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Combat Modeling with Partial Differential Equations

V. Protopopescu
R. T. Santoro
J. Dockery
R. L. Cox
J. M. Barnes

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COMBAT MODELING WITH PARTIAL DIFFERENTIAL EQUATIONS

V. Protopopescu and R. T. Santoro
Engineering Physics and Mathematics Division

and

Center for Studies of Nonlinear Phenomena

Oak Ridge National Laboratory

J. Dockery
Organization of the Joint Chiefs of Staff

R. L. Cox and J. M. Barnes
Computing and Telecommunications Division

Oak Ridge National Laboratory

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ABSTRACT

A new analytic model based on coupled nonlinear partial differential equations is proposed to describe the temporal and spatial evolution of opposing forces in combat. Analytic descriptions of combat have been developed previously using relatively simpler models based on ordinary differential equations (e.g., Lanchester's equations of combat) that capture only the global temporal variation of the forces, but not their spatial movement (advance, retreat, flanking manuever, etc.). The rationale for analytic models and, particularly, the motivation for the present model are reviewed. A detailed description of this model in terms of the mathematical equations together with the possible and plausible military interpretation are presented. Numerical solutions of the nonlinear differential equation model for a large variety of parameters (battlefield length, initial force ratios, initial spatial distribution of forces, boundary conditions, type of interaction, etc.) are implemented. The computational methods and computer programs are described and the results are given in tabular and graphic form. Where possible, the results are compared with the predictions given by the traditional Lanchester equations. Finally, a PC program is described that uses data downloaded from the mainframe computer for rapid analysis of the various combat scenarios.

1. INTRODUCTION

Since World War II, there has been a significant increase in the use of scientific methods for the study and solution of military problems. Mathematical methods have been widely used for the analysis and assessment of tactical and strategic warfare and related issues. In particular, Operations Research (OR) has been used extensively by the Department of Defense (DOD) and has influenced the process whereby military planners and analysts collect data and formulate decisions for the myriad of increasingly complex military issues. Kimball and Morse[1] introduce their text with an illuminating discussion of the evolution of OR methods in the DOD as a tool for guiding commanders in troop disposition and logistics requirements in specific tactical situations.

Tactical and strategic combat has been studied using one of three mathematically based methods: war games, simulations, and analytic models. Each of these methods has capability for providing planning guidance and essential elements of information for conventional, unconventional, and nuclear warfare. Each method has specific objectives and applications, and, depending on the problem, certain advantages and disadvantages.

War games have been used for both operations planning and conflict analysis and as one of the principal tools for training new officers in the principles of tactics and strategy. In general, war games must be carried out with the aid of computers. They are very laborious and do not, for the most part, run in real time. Meaningful solutions can be achieved but usually after long execution times. For example, a war game that simulates a battle of a few hours duration takes tens of hours to reach the expected conclusion. Identification of critical issues can be realized, but because war games rely on human intervention during play, the outcome is often determined by the experience of the players or by the judgment of the umpires who monitor the game play. Since war games require long execution times, they are not appropriate tools for performing parametric analyses or sensitivity studies.

Simulations are the most widely used of the methods for studying and analyzing combat and engagement scenarios. Of the numerous kinds of simulations, the most effective is the Monte Carlo method which allows for detailed treatment of problems. As is the case for all modern Monte Carlo codes, large computers are required for execution and long running times are necessary to achieve good statistics in the answers to even modestly complicated problems. The sampling techniques, probability distributions, execution times, and the data used to model the various aspects of the conflict can introduce large statistical variations in the results. Monte Carlo codes are costly to build and require a dedicated staff to maintain them to assure that relevant solutions are obtained. As in the case of war games, these

types of codes also require long execution times which, combined with modeling limitations, preclude parametric and sensitivity analysis.

Finally, there are analytic models where mathematical algorithms and formulae are used to describe the state of an engagement. The equations vary in complexity, but generally are characterized by a high level of abstraction, flexibility, speed, convenience, and reproducibility and can be solved either manually or with aid of computers. One of the most widely used mathematical treatments has been the Lanchester differential equation model.[2] These equations predict the time dependent state of a battle based on attrition. Taylor[3] has published a comprehensive text on these differential equation models of attrition in force-on-force combat, alluding also to various OR methods that have been used historically in the study of military problems.

Lanchester's equations provide commanders and combat analysts with a means for studying global issues concerning the numbers of combatants on each side. Issues such as who will win or what losses will be suffered can be addressed. The equations provide a means for estimating the force ratio required to achieve victory while also yielding the time dependence of the strengths of the opposing forces and the duration of the engagement. However, there are shortcomings of the Lanchester model that severely limit its use for detailed battle planning and analysis; they do not address stochastic effects which can dominate local combat interactions and they are static in space variables. Critical questions on force movement, spatial variations in troop density, and replacements and/or withdrawals, therefore, cannot be answered. These deficiencies, combined with the failure of the model to treat terrain and other spatial factors, result in a limited analysis of conflict. Other relevant military issues including tactical decision making, intelligence, command, control, and communications, and logistics requirements cannot be modeled within the framework of the classical Lanchester equations. These processes are essential for planning and successfully meeting mission requirements in battle. Failure to include these factors in a combat model seriously limits its use in tactical and strategic planning.

A serious attempt to expand the capability of the Lanchester equations must include means for ameliorating these and other deficiencies. A more comprehensive analytic treatment should take into account

a. random effects in the attrition process,

b. attrition structures other than those treated in the classical Lanchester model,

c. time and force size dependent attrition rate coefficients,

d. non-combat attrition (surrenders, desertion, sickness, etc,) and replacements and/or withdrawals, and

e. engagement of heterogeneous forces (infantry, armor, mechanized infantry, air assault, etc.).

The list of additions or modifications to the Lanchester or any equivalent model to describe combat is limited only by the imagination of the model developer. The goal, however, is clear: to create an analytic tool that can provide a commander with an efficient means for planning or evaluating battle that includes the critical elements to maximize mission success.

The point to note is that modeling, in one form or another, can be an essential part of the military planning and analysis process. Strategic and tactical mission planners are relying more heavily on the use of these models for a wide variety of anticipated conflict scenarios. As weapon systems have become more complex, the capabilities of conflict models to describe combat from the local to global arenas have been severely stressed. Advances in military technology have widened the gap between current military planning experience and future battlefield situations. The complexity of the battlefield is changing rapidly and the strategy and tactics of future wars may have to be planned with virtually no extrapolation from experience gained in recent conflicts. Since experience may be lacking, planning may depend on information generated using combat modeling.

1.1. AN ADVANCED COMBAT MODEL FORMULATION

In this paper, we introduce a new analytic approach for describing combat that is based on non-linear differential equations. This method introduces, for the first time in military analyses and application, an *analytic* approach that yields *both* the spatial and temporal distribution of forces. The approach was motivated by the work of Lanchester—in particular, the limitations of the ordinary differential equations that he used to describe force-on-force conflict.

The Lanchester approach uses simple, but effective, ordinary differential equations (ODEs) to model the state of opposing forces and the expected outcome of the battle. According to Lanchester, "one of the great questions at the root of all strategy is that of concentration; the concentration of a whole resource of a belligerent on a single purpose or object and, concurrently, the concentration of the main strength of his (commander), whether naval or military, at one point in the field of operations." The Lanchester equations provide the data from which the time dependent results of such engagements can be obtained.

For ancient, or pre-modern combat, concentration of forces was not found to be an advantage. Man responded to man, weapon responded to weapon, i.e., engagements were a series of one-on-one duels or a direct phenomenon. For the conditions that exist in modern combat, however, the conclusions obtained from the Lanchester formulation are quite different. Modern combat includes both direct and indirect fire and engagements are essentially collective. There are both individual and collective phenomena occurring. Force concentration is clearly an advantage as revealed through Lanchester's square law. The advantages of incorporating indirect artillery fire and long-range small arms significantly modifies the battlefield scenario

and the inferior force finds itself in a dilemma where it cannot return fire-for-fire, thus infantry, armor, or artillery concentration becomes a decided advantage for the superior force.

The characteristics for the modern battlefield have changed significantly since Lanchester introduced his model equations. The conventional combat force structure (infantry, armor, and artillery) remains but there are new combat elements that have been introduced. Tactical air support alters the attrition rate variables for both sides and the commander must consider the impact of this variant on force concentration. In addition, for some expected combat zones/scenarios, the modeling of "conventional-on-conventional force" engagement concepts is rapidly being replaced by the "conventional-on-unconventional force" combat scenarios where guerilla warfare must be modeled.

Current military thinking indicates that in future conflicts the movement of the forward edge of battle area will be difficult to isolate. The introduction of mechanized units and air assault infantry will introduce more diffuse boundaries. The concept of a battlefield edge, or line, has been replaced with a concept in which the forward and "lateral" lines of troops define the areal limits of the battlefield. Mobile combat elements will find themselves engaged with the enemy in separate locations, or isolated pockets, throughout the battle area. The capability of commanders to mobilize tactical air support introduces another source of firepower that was not a significant consideration in 1914 when Lanchester proposed his equations. If the conflict changes from conventional to nuclear, then the criteria for battle management—command, control, and communications (C^3) becomes an even more severe problem to characterize and model. When the conflict is unconventional, the model equations and associated initial and boundary conditions must be further modified.

However severe the limitations encountered in the use of the Lanchester model, the most significant is the failure of this model to describe the spatial extent of the battlefield and its essential consequences on the development of the battle. The Lanchester equations yield only time-dependent results. There is no mechanism to account for force movement either forward or in retrograde, for lateral deployment, encircling, or any other spatial displacement.

In the Lanchester models, it is also assumed that the kill rate of each weapon is constant with time. The inability to account for the spatial movement of the engaged forces does not provide a means to account for the attrition rate change as the engaged elements close on each other. As the distance between forces decreases, the effectiveness of both direct and indirect weapons becomes greater leading to larger losses on one or both sides depending upon the firepower advantage. The increased effectiveness of small arms fire and advanced artillery systems reduces indirect fire capability, troop concentration, and logistics requirements.

In the classical Lanchester equations, the composition of opposing forces are considered to be identical (homogeneous forces). However, tactical units are characterized by a mix of combat arms. Depending on the mission, terrain, and

the enemy force structure, offensive or defensive forces are tailored in organization and size to accomplish the intended mission. In addition to the mix of combat units, their spatial distribution can alter their fighting effectiveness. For example, in a combined infantry-armor configuration fighting in tank suitable terrain, the armored elements can advance at a much faster rate than the infantry. The firepower of the armor will predominate until the infantry closes and engages the enemy. In cases where the terrain is mountainous or swampy, the armor may be slowed or delayed until appropriate means are taken to expedite crossings or reduce mountainous obstacles. In this case, the infantry may outrun the armor and the effectiveness of the tracked firepower is reduced, or possibly even eliminated. Expanding the model equations to take the spatial extent of the battlefield into account will allow for an assessment of the impact of terrain on the course of the battle. Correspondingly, the number of equations must also be increased to include different mixes of combat elements facing an opposing force made up of another mixture of elements. Depending on the force composition, the force mixture combined with terrain modeling via the inclusion of spatial terms in the equations would significantly alter the predicted attrition rates.

Success in combat depends very strongly on the level and maintenance of logistics. Models of combat that fail to include this essential data are also severely limited in their predictions. If attrition rates are to be treated appropriately, the replacement rate must also be included. In addition, if the model assumes infinite firepower capability rather than firepower based on available ammunition and ammunition consumption and resupply rates, then the model is even more constrained. When logistics includes replacement of combatants, the model must treat both the operational loss rates and the combat loss rates. Operational loss rates are those due to non-combat mishaps: desertions, sickness, disease, etc., while the combat loss rate defines the losses that occur during actual engagement with the enemy. Operational loss rates are caused by a number of physiological and psychological factors. The loss rates are caused by the psychological factors of fear, apathy, etc., and may include educational factors such as failure to be prepared, lack of ability, etc., and goal oriented factors like suicide charges or loss of motivation. The operational and combat loss rates in conventional warfare are markedly different than those in unconventional (guerilla) warfare so advanced models must take these factors into account. In the work proposed here, both of these factors will be included in the mathematical model. This more comprehensive mathematical treatment will yield a more detailed description of combat than can be achieved presently using Lanchester's ODEs. Namely:

(1) The inclusion of the terms to account for the spatial dependence will provide the analyst with a means for following troop movement.

(2) As the model is expanded to include more than one spatial dimension, effects of terrain and obstacles can be treated and the impacts of delays on the relative movement of infantry and armor can be included in battle planning.

(3) The spatial terms will also account for maneuvers by enemy forces.
(4) When the model is further expanded to include sufficient terms for representing heterogeneous force mixtures, then the impact on the battle of mixes of conventional force structures and/or unconventional warfare can be studied.

As the model becomes more comprehensive, solutions to the equations will depend on the capability of the computers that are used for attaining solutions. Inclusion of more than one spatial term and increasing the number of equations to model heterogeneous force structure will require longer execution time than for the relatively simple expressions discussed below. Nonetheless, these solutions may be achievable in running times that are much shorter than the running times for equivalent solutions from war games or Monte Carlo simulations; moreover, they have the added advantage of allowing for parametric and sensitivity analysis provided the appropriate computer is selected.

A discussion of the mathematics of competition/antagonism/combat and the specific partial differential equations (PDEs) used to describe the temporal and spatial distributions of engagement between homogeneous opposing forces is presented in Section 2. The method of solution and computer codes used to obtain the various results of engagement are discussed in Section 3. The results of various engagement scenarios and parametric studies are presented and discussed in Section 4.

A desk-top/PC computer code developed to generate graphic displays and summary data of the combat scenarios presented in Section 4 is included in the Appendix.

2. MATHEMATICAL MODELING

A mathematical treatment is indispensable if the dynamics of the system is to be analyzed and predicted quantitatively. Through trial and error, the use of mathematical models will eventually lead to the establishment of laws and basic equations. While recognizing the value of mathematical models, an overemphasis may be dangerous and/or misleading. Despite their appeal, they must be considered as sometimes effective and always improvable tools rather than ultimate sources of knowledge. Mathematical models can be roughly divided into two categories: educational (exemplary) and practical. Some models share the two features. Educational models are based on a small number of simple assumptions and are analytically tractable. The method employed by such models involves the investigation of one or two processes, considered to be essential, that have been simplified and isolated from the complexity of the whole system. The danger is that the analysis, although feasible, may be divorced from reality. The virtue of such models lies in their providing a paradigm for gaining insight, expressing ideas, and forming a basis for more sophisticated and complex models. Practical models are based on realistic assumptions and therefore often involve an intricate parameterization of numerous interrelationships among a large number of variables. With numerous parameters and numerous (usually nonlinear) evolution equations, analytical treatment becomes impossible and one must rely on computer calculations. The role of the computer is, again, that of a tool, since the deduction of general rules from these complex systems is progressively more difficult.

Mathematical models can also be delineated as either deterministic or stochastic. Often educational models employ deterministic methods while practical models tend to favor the stochastic approach. However, in an overwhelmingly large number of situations, one may speak about visualizing the real processes as deterministic on the average and treat them as deterministic in the first approximation. Of course, the next approximation must take into account the stochastic fluctuations about the average.

Every mathematical model has numerical parameters (rates and probabilities of various occurrences, coefficients in differential equations, the dimensions of spaces, etc.). Even functional, rather than numerical, parameters are common. In military situations, parameters are typically difficult to measure in specific instances, but even without them, some qualitative conclusions can sometimes be deduced.

In military modeling, qualitative results for models are the most important kind, for accurate quantitative results can only be expected occasionally, if then. The main reason is that such models here are much more idealized than in physics, chemistry, or ecology. What one can often hope for, however, is some indication as to the effect, within the total picture, of the few factors and influences specifically

being accounted for in the model. Other factors will also leave their marks on the behavior of actual populations. The ideal assumption that these other influences do not exist is expressed by the deterministic form of the equations.

Our approach in this chapter will be first, to describe hypotheses under which a phenomenon can be modeled by a system of partial differential equations. This leaves unanswered the question of which coefficients and reaction functions are appropriate for a given situation. Some discussion of this aspect of modeling will be given in later sections. In the following, the emphasis will be on methods of qualitative analysis of the equations; that is, on discovering conditions under which solutions with certain given properties exist.

2.1. GENERAL

Although the classical Lanchester model provides an oversimplified representation of two interacting armies, calculations based on the model have provided much insight and have proven to be of practical value. It seems appropriate to extend the model by including diffusional and convection effects, allowing for the investigation of the dynamics of interacting populations with spatial dispersion.

The motivation for introducing a new mathematical model to replace the Lanchester's ODE of combat was influenced, in part, by the existence of algorithms that address similar phenomena and include the treatment of nonlinear phenomena that account for the "true" behavior of adversaries in conflict. In addition, analytic models of combat that do not include mechanisms for describing the spatial extent of the battle area are severely limited. In this section, related mathematical treatments of competitive phenomena are reviewed. In Section 2.2, the treatment of combat including convection and diffusion terms based on nonlinear PDEs that were developed for this study are reviewed.

The usual analysis of competition (antagonism, combat, etc.) between two species (populations, armies, economies, players, etc.) in spatially uniform distribution starts with the ODEs of Lotka-Volterra type[4]

$$\frac{du_1}{dt} = (a_1 - b_{11}u_1 - b_{12}u_2)u_1 \tag{2.1}$$

$$\frac{du_2}{dt} = (a_2 - b_{21}u_1 - b_{22}u_2)u_2. \tag{2.2}$$

The sign of the coefficients a, b determines exactly the type of competition. For armies in combat, b_{12} and b_{21} are non-negative constants, the rest of the coefficients being zero. In ecological models, all the coefficients are non-negative constants; a_1 and a_2 are the intrinsic growth rates of the populations, b_{11} and b_{22} are the coefficients of *intraspecific competition*, and b_{12} and b_{21} are those of *interspecific*

competition. Coexistence of these two populations becomes possible only when the conditions $b_{11}/a_1 > b_{21}/a_2$ and $b_{22}/a_2 > b_{12}/a_1$ are satisfied; otherwise, one population becomes extinct.

In the field of ecology and population dynamics, there has been considerable interest in models of systems where the components are interacting while undergoing convection and diffusion (see, for instance, Ref. 4 and references therein). Such systems are described by equations of the form

$$u_t + \nabla \cdot \vec{J} = f(u) \tag{2.3}$$

for the density vector $u = (u_1, u_2, u_3, ..., u_n)$. In Eq. (2.3), $\vec{J} = (\vec{J}_1, \vec{J}_2, \vec{J}_3, ..., \vec{J}_n)$ is the flux associated with the density u, and $f = (f_1, f_2, f_3 ... f_n)$ describes the kinetics of the process. The current $\vec{J}$ can be classified into three types: (i) Fickian expression: $\vec{J}^f = -D_i \nabla u_i$, (ii) repulsive expression: $\vec{J}^r = -\nabla(D_i u_i)$, or (iii) attractive expression: $\vec{J}^a = -D_i^2 \nabla(u_i/D_i)$, where $D_i(> 0)$ is called the coefficient of diffusion. If it is assumed that diffusion is constant and isentropic, it follows that $\vec{J}_i^f = \vec{J}_i^r = \vec{J}_i^a$ and then Eq. 2.3 is reduced to a system of reaction-diffusion (RD) equations

$$u_t = D\Delta u + f(u), \tag{2.4}$$

where D is a diagonal positive constant matrix with elements D_i and Δ is the Laplace operator in the spatial coordinates. On the other hand, if D_i depends on x, then, although $\vec{J}_i^f$ is always directed from high density to low density as expected, $\vec{J}_i^r$ and $\vec{J}_i^a$ are not necessarily directed down the density gradient. It has been noted[3] in animal ecology that certain characteristics of animal diffusion and taxis can be explained with the use of flux J_i^r or J_i^a.

Competition between two populations considered over two patchy habitats where interhabitat migration is allowed shows that coexistence of competing species that would otherwise exclude each other is possible provided the rate of migration is small enough. When the migration rate is too high, rapid mixing produces effectively a single patch situation and coexistence is no longer possible.

By numerical computation, it was shown that the spatial segregation effect caused by the advection-diffusion mechanisms can stabilize the competing populations and lead to coexistence.[5] Thus, if environmental heterogeneity and the nonlinear dispersal are taken into account, the coexistence of two similar and competing species becomes possible at least under some conditions, even though the two species have the same affinity for the environment.

In general, *a diffusion process in an ecosystem tends to give rise to a uniform density of population in space.* As a consequence, it may be expected that diffusion, when it occurs, plays the general role of increasing stability in a system of mixed populations and resources.

However, there is an important exception known as *"diffusion-induced instability"* or *"diffusive instability"* when the combined presence of diffusion and certain types of nonlinearities induce the opposite effect and lead to pattern formation (aggregation, segregation, etc.), spatially non-uniform stationary states and oscillations, spatio-temporal quasi-periodicity, and chaos.

For example, with the bistable nonlinear diffusion equation very general assumptions on the nature of the function appearing in the equation lead to the existence of stable traveling waves. And, for nonlinear diffusion systems with an equilibrium point, quite general conditions lead to the bifurcation of small amplitude wave trains.

The use of these equations can be rightly subject to dispute in some cases, so it is appropriate to attempt to elucidate the assumptions and rationale on which they are based in *military* contexts.

In the general case, the basis of these models are equations of the form

$$\frac{\partial u}{\partial t} = \nabla D \nabla u + f(u, \nabla u, x, t; \lambda) \tag{2.5}$$

where $u = (u_1, u_2, ..., u_n)$ is a n-vector, D is an $n \times n$ matrix whose elements may depend in general on $u, x \in \Omega \subset \mathbb{R}^m$, and $t > 0$. Here, ∇ is the spatial gradient operator and λ is a generic parameter family, $\lambda \in \Lambda \subset \mathbb{R}^l$. The vector $f = (f_1, ..., f_n)$ is a catch-all term describing reactions and interactions. These systems known under the generic name of reaction-diffusion (RD) equations are common to a large number of disciplines and are related to many applications.[6]

In particular, these equations occur widely as models for the dynamics of large multispecies populations whose individuals interact (to produce and/or eliminate individuals) and are also capable of orderly and/or random spatial migration. In this context x represents position in space (Δ is the Laplacian in x), t is time, and at least some of the components of u represent spatial densities of the various species. In the prototypical example of this modeling, the species are chemical species, and the individuals of the population are molecules. Their random migration through the chemically reacting medium is called diffusion.

Eq. (2.5) also appears as a model for problems outside of population or chemical dynamics, one important example being the models of nerve conduction.

For population problems, the term $D\Delta u$ is a convenient approximation, justifiable under various circumstances, for the rate of change of the population u at any given position and time, due to random spatial migration. The term f measures the rate of change due to "reaction processes" in the medium, *such as chemical reactions, reproduction processes or deaths* in a biological population, and *material transfer.*

Reaction-diffusion systems (2.5) include two very important extreme cases: (1) $D = 0, f$ independent of x and of ∇u:

$$du/dt = f(t, u). \tag{2.6}$$

This is called the system of "kinetic equations," associated with (2.5).

(2) $n = 1, f = 0, D > 0$:

$$\partial u/\partial t = D\Delta u \qquad (u \text{ scalar}) \tag{2.7}$$

This is the well-known scalar diffusion, or heat, equation.

To say that ordinary differential equations and diffusion equations have each been studied extensively in their own rights would be an understatement. Though both represent evolution processes, the two theories have relatively little in common. In some sense, the theory of RD equations attempts to bridge the gap between them. In pursuing this subject, an interesting question to keep in mind is, how the presence of diffusion effects modifies the rich dynamical theory available for the kinetic equations (2.6). This point of view is relevant to studying wave fronts, which connect two rest states of (2.6), and wave trains, some of which modify periodic solutions of (2.6).

Since real systems must date back to some previous time and have a finite spatial extent, it is necessary to specify the initial and boundary conditions that the system satisfies in order to solve the advection-diffusion-reaction equations. In some cases, however, these conditions are considered to be irrelevant to the problem. Thus, if the equation admits a unique stable steady-state solution to which the system will tend for all initial conditions, the initial conditions used to obtain it may be ignored. Also, when we are interested in the dispersal pattern in an early period during which the density remains away from boundaries, or when the site of the medium is large or the diffusion effect is small, the presence of the boundary may be ignored and the domain can be treated as if it extends to infinity. Then the effect of the boundary is often confined to a boundary layer. Also, wave trains and pulses, though strictly defined only for an infinite medium, still actually reflect phenomena occuring in bounded geometries.

While initial conditions are rather easy to specify, in a universal form

$$\lim_{t \to 0} u_i^o(x, t) = u_i^o(x), \tag{2.8}$$

spatial boundary conditions at the boundary $x = x_b$ can be specified in various manners, such as those described below.

1) *Prescribed population densities* at the boundary; these densities may be constant or functions of time.

$$u_i(x, t) = u_i(t) \qquad \text{at } x = x_b \tag{2.9}$$

A population reservoir at the boundary may be represented by the condition (2.9). When a habitat is surrounded by a completely hostile environment, the population density can be considered to be zero at the boundary, i.e., an *absorbing* boundary. In this particular case,

$$u_i(x,t) = 0 \qquad \text{at } x = x_b.$$

In military situations, this BC corresponds to an actual removal of troops out of the engagement zones when reaching a preassigned spatial threshold.

2) *Prescribed flux* across the boundary; the flux may be constant or a function of time.

$$J_i = H_i(t) \qquad \text{at } x = x_b. \tag{2.10}$$

If the flux is represented by a diffusion process, the BC can be written as

$$-D_i \partial u_i / \partial x = H_i(t) \qquad \text{at } x = x_b. \tag{2.11}$$

Migration across the boundary may be represented by the condition (2.10) or (2.11). When a habitat boundary is completely closed to a population, e.g., a fenced population, its flux can be considered to be zero across the boundary, i.e., a *reflecting* boundary. In this particular case,

$$J_i = 0 \qquad \text{at } x = x_b. \tag{2.12}$$

In military situations, this corresponds to a stationary level of troops at the boundary, i.e., reinforcement coming in at the same rate as troops are moving out of the engagement area.

3) *Radiation (mixed) boundary conditions.* If the flux across the boundary of a domain of interest is proportional to the difference between the population density in the domain extrapolated to the boundary and the population density of the surroundings of the domain extrapolated to the boundary, we may write

$$J_i = h_i \Big\{ u_i(x_b, t) - u_{io}(t) \Big\}. \tag{2.13}$$

Here $u_{io}(t)$ is the population density of the surroundings, and h_i is an exchange constant. Condition (2.13) is a weighted combination of (2.9) and (2.10) and as

$h_i \to 0$, this condition tends to a form of 2), while as $h_i \to \infty$, it tends to a form of 1).

4) *Continuity conditions at a boundary* separating two regions of interest. This BC expresses that the population density and its flux must be continuous at the boundary of separation between regions 1 and 2. Thus, the conditions at a common boundary are expressed mathematically as follows:

$$\begin{aligned} (u_i)_1 &= (u_i)_2 \\ (J_i)_1 &= (J_i)_2. \end{aligned} \tag{2.14}$$

2.2. MATHEMATICAL MODEL OF COMBAT

In 1914, Lanchester[6] presented arguments that led to the formulation of ordinary differential equations describing combat between two forces. The generic Lanchester equations are given by

$$\begin{aligned} \frac{du_1}{dt} &= c_1 u_1 u_2 + d_1 u_2 + e_1 \\ \frac{du_2}{dt} &= c_2 u_1 u_2 + d_2 u_1 + e_2 \end{aligned} \tag{2.15}$$

where the negative coefficients c_1, c_2, d_1, d_2 represent mutual attritions of two opposing forces u_1, u_2 depending only on the duration of the battle, t. The terms e_1, e_2 represent autonomous sources. For modern combat, the first two terms have come to be known as area and aimed fire for static forces. The model equations (2.15) have been used extensively to predict the outcome of battle, namely to answer questions of military interest: Who will win? What initial force ratio is required to guarantee victory? How long will the battle last? How many survivors will the victor have? How do changes in parameters affect the development of the battle? etc. A considerable body of knowledge and expertise exists in the use of (2.15), and their extensions, which is well summarized by Taylor[2]. The classical Lanchester equations with additional terms attributable to smart weapons have been discussed in the framework of catastrophe theory by Woodcock and Dockery.[7]

Independent of their specific form, the Lanchester-type equations face a fundamental shortcoming: they do not account for the movement of the opposing forces on the battlefield. In addition, command and control (C^2) are likewise not treated. An extensive account of other shortcomings is listed in the text by Taylor.[2]

The motivation for the work reported here is the assertion that combat modeling requires a full <u>time</u> and <u>space</u>-dependent nonlinear formulation. Without spatial dependence maneuver is impossible; without nonlinear effects the larger force

always wins. Alternatively, stalemate ensues when firepower is relatively evenly matched and there is no motion, e.g., trench warfare. Linking C^2 to combat causes opportunities to be created in which nonlinear effects may predominate. Of course, a more general formulation of combat modeling ought to sort out what is due to chance (random effects) and what is due to choice. The formulation should also say what happens when choice sows chance as in an ambush. A complete mathematical description is also rather likely to substantiate the observation that combat appears locally chaotic, but still exhibits long-range order at least over certain periods of time. In this paper, only deterministic models with crisp information, variables, and coefficients are considered.

In the standard formulation of Lanchester equations, one deals with ordinary differential equations (ODE). In the present context, the problem is cast in the more general framework of partial differential equations (PDE). The PDE formulation introduces the effects of spatial dependence thus rectifying the utmost shortcoming in the ODE form of combat equations. The PDE formalism assures that the two important elements for the formulation of combat, i.e., movement in both time and space receive the proper treatment. The representation of the movement of the front line of battle or flanking maneuvers can be accounted for without recourse to *ad hoc* tables based on time-dependent attrition alone.

In order to understand modern combat and its interaction with command and control, an assumption was made that all elements of combat, ancient and modern, coexist on the modern battlefield. Therefore, if command and control are to be correctly modeled, the heterogeneous combat force with all fundamental motions has to be modeled as well. In the absence of this systematic model, only *ad hoc* attribution of the effects of C^2 on combat can be made, especially where high technology weapons are employed.

A single component of a heterogeneous force (or, equivalently, a homogeneous force) moving in one dimension has been selected for the first attempt at representation. The original Lanchester equations

$$\begin{aligned}(u_1)_t &= F_1(t, \lambda; u_1, u_2)\\ (u_2)_t &= F_2(t, \lambda; u_1, u_2)\end{aligned} \tag{2.16}$$

are replaced by the more general partial differential equations

$$\begin{aligned}(u_1)_t &= F_1(t, x, \lambda; u_1, u_2, (u_1)_x, (u_2)_x, (u_1)_{xx}, (u_2)_{xx})\\ (u_2)_t &= F_2(t, x, \lambda; u_1, u_2, (u_1)_x, (u_2)_x, (u_1)_{xx}, (u_2)_{xx}).\end{aligned} \tag{2.17}$$

Here, u_1, u_2 are the troop levels, F_1, F_2 represent generic nonlinear interaction laws. At this point, these functions are arbitrary as to their specific functional forms, but not as to some general properties they must have in order to represent a competitively interactive system equally able to describe the more acute phases of

conflict and combat. The determination of the exact functional form of F_1, F_2 (in either the ODE or the PDE case) requires additional empirical and/or model-related information. The subscripts t and x denote (partial) derivatives with respect to the corresponding variable. The independent variables are now the time, t, and the position, x, taking values in a certain spatial domain $[-L, L]$. The interval $[-L, L]$ is interpreted as the length of the battlefield for the war between the forces u_1 and u_2. By λ we denote other parameters entering the problem. Unlike Eqs. (2.16), the new PDE system (2.17) has to be supplemented, not only with initial conditions,

$$\begin{aligned} lim_{t\to 0} u_1(t,x,\lambda) &= u_1^o(x,\lambda) \\ lim_{t\to 0} u_2(t,x,\lambda) &= u_2^o(x,\lambda)\,, \end{aligned} \tag{2.18}$$

but also with boundary conditions at the extremities of the battlefield.

Specifically, we propose the following combat model equations[8]

$$\begin{aligned} (u_1)_t &= (D_1(u_1)_x)_x + (C_1u_1)_x + u_1(a_1 + b_1u_1 + \int c_1(x-y)u_2(y)dy) + d_1u_2 + e_1 \\ (u_2)_t &= (D_2(u_2)_x)_x + (C_2u_2)_x + u_2(a_2 + b_2u_2 + \int c_2(x-y)u_1(y)dy) + d_2u_1 + e_2 \end{aligned} \tag{2.19}$$

with mixed boundary conditions. The actual interpretation of (2.19) depends primarily on the sign of the coefficients (functions) $a, b, c, d, ...$ and of the type of interaction (local, nonlocal, quadratic, logistic, sigmoidal, etc.). For simplicity, here we have considered that the nonlocality appears only in the interaction terms $u_i * u_j \stackrel{\text{def}}{=} u_i(x) \cdot \int c_i(x-y)u_j(y)dy, \quad i,j = 1,2$, but it is quite obvious that all the other terms can be considered in the nonlocal formulation. Also, in order to draw an immediate parallel to the Lanchester model, we treat here only quadratic nonlinearities, but other types can be easily included if necessary. A combat model interpretation of the terms appearing in (2.19) is described in the following display.

In general, the "diffusion coefficients" D_1, D_2, the "velocities" C_1, C_2, and different attrition, supply, and enhancement rates $(a_1, a_2, b_1, b_2...)$ will depend on space, time, and even on the troop levels themselves. Also, the relative importance of different terms is determined by factors such as preparation, motivation, efficacy, time elapsed since engagement in the battle, exhaustion, sudden change in plans due to newly acquired intelligence, etc. To our knowledge, even a roughly reasonable mathematical modeling of these factors is still lacking. Further sophistication can be achieved by including other types of nonlinear interactions and by taking advantage of the parameters λ. A caveat though: for the type of problem we study, the robustness of the modeling to the type of nonlinearity is essential in order to achieve any degree of predictability.

EXPRESSION	INTERPRETATION
$(D_1(u_1)_x)_x$	The (Fickian) diffusion term expresses the natural tendency of any force, ancient or modern, to lose its initial configuration as it moves, fights, etc., or simply as just time goes by, due to fatigue, loss of concentration, loss of motivation, etc. Other types of diffusion (repulsive, attractive, cross, anisotropic) may be considered according to empiric or model related indications. The inclusion of this term is a recognition of those factors that tend to disperse troop forces, as by some troops being sent ahead to reconnoiter and others struggling behind because of failure or inability to maintain the pace of the main body.
$(C_1 u_1)_x$	The advection term describes the large-scale, ordered "flow" of troops on the battlefield as opposed to the "chaotic," small-scale movement represented by diffusion. If only this term were present and C_1 were constant, the initial force profile would be convected undistorted along the x-axis at a speed C_1. This term simulates the advance or retreat of a body of troops as by foot or mechanized vehicles with C_1 clearly set to reflect the attitude of the troops and the mode of transport.
$a_1 u_1$	Represents re-supply of the force u at the rate $a_1 > 0$ (as for example, by paratrooper reinforcements).
$b_1 u_1^2$	For $b_1 < 0$, it is intended to model self-repressing effects due to crowding, saturation, etc.
$u_1 \int c_1(x-y)u_2(y)dy$	Is a typical interaction term between the two opposing forces and represents a variety of hypothesized attrition modes. For $c_1(x-y) = \delta(x-y)$, it reduces to the Lanchester form and models a local (man-to-man or one-against-one) type of combat. For $c_1(x-y) \equiv 1$, it models a global (all-against-one) interaction. A general function $c_1(x-y)$ will describe a weighted type of interaction (combat) and can be adjusted to model any given (or perceived) situation. For instance, it can model attrition losses which vary as a function of the distance between the combatants, this variation occurring because of the change in effectiveness of firepower as the distance between forces changes. Obviously, accuracy in this last term is crucial in realistic modeling because of the importance of long-range weapon delivery systems in modern warfare.
$d_1 u_2 + e_1$	Reproduces the other terms in the classical Lanchester form (2.15).

There are several parameters entering the problem: the length of the battlefield, the exact form of the boundary conditions, the initial force levels, their spatial distribution (shape), the attrition rates, the criteria for disengagements and termination, the option pursuit/nonpursuit, etc. For illustration, we report here a few cases showing the influence of different factors on the outcome of the proposed battle model.

3. NUMERICAL COMPUTATIONS

3.1. COMPUTATIONAL METHODS AND COMPUTER PROGRAMS

The combat models as presently envisioned are mathematically represented by linear parabolic initial value-boundary value PDEs. In general, the numerical solution of PDEs involves specialized and diverse methods that are highly problem (i.e., dimensionality, interaction, boundary-conditions) dependent, with the consequences that little software for general classes of PDE problems is available and the analyst is often forced to undertake the difficult and time-consuming task of developing a computer program tailored to his own needs. However, a general method, of relatively recent vintage and applicable to the class of PDEs under investigation, is the so-called method of lines. This method is specifically designed to take advantage of the advances made in recent years in both the theory and the computer implementation of techniques for solving ODEs. In particular, there now exists very capable, robust, and proven software for nonlinear initial value ODE problems. Recent notable developments include software for the heretofore very difficult stiff ODEs and reliable, efficient algorithms for dynamically changing step size and method order to maintain mathematical stability and a user-specified accuracy during the course of the integration.

The method of lines consists of discretizing the space variable(s), thus converting the PDEs into a semidiscrete approximating system of ODEs. If a stringent time integration error tolerance is chosen for the ODE solver, any error in the calculations relative to the exact solution may be attributed to the spatial discretization and these errors may be investigated by comparing the results for different spacings of the mesh points. The ODE solver chosen should be one designed to handle stiff sets of ODEs because classical nonstiff integrators are generally inadequate even for simple PDEs. This is because the problem, if not already stiff, can become stiff as the number of spatial points increases. Use of a stiff solver avoids the very small integration time steps needed to maintain stability and accuracy when stiff problems are attacked using nonstiff methods.

The code MOL1D (method of lines, one dimension) written by Hyman[9] has been used to solve the combat modeling PDEs. Formation of the ODE system is automatically performed based upon an input description of the PDE equations, the initial and boundary conditions, and the desired spatial mesh. Spatial derivatives may be approximated by the user's choice of second, fourth, and sixth order symmetric differences. After forming the system of ODEs, MOL1D uses the

stiff/nonstiff ODE solver GEARB written by Hindmarsh[10] to solve the resulting system.

3.2. COMPUTATIONAL COMPLEXITIES

In general, stiff ODE solvers are marked by the need to solve a set of nonlinear equations at each time step because the formulas used in advancing the solution in time are implicit functions of the dependent variables at the new time for which the dependent variables are to be determined. This is in contrast to many classical nonstiff methods, such as the classical Runge-Kutta formulas, which involve explicit formulas that contain only the values of the solution at previous time steps. It may be noted that implicit methods are also often used to solve nonstiff sets of equations because they usually permit the use of larger time steps than do the explicit methods. But while the system of nonlinear equations arising in the case of implicit methods applied to nonstiff problems can often be solved by functional integration, the systems of equations arising in the case of stiff problems almost invariably require the use of the more powerful Newton's method in order to obtain convergence. Newton's method, in turn, requires the solution of sets of linear equations involving the Jacobian matrix of the ODE system. It is, in fact, the formation and solution of these linear equations that is the most time-consuming part of the algorithm. The construction of the Jacobian matrix, the first derivative matrix of the right-hand side of the ODE system with respect to the dependent variables, requires that the needed derivative either be computed analytically or approximated by numerical differentiation. As the ODE system itself is generated by discretization of the spatial variables, the complexity of the ODEs almost necessitates that the Jacobian be approximated numerically; this is accomplished automatically in MOL1D by invoking a numerical differentiation option in the GEARB package.

Not only are the linear algebra operations involved time-consuming, but the storage of the associated coefficient matrices accounts for much of the memory requirement of the computer program. For a general stiff ODE system of n equations, the Jacobian matrix has n^2 elements. The number of ODEs to be solved using the method of lines is equal to the product of the number of PDEs (NPDE) and the number of spatial mesh points (NPTS). For the present problem, the model consists of two partial differential equations while based on results to date, 400 to 800 spatial mesh points appear to be needed to ensure accurate results. Hence, 800 to 1600 ordinary differential equations must be integrated to effect a solution of the two original partial differential equations. Thus, ostensibly, storage of the Jacobian would appear to require the storage of 800^2 to 1600^2 matrix elements, a formidable memory requirement. Fortunately, however, in the present problem, the Jacobian is a banded matrix of band width less than nine times the number of PDEs so that the number of nonzero Jacobian elements is approximately 9*NPDE*NPTS rather

than (NPDE*NPTS)^2; that is, 14,400 elements rather than 2,560,000 elements if NPTS= 800. Specialized algorithms for solving linear equations with a banded coefficient matrix and which store and manipulate only the nonzero elements are applicable allowing a great reduction in both computer memory and computational time. The B in the GEARB program name denotes Hindmarsh's variant of his original GEAR code which is specifically designed to exploit the sparsity of banded Jacobian matrices.

As will be shown later, the mathematical model contains both a "wavelike" part due to a convective term and a "diffusive" contribution with these two effects, in a sense, competing to govern the solution behavior. Moreover, the discrete approximations to the convection and diffusion terms compete in the numerical methods for the approximate solution with the consequence that the manner in which these terms are treated influences the behavior of the approximate solution. If convective effects are strongly dominant in the model, the numerical solution may exhibit inappropriate oscillatory behavior (relative to the true solution) due to dispersion of high-frequency errors that are essentially undamped numerical oscillations. This "numerical dispersion" causes both spurious oscillations in the solution and phase errors in the propagation of wavefronts. On the other hand, if diffusion is dominant in the model, the approximation may be too strongly damped, particularly in regions where there are sharp gradients or discontinuities in the exact solution. This "numerical dissipation" causes the solution to be "smeared" over several mesh intervals.

Although unwarranted oscillatory behavior has been observed in the solution during some scoping calculations in which the convection term was strongly dominant (with diffusion coefficients one to two orders of magnitude smaller than realistic values), such oscillations do not appear to be of material concern in the practical computations done to date. To some extent, the aforementioned computational difficulties have been obviated by the fine mesh which has been employed since a finer mesh is, indeed, one approach used in overcoming these difficulties. The original motivation, however, for the fine mesh was to allow accurate representation of initially steep forces profiles (such as close approximations to square waves). While steep gradients can of themselves exacerbate the difficulty of accurately tracking the solution without introducing spurious components, it appears that in the present problem this effect has been outweighed by the greater stability introduced by the fine mesh. However, while a certain fineness of nodal spacing is inherently necessary just to ensure accurate representation of the initial and developing force profiles, it is, in general, undesirable to continue refining the mesh because of the concomitant increase in the number of ODEs which must be solved. For this reason, other methods such as the use of "upwind" (backward spatial difference) approximations for the convection term or the use of artificial viscosity techniques may need to be exploited if computational difficulties arise and further refinement of the mesh would cause the number of nodal points (and hence equations) to be excessive.

3.3. COMPUTATIONAL MEASURES

A relative error tolerance of 1×10^{-5} was specified for the time integration of the ODE system ensuring that the weighted single step error estimates were kept less than this value in the root–mean–square norm (the Euclidean norm of the dependent variable vector divided by the square root of the number of variables). It could then be reasoned that errors (relative to the true solution) larger than this value could be attributed to the spatial discretization. Since, obviously, the true solution is unknown, the magnitude of the spatial discretization errors was investigated by comparing solutions for various mesh spacings in order to find the spacing at which the spatial discretization errors were of the same magnitude as those for the temporal integration. It was found that the accuracy desired was obtained when the number of spatial node points exceeded about 400.

With 800 nodal points and an error tolerance of 1×10^{-5} specified to the ODE integrator, the time required for the computations done to date has ranged from about 10 seconds to approximately one minute on a CRAY XM-P computer, depending upon the number and complexity of the coefficients and ancillary functions involved in the evaluation of the terms of the PDEs and the values of the parameters governing the unfolding and termination of the battle. While these computational times are quite modest, it must not be overlooked that the computations were performed on a supercomputer. Rather rough estimates of the time multiples if the calculations were performed on other machines are 4 for an IBM 3033, 7 for a DEC VAX 8600, and 300-500 for an IBM PC-AT. Though there exists the possibility of reducing the computational times on the CRAY by increased vectorization of the algorithms, offsetting this savings will almost certainly be the desire to introduce additional terms in the evolution equations and more complex strategies in the battle management. Further, while neither memory requirements nor computational time make it imperative to perform the calculations on a supercomputer for the one-dimensional problem, this is unlikely to be the case if the analysis is extended to two spatial dimensions because of the large increase in the number of equations to be solved. In summary, while some computational efficiencies are probably feasible, desire for additional complexity in the modeling is likely to result in a net increase in demand for computer resources so that the problem will remain computationally intensive.

Thus far, the PDEs have proven quite tractable when approached using the method of lines. Partly, this is attributable to the powerful ODE solvers incorporated into MOL1D, in particular, the inclusion of a state-of-the-art stiff ODE solver. The present problem, while not so stiff as to preclude solution by nonstiff methods, is much more efficiently solved using a stiff ODE solver as evidenced by a limited number of timings which showed a nonstiff solver requiring almost twice as long as a stiff solver to integrate the same problem.

4. COMPUTATIONAL RESULTS

4.1. SCENARIOS AND PARAMETERS

We present here some classical situations of opposing force postures in order to illustrate the integration of Eq. (2.19). Examples are chosen to emphasize the effects of introducing explicit spatial dependence via advection, diffusion, and space-dependent attrition for forces which are closing on one another. None of these phenomena is present in the classical Lanchester equations formulation.

In obtaining results for the foregoing postures, we integrated a simplified version of Eq. (2.19), namely:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= (D_1(u_1)_x)_x + (C_1 u_1)_x + u_1(x) \int c_1(x-y) u_2(y) dy \\ \frac{\partial u_2}{\partial t} &= (D_2(u_2)_x)_x + (C_2 u_2)_x + u_2(x) \int c_2(x-y) u_1(y) dy \end{aligned} \tag{4.1}$$

We remark that for $c_1(x-y) = c_2(x-y) = \delta(x-y)$ the nonlinear term in (4.1) reduces to the form used in the "area-fire" Lanchester equations. Another choice of the coefficients in (2.19), namely $a = b = c = e = 0, d \neq 0$, would lead after appropriate simplifications to the "modern-warfare" form of the Lanchester equations.[3] The analysis of that model can be performed in a similar way, but we preferred to study the nonlinear interaction term. A description of the case studies may perhaps be best begun by detailing the parameters which both must be set in defining a particular problem and which indicate the modeling possibilities afforded the analyst. The parameters can be categorized as follows:

1. parameters defining the initial and boundary conditions
2. coefficients of the evolution equations
3. parameters related to externally applied conditions on the system evolution.

The initial conditions entail a specification of the strengths and dispositions of the forces. Subroutines are provided to permit construction of rectangular, triangular, and spike force profiles with the height, width, and locus of a profile specified by parameters set in the input data. Boundary conditions are specified in the general form

$$\begin{aligned} \alpha_i^{(1)} u_i + \beta_i^{(1)} \frac{\partial u_i}{\partial x} &= \gamma_i^{(1)} \quad \text{at } x = -L \\ \alpha_i^{(2)} u_i + \beta_i^{(2)} \frac{\partial u_i}{\partial x} &= \gamma_i^{(2)} \quad \text{at } x = L \end{aligned} \qquad t > 0;\ i = 1,2 \tag{4.2}$$

where

$$\begin{aligned} u_i &= \text{force density of force } i \\ x &= \text{spatial coordinate} \\ -L, L &= \text{left and right boundaries of battlefield} \\ t &= \text{time} \end{aligned}$$

$\alpha_i^{(1)}, \beta_i^{(1)}, \gamma_i^{(1)}, \alpha_i^{(2)}, \beta_i^{(2)}, \gamma_i^{(2)}, \qquad i = 1,2$ are constants.

This general form allows specification of Dirichlet boundary conditions ($\alpha_i \neq 0, \beta_i = 0$), Neumann boundary conditions ($\alpha_i = 0, \beta_i \neq 0$), and mixed boundary conditions ($\alpha_i \neq 0, \beta_i \neq 0$). The type of boundary condition may change with respect to time as well as for each force and from left to right boundary.

The basic battle scenario is envisaged as follows: the forces advance toward each other until a maximum allowable interpenetration fraction (PENLIM) is achieved, the troops then stand and fight until one of the forces is reduced to a specified force fraction (DENG) at which point disengagement occurs, the losing force retreats and the winning force either pursues or remains stationary, the battle continues until one of the forces has been reduced to the specified force fraction (TERM) for battle termination. The values of the various parameters will, however, affect this basic scenario since, for example, if the effectiveness of firepower is high the battle could be terminated before any interpenetration of the forces occurred.

The length of the battlefield is characterized by the specification of the half-length of the battlefield (HL) while the speed at which the losing force retreats and the winning force pursues is defined in terms of a multiple (DVF) of the initial velocities. In order to establish a base case for testing the computer program and a starting point for exploring the "interesting" values for the various parameters, "reasonable" values for each of the parameters were selected. The following values were chosen for the parameters related to the externally applied conditions:

$$\begin{aligned} \text{PENLIM} &= 0.15 \qquad & \text{HL} &= 5.0 \text{ miles} \\ \text{DENG} &= 0.80 \qquad & \text{DVF} &= 0.5 \\ \text{TERM} &= 0.60 \end{aligned}$$

In order to differentiate more easily between convective, diffusive, and attrition effects, only one of the attrition terms in the evolution equations, namely the last (the integral term), was taken to have a nonzero coefficient. For the primary sequence of calculations presented in this report, the values of the coefficients of the PDEs were:

$$\begin{aligned} C_1 &= 5 \text{ miles/hr}, & D_1 &= 0.01 \text{ (miles)}^2/\text{hr}, & c_1 &= -2.4 \times 10^{-4} (\text{miles} \cdot \text{hr})^{-1} \\ C_2 &= -6 \text{ miles/hr}, & D_2 &= 0.02 \text{ (miles)}^2/\text{hr}, & c_2 &= -3.0 \times 10^{-4} (\text{miles} \cdot \text{hr})^{-1} \end{aligned} \tag{4.3}$$

The negative sign for C_2 represents the fact that force 2 is initially advancing to the left along the x-axis to confront force 1 which is moving to the right in the positive x-direction. The values of the velocities, of course, should be chosen to realistically model the rate of advance of the forces being modeled. The values chosen ($C_1 = 5$ miles/hr, $C_2 = -6$ miles/hr) which are in miles per hour certainly are too large for forces advancing entirely on foot, but could well be low for mechanized forces, and hence should be viewed as average values, as for a mixture of infantry and armor.

The diffusion coefficients D_i are a measure of the rate of diffusion or dispersion of the forces. In the absence of diffusion (and attrition), the initial force profiles would be unchanged over time and would simply be translated at their respective velocities. The diffusion terms attempt to account for the fact that the forces will naturally tend to disperse as a consequence of portions of the force moving faster and slower than the main body. The values assumed above for the D_i are small enough so the forces remain tightly massed relative to the initial force profiles. Though the values of D_i used may be unrealistically low (guidance is needed on this point), they do have the virtue of suggesting that, since the computations are numerically stable for these values, the computations will be assuredly stable for larger values of the D_i because the convection term will be less dominant.

The attrition factors $c_i(x-y)$ have been chosen of the form $c_i\phi_i(x-y)$. The coefficients c_i have been chosen completely on the basis of providing attrition rates commensurate with accomplishing the desired force reductions in the desired time and spatial reference frames. That is, the values of the c_i are selected so that the complete scenario of advance, interpenetration, disengagement, pursuit/retreat, and battle termination occurs within the confines of the defined battlefield and within both a reasonable real time and computational time. The factors $\phi_i(x-y)$, which will be termed the "firing function," was taken to have the form $e^{-\nu_i|x-y|}$, thus accounting for the expected result that the effectiveness of a weapon delivery system will decrease with distance from the target. In the computations to date, the $\nu_i (i = 1, 2)$ have arbitrarily been set to a value of 1/3. Clearly, realistic values of the c_i and ν_i (indeed, even the mathematical form of the firing functions) require the guidance provided by experimental data or theoretical predictions.

The numerical integrations required in the evaluation of the integral term represent a significant fraction of the computational time as the integrals are functions of the independent spatial variable x and must be reevaluated for each mesh point. Further, each integral evaluation is itself computationally costly because of the need to use multi-point quadrature formulas in order to accurately integrate steep force profiles. Some savings in computational time has been achieved, however, by avoiding the integration if the premultiplier u_i of the integral is so small as to essentially make the term vanish. This is achieved by testing the premultiplier to ensure that it exceeds a specified cutoff value before proceeding with the integral evaluation.

In order to show the changes introduced by spatial effects, we compared our results with the results obtained in the "equivalent Lanchester scenarios": namely, we compared $U_1(t) \stackrel{\text{def}}{=} \int_{-L}^{L} u_1(x,t)dx$ and $U_2(t) \stackrel{\text{def}}{=} \int_{-L}^{L} u_2(x,t)dx$ obtained from Eq. (4.1) with the solutions of the "equivalent Lanchester equations"

$$\begin{aligned} \frac{dU_1}{dt} &= c_{10}U_1U_2 \\ \frac{dU_2}{dt} &= c_{20}U_1U_2 \end{aligned} \tag{4.4}$$

with compatible initial conditions, i.e.,

$$U_1(t=0) = U_1^o, U_2(t=0) = U_2^o\,. \tag{4.5}$$

Different versions of the system (4.1) corresponding to the postures and scenarios presented in Table 1 were integrated by using the routine MOL1D. Running times on a Cray XMP were typically 1 minute for 40-80 minutes of combat. In order to present this output in a usable form with good graphics, results were downloaded for presentation on desktop computer configuration. In particular, a graphics program for the IBM PC produced a capsule video display in which the forces are seen to march toward one another as a function of time.

Because this is a one-dimensional case, the display in Fig. 1 shows the time ordered development of two initially sharp distributions moving along an axis; the third axis is time. The situation is completely academic since the forces are simply passing through each other.

Figure 2 represents a typical scenario of battle. The ratio of the total initial forces is 1:1 and the length of the battlefield is 10 miles. The forces start moving toward each other while shooting at each other with a space-dependent fire power. After engagement, they continue to fight until one force has suffered 20% casualties. Disengagement proceeds by slow pursuit of the losing force by the winning force

POSTURE	ASSUMPTIONS AND RATIONALE FOR CHOICE
Forces pass through each other.	This is a purely theoretical engagement to demonstrate effects of diffusion and attrition on the initial "shape" of the forces.
3-1 and 1-1 troop ratio. Both forces closing.	Forward movement due to advection ceases when 15% of the smaller force interpenetrates the other. Withdrawal begins when casualty figures reach 20% on a side. Disengagement proceeds by either slow pursuit or no pursuit and continues until the termination of battle (40% casualties on a side). In principle, 1-1 should stalemate, and 3-1 produce a winner.
6-1 troop ratio. Only one force moving. Smaller force may dig in.	The large force is moving and smaller force is stationed in a trench. There are different attrition rates for area and aimed fire for each side. Defending force may sag during engagement (interpenetration). No pursuit is permitted. Usually, the 1-3 defense has a chance, but at 1-6 will be rapidly overrun.

Table 1. Postures and scenarios selected for the numerical study of Eq. (4.1).

and continues until the losing force reaches 40% casualties. At this point, the battle is declared terminated.

In Figs. 3, 5, and 6 we represented the total forces as functions of time in three different scenarios: PDE with pursuit, PDE without pursuit, and Lanchester. In both PDE cases the attrition is space-dependent, i.e., for the first time, attrition of forces on the move has been shown. We also noted that if we run the PDE cases with attrition independent of space, $c_1(x) = c_{10}, c_2(x) = c_{20}$, (these cases are not shown in Fig. 2) the results are, initially, practically identical to those given by the Lanchester model and differ significantly only when boundary effects come into play. This is to be expected as the effects on combat of the forward motion will be cumulative.

Dirichlet B.C.

Force 1 Profile	Force 2 Profile	Termination time, hr.	Survival fraction force 1	Survival fraction force 2
Rectangular	Rectangular	1.46	0.680	0.600
Rectangular	Triangular	1.44	0.680	0.600
Rectangular	Spike	1.42	0.680	0.600
Triangular	Rectangular	1.43	0.680	0.600
Triangular	Triangular	1.40	0.680	0.600
Triangular	Spike	1.40	0.680	0.600
Spike	Rectangular	1.42	0.680	0.600
Spike	Triangular	1.39	0.680	0.600
Spike	Spike	1.38	0.680	0.600

Neumann B.C.

Force 1 Profile	Force 2 Profile	Termination time, hr.	Survival fraction force 1	Survival fraction force 2
Rectangular	Rectangular	1.46	0.680	0.600
Rectangular	Triangular	1.44	0.680	0.600
Rectangular	Spike	1.42	0.680	0.600
Triangular	Rectangular	1.43	0.680	0.600
Triangular	Triangular	1.40	0.680	0.600
Triangular	Spike	1.40	0.680	0.600
Spike	Rectangular	1.42	0.680	0.600
Spike	Triangular	1.39	0.680	0.600
Spike	Spike	1.38	0.680	0.600

Table 2. Influence of initial shape on the termination time and fraction survivors at fixed interaction, scenario, and initial force ratio, at two different B.C.'s. The evolution in equations are (4.1) with the parameter values given by (4.3).

Force 1 coefficients	Force 2 coefficients	Boundary condition	Termination time, hr.	Fraction survivors force 1	Fraction survivors force 2
1	1	Dirichlet	3.04	0.797	0.600
1	1	Neumann	3.04	0.797	0.600
1	2	Dirichlet	0.85	0.999	0.600
1	2	Neumann	0.85	0.999	0.600
1	3	Dirichlet	1.27	0.911	0.600
1	3	Neumann	1.27	0.911	0.600
2	1	Dirichlet	1.13	0.600	0.811
2	1	Neumann	1.13	0.600	0.811
2	2	Dirichlet	1.04	0.680	0.600
2	2	Neumann	1.04	0.680	0.600
2	3	Dirichlet	1.28	0.600	0.656
2	3	Neumann	1.28	0.600	0.656
3	1	Dirichlet	1.58	0.600	0.720
3	1	Neumann	1.58	0.600	0.720
3	2	Dirichlet	0.93	0.813	0.600
3	2	Neumann	0.93	0.813	0.600
3	3	Dirichlet	1.46	0.680	0.600
3	3	Neumann	1.46	0.680	0.600

Table 3. Influence of different interaction forms on the termination time and fraction of survivors at fixed initial shape, initial ratio, pursuit scenario, and two B.C.'s.

Half-length of battlefield, miles	Boundary condition	Scenario	Termination time, hr.	Survival fraction force 1	Survival fraction force 2
5	Dirichlet	pursuit	1.46	0.680	0.600
	Dirichlet	no pursuit	1.96	0.680	0.600
	Neumann	pursuit	1.46	0.680	0.600
	Neumann	no pursuit	1.96	0.680	0.600
	Mixed	pursuit	1.46	0.680	0.600
	Mixed	no pursuit	1.96	0.680	0.600
4	Dirichlet	pursuit	1.46	0.680	0.600
	Dirichlet	no pursuit	1.80	0.689	0.600
	Neumann	pursuit	1.46	0.680	0.600
	Neumann	no pursuit	1.80	0.689	0.600
	Mixed	pursuit	1.46	0.680	0.600
	Mixed	no pursuit	1.80	0.689	0.600
3	Dirichlet	pursuit	1.44	0.683	0.600
	Dirichlet	no pursuit	1.52	0.708	0.600
	Neumann	pursuit	1.44	0.683	0.600
	Neumann	no pursuit	1.52	0.708	0.600
	Mixed	pursuit	1.44	0.683	0.600
	Mixed	no pursuit	1.52	0.708	0.600

Table 4. Influence of different B.C.'s on the termination time and survival fractions at fixed interaction, initial shape, and initial ratio for two scenarios.

I. Dirichlet Boundary Condition

Initial Force Ratio F_1/F_2	Scenario	Duration of battle, hr.	F_1 survival fraction	F_2 survival fraction
1	pass through	1.22	0.818	0.600
1	pursuit	1.36	0.679	0.600
1	no pursuit	1.64	0.680	0.600
3	pursuit	0.60	0.893	0.600
3	no pursuit	0.63	0.893	0.600
6	force 2 entrenched	0.54	0.946	0.600

II. Neumann Boundary Condition

Initial Force Ratio F_1/F_2	Scenario	Duration of battle, hr.	F_1 survival fraction	F_2 survival fraction
1	pass through	1.22	0.818	0.600
1	pursuit	1.36	0.679	0.600
1	no pursuit	1.64	0.680	0.600
3	pursuit	0.60	0.893	0.600
3	no pursuit	0.63	0.893	0.600
6	force 2 entrenched	0.54	0.946	0.600

III. Lanchester

Initial Force Ratio F_1/F_2	Duration of battle, hr.	F_1 survival fraction	F_2 fraction
1	1.05	0.679	0.600
3	0.30	0.893	0.600
6	0.15	0.946	0.600

Table 5. Influence of different scenarios and initial force ratios on the termination time and survival functions at fixed interaction and initial shape, different B.C.'s.

Table 2 shows a mild effect of the spatial shape of the initial distributions and no effect of the B.C.'s. The latter is due to the relatively large extent of the battlefield.

In Table 3, the ratio of the total initial forces is 1:1 (2000:2000); the initial shapes are rectangular. Here we have analyzed the evolution equations:

$$\frac{\partial u_1}{\partial t} = D_1 \frac{\partial^2 u_1}{\partial x^2} + C_1 \frac{\partial u_1}{\partial x} - b_{12}\, u_1\, u_2 - c_1\, u_1 \int_{-L}^{L} e^{-\nu_1 |x-y|}\, u_2(y)\, dy$$

$$\frac{\partial u_2}{\partial t} = D_2 \frac{\partial^2 u_2}{\partial x^2} + C_2 \frac{\partial u_2}{\partial x} - b_{21}\, u_1\, u_2 - c_2\, u_2 \int_{-L}^{L} e^{-\nu_2 |x-y|}\, u_1(y)\, dy$$

that are a variant of (2.19).

Force 1

1	$b_{12} = 0.00024$,	$c_1 = 0$
2	$b_{12} = 0$	$, c_1 = 0.00024\,,\ \nu_1 = 0$
3	$b_{12} = 0$	$, c_1 = 0.00024\,,\ \nu_1 = 0.333$

Force 2

1	$b_{21} = 0.00030$,	$c_2 = 0$
2	$b_{21} = 0$	$, c_2 = 0.00030\,,\ \nu_2 = 0$
3	$b_{21} = 0$	$, c_2 = 0.00030\,,\ \nu_2 = 0.333$

The table shows a relatively strong effect of the interaction on the termination time, and again no effect of the B.C.'s. We notice that the battle lasts longest if the interaction is man-to-man ($b_{12}, b_{21} \neq 0$; $c_1, c_2, \nu_1, \nu_2 = 0$); it lasts less time if the interaction is spatially weighted ($b_{12} = b_{21} = 0$; $c_1, c_2 \neq 0$; $\nu_1 = \nu_2 = 1/3$) and is shortest in the case when the loser is annihilated in a "one-against-all" interaction ($b_{21} = 0, c_2 \neq 0, \nu_2 = 0$).

In Table 4, the ratio of the total initial forces is 1:1 (2000:2000). The initial shapes are rectangular. The evolution equations are (4.1) with the parameter values given by (4.3). One notices a sizable difference between the pursuit and no-pursuit scenarios, but only a very slight one between different B.C.'s. The B.C.'s are felt only when the length of the battlefield is small enough. Even for $L = 3$ mi, one does not see any difference between Dirichlet, Neumann, and mixed B.C.'s for a given scenario.

In Table 5, the initial shapes are rectangular. The evolution equations are (4.1) with the parameter values given by (4.3). Some of the situations analyzed here are also graphically depicted in Figs. 1–11.

These results are preliminary in nature. Much more work remains to be done in order to obtain complete parametric studies and to extend the model to more realistic situations. For instance, taking $n > 2$ in Eq. (2.5) would allow representation of heterogeneous forces and, accordingly, accounting for competitive/cooperative interactions. Taking $m > 1$ would allow for more realistic spatial descriptions. An extension to a two dimensional spatial representation ($m = 2$) is in progress.[11] This will permit a representation of opposing forces in which envelopment can be analytically modeled for the first time. The foregoing is still combat on a tabletop. In order to deal with actual map terrain one may or may not consider an extension to three dimensions ($m = 3$): indeed, natural barriers and elevations should be capable of representation by either/both boundary conditions and alterations to the advection and diffusion terms in the two dimensional case. Effort will also be directed toward including heterogeneous force analysis and stochasticity into the model.

ACKNOWLEDGEMENTS

The authors acknowledge the expert typing of D. C. Human and the technical support of M. S. Smith who converted into TEX sometimes illegible scribbles.

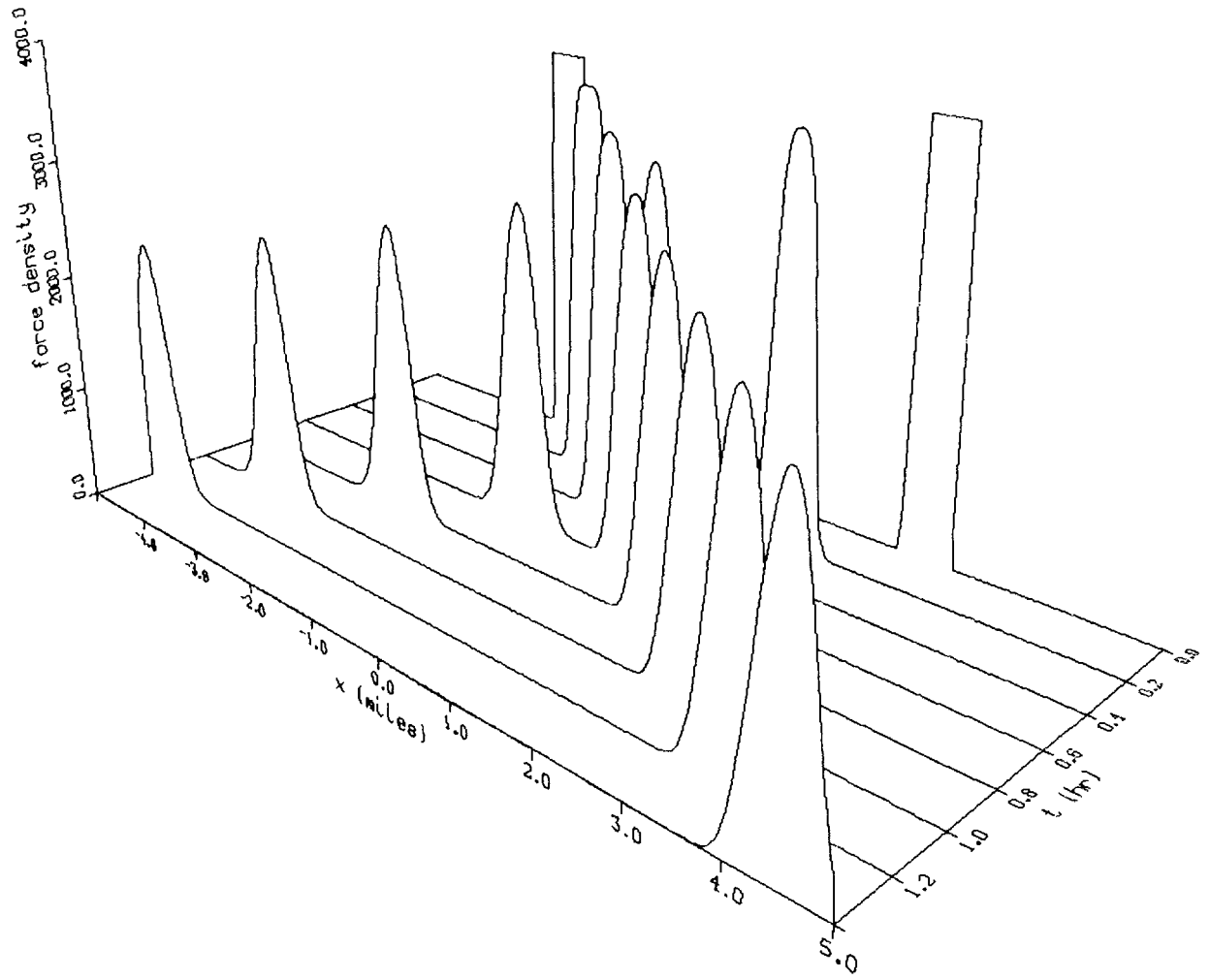


Fig. 1. Spatio-temporal evolution of two competing forces evolving according to Eq. (4.1) under Dirichlet or Neumann B.C. Scenario: forces go through. Ratio of the total initial forces is 1:1.

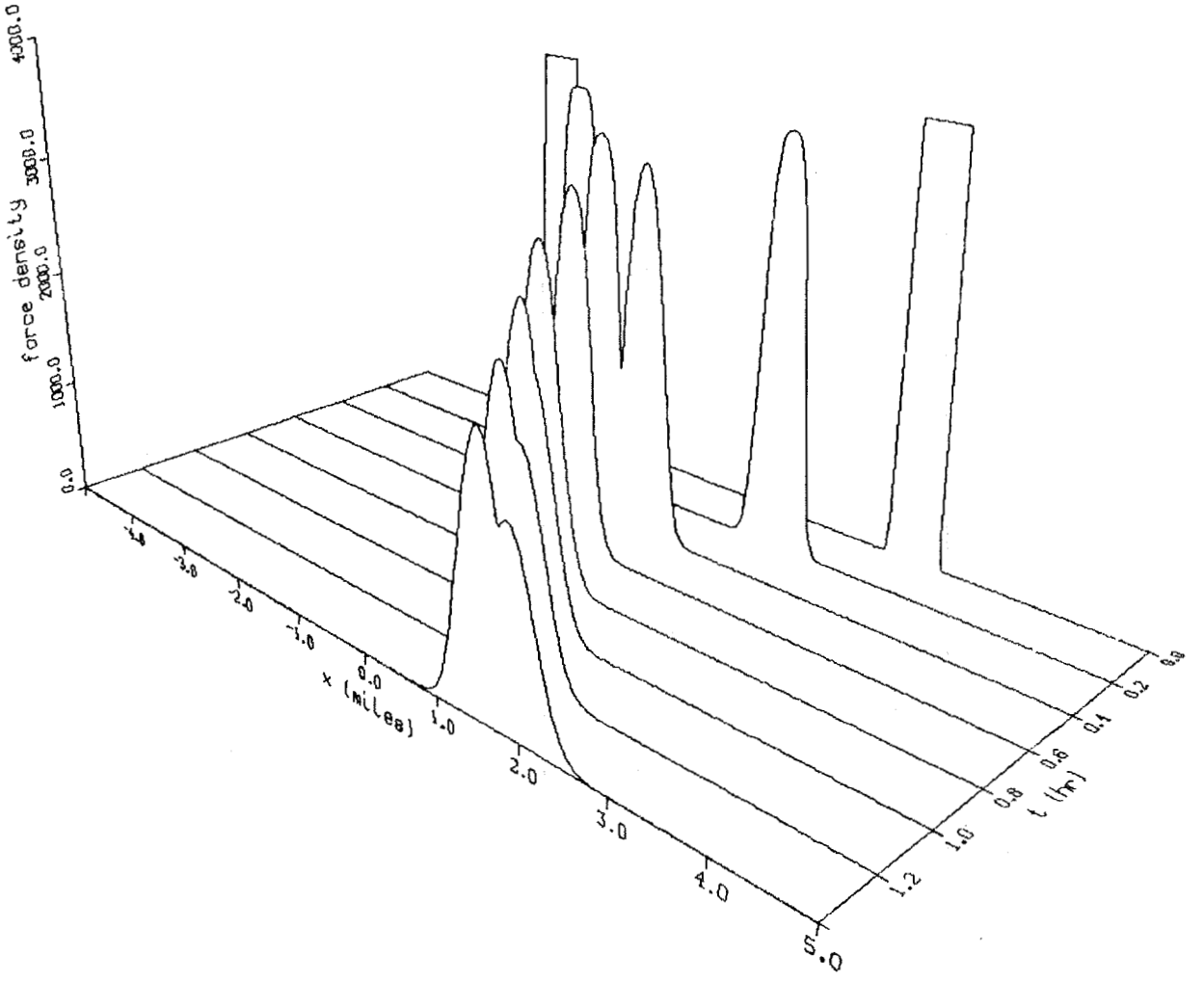


Fig. 2. Spatio-temporal evolution of two competing forces evolving according to Eq. (4.1) under Dirichlet or Neumann B.C. Scenario: pursuit. Ratio of the total initial forces is 1:1.

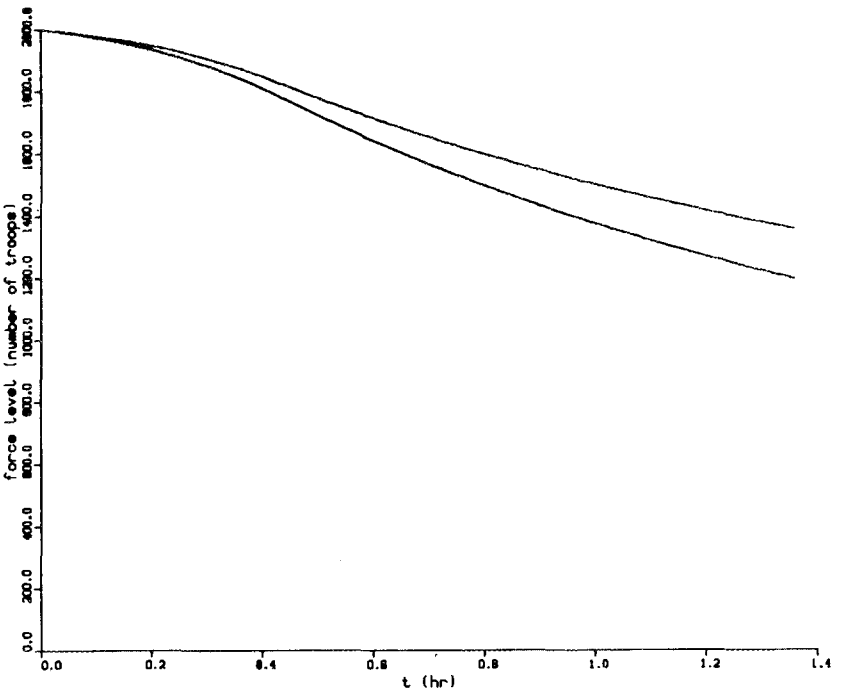


Fig. 3. Time evolution of the total forces as given by the solutions analyzed in Fig. 2.

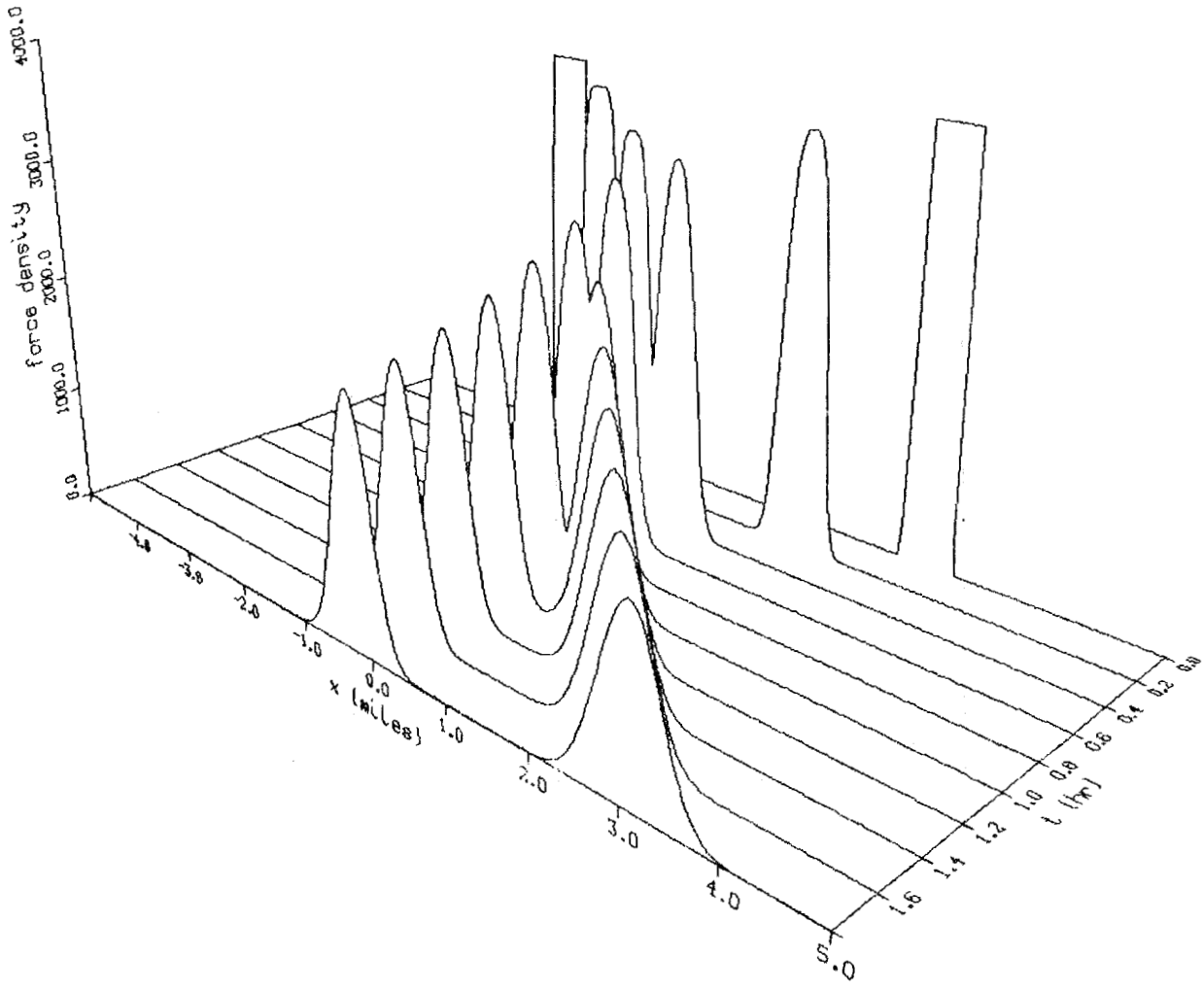


Fig. 4. Spatio-temporal evolution of the two competing forces evolving according to Eq. (4.1) under Dirichlet or Neumann B.C. Scenario: no-pursuit. Ratio of the total initial forces is 1:1.

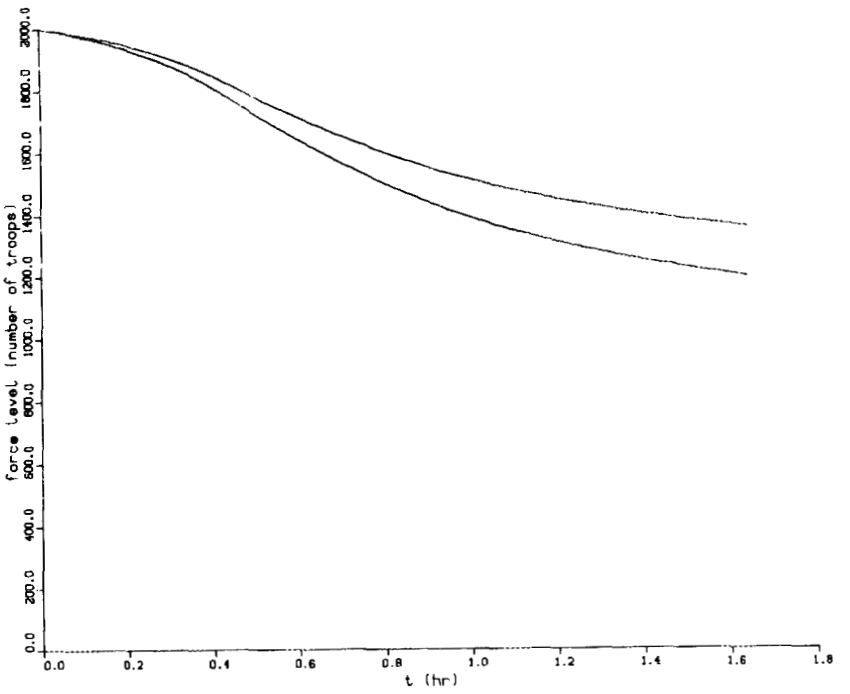


Fig. 5. Time evolution of the total forces as given by the solutions analyzed in Fig. 4.

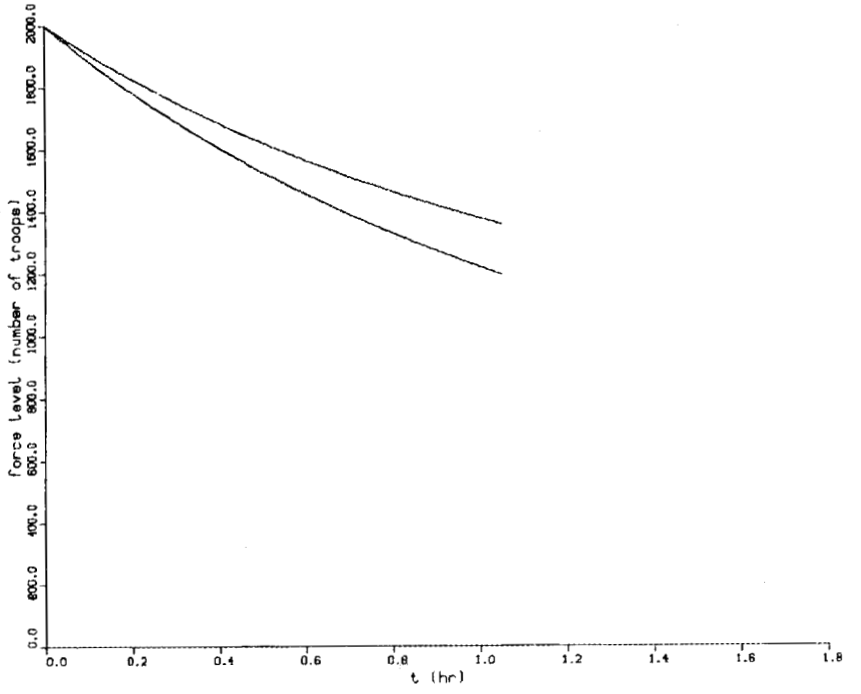


Fig. 6. The equivalent Lanchester solutions corresponding to the cases depicted in Figs. 2 and 4.

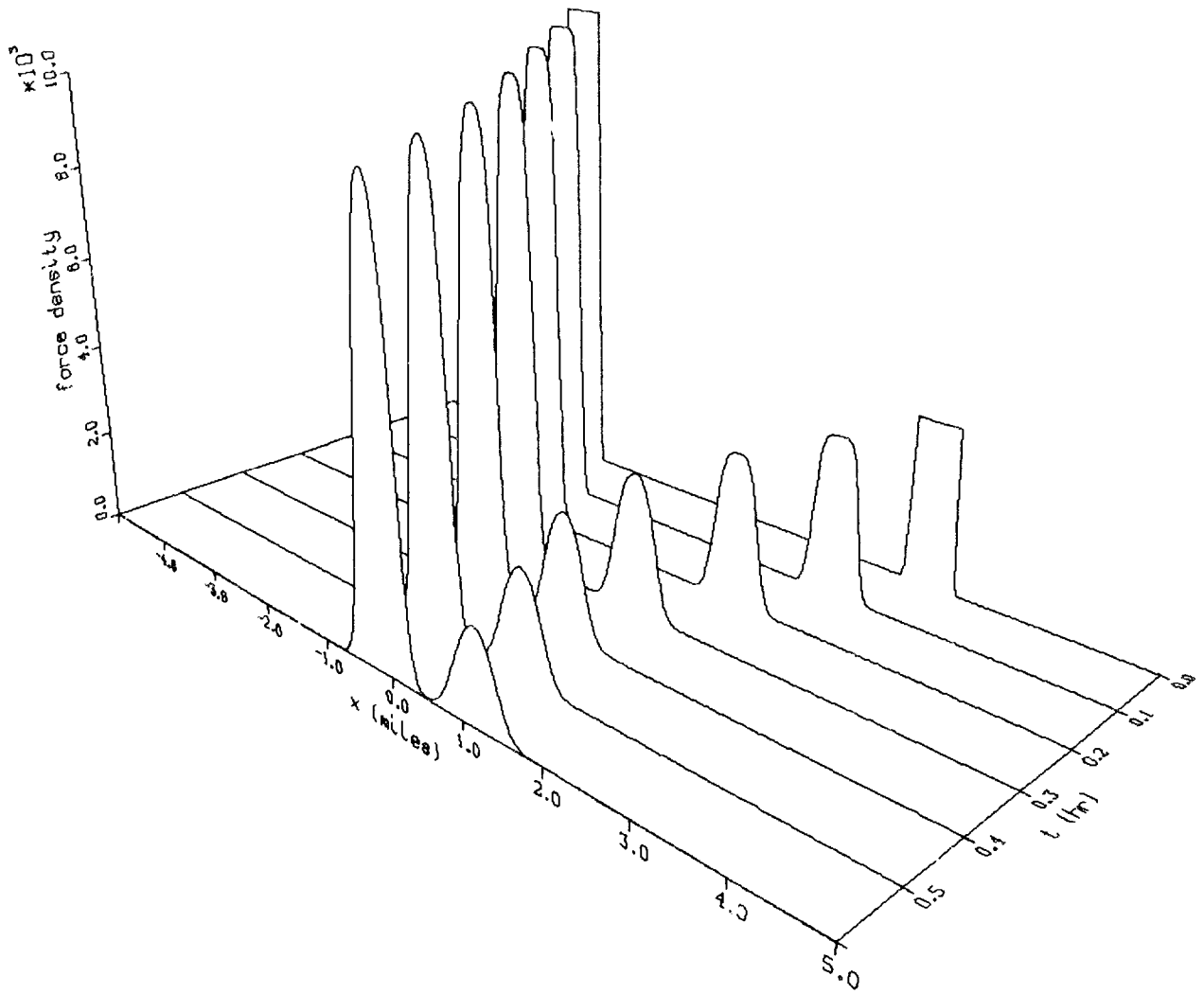


Fig. 7. Spatio-temporal evolution of two competing forces evolving according to Eq. (4.1) under Dirichlet or Neumann B.C. Scenario: pursuit. Ratio of the total initial forces is 3:1.

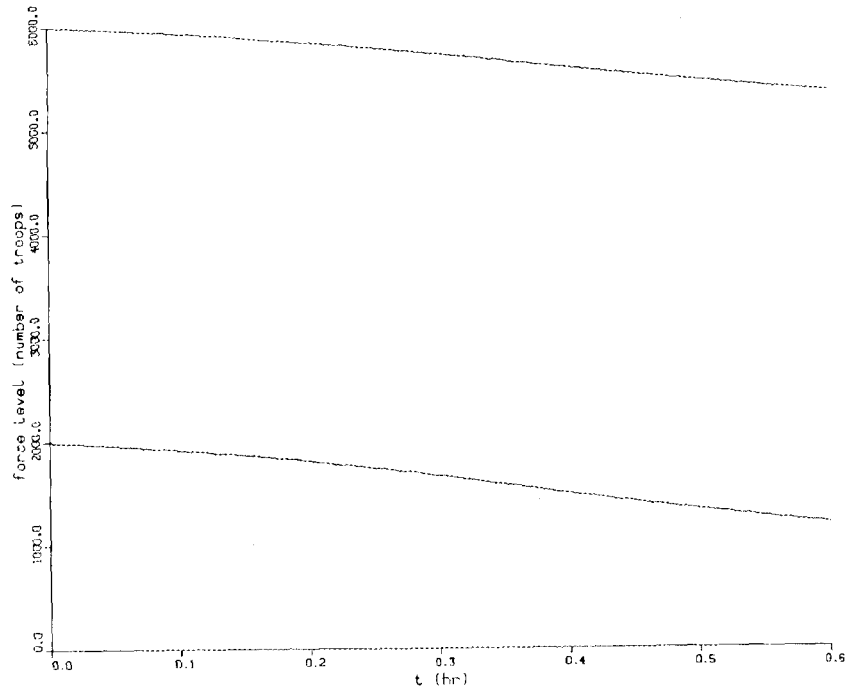


Fig. 8. Time evolution of the total forces as given by the solutions analyzed in Fig. 7.

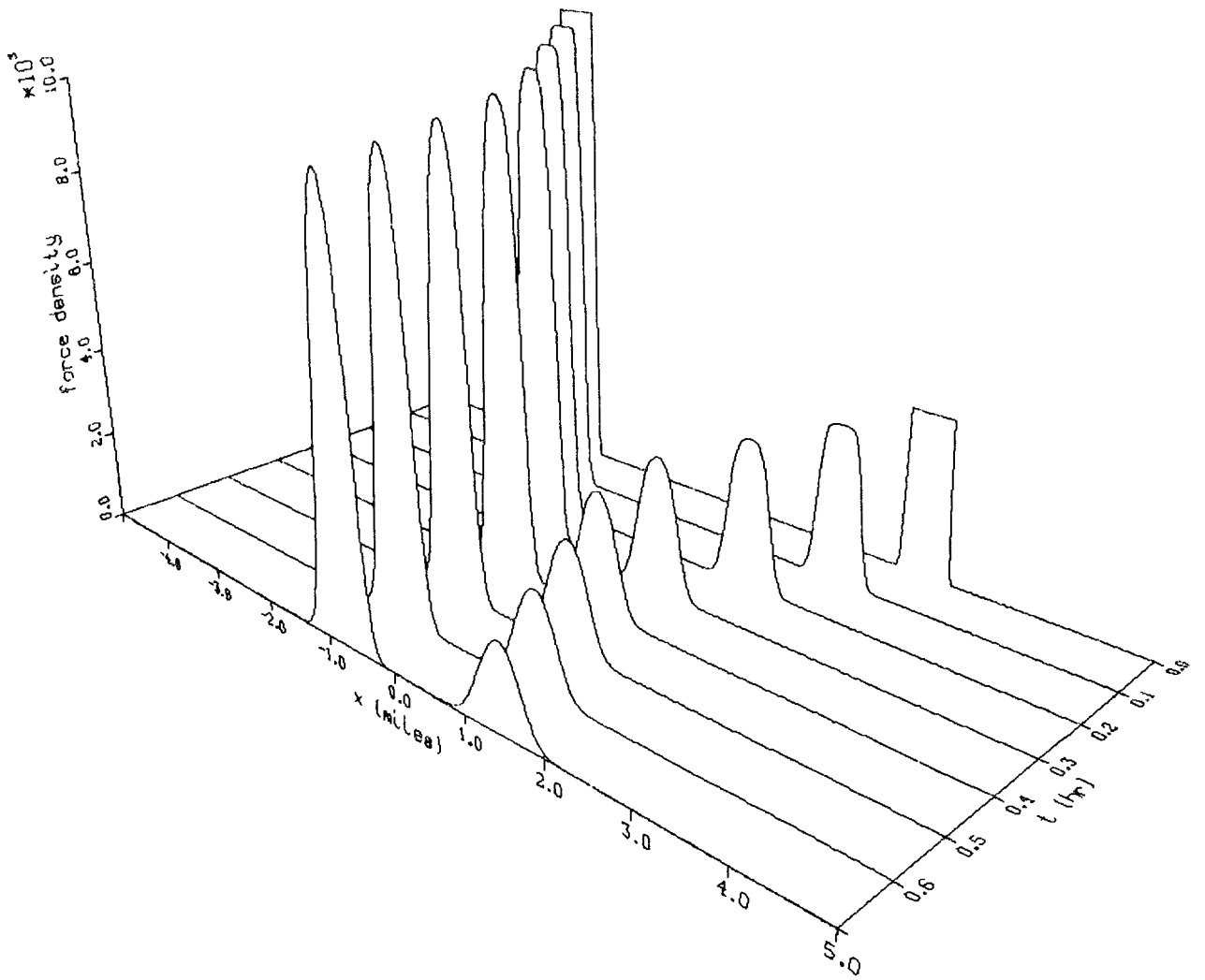


Fig. 9. Spatio-temporal evolution of the computing forces evolving according to Eq. (4.1) under Dirichlet or Neumann B.C. Scenario: no-pursuit. Ratio of the total initial forces is 3:1.

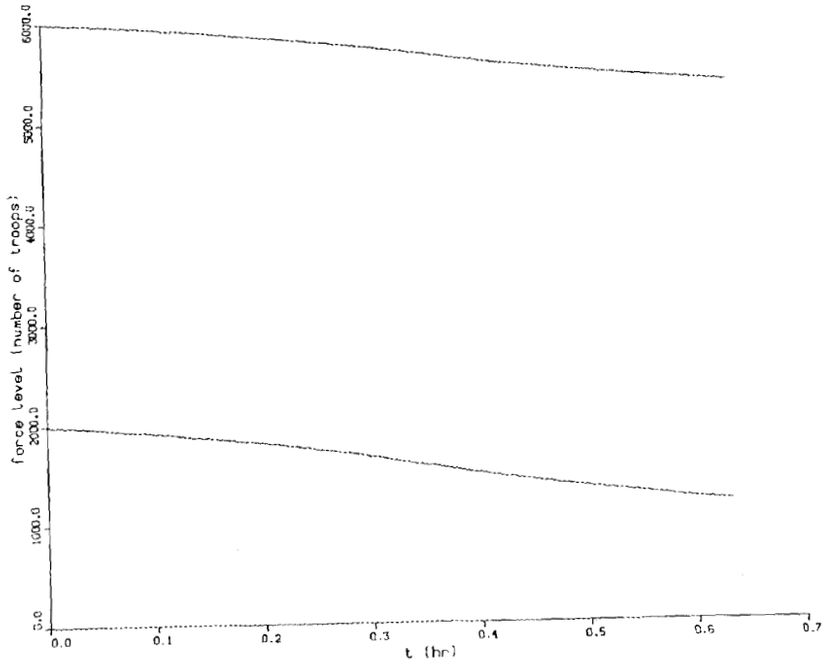


Fig. 10. Time evolution of the total forces as given by the solutions analyzed in Fig. 9.

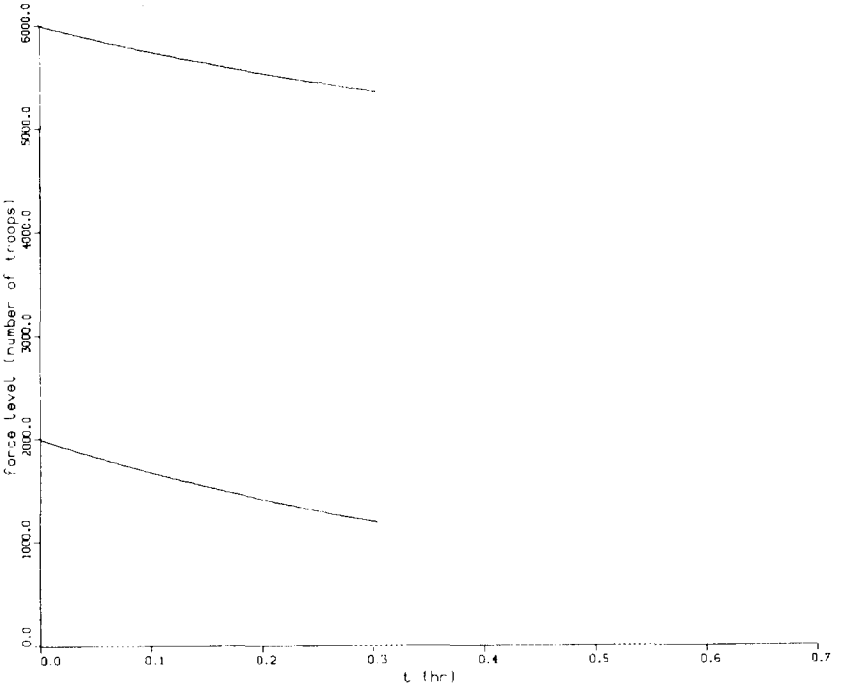


Fig. 11. The equivalent Lanchester solutions corresponding to the cases depicted in Figs. 7 and 9.

APPENDIX A

COMMOD, A Menu-Query PC Program for Rapid Analysis of Engagement Scenarios

A.1. INTRODUCTION

The numerical solutions of the engagement equations yield time and spatial distributions of troop concentrations that are described in fine mesh intervals. Each engagement scenario contains considerable data. The analysis and assessment of these data and their display graphically requires a computer with large storage capability. To circumvent this problem, and to have available a means for easy inspection and analysis of these data, a PC computer program, COMMOD, has been developed for displaying these distributions.

COMMOD is a menu-query driven program that runs on an IBM-PC (or equivalent) computer. The input data are the space-time distributions of the engagement equations that have been post-processed to reduce the number of time and space mesh intervals. COMMOD uses these data to generate summary data and graphic displays of the simulated engagements. The program is written in FORTRAN-77, Version 3.21 using graphics subroutines taken from plotting packages developed by Microcompatible, Inc.

A.2. PC REQUIREMENTS

COMMOD will execute on an IBM-PC, XT, or AT, or compatible computer having the following hardware.

- 256K memory
- 360K diskette or a fixed disc
- IBM CGA Graphics Board
- Color monitor
- Microsoft or PC DOS Version 2.1, or higher.

The code will also execute with a monochrome monitor or with an EGA Graphics Board with minor modifications.

A.3. INPUT DATA

The input data to COMMOD are initially generated using a CRAY X-MP computer. Since large amounts of data are generated for each engagement simulation, post processing subroutines are used to reduce the space-time mesh size for compatibility with the limited storage of the PC system. For example, the solutions of the engagement equations yield

$$NPTS \cdot NPDE \cdot T \tag{1}$$

time and spatial points. In this equation,

> $NPTS$ is the number of spatial mesh points,
> $NPDE$ is the number of partial differential equations used to describe the problem, and
> T is the number of time steps over which the engagement is being analyzed.

These data are reduced to an equivalent distribution containing 401 spatial intervals for each time step T. These data are downloaded to the PC using the Columbia University Program, KERMIT, and permanently stored on 360K diskettes.

A.4. EXECUTION

COMMOD is loaded into a PC where it resides in core until execution is completed or an unrecoverable error occurs. Although some error checking and monitoring with default overrides have been programmed into COMMOD, failures may occur and some errors cannot be easily detected and corrected. In the event of failure of the program, COMMOD must be reloaded.

The program is initiated from DOS using the diskette that contains the executable modules. After the program is loaded into core, the diskette is replaced by the diskette containing the reduced engagement scenario data.

The program searches the default/assigned drive for the INTRODUCTION and SUMMARY data files. These files are identified by a descriptive root name specified by the user and with the extensions .INT or .SUM. If the introductory and summary data files are not present, then an entry of .NUL will cause the program to omit them. Even though these data files are suppressed, a limited amount of introductory and summary are generated by the code internally.

After the space-time data and the introductory and summary data have been loaded, a menu appears on the screen and the user is prompted to respond using function codes. Following execution of the specified function, the menu reappears and the user may continue or terminate execution.

A.5. GRAPHICS

COMMOD is the driver program for a host of subroutines contained in the grafmatic library. Two- and three-dimensional plots are limited to a fixed working space called windows. These are defined by fixed pixel locations and a fixed aspect ratio to minimize distortion. The code was written for a system without extended graphics capability so the screen option was fixed at 640 column by 200 rows black and white equivalent pixels. These dimensions were selected to maximize clarity in the displays and maximizing the use of the display screen.

Two-dimensional (x, y) graphics use 360 columns by 150 rows of pixels and an aspect ratio of 1.5 while the three-dimensional (xyz) displays use 150 columns by 149 rows and an aspect ratio of 0.98. These dimensions allow for sufficient room to include labels and titles on the graphs.

The three-dimensional graphics incorporate hidden lines routines which tend to impede execution and initial display of these data. However, subsequent graphs at different rotations of the axes are displayed in very short times.

A.6. OPTIONS

Menu

CODE	FUNCTION
1	CHANGE FILE NAME (default file name is R1P11)
2	INTRODUCTORY DATA
3	2-D STEP GRAPH WITH OPTIONAL TIME SLICES
4	3-D PLOT
5	SYNOPSIS OF DATA FILE
6	SUMMARY OF SCENARIO
7	TERMINATE RUN WITH OPTIONAL SUMMARY

```
Please enter function code -
```

Example of function code 1

Result of executing function code 1 with correct input.

```
Please enter new INPUT filename -

Please enter new INPUT filename - A:R1P31
```

Result of inputting wrong data set name or data set is not on default drive.

```
Please enter new INPUT filename - R1P32
Following file was not found - R1P32
Please enter new INPUT filename - R1P31
Following file was not found - R1P31
Please enter new INPUT filename - R1S11
Following file was not found - R1S11
```

Result of inputting correct data set name and drive, but summary and introduction files are missing.

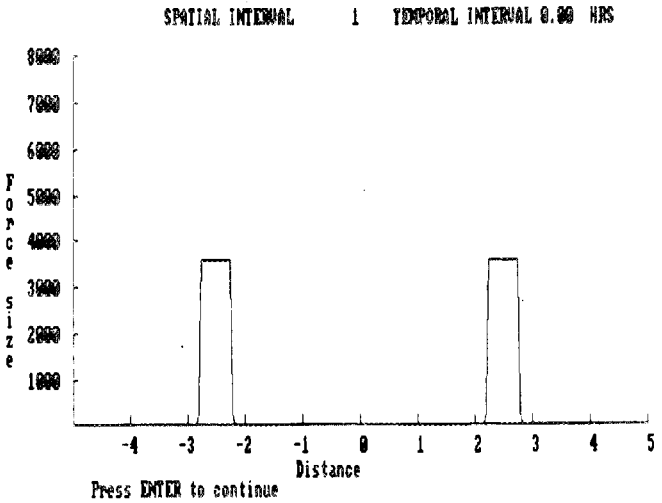
```
 Please enter new INPUT filename - A:R1P31
Following file was not found - A:R1P31.INT
 Please enter new INPUT filename - NUL
Following file was not found - A:R1P31.SUM
 Please enter new INPUT filename - NUL
```

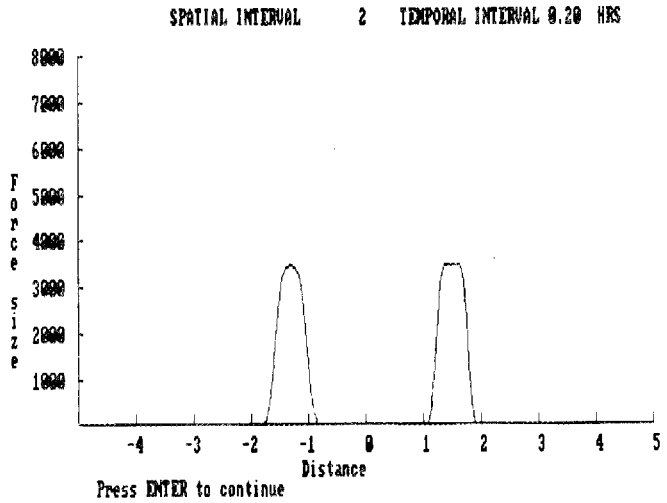
Example of function code 2

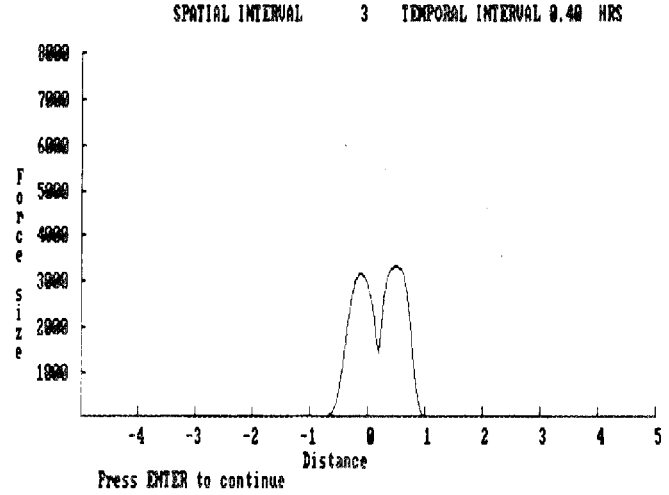
Introductory data generated from data file

```
                    * * * INTRODUCTION * * *

  FILE NAME                          A:R1P31
  BATTLE TYPE                        PURSUIT
  INITIAL TROOP DISTRIBUTION         RECTANGULAR
  FORCES RATIO                         3 TO    1
  MAXIMUM FORCE DENSITY               3600.  B= 26246.
  BATTLE TEMPORAL MESH                  .2000 hrs.
  BATTLE ZONE                        10.0000 mi
  BATTLE TIME                         1.4000 hrs.

Pause
Please press <return> to continue.
```

Example of function code 3







Example of function code 3 continued

```
THAT WAS THE LAST STEP GRAPH

WOULD YOU LIKE TO SEE A TIME SLICE?
Enter 1=yes or <cr>
```

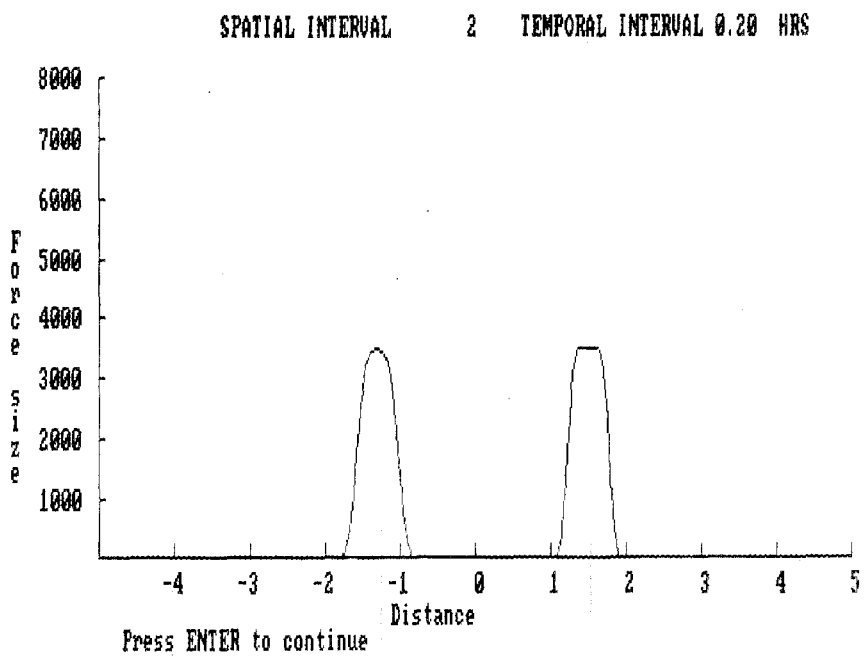
If a carriage return is entered, the program returns to the menu.

```
THAT WAS THE LAST STEP GRAPH

WOULD YOU LIKE TO SEE A TIME SLICE?
Enter 1=yes or <cr> 1
```

If a 1 is entered –

```
Please enter Time Slice number (12) - 2
```

A 2 was entered



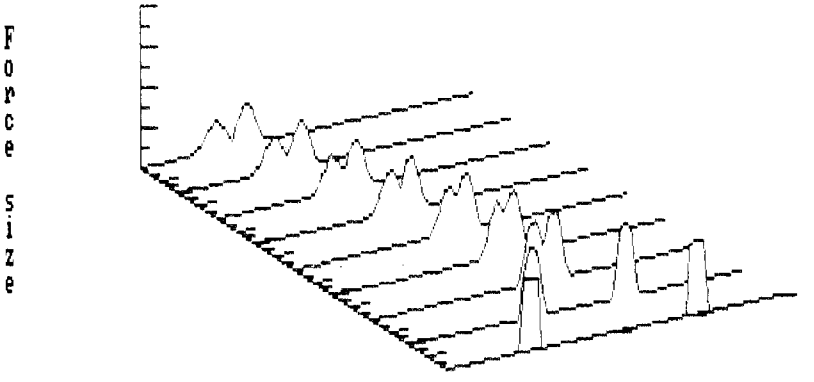

```
WOULD YOU LIKE TO SEE ANOTHER TIME SLICE?
Enter 1=yes or <cr>
```

You may enter a 1 to see another time slice or a carriage return to return to the menu.

Example of function code 4

The following message will appear on your screen to warn you of the delay you will experience while the program is interpolating.

```
COMPUTING - Please wait

Program is performing a PARABOLIC interpolation to reduce
the number of data points to be used by 3-D PLOT. The
reduced number of points is needed to expedite the
HIDDEN LINES routine.

THANK YOU
```



Press ENTER to continue

Example of function code 4 continued

```
IS VIEW OK < 1=yes, cr=no> ?
```

An entry of 1 will return you to the menu.

An entry of a carriage return will display the following message:

```
 Present values are:
PHI - angle (in DEGREES) with respect to X-axis - -30.00
THETA - angle (in DEGREES) with respect to Z-axis - 70.00

 Please enter new values for PHI and THETA
```

```
IS VIEW OK < 1=yes, cr=no> ?
```

Results of entering a carriage return

```
 Present values are:
PHI - angle (in DEGREES) with respect to X-axis - -35.00
THETA - angle (in DEGREES) with respect to Z-axis - 65.00

 Please enter new values for PHI and THETA
35 60
```

New values or the old values must be entered. It is not necessary to enter negative numbers because the program checks the entered values to be sure the graph is plotted in the correct quadrant.

Example of function code 5

Results from executing function code 5

```
FILE NAME KEY:
   2nd Char                3rd Char              4th & 5th Char
(type PDE solution)      (battle type)       (force density ratio)

1  RECTANGULAR       E - 1-FORCE IN DEFENSE  11  RATIO 1 : 1
2  TRIANGULAR        P - 1-FORCE IN PURSUIT  13  RATIO 1 : 3
3  TRAPEZOIDAL       S - OFFENSIVE           16  RATIO 1 : 6
                     T - PASSAGE OF LINES

FILES AVAILABLE:
                     1  R1P11
                     2  R1S11
                     3  R1P31
                     4  R1S31
                     5  R3E16
                     6  R1T11

      Press and key to continue -
```

Example of function code 6

Result of executing function code 6 and the summary file is not available. This data is generated internally.

```
                    * * * SUMMARY * * *

FILE NAME                          A:R1P31
BATTLE TYPE                        PURSUIT
INITIAL FORCE DISTRIBUTION         RECTANGULAR
FORCES RATIO                         3 TO    1
MAX. REMAINING FORCE                3187.
BATTLE TEMPORAL MESH                  .2000 hrs.
BATTLE ZONE                        10.0000 mi
BATTLE TIME                         1.4000 hrs.

Pause
Please press <return> to continue.
```

Return to menu.

Example of function code 7

Results of entering a carriage return.

```
Do you want a summary <enter 1=yes or cr> ?
Stop - Program terminated.

C>
```

Results of entering a 1

```
Do you want a summary <enter 1=yes or cr> ?
```

```
* * * SUMMARY * * *

FILE NAME                          A:R1P31
BATTLE TYPE                        PURSUIT
INITIAL FORCE DISTRIBUTION         RECTANGULAR
FORCES RATIO                         3 TO    1
MAX. REMAINING FORCE                3187.
BATTLE TEMPORAL MESH                  .2000 hrs.
BATTLE ZONE                        10.0000 mi
BATTLE TIME                         1.4000 hrs.

Pause
Please press <return> to continue.
```

Exit program when a carriage return is entered.

REFERENCES

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