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Supertracks: The Building Blocks of Chaos

E. M. Oblow

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ABSTRACT

A theoretical conjecture is made about the nature of chaotic behavior in systems with simple maps. This conjecture gives rise to a computational scheme based on trajectories starting from the maximum in the system map. These trajectories are called *supertracks* and their loci in the parameter phase-space indexed by iteration number are called *supertracks functions.* The chaotic regimes of two nonlinear systems with a single maximum are studied using this scheme. It is found that the behavior in these regimes can be characterized recursively by supertrack functions. At low recursion order these functions analytically describe the gross features of the chaotic regime (i. e. bounding envelopes, stable cycles, star points, etc.). They can be iterated to higher orders to study higher order features. The methodology employed is general enough to be used to study other nonlinear systems. It is also has potential for identifying chaotic behavior experimentally. Universality is the basis for theoretically understanding the results.

1. INTRODUCTION

This paper describes the results of an investigation into the chaotic behavior of some simple nonlinear systems. In particular, the classic quadratic system studied by Feigenbaum[1–3] and others[4–6] will be the main focus of this effort. A description of the behavior of a system with a linear cusp will be included to highlight the role played by universality in these systems having a single extremum. The goal of the work is to use the understanding provided by Feigenbaum about the approach to chaotic behavior to characterize the full chaotic regime. A theoretical conjecture about the nature of chaotic phenomena in these systems is the basis for the developments to follow.

To begin, let us look at the simple quadratic system given by the following recursive equation

$$x_{n+1} = 4\lambda x_n(1 - x_n) = f(\lambda, x_n)\,. \tag{1}$$

A graphical display of the asymptotic behavior of this system as a function of the system parameter λ is given in Figs. 1 and 2.

These figures display the now classic behavior of this system in its approach to chaos (Fig. 1) and in the full chaotic regime (Fig. 2). The boundary separating these two regimes is given by the critical parameter of the system $\lambda_c \approx 0.8925$.

While much is understood about the approach to chaos in this system and its universality[1–6], questions about the behavior in the chaotic regime still remain unanswered. It is quite apparent from Fig. 2 that much regularity exists in this region (i. e. the occurrence of stable and unstable cycles). While a detailed analysis of such features has been offered (see Refs. 4 and 6), no full theory is available to tie all this information together. Is there any unifying principle behind all this apparent chaos? Can such behavior be simply understood and easily predicted?

The rest of this paper is devoted to developing a methodology and a theoretical basis which provides some answers to the questions raised. As will be seen, this approach allows the chaotic regime for this quadratic system to be characterized in terms of a recursive series of polynomials which are only a function of the system parameter λ. The gross features of the chaotic regime, as well as the fine details to arbitrary order, are displayed by these polynomials.

The results presented support the argument that Feigenbaum's universality[1–3] describes the chaotic regime as well as its approach via bifurcations. This theory predicts the approach to chaos with a universal function dependent only on the behavior of the system map in the vicinity of its maximum. The work presented in this paper extends these arguments to show that the chaotic regime can be characterized by the behavior at the maximum itself.

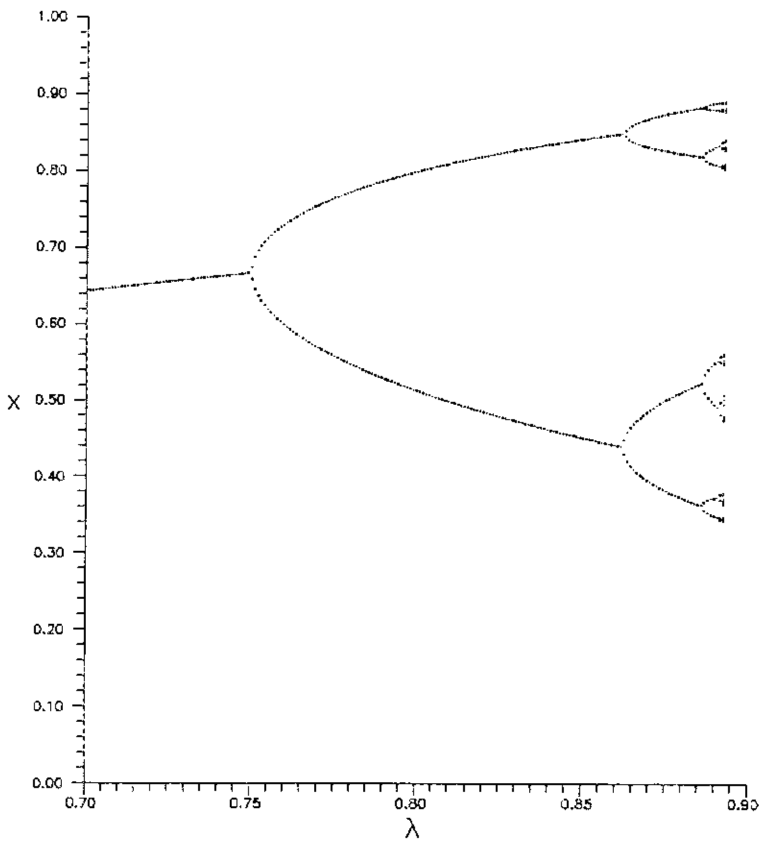

Figure 1. Approach to chaos in a quadratic system.

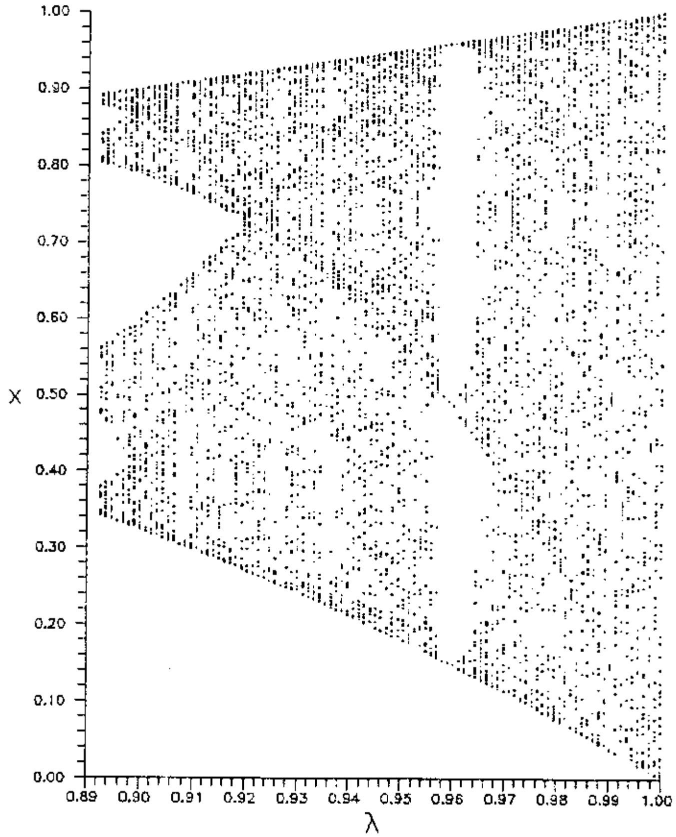

Figure 2. Chaotic regime in a quadratic system.

2. GENERAL APPROACH

In seeking a solution to the quadratic map problem, several key ideas provided motivation for making the particular theoretical conjecture which forms the basis of this work. These ideas are summarized by four important characteristics of the problem.

1. Universality considerations allow the approach to chaos to be characterized by the behavior of the system map f in the immediate vicinity of the quadratic maximum[3].

2. Superstable cycles (i. e. those cycles involving the maximum point of the map) appear in every successive bifurcation regime in the approach to chaos. These *supercycles* represent maximal bounds in these regions. They are also the basis for deriving the universal parameters which describe the system behavior[3,6].

3. In the approach to chaos, the system generates increasingly large numbers of unstable cycles which ultimately drive the iteration process to some form of bounded behavior. Boundaries appear to be a characteristic feature of the chaotic regime.

4. The chaotic regime is composed of infinitely many stable and unstable cycles. The aperiodic behavior which is also possible is unmeasurable and, therefore, need not be considered[4,6].

These four observations lead to the conjecture that chaos can be described as the bounded behavior of a system, derived from the extremal points of the system map. These extrema define chaotic attractors in the system phase-space which govern its chaotic behavior.

Chaos is achieved in this scenario when the system looses all internal stability locally in the parameter phase-space. As each stable state becomes unstable it acts as a repeller instead of an attractor. With no stable states and a collection of repellers driving it, system behavior should be characterized by its boundaries. The extrema in the system map play an essential role in defining such boundaries.

Bounded system behavior, defined by a system extremum that acts as its chaotic attractor, is thus the conjectured basis of chaos. If this conjecture is correct, then the behavior of a system map having only a single maximal point (or one dominant one) should be related only to the characteristics of this maximum. In essence, it determines the outer boundary of system behavior at all iteration orders by functional composition.

For notational simplicity, the iterative trajectory emanating from a system maximal point will be called a *supertrack*. The loci of these tracks in the system parameter phase-space, indexed by iteration number, will be called *supertrack functions*. They will be denoted by $s_n(\lambda)$. This notation allows distinctions to be made between *supercycles* and other system behavior emanating from the maximal point. The necessity for such distinctions will become clear later. The computational methodology for generating these trajectories and their associated functions comprise the rest of this paper. They will be used to test the conjecture made about chaotic behavior.

2.1. COMPUTATIONAL METHODOLOGY

In order to develop the computational methodology needed for this study, one additional factor is needed. This factor is found by looking at the derivatives of the iterates as functions of the initial condition

$$\frac{dx_{n+1}}{dx_0} = \frac{df}{dx_n}\frac{dx_n}{dx_0}\,. \tag{2}$$

For stable limiting behavior, these derivatives should converge to either zero for a system fixed point or cyclic behavior for system cycles[3,6]. The mark of chaotic behavior, on the other hand, has been extreme sensitivity to initial conditions[6].

It is clear, however, that if during iteration, the condition

$$\frac{df}{dx_n} = 4\lambda(1 - 2x_n) = 0 \tag{3}$$

happens to be satisfied, the system will discontinuously loose its functional dependence on initial conditions. This point of discontinuity is the mark of a region of qualitatively different system evolution. Transient behavior surely ends at this singularity, because what follows has lost all relationship to the initial state of the system.

This transition point is conjectured here to be the essence of the difference between chaotic and transient behavior in this problem. Transients have sensitivity to initial conditions, the bounding behavior of chaos does not. This distinction is a point of departure from conventional thinking on chaotic behavior.

The role of the system maximum is evident here. The condition in Eq. (3) is satisfied by the maximum in the system map. In general this maximum point is a function of the system parameter, but here it is simply the constant $x = 1/2$. In the iterative procedure any trajectory that passes through this point characterizes the onset of a supertrack. The evolution of such a supertrack should display the qualitative differences between chaotic and transient behavior. If such a supertrack is cyclic in nature (i. e. it is a supercycle), then cyclic behavior in the chaotic regime should be displayed.

This analysis makes it clear that the behavior of supertracks can be investigated easily by insuring that the condition in Eq.(3) is met at the outset of the iteration procedure. That is, all supertracks can be generated by iterating on Eq.(1) starting with $x = 1/2$ as the seed.

If the ideas being explored are correct, then the nature of chaos can be studied at low iteration orders, not at high order limits. This is another departure from current thinking. This ability to study chaotic behavior at low iteration orders has great benefits. It allows a system to be studied more easily over a range of system parameters and it allows analytic estimates to be made of gross behavior.

The important point here for the quadratic problem, is that the investigation of supertracks eliminates the functional dependence of Eq.(1) on the initial condition. Such trajectories are only a function of the system parameter λ. Every system iterate is also a continuous function of this parameter. Lock-stepping the entire iteration process by starting all iterations at the maximum value gives these supertrack functions their importance in characterizing chaos.

Since the seed $x = 1/2$ is the generator of all supertracks these trajectories are, therefore, characterized solely by the maximal point in the system. It is clear then that Feigenbaum's universality is the basis of this approach. The premise of universality is that the chaotic behavior of the system is fully characterized by the behavior of the system near its maximum. The concept was derived by studying supercycle scaling[3]. The behavior of nonlinear systems starting at this maximum should then be the crux of universality.

3. CHAOS FOR A QUADRATIC MAXIMUM

The conjectures made in the last section can be tested by studying supertrack behavior for the quadratic problem. This will be done by constructing a set of supertrack functions in λ from the seed $x = 1/2$, which is the maximal value of the system function in this case. These continuous supertrack functions $s_n(\lambda)$, are generated recursively from

$$s_{n+1}(\lambda) = 4\lambda s_n(\lambda)[1 - s_n(\lambda)] \qquad s_0 = 1/2\,. \tag{4}$$

They represent the locus of supertracks at a given iteration number n over the range of the system parameter λ.

3.1. BEHAVIOR OF FIRST FOUR POLYNOMIALS

A plot of the first four of these supertrack functions together with the asymptotic chaotic results appear in Figs. 3 and 4. These figures are given one after the other on separate pages to facilitate comparisons between them.

The remarkable properties of these functions even at such low orders is immediately apparent. First, it is clear that $s_1(\lambda)$ and $s_2(\lambda)$ completely specify the outer boundary of the chaotic regime starting at $\lambda_c \approx 0.8925$. They also form the bounds of the bifurcating cycles that characterize the approach to chaos in the region from $\lambda = 0.5$ to $\lambda_c \approx 0.8925$. The major open region and the apparent darker curves in the chaotic regime (see Fig. 4) are seen to be described by $s_3(\lambda)$ and $s_4(\lambda)$.

The major 3-supercycle[4,6] is also predicted by these functions. This stable feature in the chaotic regime is a supertrack which is cyclic in nature. It occurs here at $\lambda = 0.958$. Feigenbaum's analysis showed that all such stable cycles are characterized by equal derivatives of the iterates at their points of intersection[3,6]. This cycle is thus noted by the point at which $s_4(\lambda)$ intersects $s_1(\lambda)$. Since this is a supercycle, it must also include the point $x = 1/2$. This point is seen to occur at the intersection of $s_3(\lambda)$ and the line $x = 1/2$. A 4-supercycle is thus indicated by the intersection of $s_4(\lambda)$ with $x = 1/2$ at $\lambda = 0.990$.

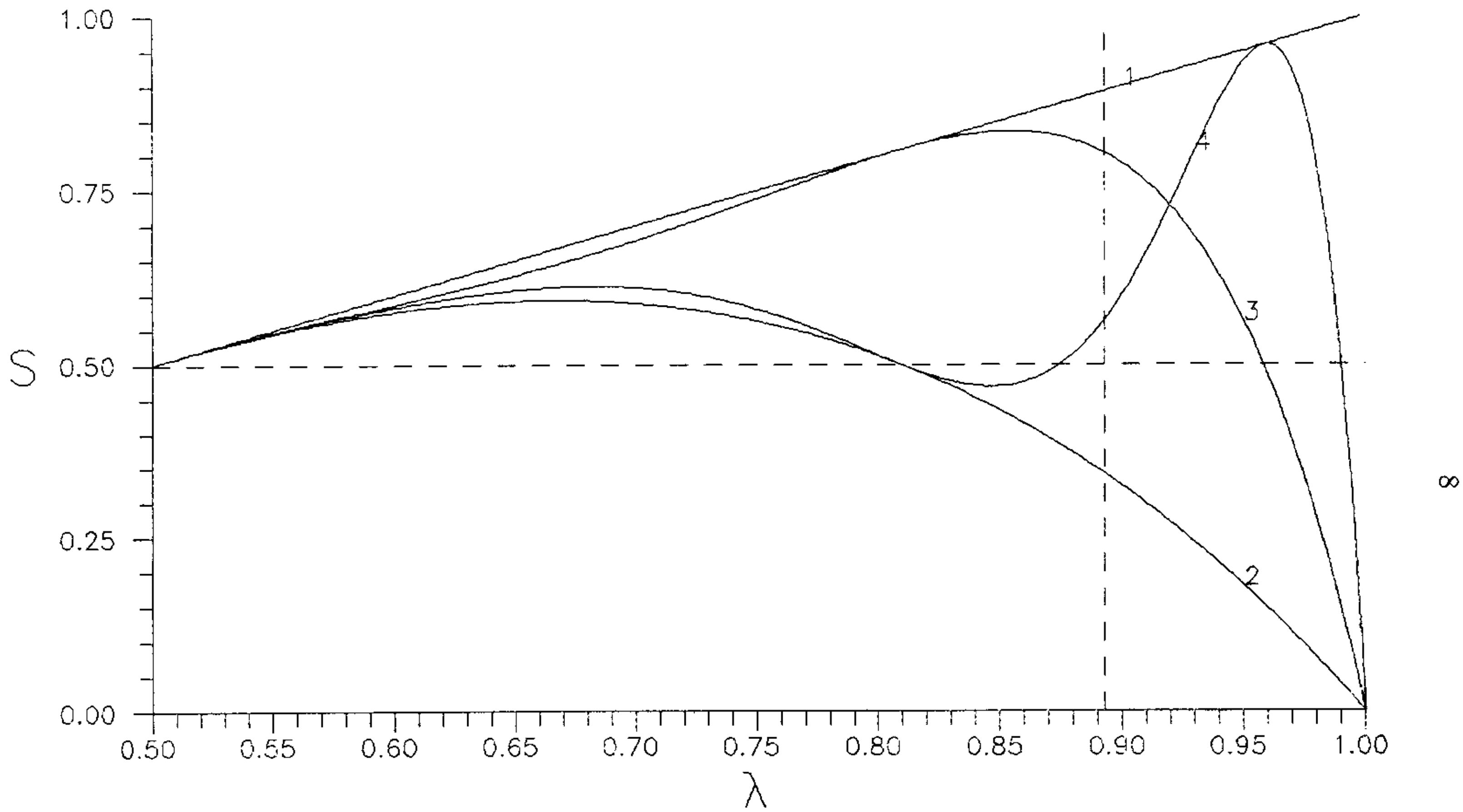

Figure 3. First four supertrack functions for quadratic system.

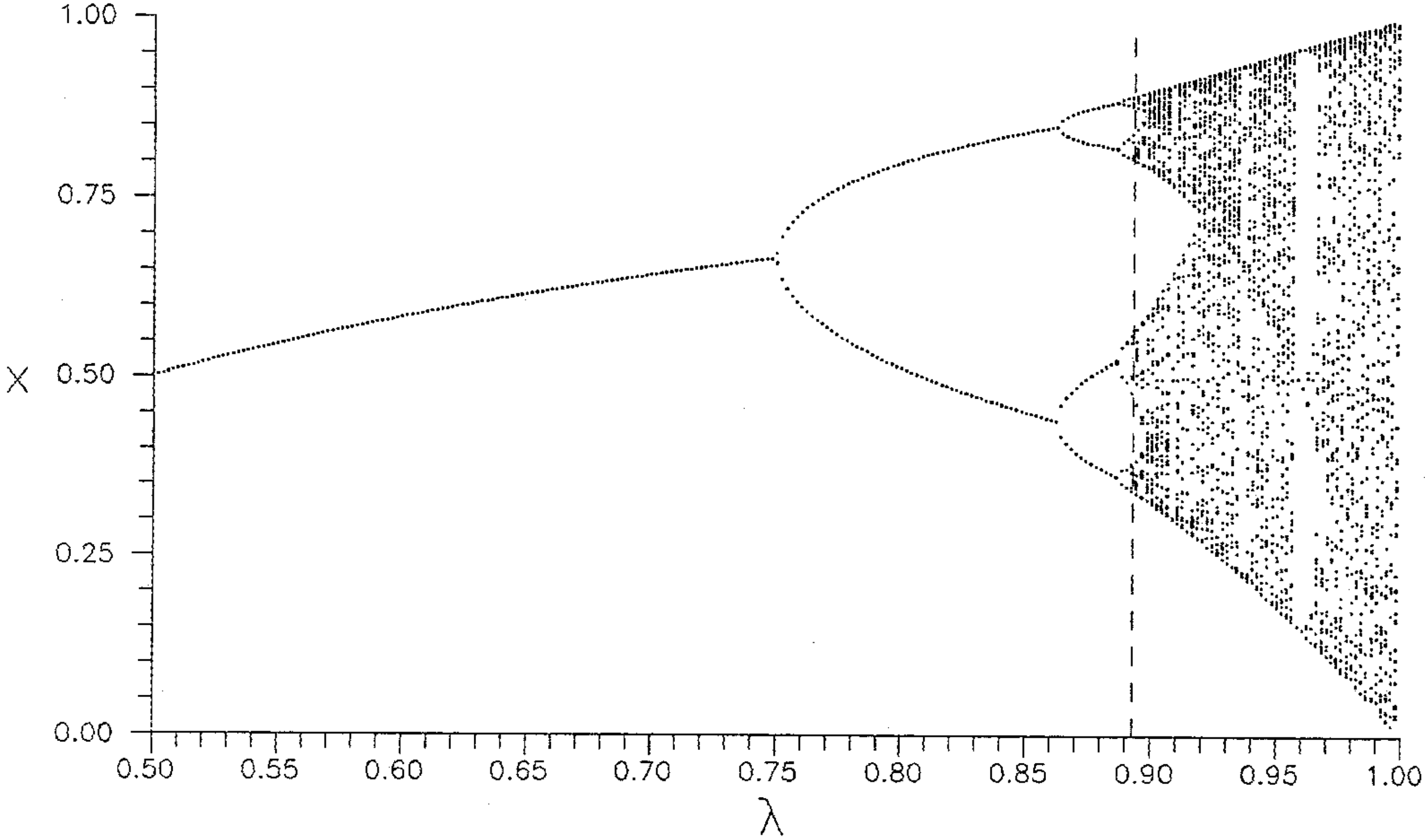

Figure 4. Quadratic system behavior.

Such intersection and equal slope conditions are common to all supercycles. Denoting a point of intersection of general n-supercycle with $x = 1/2$ by λ_n, it is easy to show that they can be derived from the following relationships:

$$s_{n+m}(\lambda_n) = s_m(\lambda_n) \qquad \forall\lambda_n \quad \text{for which} \quad s_n(\lambda_n) = 1/2\,, \tag{5}$$

and

$$\frac{ds_{n+m}}{d\lambda}(\lambda_n) = \frac{ds_m}{d\lambda}(\lambda_n) \qquad \forall\lambda_n \quad \text{for which} \quad s_n(\lambda_n) = 1/2\,. \tag{6}$$

Another major feature resembling a *star* also appears at these low orders. A star is a supertrack which ends abruptly at a single point. This star coincides with the intersection of the $s_3(\lambda)$ and $s_4(\lambda)$ boundaries at $\lambda = 0.920$. The point at which they intersect is characterized by unequal slopes and therefore represents an unstable singularity in the chaotic regime (i. e. a singular supertrack). Since it lies along the curve $f = 1 - 1/4\lambda$, it represents the extended behavior of the fixed-point of Eq.(1) as a function of λ in the chaotic regime. Its appearance as an unstable singularity is thus easily understood. This 3-star corresponds to the point at which odd order cycles begin in the quadratic map problem[4].

It is clear that the behavior of these supertrack functions below $\lambda_c \approx 0.8925$ are transient in nature, except for the points with stable supercycles. This is true for any extended region of stability. Since supercycles play an important role in universality theory, however, supertrack functions are also useful outside of the chaotic regime. In particular, they serve as bounding functions in the approach to chaos. This bounding begins at $\lambda = 0.5$, the point at which the 1-supercycle is located, and is traced by the curves for $s_1(\lambda)$ and $s_2(\lambda)$ up to the chaotic regime at $\lambda_c \approx 0.8925$.

3.2. HIGHER ORDER POLYNOMIALS

Continuing the analysis of these supertrack functions, we display in Fig. 5 the behavior of all of the first eight of these functions. Fig. 6 expands this view in the chaotic regime and Fig. 7 shows the characteristic asymptotic chaotic behavior at this scale also.

It is even clearer here how the bounding structure of chaos is evolving. Several major new supercycles appear at this level of resolution. These are noted by the intersection of each of the supertrack functions with either $x = 1/2$ or the bounding curves $s_1(\lambda)$ and $s_2(\lambda)$. Both the 3-star and the 3-supercycle have taken on more concrete shape. Closer to $\lambda_c \approx 0.8925$, bifurcated versions of these structures have also appeared (i. e. the 6-star at $\lambda = 0.898$ and the 6-supercycle at $\lambda = 0.907$).

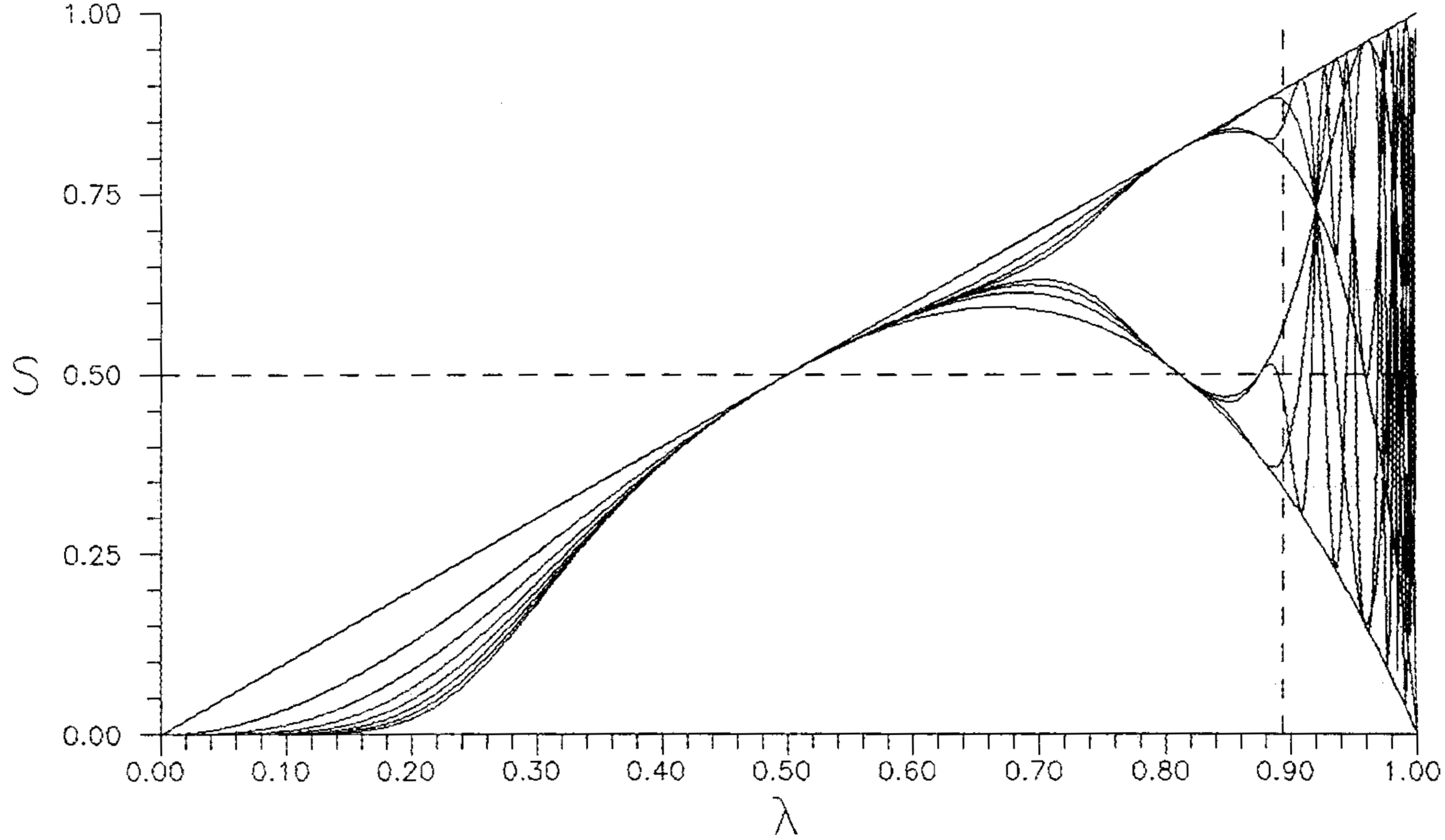

Figure 5. Supertrack functions for full range of the quadratic system.

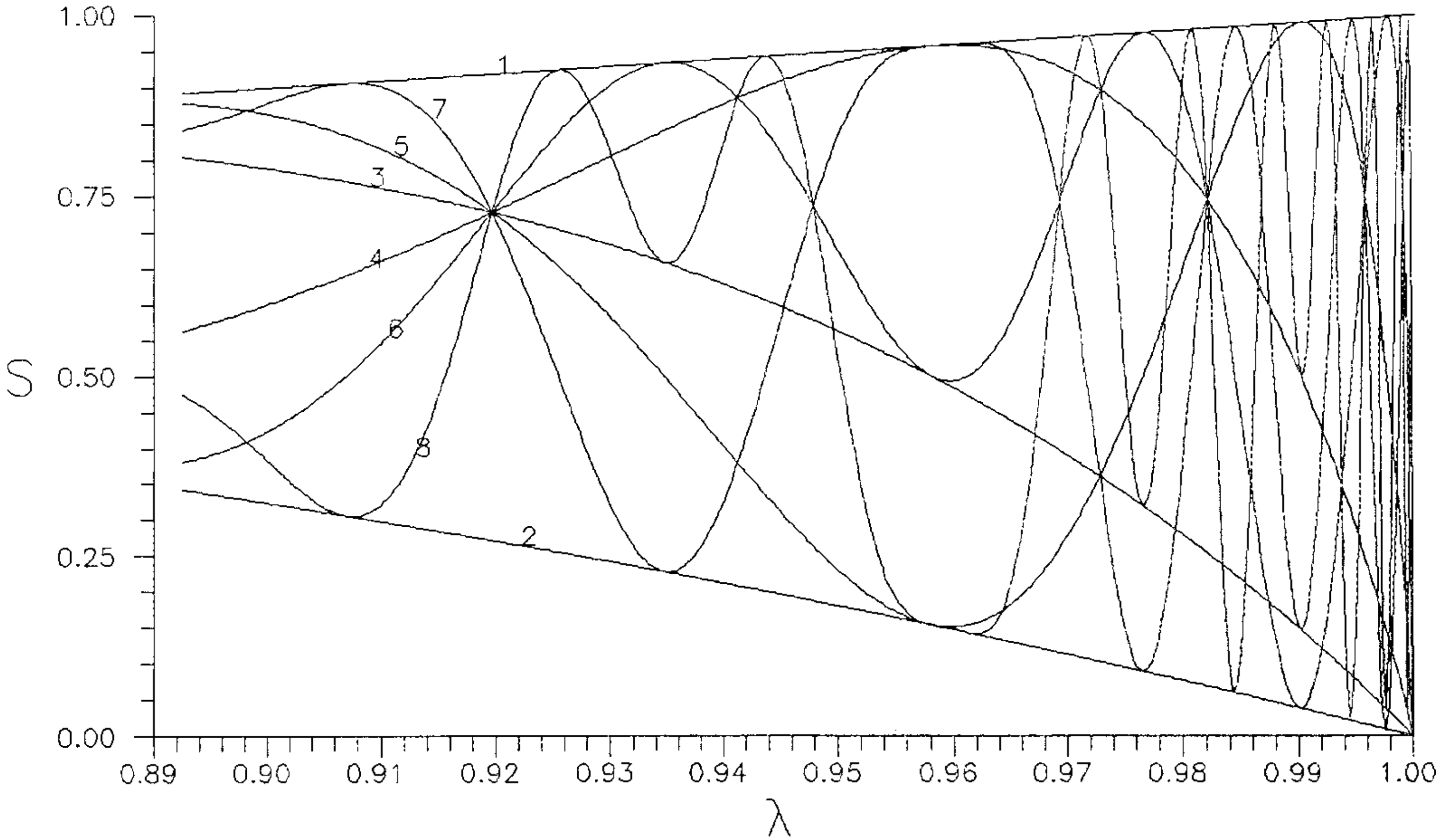

Figure 6. Supertrack functions for chaotic regime of quadratic system.

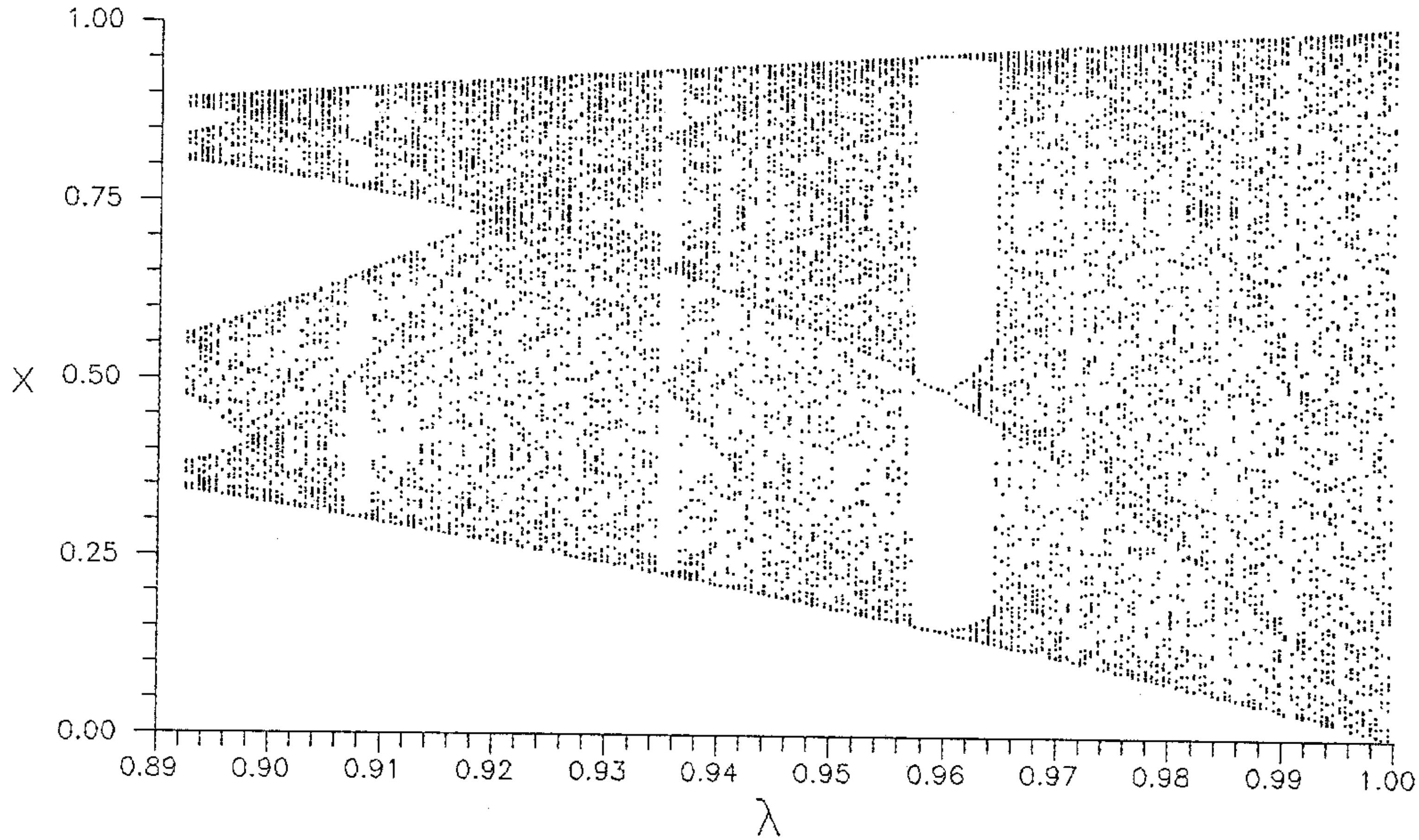

Figure 7. Chaotic regime in a quadratic system.

Since others have studied this region extensively[4-6] (and the results speak for themselves), we will not belabor the analysis any further. Several observations can be immediately made, however.

1. The supercycles are becoming combinatorially more dense and they appear interspersed over the entire chaotic regime. High and low order supercycles appear closer and closer to each other. In this pattern stars and supercycles systematically alternate with each other. This is also true about the approach to chaos, except that the stars here can be loosely interpreted as stable regions. Some of the major structures predicted to exist in the chaotic region have already appeared at these low orders.

2. The entire chaotic regime displays a reverse cascade of bifurcations away from $\lambda_c \approx 0.8925$ similar to that observed in the approach to chaos. Some simple numerical calculation even at this point confirm that the universality predicted in this reverse process is indeed present[4-6]. The universal behavior is contained in the roots of the supertrack functions with respect to $x = 1/2$. The succession of these λ_n for different cycles can easily be used to derive the Feigenbaum constant δ and the universal scaling number α. These numbers thus describes both the approach to chaos through supercycles and the bifurcating cascade of the major supercycles of all orders away from $\lambda_c \approx 0.8925$ in the chaotic region.

3. By starting at the system maximal point, the apparent disorder of the chaotic regime and its sensitivity to initial conditions are both eliminated. This maximal seed orders the entire chaotic regime in such a manner that it can be described by polynomials. Such ordering is the result of treating the maximal point as a chaotic attractor. This attractor, once reached, determines all future system behavior. The stable range of the attractor in this problem is seen to lie between $\lambda_c \approx 0.8925$ and $\lambda = 1.0$. Since all supertracks are bounded by $s_1(\lambda)$ and $s_2(\lambda)$ for this case, these curves describe the range of influence of the chaotic attractor at any value of the system parameter.

4. The only infinite supertracks in this picture appear at $\lambda_c \approx 0.8925$ and the accumulation points of the other n-supercycle. At these points an infinite number of these supertrack functions exist without crossing each other. These infinite sequences are important because they are directly related to Feigenbaum's universal function[6]. The universal scaling parameter α can calculated from these sequences by setting the system parameter in Eq.(4) equal to the critical point value and iterating. The universal scaling law for any sequence can be calculated to any desired accuracy from

$$\lim_{n\to\infty} \frac{s_n(\lambda_c) - s_0}{s_{2n}(\lambda_c) - s_0} = \alpha\,. \tag{7}$$

Extrapolating these observations to increasingly higher order behavior we can conclude that the chaotic regime in this problem is characterized by systematically alternating stable supercycles and unstable stars. These latter features follow paths corresponding to the continuation of the functional dependence of stable cycles on λ found in the approach to chaos. The orders of both of these structures are bounded by the order of the highest supertrack function being considered.

The two universal constants δ and α, and the map maximum $x = 1/2$, thus characterize the chaotic attractor of the quadratic problem. The approach to the $x = 1/2$ attractor and the chaotic behavior generated by it, both clearly obey universality relations[4-6]. This is as it should be, since they are both are governed by the maximal point of the map.

4. CHAOS FOR A LINEAR CUSP MAXIMUM

To complete this discussion, the supertrack approach will be used to briefly describe the behavior of another simple system - linear cusp or triangle maximum[4,5]. The equation for the iterative behavior of this nonlinear system is given by

$$x_{n+1} = f(\lambda, x_n) = \lambda\bigl(1 - |1 - 2x_n|\bigr)\,. \tag{8}$$

Being composed of piecewise linear sections, this functional form increases only linearly in order of complexity at each iteration. Many of the salient features of the problem are thus amenable to analytic verification[4,5]. In particular, $\lambda = 1/2$ represents a singularity in the system above which there are no stable fixed points.

This problem was brought up here because the cusp system has one new feature that makes it an important test of the general concept of universality and the supertrack hypothesis. This feature is the singular nature of the peak in the map. If universality holds, then the change from a maximum with zero slope to one with a discontinuity and a slope that precludes stability[4], should be directly reflected in the chaotic regime for this map. This change is indeed borne out by the results which follow.

4.1. RESULTS

The asymptotic chaotic behavior of the iterates of the cusp map in the interesting region from $\lambda = [1/2, 1]$ is shown in Fig. 8. Using a seed of $x = 1/2$, the supertrack functions for this system can be generated from Eq.(8). The first eight of these are shown for the same region from $\lambda = [1/2, 1]$ in Figs. 9.

It is apparent that there are many similarities between these results and the ones derived for the quadratic case (see Figs. 6 and 7). Topological similarities were expected[5,6]. Their computational reality here can be discussed at length, but this will not be done. The important point to be noted is the changed nature of the chaotic regime.

As is evident from the results in Fig. 8, the entire region $\lambda = [1/2, 1]$ is chaotic. The supertrack analysis results in Fig. 9 bear this out. Since all the supertrack functions attain maxima only as cusps with unstable slope conditions, all the supertracks are unstable and singular. The entire chaotic regime is made up of such singular supertracks.

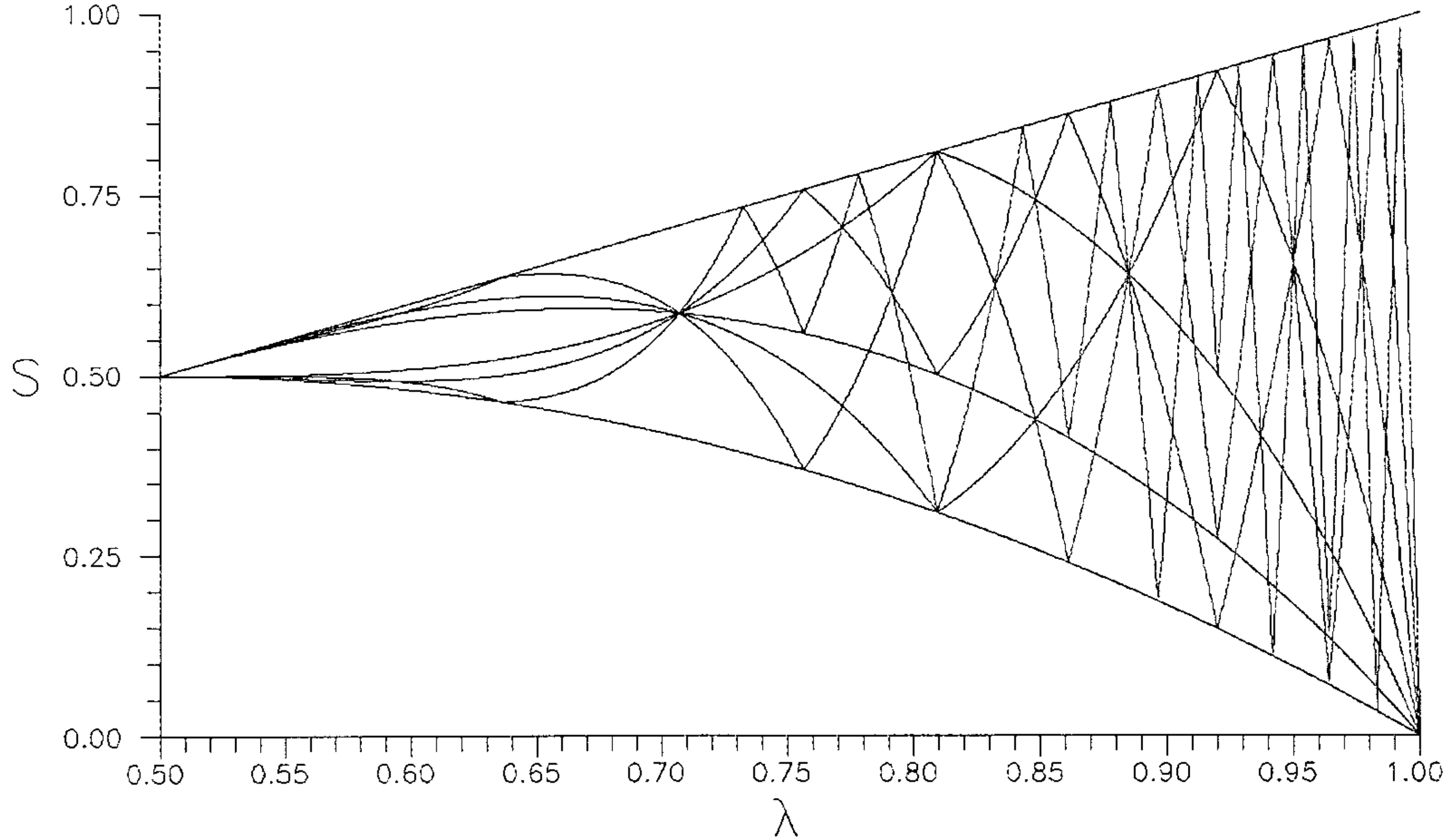

Figure 8. First eight supertrack functions for cusp system in chaotic regime.

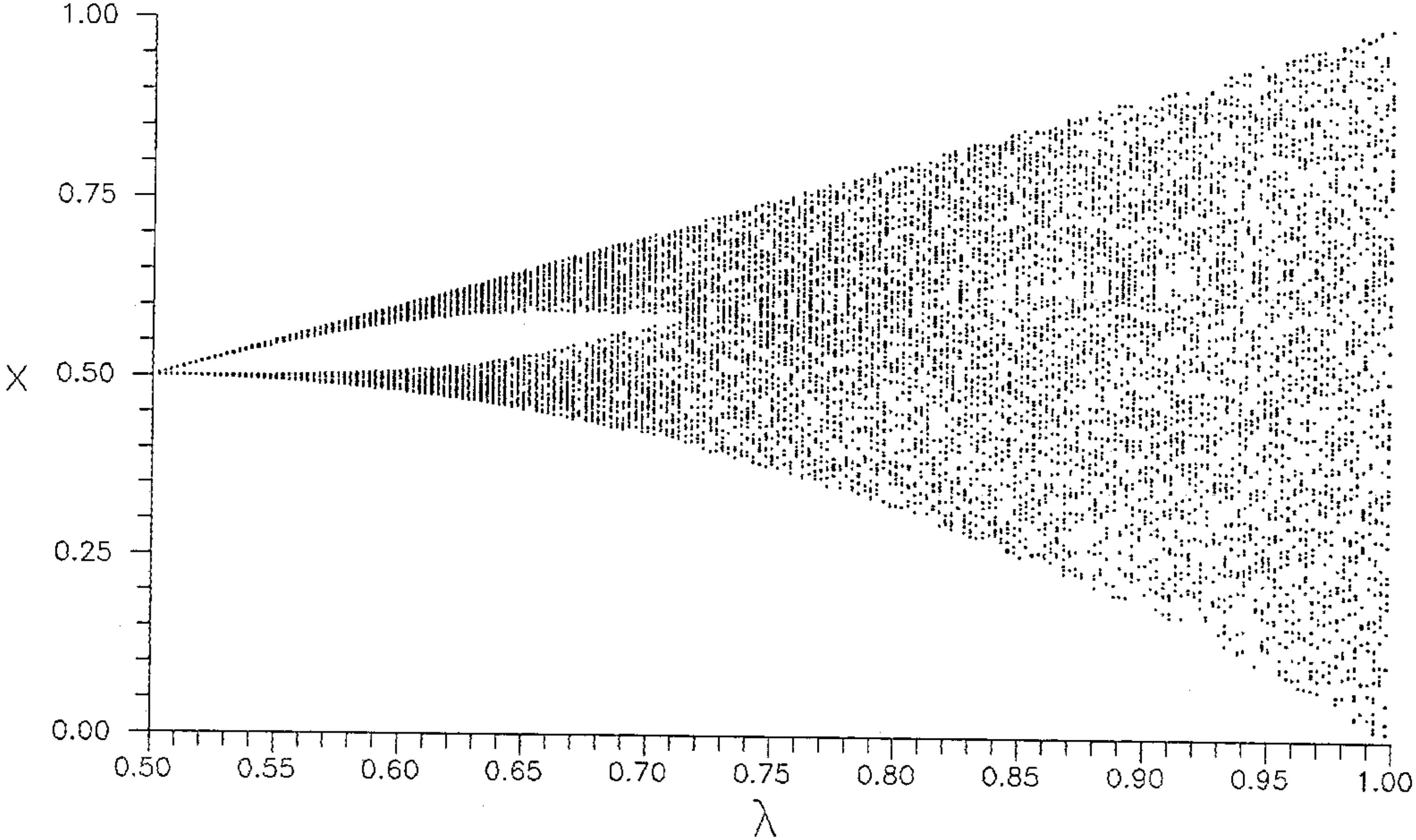

Figure 9. Chaotic regime of a system with a cusp maximum.

In particular, there is neither stability nor any approach to stability anywhere in the chaotic regime. The point $\lambda_c = 1/2$ is the critical point for this problem and it too is singular, in the sense that it is approached discontinuously. The system behavior degenerates from a single stable fixed point solution of $x = 0$ for $\lambda < 1/2$ into immediate chaos at $\lambda_c = 1/2$. Only the bounding nature of the supertrack functions are useful in analytically describing the features of this regime. The only open window is fully described by $s_3(\lambda)$ and $s_4(\lambda)$, as before.

These results clearly support the concept of universal behavior. The change from a quadratic to a cusp maximum resulted in the elimination of all stable system behavior in the chaotic regime. The singular nature of the cusp is reflected in the singular nature of the chaos produced.

5. CONCLUSIONS

The conclusion that can be drawn from this work is that chaotic behavior can be studied as a bounding phenomenum. It can also be characterized by a chaotic attractor which is an extremum in the system map. The evolution of system behavior starting at this point characterizes the chaotic regime as a function of the system parameter. This characterization can be studied at low iteration order by following the system behavior as it evolves directly from the attractor point.

Although a lot more work needs to be done to theoretically understand the behavior of the recursive supertrack functions developed, it is clear that they reveal important aspects of nonlinear system behavior in the chaotic and non-chaotic regimes. These functions faithfully represent the approach to a chaotic attractor in their transient behavior at high orders. They characterize both the stable and unstable trajectories present in the chaotic regime at all orders. In addition, they can be generated easily and at low orders they describe the gross features of chaos. Their relationship to other methods of characterizing chaotic behavior is under investigation.

The theoretical conjecture used to derive the supertrack functions also provides a means for extending this computational methodology to the study of more complex problems. It appears to be general enough to apply to more complex one dimensional maps and to multidimensional nonlinear maps as well. The methodology might be used to identify the appearance of chaotic behavior experimentally by studying the bounded behavior of systems in a fully chaotic regime.

At the heart of all of these matters is Feigenbaum's universality and the theory that chaotic phenomena are tied closely to the behavior of nonlinear systems in the vicinity of a maximal point. The success of universality in a range of other problems suggests that supertracks might be useful there also.

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