ORNL/TM-10973

ornl

OAK RIDGE NATIONAL LABORATORY

MARTIN MARIETTA

# Two-Dimensional Combat Modeling with Partial Differential Equations

Petre Rusu



OPERATED BY
MARTIN MARIETTA ENERGY SYSTEMS, INC.
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
NTIS price codes—Printed Copy: A02 Microfiche A01

ORNL/TM-10973

Engineering Physics and Mathematics

# Two-Dimensional Combat Modeling with Partial Differential Equations

Petre Rusu
Physics Department
University of Colorado
Boulder, Colorado

DATE PUBLISHED —DECEMBER 1988

| NOTICE: | This document contains information of a preliminary nature. It is subject to revision or correction and therefore does not represent a final report. |
|---|---|

Prepared by the
OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37831
operated by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
for the
U.S. DEPARTMENT OF ENERGY
under contract DE-AC05-84OR21400

# CONTENTS

# ABSTRACT

The system of partial differential equations introduced in Reference 2 to model combat in one spatial dimension has been extended to include two spatial dimensions and has been numerically integrated to demonstrate its capability to describe maneuver. The analysis of a turning maneuver is demonstrated.

# 1. INTRODUCTION

In 1914, F. W. Lanchester[1] introduced the first successful mathematical model for describing the military combat. His model is a system of nonlinear ordinary differential equations (ODE)

$$\begin{aligned} \frac{du}{dt} &= c_1 uv + d_1 v + e_1 \\ \frac{dv}{dt} &= c_2 uv + d_2 u + e_2 \end{aligned} \tag{1}$$

giving the time evolution of the total number of troops, $u$ and $v$, during a combat. In this model the mutual attrition is controlled by the negative coefficients $c_1, c_2, d_1, d_2$ and for sources or sinks for the participating troops are represented by the free terms $e_1$ and $e_2$.

Lanchester's equations and their direct generalizations have been used for more than 70 years to study combat and guide military researchers in the assessment of concentration of troops in combat. Force concentration constituted the base for the "Lanchester square law" which states that the strength of a combat force is proportional to the square of the number of the combatants entering the engagement.

However, the model proved to be too simple to describe the modern forms of warfare. A model for studying modern combat should take into account the movement of the opposing forces on the battlefield, the nonhomogeneous character of the modern army, the importance of the principle of command and control, and the impact of logistic and intelligence.

Based on these considerations, a new and more comprehensive mathematical model was introduced[2] recently to account for the dynamics of the modern tactical situations. This new model is based on partial differential equations (PDE) and contains the Lanchester model as one of its limiting cases but goes beyond any generalization of Eq. (1) that has been tried over the years. It is a new and original tool for military research. In Ref. 2, the model was introduced and analyzed for one spatial dimension; the two-dimensional version has subsequently been developed.[3]

In the case of two forces engaged in combat, the PDE model in two spatial dimension reads:[2]

$$\begin{aligned} \partial_t u_1 &= \partial_{\vec{r}}(\hat{D}_1 d_{\vec{r}} u_1) + \partial_{\vec{r}}(\vec{c}_1 u_1) + u_1(a_1 + b_1 u_1 + k_{12} * u_2) + d_1 u_2 + e_1 \\ \partial_1 u_2 &= \partial_{\vec{r}}(\hat{D}_2 d_{\vec{r}} u_2) + \partial_{\vec{r}}(\vec{c}_2 u_2) + u_2(a_2 + b_2 u_2 + k_{21} * u_1) + d_2 u_1 + e_2 \end{aligned} \quad . \tag{2}$$

To be well posed, this problem for the two nonlinearly coupled diffusion-convection equations also includes initial conditions $u_i|_{t=0} = u_{i0}(x)$, $i = 1, 2$; and boundary conditions (B.C.)

$$(\alpha_i u_i + \vec{\beta}_i \partial_{\vec{r}} u_i)|_{\vec{r} \in \partial\Omega} = r(\vec{r}) \;\; i = 1, 2 \; . \tag{3}$$

For a detailed explanation of symbols and notation, see Ref. 3. The nonlinear interaction in the Eq.(2) is represented by the convolution term $u_i k_i * u_j$ given by

$$u_i k_i * u_j \equiv u_i(\vec{r}, t) \int k_{ij}(\vec{r} - \vec{r}\,', t) u_j(\vec{r}\,', t) d\vec{r}\,' \tag{4}$$

where $i, j \in \{1, 2\}$, $i \neq j$. The kernel $k_{ij}$ represents the attrition inflicted on force $u_i$ by force $u_j$ during engagement. The coefficients $d_i$ and $e_i$ have the same meaning as in Eq. (1), and the coefficients $a_i$ and $b_i$ model the natural birth and death phenomena. The diffusion coefficients $D_i$ account for the local loss of order during the battle or only the displacement of troops and $c_i$ are the convection velocities introducing in the model the effects of the command and control and the tactical principles governing the movement of the troops on the battlefield.

The aim of the paper is to assess the capability of the two-dimensional model to describe the tactical aspects in mid-intensity conflict. By using numerical methods, the qualitative behavior of the model has been studied for different test cases. Here we give explicit results for the turning movement.

# 2. COMBAT MODELING IN TWO SPATIAL DIMENSIONS

In the remainder of this paper, we demonstrate the capability of the PDE model to deal with a standard combat maneuver. Being the first study of the two dimensional case, the purpose was only to prove that the model correctly describes the tactical process from a qualitative point of view. We did not try to provide a quantitative and detailed description of maneuver. For such an analysis, real data regarding attritions, speeds, diffusions, etc. need to be taken into account. These data are usually not readily available or easily derived from engagement histories.

The integrator chosen to solve this problem covers a large class of PDE systems with different initial (I.C.) and boundary conditions (B.C.). Thus it was necessary to investigate and judge the reliability and precision of the software for the narrow class of problems to be solved. Extensive testing of the program was carried out on a large set of relevant cases related to the problem. This preliminary work involved different versions of the integrator code, as well as analytic work that produced exact solutions for comparison with the numerical results. The main purpose of this testing was to determine numerical values for the coefficients in Eq.(2) to be used in the modeling of the important nonanalytically solvable cases.

The software used to numerically integrate the system (2) is based on the method of lines. The integrator codes are due to Sincovec, et al[(4),(5)]. Driver programs for these integrators were written to include the tactical information needed to simulate the specific maneuvers.

The method of lines on which the integrator is based consists of two parts. The first one is the discretization of the spatial differentiation terms in the system (2); it generates an extended ODE system for the time evolution of the troop densities. The second part consists of the integration of this ODE system using the powerful numerical techniques that have been independently developed to solve this kind of problem. The number of ODE's generated by the method, $n_{\mathrm{ODE}}$, is usually very large. It is the product of the following three: $n_{\mathrm{PDE}}$ = the number of PDE's in the system, $n_x$ = the number of points in the x-direction of the spatial grid, and $n_y$ = the number of points in the y-direction of the grid.

The study of the temporal stability of the one-dimensional stationary solution for the pure diffusion-convection equation was one of the first things we performed. This was done to clarify the implications of different specific choices of B.C.'s on the accuracy of the software. When taken as the initial condition for integrating one uncoupled equation from system (2) with all the coefficients except d and c taken equal to zero, this solution is conserved in the exact analytic sense if convenient mixed B.C.'s are considered. Computationally, however, the situation is very different. The numerical solution varies slowly in time, but the rate of change can be decreased if the spatial grid is made finer. Nevertheless, that numerical imprecision cannot be totally avoided and the set of parameters for which the stationary, and, implicitly, the total number of troops, is conserved in an acceptable proportion for intervals of time comparable to those intended to follow the evolution in the relevant two-dimensional cases was sought.

From the point of view of numerical computation, there is a fundamental time $\tau$ associated with each combat situation. This is the time needed by the troops moving with their average or normal speed to travel across the battlefield. The battlefield was chosen to be a square with each side taken as the fundamental unit of length. The value of $\tau$ is numerically equal to the inverse of the average speed of the moving troops. For intervals of time $\Delta t$ that are of the same order of magnitude as $\tau$, the conservation of the total number of troops must be satisfied as well as possible in the absence of attrition.

The theoretical device to insure perfect conservations is to impose mixed B.C.'s that would cancel the individual diffusion-convection currents at the boundary.

$$\vec{j}_k = \hat{D}_k \partial_{\vec{r}} u_k + \vec{c}_k u_k \quad k = 1, 2 \ldots \tag{5}$$

The cancellation of $\vec{j}_k$ on $\partial\Omega$, the $\hat{D}_k$ boundary of the domain $\Omega$, is equivalent to the choice $\vec{\tau}_k = 0$, $\vec{\alpha}_k = -\vec{c}_k$, $\hat{\beta}_k = \hat{D}_k$ in the corresponding B.C.'s. The consequences of this choice for the BC's in the case $n = 1$ was studied because this situation was not previously investigated despite its importance for the confinement of combating armies to the battlefield during the engagement and the results of such a test are relevant for the two-dimensional modeling.

The parameter controlling the numerical stability and the accuracy of the solution for the simulations where the attrition was turned off is the ratio $\rho = \frac{c}{D}$. If one decides that "quantitatively good results" mean, in fact, boundary generated losses smaller than 2-3% in the total number of troops for intervals of time $\Delta t \sim O(\tau)$, the conclusion of these tests is that "good results" can be obtained only if $n_x \sim O(100)$. For the cases where $n = 2$ case this means $n_{\mathrm{ODE}} \sim O(10^4)$ which is too large to allow a reasonable numerical treatment of the problem.

However, the correct qualitative behavior is visible in our modeling based on the Eq. (2), even with such a small number of points per direction as 32. Although the losses due from boundary crossings would be larger in this case, parallel runs of the code with the attrition turned on and off can distinguish between the losses due to engagement and those due to numerical imprecision. For this rough grid, and $n = 2$ it follows that a $n_{\mathrm{ODE}} = 2048$ for a two-forces combat, and $n_{\mathrm{ODE}} = 3072$ for the cases when three forces are involved (in this last case there might be in fact two forces fighting but one is made up of two more or less independent parts). For these calculations, a good value for $\rho$ was found to be $2.5 \times 10^2$. This value for $\rho$ insures sufficient stability during evolution for a Gaussian shaped initial distribution of forces. It should be pointed out, however, that for "square" type initial distributions, the combination between the small number of points per direction in the spatial grid and the value of $\rho$ given above proved to be unsatisfactory. During the evolution, the distribution of troops developed oscillating tails which are unacceptable in combat modeling because the density of troops cannot take local negative values.

A detailed and more complex combat situation can be modeled only if the spatial grid is made fine enough to account for the local situation on the battlefield. This has as a consequence the increase of the number of ODE's to be solved, and thus generates an increase of the computational time. At the same time there are indications that for the same problem, the increase in $n_{\rm ODE}$ through the increase of $n_x$ and $n_y$ tends to transform the ODE system into a stiff one making its integration a difficult task and increasing the computational time even more.

We have considered for our numerical study the following version of Eq. (2):

$$\partial_t u_i = \partial_{\vec r}[\hat{D}_i \partial_{\vec r} u_i + \vec{c}_i u_i] + \sum_{j \neq i} a_{ij} u_i u_j \qquad i = 1, 2, 3 \; , \tag{6}$$

in which diffusion, convection, and attrition terms are accounted for, as a basic model of tactical maneuvers. This system of equations have been supplemented with the initial condition

$$u_i|_{t=t_0} = u_{i0}(\vec r) \qquad i = 1, 2, 3, \tag{7}$$

and the mixed B.C.'s

$$(-\vec{c}_i u_i + \hat{D}_i \partial_{\vec r} u_i)|_{\vec r \in \partial\Omega} = 0 \qquad i = 1, 2, 3. \tag{8}$$

Here, $u_1$ and $u_3$ are two force densities representing the same combatant. For this reason in the matrix of the attrition coefficients, we have $a_{31} = a_{13} = 0$. The diffusion tensor is diagonal $\hat{D}_{ij} = D_i \delta_{ij}$ and the magnitude of $D_i$ has been chosen 0.01. The convection velocities $c_i$ have been variable with $t$, their magnitude taking values between 0 and 2.5. The choice of a local attrition term in Eq.(6) was made to simplify the numerical computation. This choice is not so restrictive as it may seem if only a qualitative study of the problem is made.

The best choice for the initial troop distributions $u_{i0}$ proved to be the Gaussians in two-dimensions. Flat rectangular functions have been tried as well but, as already mentioned, the oscillations present in the tail of the density during the evolution indicated this choice was not consistent with the value of $\rho$ used in the runs ($\rho = 2.5 \times 10^2$).

The use of the mentioned software to integrate the PDE system required the writing of a driver program which would define the specific equations to be integrated, the initial condition and the boundary conditions attached to the problem. Detailed description of the driver programs can be found in the papers presenting the integrator[(4),(5)]. We point out that the essential subroutine of the driver which has built in it all the features allowing for the modeling of a specific combat situation is the subroutine defining the convection field. This subroutine returns upon its call all the values of the convection velocities $\vec{c}_i$ for any values of the independent variables $t$ and $x$. It works in conjunction with the main program.

Testing the computer code by using different point dependent convection fields has shown that sensible losses through the boundary cannot be avoided. A simple uniform convection field with a time dependent direction proved to be the best choice as it ensured an acceptable conservation of the total number of troops in case the attrition was turned off. The boundary conditions have been modeled according to the instantaneous convection velocity which is entered as a coefficient within the B.C.'s. This approach proved to be at the same time the simplest and the most efficient for allowing a reasonably simple version of the numerical code, besides the realization of a good numerical conservation of the number of troops during the runs.

The combat situation that was successfully modeled was a turning maneuver which required the presence of only two equations in the system (6). The classical schematic way this maneuver is described in the military literature is presented in Fig. 1. The numerical simulation did clearly demonstrate the movement of troops on the field, the active phase of the battle when the direct contact is realized and the mutual attrition decreases the number of troops engaged in combat, and the retreat of the defeated force (not shown in Figure 1). If the defeated force is the one that made the attack, this last phase of the battle takes place on the same track as the one used for engagement but in reversed direction. If the entrenched force is the defeated one its retreat is made on some new track conveniently chosen. The loss of the total number of troops of one combatant that triggered the retreat was arbitrarily set at 15%. The specific results obtained through numerical simulation are well illustrated in a companion paper.[6]

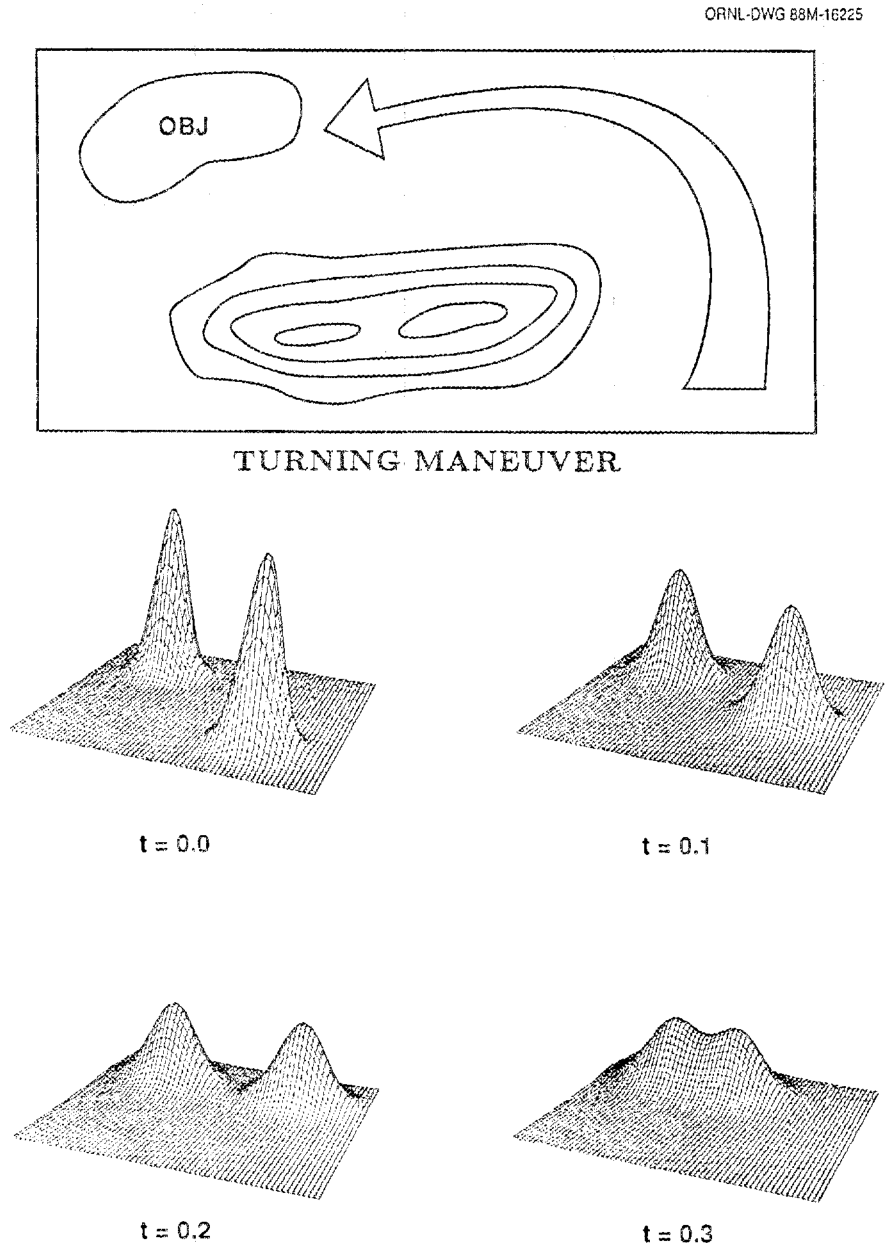


Figure 1. Turning Maneuver.

## 3. CONCLUSIONS

The numerical study of the PDE combat model has proved its value in dealing with some of the complex aspects of the modern warfare. The results presented here clearly show that the tactical aspects of certain forms of maneuver are accurately described by the two-dimensional version of the model. Finally, a good knowledge of the software that can be used for the integration of this type of equations has resulted. This has generated a more realistic point of view regarding the expectations of a researcher working in the combat modeling field.


### ACKNOWLEDGEMENTS

The author would like to thank V. Protopopescu and R. T. Santoro for their assistance during this study.

# REFERENCES


1. F. W. Lanchester, "Aircraft in Warfare, The Down of the Fourth Arm – No. V., The Principal of Concentration," *Engineering* **98**, 422 (1914).

2. V. Protopopescu, R. T. Santoro, and J. Dockery, "Combat Modeling with Partial Differential Equations," *European J. Oper. Res.* (1988); for a detailed version, see also V. Protopopescu, R. T. Santoro, J. Dockery, R. L. Cox, and J. M. Barnes, "Combat Modeling with Partial Differential Equations," ORNL/TM-10636, November 1987.

3. V. Protopopescu, R. T. Santoro, and J. Dockery, "A Two-Dimensional Version of the Partial Differential Equation Combat Model," in preparation.

4. R. F. Sincovec, N. K. Madsen, "Software for Nonlinear Partial Differential Equations," *ACM Trans. Math. Software* 1, 232 (1975).

5. D. K. Melgaard, R. F. Sincovec, "General Software for Two Dimensional Nonlinear Partial Differential Equations," *ACM Trans. Math. Software* **7**, 106 (1981).

6. R. T. Santoro, P. Rusu, and J. M. Barnes, "Mathematical Solutions of Offensive Combat Maneuvers," Oak Ridge National Laboratory, ORNL/TM-11000.

## INTERNAL DISTRIBUTION

1. B. R. Appleton
2. Y. Y. Azmy
3. J. M. Barnes
4. T. J. Burns
5. C. M. Haaland
6. F. C. Maienschein
7. D. L. Moses
8–12. V. Protopopescu
13–17. R. T. Santoro
18. M. S. Smith
19. EPMD Reports Office
20–21. Laboratory Records Department
22. Laboratory Records, ORNL-RC
23. Document Reference Section
24. Central Research Library
25. ORNL Patent Section

## EXTERNAL DISTRIBUTION

26. J. Dockery, Office of the Joint Chiefs of Staff, J6F, Pentagon, Washington, D.C. 20301
27. J. J. Dorning, Department of Nuclear Engineering and Engineering Physics, Thornton Hall, University of Virginia, Charlottesville, Virginia 22901
28. R. M. Haralick, Department of Electrical Engineering, University of Washington, Seattle, Washington 98195
29–33. Petre Rusu, Physics Department, University of Colorado, Campus Box 390, Boulder, Colorado 80309
34. Office of the Assistant Manager for Energy Research and Development, Department of Energy, Oak Ridge Operations, P.O. Box 2001, Oak Ridge, TN 37831
35–44. Office of Scientific and Technical Information, P.O. Box 62, Oak Ridge, Tennessee 37830