/D \ . \tag{19}$$

Using the state variables, the equations that determine the motion of the robot are:

$$\dot{x}_1 = x_4 - x_5 \tag{20}$$

$$\dot{x}_2 = -(x_4 + x_5)\sin x_1 \tag{21}$$

$$\dot{x}_3 = (x_4 + x_5)\cos x_1 \tag{22}$$

$$\dot{x}_4 = u_1 \tag{23}$$

$$\dot{x}_5 = u_2 \ , \tag{24}$$

where the u_i are control variables that are bounded:

$$|u_i| \leq b \ . \tag{25}$$

3. OPTIMAL TRAJECTORIES

Since its discovery in 1956, the Pontryagin Maximum Principle has been used to solve a wide variety of optimization problems. This section will summarize the maximum principle and apply it to our problem. In vector notation, the motion of the system is described by:

$$\dot{x} = f(x, u) \ . \tag{26}$$

The optimization problem is to find control variables $[u_i(t)]$ that will move the system from an initial state x_0 to a final state x_1 and minimize a functional. For our case, the goal is to minimize the time.

Pontryagin introduced a system of dual variables $[\psi_i]$. The dual variables satisfy the following system of differential equations:

$$\dot{\psi} = -\sum_j \psi_j \partial f^j / \partial x_i \ , \quad i \text{ and } j = 1, \ldots, n \tag{27}$$

Using the dual variables, Pontryagin defines a Hamiltonian function, H:

$$H = \sum_j \psi_j f^j \ . \tag{28}$$

The optimal set of control variables maximizes the Hamiltonian function. For our problem, the $f^j(x, u)$ are the right sides of Eqs. (20) to (24) and the only f^j that depend on the control variables are f^4 and f^5. Thus, the control variables are at their upper limits when the corresponding dual variables are positive and at their lower limits when the dual variables are negative. In the jargon of optimal control, the optimal trajectory is bang-bang. Furthermore, for an optimal trajectory, the Hamiltonian is constant and non-negative.

For our problem, the dual variables satisfy the following equations:

$$\dot{\psi}_1 = [x_4 + x_5] \ [\psi_2 \cos x_1 + \psi_3 \sin x_1] \tag{29}$$

$$\dot{\psi}_2 = 0 \tag{30}$$

$$\dot{\psi}_3 = 0 \tag{31}$$

$$\dot{\psi}_4 = -\psi_1 + \psi_2 \sin x_1 - \psi_3 \cos x_1 \tag{32}$$

$$\dot{\psi}_5 = +\psi_1 + \psi_2 \sin x_1 - \psi_3 \cos x_1 \tag{33}$$

Our objective is to move from an arbitrary initial posture to an arbitrary final posture. By a coordinate transformation, we can move the initial posture to the origin. Thus, our initial conditions are that all five of the state variables are zero. We will assume that the final posture is in the first quadrant; if we can reach any posture in the first quadrant, simple sign transformations will allow the robot to reach any posture in the other three quadrants.

The requirement that the Hamiltonian is a constant often provides initial or final conditions on the dual variables. For our problem, the wheel velocities are zero at the beginning and end of the optimal trajectory. Consequently, the first three f^i are zero at both ends of the trajectory and the values of the ψ_i are arbitrary. The initial values of the dual variables $[c_i]$ will characterize a trajectory:

$$\psi_i(0) = c_i \ , \ \text{for} \ i = 1, ..., 5 \ . \tag{34}$$

4. TRANSLATION AND ROTATION

In this section, we will derive simple analytical solutions for pure translation and pure rotation. We will also derive the conditions for combined motion (translation and rotation).

For pure translation in the first quadrant, both wheels have maximum positive acceleration for half of the trip and maximum negative acceleration for the second half of the trip:

$$\dot{x}_4 = b \text{ for } 0 \le t \le T/2 \tag{35}$$

$$\dot{x}_4 = -b \text{ for } T/2 \le t \le T \tag{36}$$

The velocity increases for half of the trip and decreases for half of the trip:

$$x_4 = bt \text{ for } 0 \le t \le T/2 \tag{37}$$

$$x_4 = b(T-t) \text{ for } T/2 \le t \le T \tag{38}$$

Similarly, $x_5(t) = x_4(t)$. Without rotation, the angle is constant and equal to its initial value $[x_1 = 0]$. Thus, there is no motion in the x direction $[x_2 = 0]$. The y displacement increases with the square of the time:

$$\dot{x}_3 = 2x_4 \tag{39}$$

$$x_3 = bt^2 \text{ for } 0 \le t \le T/2 \tag{40}$$

$$x_3 = 2bTt - bt^2 - bT^2/2 \text{ for } T/2 \le t \le T \tag{41}$$

$$x_3(T) = bT^2/2 \tag{42}$$

The solution for the dual variables that determine the motion of the wheels (ψ_4 and ψ_5) should be positive initially, decrease to zero at $T/2$, and be negative for the last half of the trip. We assume that $\psi_1 = \psi_2 = 0$. Thus, $\psi_1 = 0$ and both ψ_4 and ψ_5 satisfy the same equation:

$$\dot{\psi}_4 = -c_3 \ . \tag{43}$$

Integrating Eq. (43), we obtain the dual solution for the right wheel:

$$\psi_4 = c_4 - c_3 t \tag{44}$$

The constants are chosen to switch from positive to negative at $T/2$:

$$\psi_4 = T/2 - t \tag{45}$$

For pure rotation, the wheels always move in opposite directions $[(x_4 + x_5) = 0]$ and the location of the robot does not change $[x_2 = x_3 = 0]$. For a clockwise rotation, the right wheel has negative rotation while the left wheel has positive rotation:

$$\dot{x}_5 = b \text{ for } 0 \le t \le T/2 \tag{46}$$

$$\dot{x}_5 = -b \text{ for } T/2 \le t \le T \tag{47}$$

The velocity increases for half of the trip and decreases for half of the trip:

$$x_5 = bt \text{ for } 0 \le t \le T/2 \tag{48}$$

$$x_5 = b(T-t) \text{ for } T/2 \le t \le T \tag{49}$$

The angle decreases with the square of the time:

$$\dot{x}_1 = -2x_5 \tag{50}$$

$$x_1 = -bt^2 \text{ for } 0 \le t \le T/2 \tag{51}$$

$$x_1 = bT^2/2 + bt^2 - 2bTt \text{ for } T/2 \le t \le T \tag{52}$$

$$x_1(T) = -bT^2/2 \tag{53}$$

The dual solution for each of the wheels should have opposite signs. For the left wheel, the dual solution should be positive initially, decrease to zero at $T/2$, and be negative for the last half of the trip. For pure rotation, ψ_1 is a constant. We assume that $c_2 = c_3 = 0$. Thus, $\psi_2 = \psi_3 = 0$ and ψ_4 and ψ_5 satisfy the same equation with opposite signs:

$$\dot{\psi}_4 = -c_1 \ . \tag{54}$$

Integrating Eq. (54), we obtain the dual solution for the right wheel:

$$\psi_4 = c_4 - c_1 t \tag{55}$$

The constants are chosen to switch from negative to positive at $T/2$:

$$\psi_4 = t - T/2 \tag{56}$$

Pure rotation and pure translation are optimal trajectories with one switch point for each wheel. In the remainder of this section, we will show that we can have optimal trajectories with a maximum total of four switch points. Thus, we could have two for one wheel and one for the other, or two for one wheel and two for the other, or three for one wheel and one for the other.

Switch points occur when one of the dual variables [ψ_4 or ψ_5] is zero. To derive expressions for the dual variables, we will define some auxiliary state variables [z_i]:

$$\dot{z}_1 = \sin x_1 \tag{57}$$

$$\dot{z}_2 = \cos x_1 \tag{58}$$

$$\dot{z}_3 = x_2 \tag{59}$$

$$\dot{z}_4 = x_3 \tag{60}$$

The initial conditions for the z_i are $z_i(0) = 0$.

Using the state variables and the auxiliary state variables, the dual variables satisfy:

$$\psi_1 = c_1 + c_2 x_3 - c_3 x_2 \tag{61}$$

$$\psi_4 = c_4 - c_1 t + c_2(z_1 - z_4) + c_3(z_3 - z_2) \tag{62}$$

$$\psi_5 = c_5 + c_1 t + c_2(z_1 + z_4) - c_3(z_2 + z_3) \tag{63}$$

The dual variables that control the switch points depend on five constants $[c_i]$. At each switch point, either ψ_4 or ψ_4 or both are equal to zero. Since the equations are homogeneous, we can assume that one of the constants is known $[c_4$ or $c_5]$ and solve for the other four constants. Thus, we can have a maximum of four switch points. We will explore the solutions of these equations in the next section.

5. NUMERICAL RESULTS

The robot can reach any point by a rotation followed by a translation. If we assume that $b = 0.2$ and use Eq. (53), a 90 degree rotation requires 3.96 seconds. We will present results for two time intervals (6 and 10). Using Eq. (42), the robot can travel 3.6 when $T = 6$ and 10.0 when $T = 10$. (Recall that the x and y coordinates have been divided by half the distance between the wheels [see Eqs. (16) and (17)]). If the robot rotates 90 degrees and then moves forward, it can travel 0.4 when $T = 6$ and 3.6 when $T = 10$. The locus of points that the robot can reach when $T = 6$ is displayed in Fig. 1, while the locus for $T = 10$ is displayed in Fig. 2.

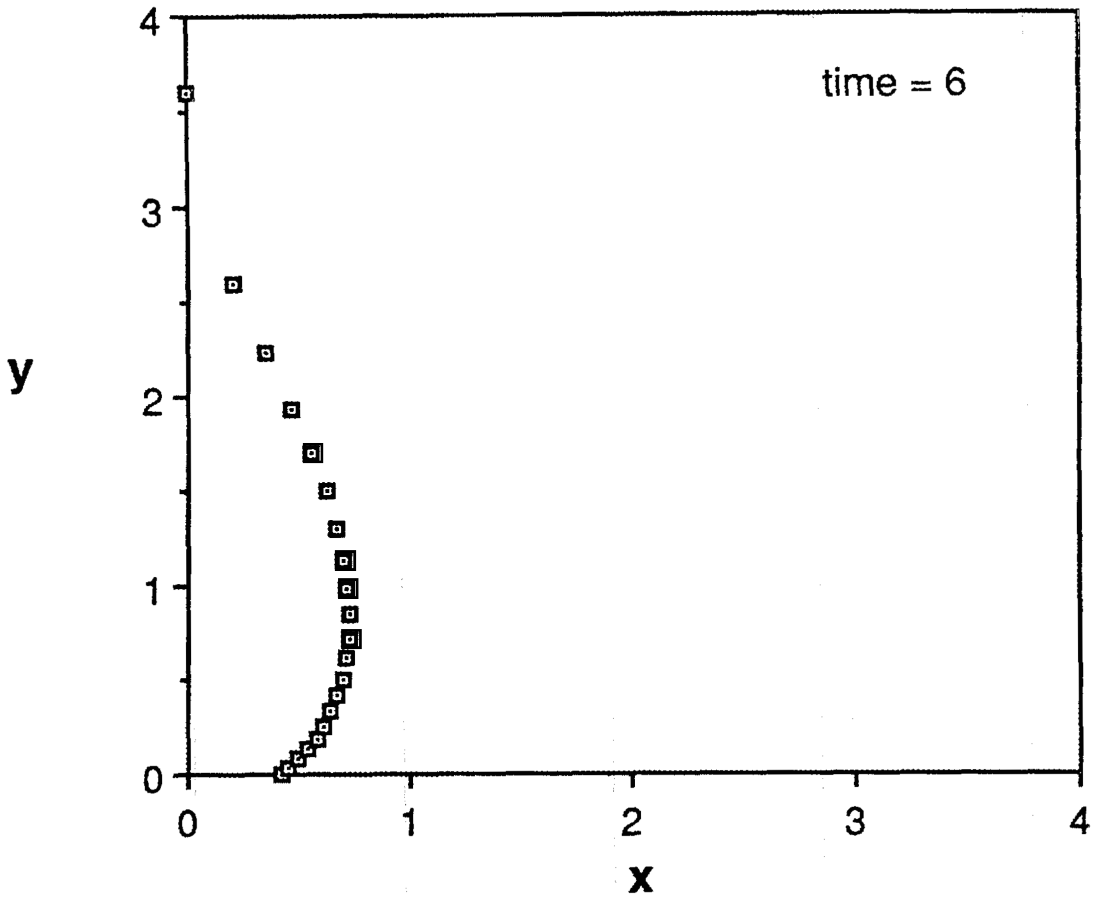

Fig. 1. The locus of points that the robot can reach when $T = 6$.

The optimal trajectories are bang-bang. During the trajectories, each wheel is always either ramping up or ramping down at the maximum rate. Since the initial and final velocities are zero, each wheel spends half of its time ramping up and the other half ramping down. A switch point is a time when the acceleration changes sign. Switch points characterize trajectories. The most simple trajectory has one switch point (at $T/2$). A trajectory with one switch point will travel farther than

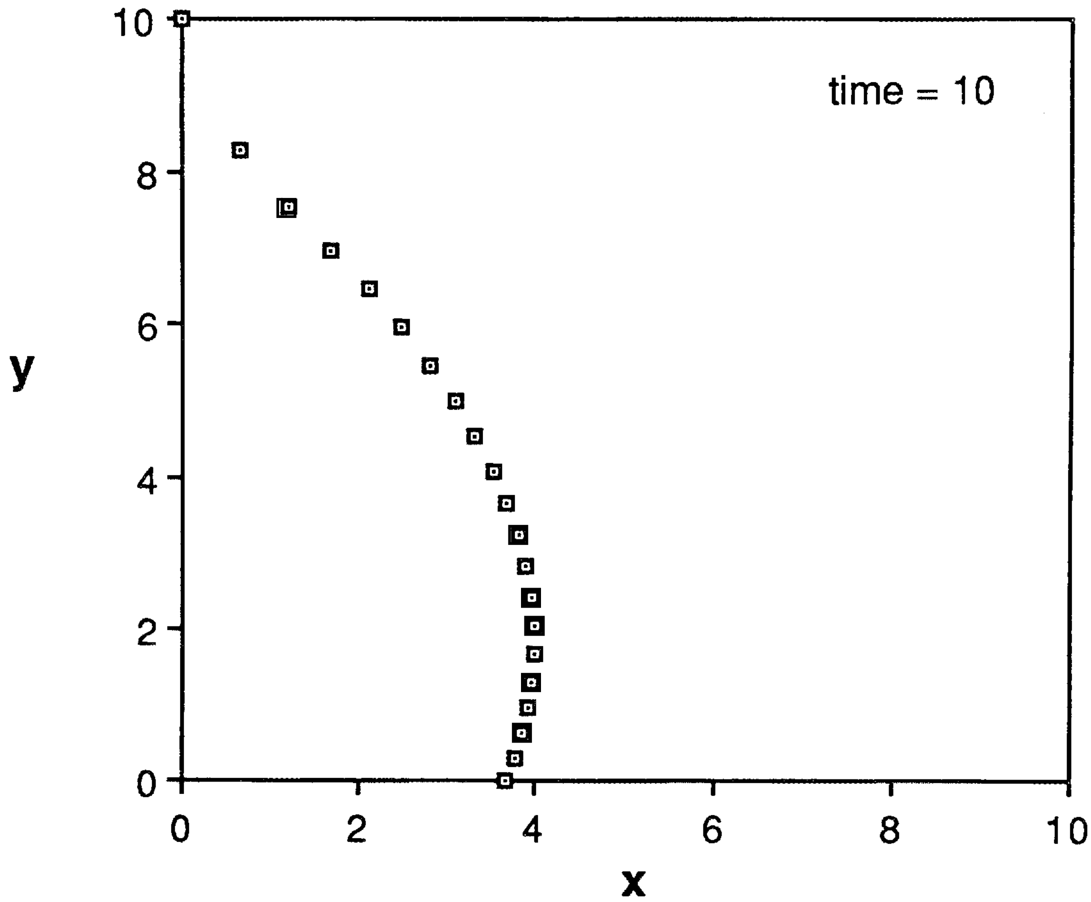

Fig. 2. The locus of points that the robot can reach when $T = 10$.

any other trajectory. The next most simple trajectory has two switch points. The first switch point (t_s) can range from 0 to $T/2$. The second switch point occurs at $t = t_s + T/2$. We will call the first switch point for the right wheel t_4 and call the left switch point t_5.

Initially, the robot is at the origin. Our objective is to move it into an arbitrary posture in the first quadrant. For the first class of trajectories that we will analyze, the left wheel has one switch point and the right wheel has two switch points. We assume that the initial acceleration on the left wheel is positive, while the acceleration on the right wheel is negative. If $t_4 = 0$, the robot will translate up the y axis. If $t_4 = T/2$, the robot will rotate without translation. As t_4 increases from zero, the final location of the robot will move through the first quadrant. Typical results for the optimal trajectories are presented in Table 1 and Fig. 3 for $T = 6$ and in Table 2 and Fig. 4 for $T = 10$.

When the total time of the trajectory is increased, the switch time required to produce a rotation through the first quadrant is decreased; as T increases from 6 to 10, t_4 decreases from 1.5 to 0.9. For each of the two values of T, the magnitude of the angle (x_1) increases in proportion to t_4.

Table 1. Optimal three switch trajectories with initial rotation for $T = 6(t_5 = 3.0)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
1	0.1	-0.1	0.35	3.46	6.90	7.34
2	0.5	-0.6	1.45	2.60	7.71	8.67
3	1.0	-1.2	2.00	1.21	8.03	9.36
4	1.5	-1.8	1.71	0.05	8.06	9.66

Table 2. Optimal three switch trajectories with initial rotation for $T = 10(t_5 = 5.0)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
5	0.1	-0.2	1.63	9.65	11.19	11.76
6	0.3	-0.6	4.49	8.16	11.89	12.88
7	0.5	-1.0	6.56	5.84	12.27	13.52
8	0.7	-1.4	7.61	3.07	12.51	13.96
9	0.9	-1.8	7.61	0.27	12.65	14.27

We have defined two performance measures: t-rad and t-tot. The first measure, t-rad, is the time required to rotate and translate to the final xy position of the robot, while t-tot is the time required to rotate, translate, and rotate to the final posture of the robot. As the total time of the trajectory increases, the performance advantage of the optimal trajectories decreases. When $T = 6$, the largest value of t-rad in Table 1 is 34% larger than T and the largest value of t-tot is 61% larger. When $T = 10$, the largest value for t-rad in Table 2 is 26% larger than T and the largest value for t-tot is 43% larger.

We have considered optimal three switch trajectories with initial rotation. The other set of optimal three switch trajectories has initial translation. The left wheel has one switch point and the right wheel has two switch points. To have initial translation, the initial acceleration on both the right and the left wheel is positive. If $t_4 = 0$, the robot will rotate without translation. If $t_4 = T/2$, the robot will translate up the y axis. As t_4 decreases from $T/2$, the final location of the robot will move through the first quadrant. Typical results for the optimal trajectories are presented in Table 3 and Fig. 5 for $T = 6$ and in Table 4 and Fig. 6 for $T = 10$.

Table 3. Optimal three switch trajectories with initial translation for $T = 6(t_5 = 3.0)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
10	2.9	-0.1	0.07	3.48	6.34	7.34
11	2.5	-0.6	0.28	2.96	6.42	8.67
12	2.0	-1.2	0.41	2.30	6.15	9.36
13	1.5	-1.8	0.43	1.65	5.73	9.66

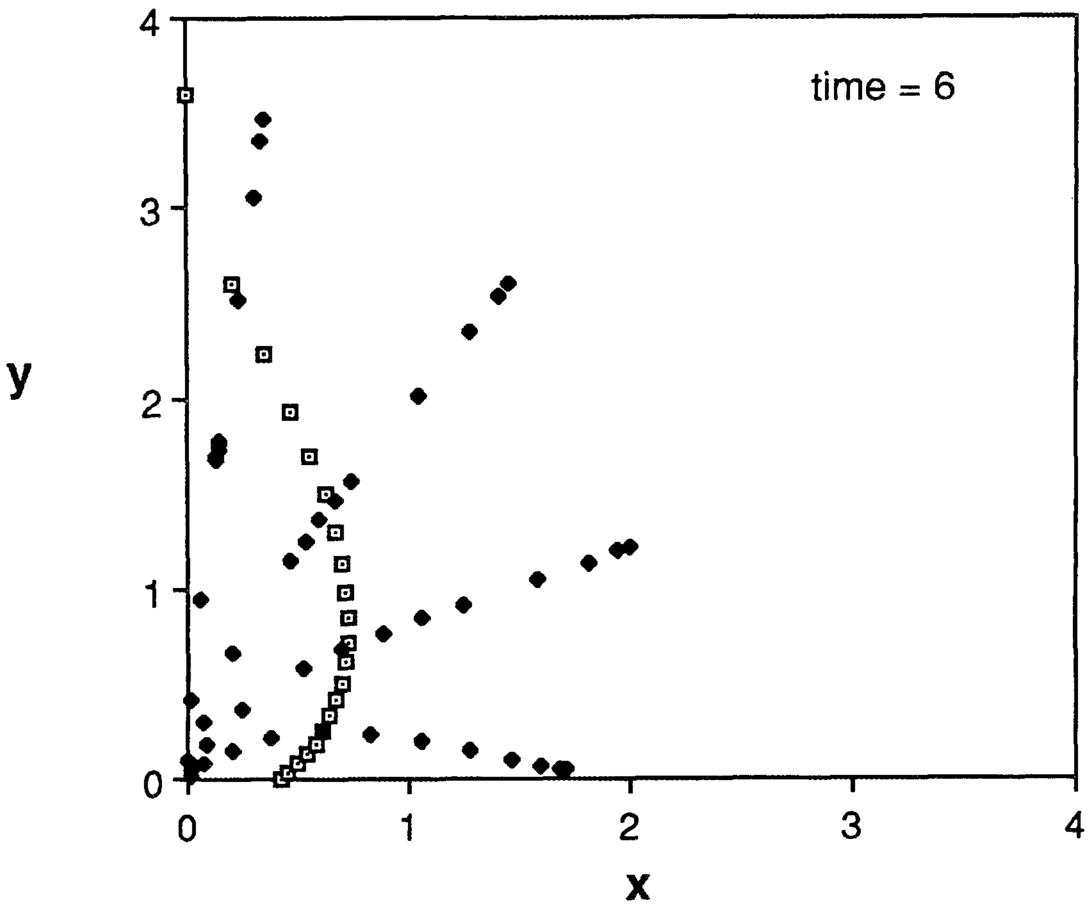

Fig. 3. Optimal three switch trajectories with initial rotation for $T = 6$.

Table 4. Optimal three switch trajectories with initial translation for $T = 10(t_5 = 5.0)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
14	4.5	-1.0	1.37	8.67	10.62	13.52
15	4.0	-2.0	2.09	7.02	10.26	14.39
16	3.5	-3.0	2.28	5.42	9.66	14.76

Unlike the trajectories with initial rotation, the trajectories with initial translation cannot reach all points in the first quadrant; the trajectories in Tables 3 and 4 lie in the sector between 67 degrees and 90 degrees. For each of the two values of T, the magnitude of the angle (x_1) increases in proportion to $(T/2 - t_4)$.

If we compare the values of the performance measures (t-rad and t-tot) in Table 1 with the corresponding $[(T/2 - t_4) = t_4]$ values in Table 3, the values of t-rad decrease while the values of t-tot are identical. Thus, optimal three switch

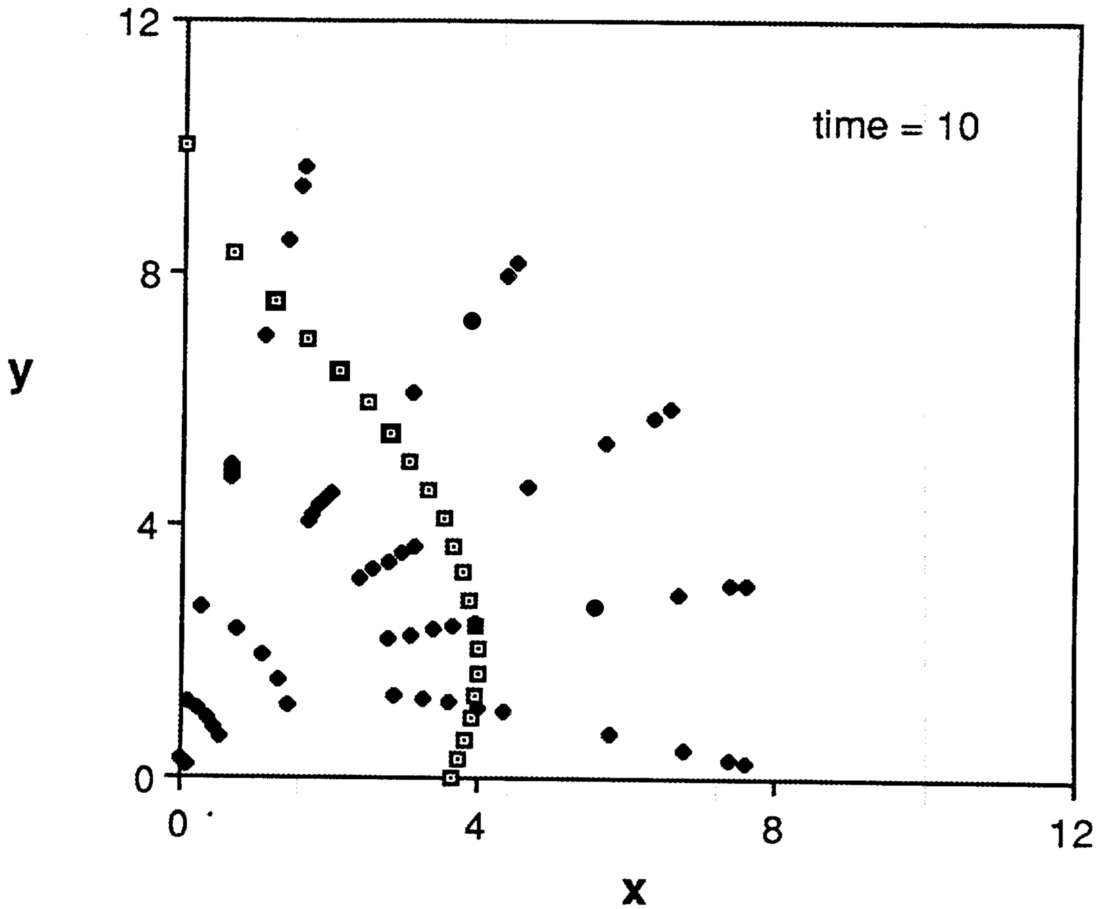

Fig. 4. Optimal three switch trajectories with initial rotation for $T = 10$.

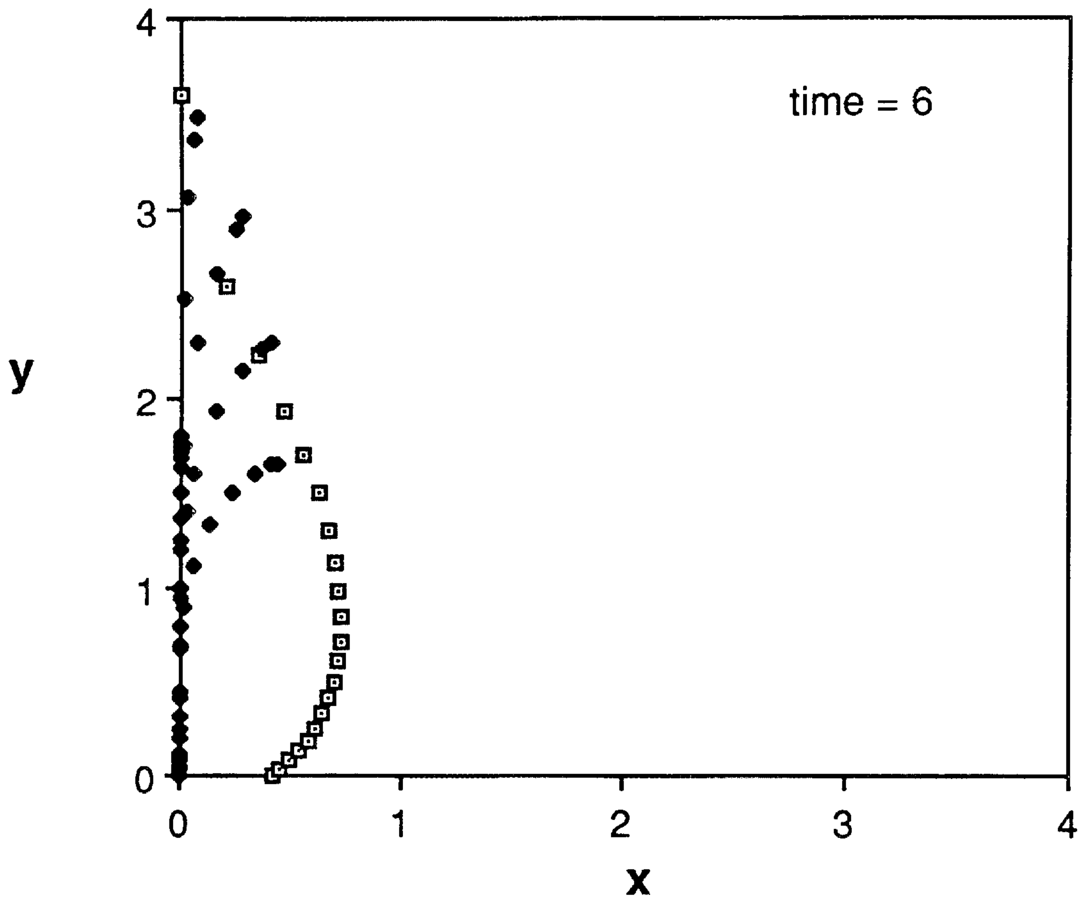

Fig. 5. Optimal three switch trajectories with initial translation for $T = 6$.

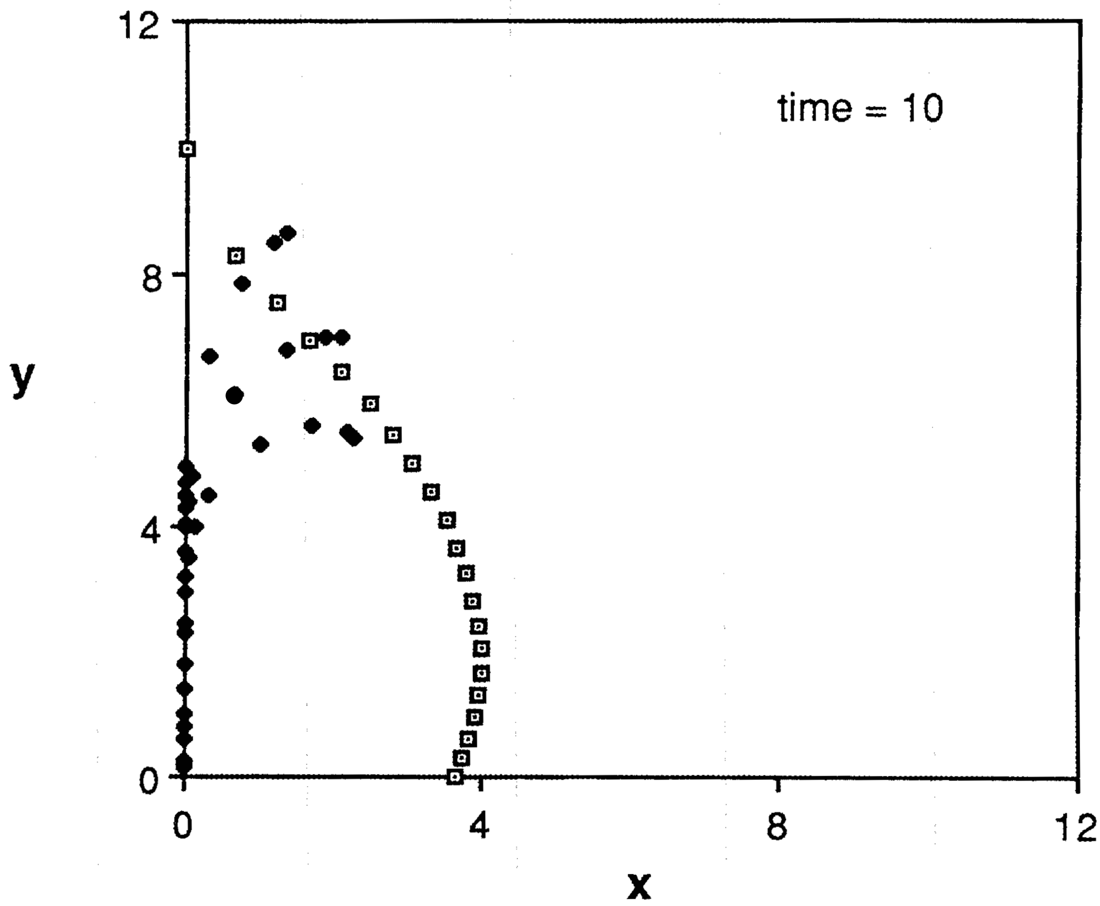

Fig. 6. Optimal three switch trajectories with initial translation for $T = 10$.

trajectories with initial translation are not well suited to cover large distances but are well suited for a trajectory with a small translation and a large rotation.

We have considered trajectories with three switch points; one on the left wheel and two on the right wheel. Next, we will examine trajectories with four switch points; two on both the right and left wheels. We will confine our attention to trajectories with initial rotation; positive acceleration on the left wheel and negative rotation on the right wheel. Sample results for optimal four switch trajectories with initial rotation are displayed in Table 5 and Fig. 7 for $T = 6$ and in Table 6 and Fig. 8 for $T = 10$.

Table 5. Optimal four switch trajectories with initial rotation for $T = 6(t_4 = 1.0)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
17	3.0	-1.2	2.00	1.21	8.03	9.36
18	2.5	-0.6	1.32	1.19	7.11	8.65
19	2.0	0.0	0.71	0.94	5.99	8.52

Table 6. Optimal four switch trajectories with initial rotation for $T = 10(t_4 = 0.5)$.

Case	t_4	x_1	x_2	x_3	t-rad	t-tot
20	5.0	-1.0	6.56	5.84	12.27	13.52
21	4.5	0.0	4.70	6.26	11.39	13.93
22	4.0	1.0	2.94	5.93	10.28	14.10

When the switch time for the left wheel (t_5) is equal to $T/2$, the four switch trajectory becomes one of the three switch trajectories that we examined previously in Tables 1 and 2. As t_5 decreases from $T/2$, the distance traveled by the left wheel decreases and the robot rotates in the positive direction. As the distance traveled by the left wheel decreases, the radial distance decreases and the radial performance measure (t-rad) decreases. For both cases, the total performance measure (t-tot) is always significantly greater than T. For the cases when $T = 6$, t-tot decreases with increased rotation while t-tot increases with increased rotation for the cases when $T = 10$.

The wheel velocities for three, four, and five switch point trajectories are displayed in Figs. 9 to 11. Figure 9 illustrates the wheel velocities for case 3 (and case 17), a three switch trajectory with initial rotation and the switch point at $t_4 = 1.0$. By integrating the wheel velocities, the right wheel travels 0.6 while the left wheel travels 1.8. The rotation of the robot during the transition from the initial posture to the final posture is the difference between the distances traveled by the two wheels; $x_1 = -1.2$.

Figure 10 presents the wheel velocities for a four switch trajectory with initial rotation and switch points at $t_4 = 1.0$ and $t_5 = 2.0$. The two wheels travel the same distance, 0.6. Thus, there is no rotation between the initial and final postures; $x_1 = 0.0$.

Figure 11 displays the wheel velocities for both a three switch trajectory and a five switch trajectory. The five switch trajectory has a rotation followed by a

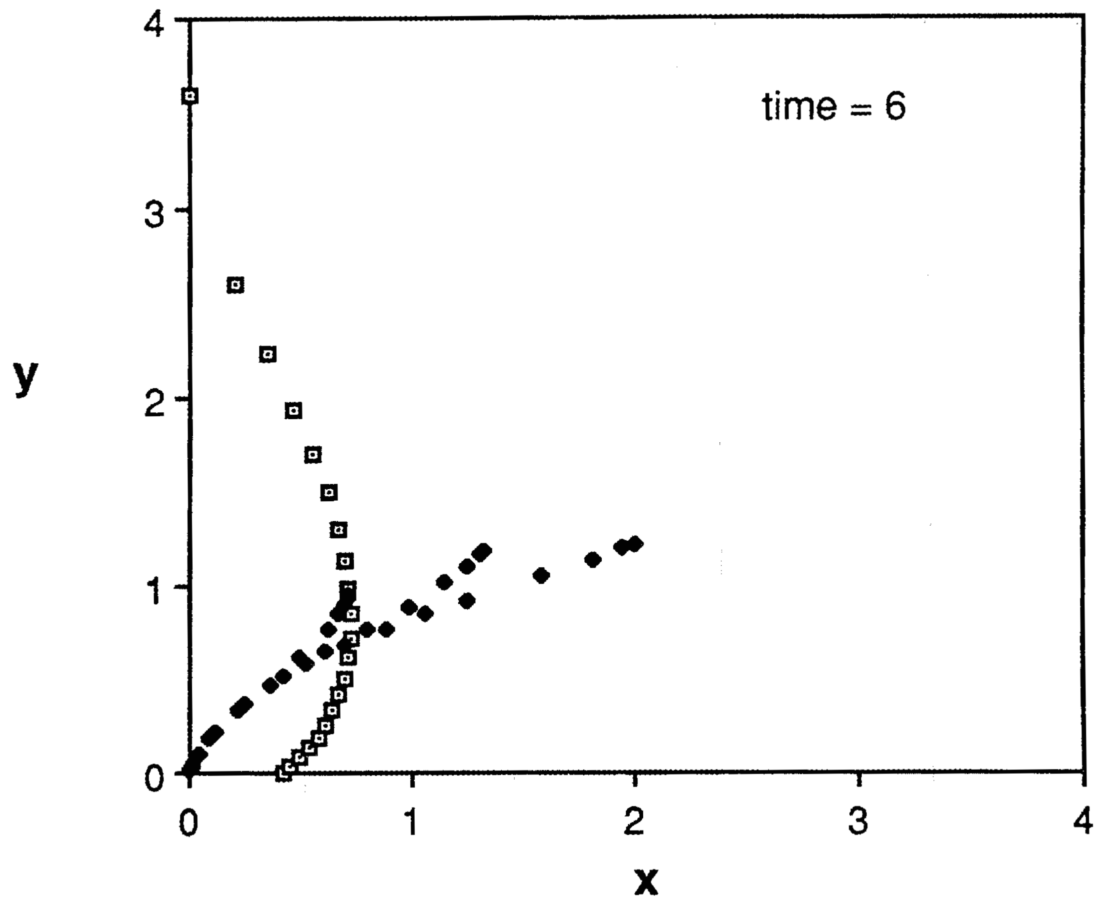

Fig. 7. Optimal four switch trajectories with initial rotation for $T = 6$.

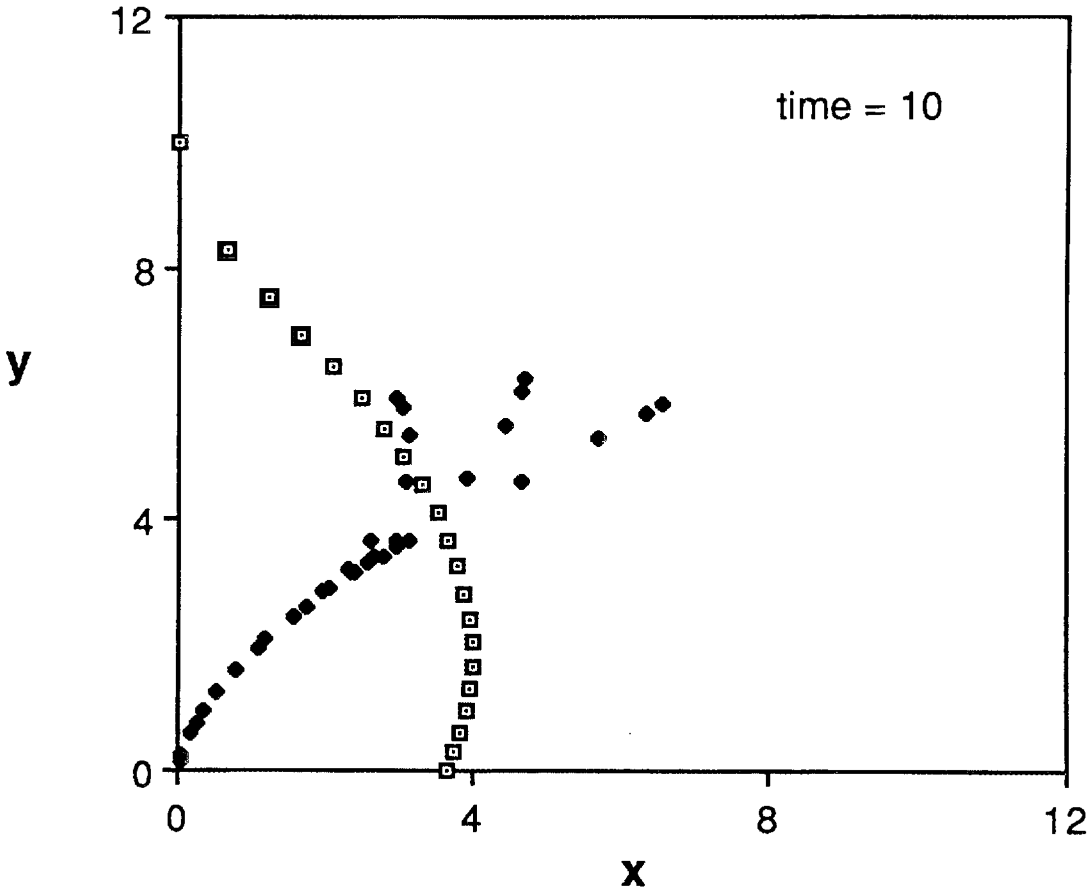

Fig. 8. Optimal four switch trajectories with initial rotation for $T = 10$.

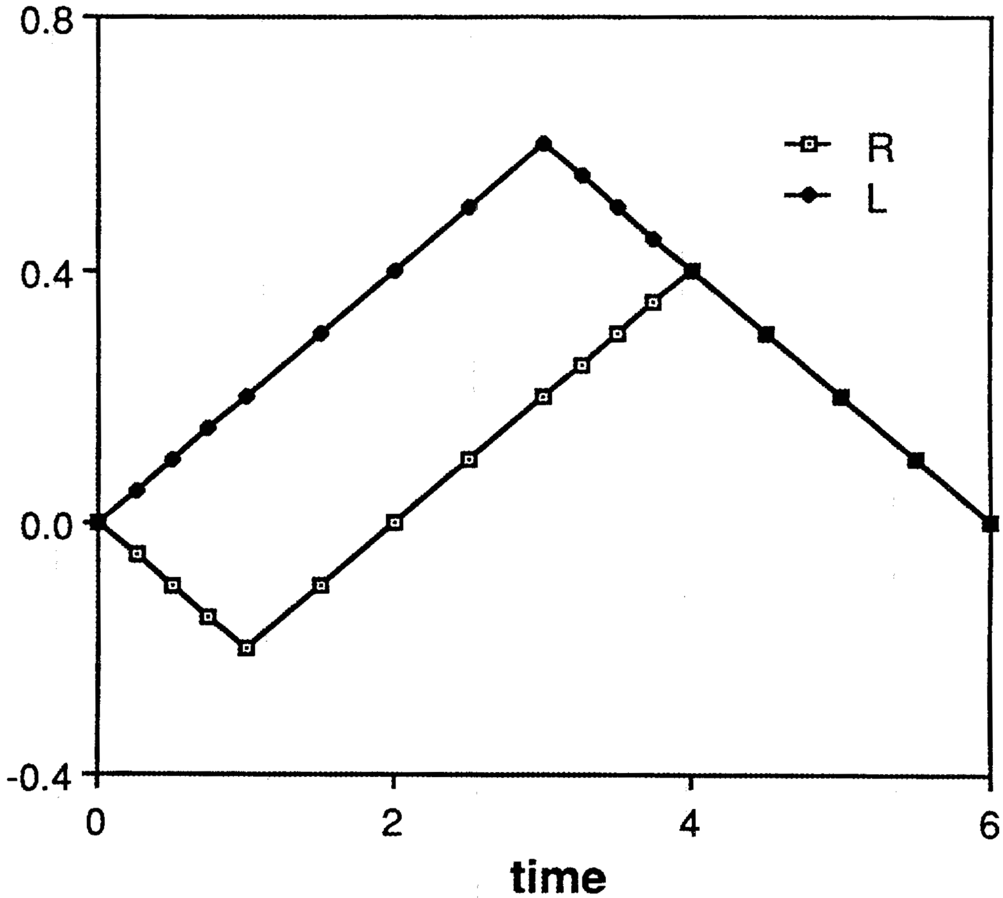

Fig. 9. Wheel velocities (x_4 and x_5) for a three switch trajectory with initial rotation.

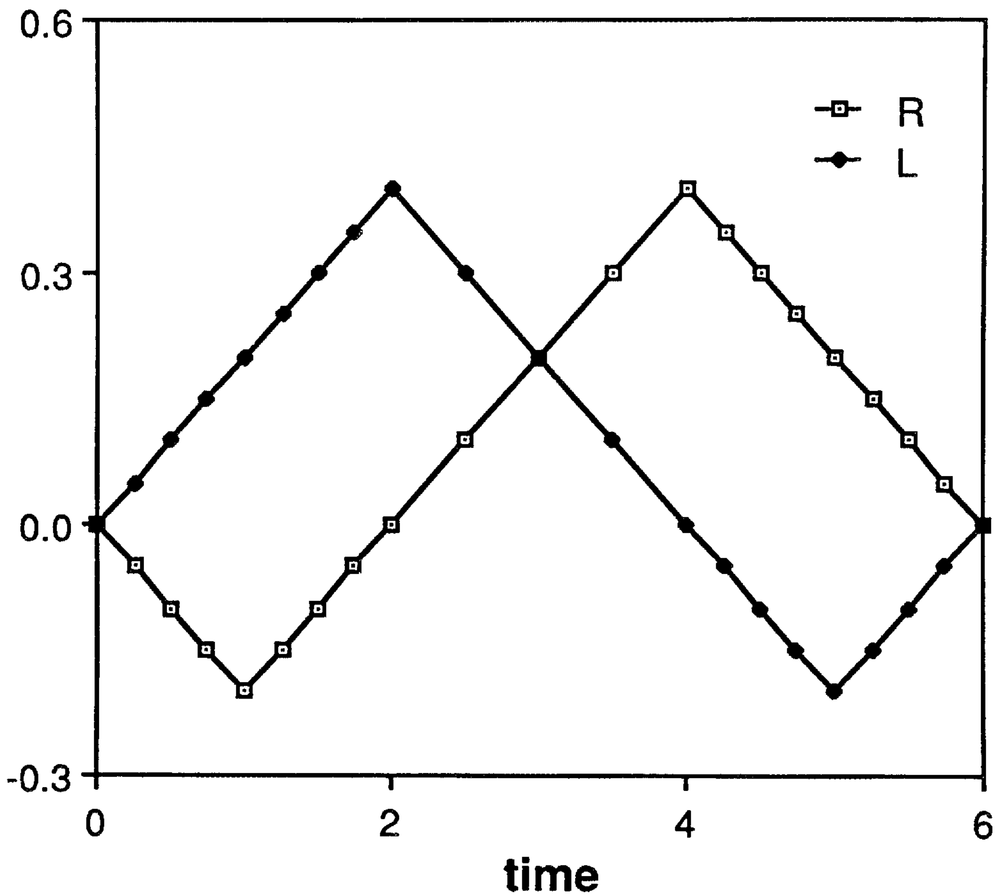

Fig. 10. Wheel velocities (x_4 and x_5) for a four switch trajectory with initial rotation.

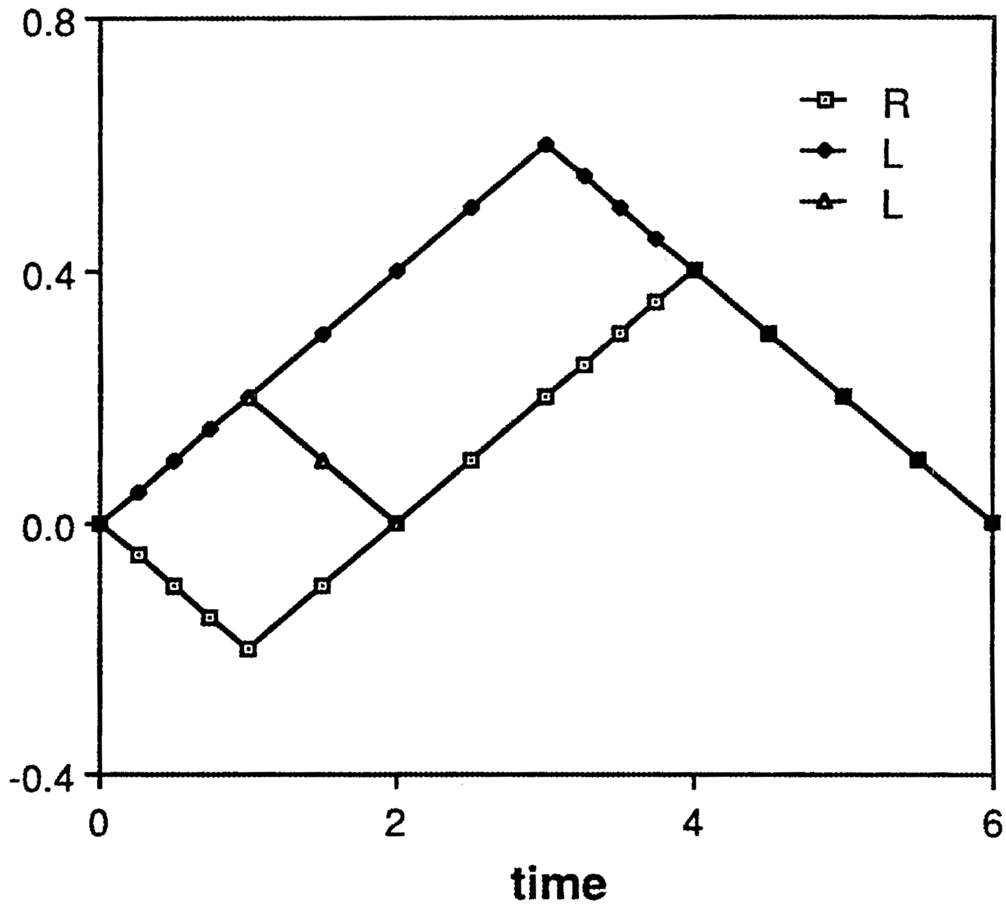

Fig. 11. Wheel velocities (x_4 and x_5) for both a three switch trajectory and a five switch trajectory.

translation. The right wheel has the same velocity profile for both trajectories. For the five switch trajectory, the left wheel ramps up and down during both the rotation (0 to 2) and the translation (2 to 6). The left wheel travels farther (1.8) during the three switch trajectory than during the five switch trajectory (1.0) because it has a higher velocity from $t = 1$ to 4.

6. CONCLUSIONS

Our objective is to move a two wheeled robot from one posture to the next in the minimum time in an environment without obstacles. We assume that the maximum acceleration on each wheel is bounded. We have used Pontryagin's Maximum Principle to find the optimal paths.

The optimal trajectories are bang-bang; at every point on the optimum path, the acceleration on each wheel is either at the upper limit or at the lower limit. We can use a coordinate transformation to move the initial posture to the origin; the midpoint between the wheels is at the origin and the vector from the left wheel to the right wheel is parallel to the positive x axis. We assume that the final posture is in the first quadrant; if we can reach any posture in the first quadrant, simple sign transformations will allow the robot to reach any posture in the other three quadrants. We can reach any point by a rotation followed by a translation. Adding a final rotation moves the robot to an arbitrary posture.

A switch point is point at which the acceleration on one of the wheels changes sign. We can characterize a trajectory by the number of switch points. A path with a smaller number of switch points will have a higher average velocity and a longer distance traveled by the wheels. The path with the smallest number of switch points has one for each wheel. However, there are only two paths with two switch points: translation and rotation. Rotation followed by translation requires five switch points, while rotation, translation, rotation has eight switch points.

We have explored paths with three and four switch points. The paths with three switch points and initial rotation can reach any point faster than a rotation followed by a translation. Paths with three switch points and initial translation or paths with four switch points are useful if the final orientation is considerably different than the direction of travel.

The real wheel controller will have a velocity constraint. The velocity constraint will reduce the velocity advantage for trajectories with a small number of switch points. The real wheel controller will choose an optimal path from a family of possible paths. In many cases, the rotate-translate-rotate path will probably be the optimal path.

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