MARTIN MARIETTA ENERGY SYSTEMS LIBRARIES

3 4456 0266331 1

ORNL/TM-10493
(CESAR-87/24)

ornl

OAK RIDGE
NATIONAL
LABORATORY

MARTIN MARIETTA

Foundations of O-Theory II: Measurements and Relation to Fuzzy Set Theory

E. M. Oblow

OAK RIDGE NATIONAL LABORATORY
CENTRAL RESEARCH LIBRARY
CIRCULATION SECTION
4500N ROOM 175
LIBRARY LOAN COPY
DO NOT TRANSFER TO ANOTHER PERSON
If you wish someone else to see this report, send in name with report and the library will arrange a loan.

CESAR

OPERATED BY
MARTIN MARIETTA ENERGY SYSTEMS, INC.
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
NTIS price codes—Printed Copy: A03; Microfiche A01

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

ORNL/TM-10493
CESAR-87/24

Engineering Physics and Mathematics Division

FOUNDATIONS OF O-THEORY II: MEASUREMENTS AND RELATION TO FUZZY SET THEORY

E. M. Oblow

Date Published — July 8, 1987

Prepared by
Oak Ridge National Laboratory
Oak Ridge, Tennessee 37831
operated by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
for the
U.S. DEPARTMENT OF ENERGY
under Contract No. DE-AC05-84OR21400

MARTIN MARIETTA ENERGY SYSTEMS LIBRARIES
3 4456 0266331 1

CONTENTS

ACKNOWLEDGMENTS v
ABSTRACT vii
1. INTRODUCTION 1
2. QUANTIFICATION FROM MEASUREMENTS 3
   2.1. REPRESENTATION OF MEASURED DATA 3
   2.2. EVALUATION OF MASSES AND MEMBERSHIP FUNCTIONS . 7
3. COMPARISON OF OPERATORS 9
4. SAMPLE PROBLEMS 13
   4.1. COMBINATORIAL ANALYSIS 13
   4.2. CONDITIONAL PROBABILITIES 18
   4.3. CORRELATED COMBINATIONS 21
5. VERIFICATION 25
6. CONCLUSIONS 29
REFERENCES 31

ACKNOWLEDGMENTS

The author would like to give thanks to F. G. Perey, F. G. Pin and B. A. Worley for many helpful discussions and comments. The continued support and encouragement of C. R. Weisbin, F. C. Maienschein, A. Zucker and our DOE sponsor O. P. Manley is also greatly appreciated. Thanks to TEX and special thanks again to my wife Ellen for providing the inspiration for this work.

ABSTRACT

A conceptual experimental framework for O-theory and the possibility form of fuzzy set theory is presented. The framework is probabilistic and derived from experimental measurements of evidential support. It is used to quantify the probabilistic masses needed in O-theory and the membership functions used in fuzzy set theory. Instances in which these two different representations of the same experimental data are equivalent are then explored. The relationships between general operators in both theories are then discussed, along with their connection to Dempster-Shafer theory. Finally, several illustrative examples of deductive inferencing under uncertainty are solved. The equivalence of the representations of the results in both uncertainty theories is demonstrated.

1. INTRODUCTION

In several previous papers[1,2,3,4], O-theory (OT), an operator-uncertainty theory was developed to bridge the gap between probability theory[5,6] (PT) and Dempster-Shafer theory[7,8] (DST) on one hand, and fuzzy set theory[9,10] (FST) on the other. In this paper, a general probabilistic measurement framework is proposed for evaluating the masses in OT. This measurement framework will also be seen to form the basis for evaluating the membership functions used in the possibility representation of FST. Use of such a measurement framework in logical inferencing applications will be the focal point of this work.

The important point being stressed in this discussion of OT foundations is the need for experimental measurement and verification procedures. These needs can not be overemphasized. Measurements form the basis for all the physical sciences and their applications. To successfully apply either OT or FST, measurements must provide the data for quantifying abstract concepts such as mass and membership. Without them, both theories remain abstract and quantitative results lose much of their meaning.

To meet these needs, we propose to tie both OT and FST to a single probabilistic-measurement basis. This will result in two alternate representations of the experimental data – OT and FST. These two representations and their associated algebras can then be used in a unified approach to applied work. The common probabilistic foundation allows more freedom to decide which theory to apply in any given circumstance. A consistent experimental basis will also be especially useful in providing a means for verifying all inferencing results. The difficulties encountered in quantifying inferencing under uncertainty make experimental verification equally essential.

In a unified probabilistic context, both theories will simply represent different views of the same measured data. The wealth of operators that exist in FST, and can be developed in OT, will be seen to form similarity classes designed to do inferencing with the same basic data. The results of such inferencing using either theory should, therefore, in principle be consistent. This consistency can be maintained in any given application by use of appropriate operator definitions in each theory. Moreover, the fact that the measurement process is probabilistic in form will lead to an equivalent interpretation of the results in classical probability theory (albeit extended in the manner originally suggested by Dempster in certain cases).

The use of probability measures as the basis for this unified approach also offers the means for experimental verification, at least in principle, if not in practice. Much of what is 'uncertain' and quantifiable, can be dealt with in probabilistic terms. A vast body of literature is available for determining probabilities in an

experimentally verifiable manner. Some of these approaches are readily available for deductive inferencing applications. While other measurement procedures are probably possible (no pun intended), it is clearly desireable to have one common framework that links together all the existing approaches (FST, DST, OT and PT). A probability basis will be seen to meet these requirements quite effectively.

2. QUANTIFICATION FROM MEASUREMENTS

2.1. REPRESENTATION OF MEASURED DATA

For the sake of clarity and brevity, the material to be presented in this section will be more visual than mathematical. This is being done so that the important aspects of representing measured data are not lost in a morass of mathematical symbols. Several examples applying this general approach (given at the end of the paper) will hopefully make up for some of the loss of rigor. As in previous papers, the problems dealt with will be characterized by finite, discrete spaces.

To begin, assume we are dealing with a possibility set $\Theta = \{x_1, x_2, \ldots, x_n\}$, representing the exhaustive and mutually exclusive possible outcomes $x_1, x_2, \ldots, x_n$ of an inferencing procedure. It is also assumed that operators are available in both OT and FST to handle the inferencing procedure in any given application. The quantified uncertainties in inferencing are represented in FST by a membership function $\mu(x_i)$, $x_i \in \Theta$ defined over Θ and in OT by a mass function $m(\mathbf{x})$, $\mathbf{x} \subset \Theta$ defined over all subsets of this set (i.e., the power set of Θ).

The basic OT algebra[1,2,3,4], consists of a union $\bigcirc\!\!\!\!\vee$, an intersection $\bigcirc\!\!\!\!\wedge$ and a complement $\tilde{}$ operator, each of which is defined in terms of mass distributions. The masses in these distributions are assigned to subsets of Θ, as opposed to elemental members, and the sum of the masses over all subsets is normalized to unity. For arbitrary mass distributions $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, with masses $m_A(\mathbf{x}_i)$, $m_B(\mathbf{x}_j)$ and $m_C(\mathbf{x}_k)$, $\forall \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k \subset \Theta$ respectively, the union operation $\mathbf{C} = \mathbf{A} \bigcirc\!\!\!\!\vee \mathbf{B}$ is defined by

$$m_C(\mathbf{x}_k) = \sum_{\substack{i,j \\ \mathbf{x}_i \cup \mathbf{x}_j = \mathbf{x}_k}} m(\mathbf{x}_i) m(\mathbf{x}_j) \qquad \forall \mathbf{x}_k \subset \Theta\,, \tag{1}$$

the intersection operation $\mathbf{C} = \mathbf{A} \bigcirc\!\!\!\!\wedge \mathbf{B}$ by

$$m_C(\mathbf{x}_k) = \sum_{\substack{i,j \\ \mathbf{x}_i \cap \mathbf{x}_j = \mathbf{x}_k}} m(\mathbf{x}_i) m(\mathbf{x}_j) \qquad \forall \mathbf{x}_k \subset \Theta\,, \tag{2}$$

and the complement is

$$m_{\tilde{A}}(\mathbf{x}) = m_A(\tilde{\mathbf{x}}) \qquad \forall \mathbf{x}, \tilde{\mathbf{x}} \subset \Theta\,. \tag{3}$$

For fuzzy sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, with membership functions specified as $\mu_A(x_i)$, $\mu_B(x_i)$ and $\mu_C(x_i)$, $\forall x_i \in \Theta$ respectively, the corresponding FST algebra is as follows[9]: the union operation $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is given by,

$$\mu_C(x_i) = \mathrm{MAX}(\mu_A(x_i), \mu_B(x_i)) \qquad \forall x_i \in \Theta\,, \tag{4}$$

the intersection operation $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ by

$$\mu_C(x_i) = \mathrm{MIN}(\mu_A(x_i), \mu_B(x_i)) \qquad \forall x_i \in \Theta\,, \tag{5}$$

and the complement is

$$\tilde{\mu}(x_i) = 1 - \mu(x_i) \qquad \forall x_i \in \Theta\,. \tag{6}$$

To start any inferencing procedure, a measurement process must be defined to assign numerical values to the respective masses and membership functions. The proposal here is to use a probabilistic basis for these assignments so that FST, OT and PT can all be unified at their foundation.

Specifically, the general proposal is to use Dempster's original conception of a multivalued mapping[7] applied to probabilistically measurable sets to represent all measured data. In this context, the measurement process is conceptually any unique, verifiable method of assigning a probability measure to experimental events. These events, when used as evidence in an inferencing model, then define support functions for definite subsets of the possibility set under investigation.

While this statement is somewhat vague, it is meant to imply that a wealth of commonly employed statistical and probabilistic analysis techniques[8,11] can be used to satisfy the criteria. More sophisticated group-theoretical methods[12] (which will be employed later in the examples) are also meant to be acceptable. One should not belabor the relevance of the methodology, so long as it is scientifically sound and experimentally verifiable.

The proposed use of a multivalued mapping, as suggested by Dempster, allows OT, DST, FST and PT to be connected in a consistent manner. Although such a mapping extends the concept of a probability distribution in a fundamental manner, it is simply just a convenient way of manipulating families of probability distributions. The operations on such families are closely related to those in random set theory[13].

The advantage offered by this added level of abstraction is its ability to handle the wider range of uncertainties found in the artificial intelligence field. This extended range is characterized by the use of both probabilistic measures and the specification of subsets of the possibility set (as opposed to an individual elements) for support of experimental evidence.

Instead of mathematically restating the definition of Dempster's mapping, a simple diagram will be used here to illustrate the measurement and quantification

procedures. Fig. 1 is, therefore, meant to define graphically the results of an experimentally measurable multivalued mapping.

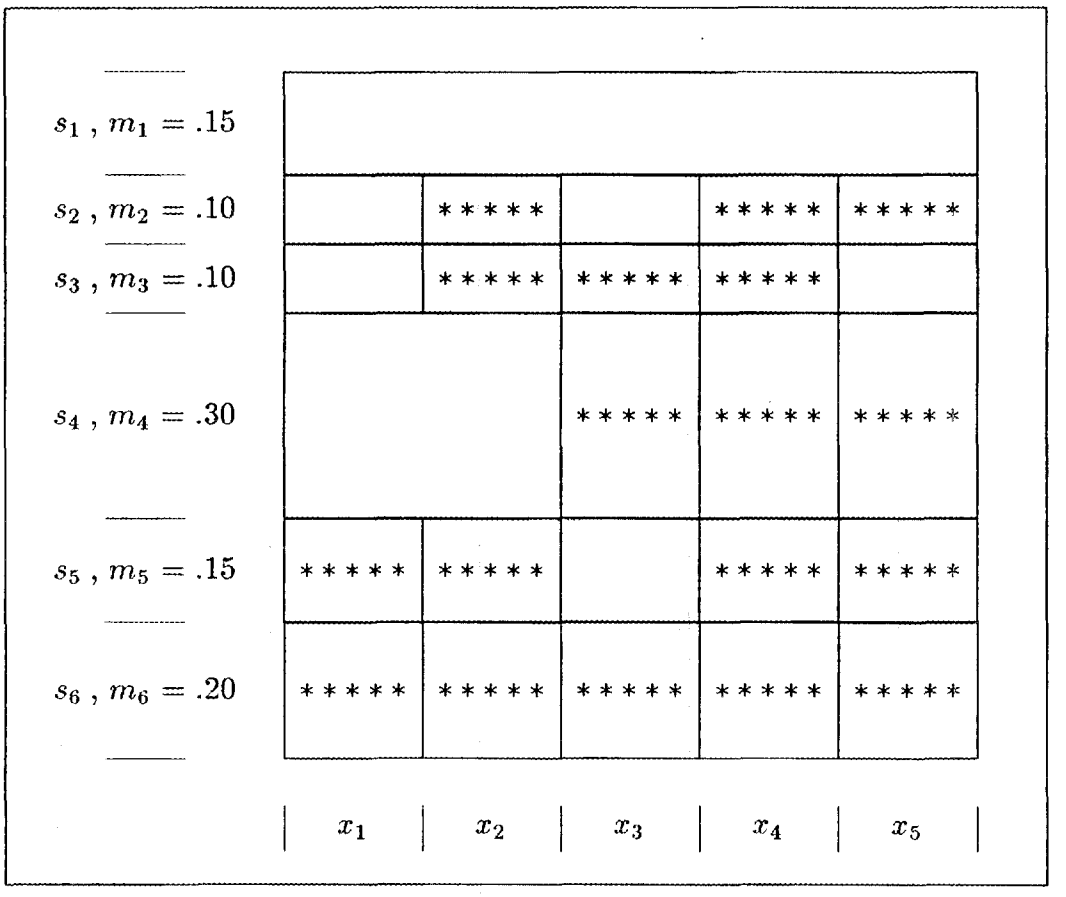

Figure 1. The results of a general multivalued mapping.

The y-axis in this diagram represents the measurement-space. It is composed of elemental operations (denoted here by s_i) which establish an element of evidence with probabilistic mass m_i. This space is a classic probability-space and any operations on its elements obey all the axioms of classical PT. The elements here are assumed to be disjoint and completely span the space. The x-axis represents the possibility set Θ.

The support generated by any event s_i is seen to associate probabilistic mass with a number of possibility elements. These elements form a subset of Θ which denotes the fact that all of its members remain possible given the evidence. These possibility elements are represented in the diagram by rectangles filled with asterisks. The height of the rectangles are proportional to the masses.

This diagram thus clearly represent Dempster's multivalued mapping. Here the mapping is from a probabilistic measurement-space to a possibility-space. The mapping is multivalued, in that it takes probability measures and assigns them to subsets (not individual elements) of the possibility-space. Without this multivalued aspect of the mapping, we simply have a classical probabilistic experimental event-space for analysis.

To make this diagram as general as possible, two important features have been explicitly included, neither of which is normally associated with experimental operations. They have been added to deal with the question of completeness of the space of probabilistic evidence. The first feature is an evidential element which supports the full set Θ and the second is one which supports the null set ϕ. Both serve to preserve probabilistic normalization of the experiment-space measures, thereby completing the space itself.

The first feature results primarily from an experimental observation that adds no relevant new information for inferencing, yet it supports all the possibilities. The fact that such an event does or can possibly occur (see the examples to follow), requires that it be given a probabilistic mass assignment. In the case that it does occur, its probability will be determined by the measurement operation. If it occurs only in principle, the probability can be assigned by logical means. That is, it is put in to complete the normalization of the probability distribution to unity. This is equivalent to saying that further evidence exists and it is possible for it to support any of the possibilities. It corresponds to the universal support statement in DST[8].

The second feature appears primarily when conflicting evidence must be dealt with. In probability theory this problem arises from logical inconsistencies in the hypotheses (or models) being used. It is, therefore, eliminated on logical grounds by scale renormalizations using conditional probabilities (i.e., conditional statements are made about the completeness of the possibility-space). The distinguishing feature of both OT and FST is that this problem is dealt with explicitly. Such evidence is represented by a probabilistic measure which supports the null possibility. This is *not* the representation of a null event, simply one that gives evidence to a possibility that lies outside the realm of the possibility set being considered. In this context, it clearly represents conflict within the set. The distinction is subtle but nonetheless important in decision making and constructing a set-theoretic inferencing algebra. While this feature appears in the diagram in the context of the discussion of experimental measurements, it does not commonly appear in the collection of measurable data. It does, however, occur frequently as a result of using inferencing operators.

The rest of the diagram represents a generic set of distinct experimental measurements that support several possibilities when used as evidence in an inferencing model. Each evidential element is derived from a single measurement operation and all are tied together probabilistically by the nature of the experimental system. That is, the entire collection of events is assumed to be probabilistic because of some symmetries present in the measurement process. These symmetries give rise to the probabilistic measures that quantify the event-space.

2.2. EVALUATION OF MASSES AND MEMBERSHIP FUNCTIONS

Using the conceptual model presented in Fig. 1, we can now define two quantitative representations of this data – one needed for OT and the other needed for FST. In OT, the most fundamental representation of the measurements is the mass distribution, denoted here as **A**. Such a distribution represents the probabilistic support given to all subsets of Θ. In this context, the masses are simply defined to *be* the experimental-space measures. The mass distribution *is* the multivalued mapping and nothing further needs to be done. The results in Fig. 1 can, therefore, be written in OT as follows:

$$\mathbf{A} = \{\, \overset{.15}{\phi}\,, \overset{.10}{(x_2, x_4, x_5)}, \overset{.10}{(x_2, x_3, x_4)}, \overset{.30}{(x_3, x_4, x_5)}, \overset{.15}{(x_1, x_2, x_4, x_5)}, \overset{.20}{\Theta}\,\}\,. \tag{7}$$

Here, the probabilistic masses of evidence appear above the respective subsets in Θ which they support. From these results, we see clearly how mass appears in the null set ϕ and the full set Θ.

The FST possibility representation of this data, denoted by $\mathcal{A}$, is also easily defined. It consists of first, treating the data as if it had all been reported in a single measurement operation and second, converting probabilities into possibilities. In the first transformation, only the degree of possibility is of interest, not the source of the measured evidence, so all evidence is summed. In the second, maximum probability is considered to be the possibility-space measure. In this context, the mass on each subset of Θ is the maximum possible mass that can be associated with any individual member of that subset.

Looking at Fig. 1, we see that suppressing evidential sources is equivalent to summing the information in the diagram by columns. The result of using maximum probability as a possibility measure is that the height of each elemental box is its possibility measure. The sum of the these measures for each individual possibility thus defines the FST membership function. That is, the membership function is defined as the sum over evidential sources of the maximum probabilities supporting any given possibility.

For this case the fuzzy set is found to be:

$$\mathcal{A} = \{\overset{.35}{x_1}\,, \overset{.55}{x_2}\,, \overset{.60}{x_3}\,, \overset{.85}{x_4}\,, \overset{.75}{x_5}\}\,. \tag{8}$$

Here, the membership values appear above their respective element in Θ. Membership in the null set is represented by a maximum possibility for any x_i that is less than unity (i.e., a possibility distribution that is unnormalized). Universal support appears in the form of a uniform non-zero measure applied to all possibilities.

While their are obvious quantitative differences between these two representations, it should be clear that they both represent the same measured data and some

similarities must exist. The differences, however, are strong. The OT mass distribution retains both the evidential and probabilistic character of the data, while the FST membership function suppresses the evidential basis and emphasizes possibilities as opposed to probabilities.

3. COMPARISON OF OPERATORS

The most obvious similarity between the OT and FST representations is in the definition of the respective complement operators. In OT, the complement operator is defined in terms of the masses as follows[1]:

$$m_{\tilde{A}}(\mathbf{x}) = m_A(\tilde{\mathbf{x}}) \qquad \mathbf{x}, \tilde{\mathbf{x}} \subset \Theta\,, \tag{9}$$

which when applied to the mass distribution derived from the data in Fig. 1, gives

$$\tilde{\mathbf{A}} = \{\, \overset{.20}{\phi}\,, \overset{.15}{(x_3)}, \overset{.10}{(x_1, x_3)}, \overset{.10}{(x_1, x_5)}, \overset{.30}{(x_1, x_2)}, \overset{.15}{\Theta}\,\}\,. \tag{10}$$

With reference to Fig. 1, this complement is simply seen to be the mass distribution representation of the subsets comprising the open area in the diagram.

If we were to use the open area in Fig. 1 to quantify a FST complement $\tilde{\mathcal{A}}$, we would get

$$\tilde{\mathcal{A}} = \{\overset{.65}{x_1}\,, \overset{.45}{x_2}\,, \overset{.40}{x_3}\,, \overset{.15}{x_4}\,, \overset{.25}{x_5}\}\,. \tag{11}$$

This is precisely the complement of the fuzzy set given in Eq.(8) derived from the general FST definition of the complement given in terms of membership functions, given by

$$\tilde{\mu}(x_i) = 1 - \mu(x_i) \qquad \forall x_i \subset \Theta\,. \tag{12}$$

In both cases what is being represented by the complement operation is a measure of what has not occurred in the experimental measurements. In the OT case, it is a representation of what the evidence does not support and in FST, it is what is not possible given all the evidence.

If we now look at other operations that can be performed on the basic experimental data, many more similarities become apparent. For instance, if the evidence is consonant, in that all the support subsets are successive subsets of each other when ordered by cardinality, then there exists a one-to-one mapping between the FST and OT representations. This isomorphism results from being able to use the *α-level sets* in the FST representation to derive an OT mass distribution[4]. The complements of the two representations also display this relationship. Basic algebraic operations with such consonant data can thus be shown to be quantitatively similar[14]. These similarities will be explored in more depth in the examples to follow.

If, in addition, all the evidence comes from a single measurement, then one-to-one relationships can be found with logical inferencing in classical PT as well.

Here, all masses will be on single subsets of the possibility set and the PT, FST, and OT operations on such subsets will be identical.

Another example of similarities, which will be discussed later, concerns the property of idempotency in the basic algebras of OT and FST. Neither possess it, but it can be restored in both by simply treating the basic experimental data as correlated by evidential source. This correlation allows union and intersection operations to be defined within each element of a fuzzy set or within each evidential element in OT. In both cases correlated operators can be defined to give

$$\mathbf{A} \textcircled{\wedge}_c \widetilde{\mathbf{A}} = \mathbf{N} \quad \text{and} \quad \mathbf{A} \textcircled{\vee}_c \widetilde{\mathbf{A}} = \mathbf{E}\,, \tag{13}$$

or

$$\mathcal{A} \cap_c \widetilde{\mathcal{A}} = \emptyset \quad \text{and} \quad \mathcal{A} \cup_c \widetilde{\mathcal{A}} = \Theta\,. \tag{14}$$

Here, $\mathbf{N}$ is the OT null distribution with $m(\phi) = 1$, $\mathbf{E}$ is the identity distribution with $m(\Theta) = 1$ and the subscript c denotes the correlated combination operation.

The importance of correlation information becomes even more evident when dealing with the MAX and MIN operators in FST and the basic OT operators previously published. It will become clear later, that in evidential-space, MAX and MIN operations are simply conventional set union and intersection rules applied to evidential subsets (see the examples). They need no further justification than these normal set operations. In OT, correlation by evidence duplicates these FST results exactly for consonant evidence. Exact results can also be obtained in other cases if FST rules are generalized to include some information about evidential sources.

In general, however, where inferencing results in OT and FST are not identical, the differences are due mainly to the different spaces in which the two theories are defined. FST operations, on the one hand, are evidence independent and deal primarily with the possibility dimension of the experimental data (i.e., the subsets of Θ). OT operations on the other hand, are designed primarily to work with the data from the evidential point of view.

If the fundamental experimental evidence is kept in mind in any particular application, however, consistency in inferencing is possible despite the differences between the two theories. This consistency can be maintained by choosing appropriate operator definitions, taking into account the given experimental data and the inferencing to be done.

The important point to remember in these comparisons is that both theories are working with the same data, albeit in different spaces – evidential and possibility. The choice of FST or OT in any given application will depend in great measure on this delineation. Consistent inferencing with the experimental data in specific applications, therefore, requires careful consideration of the character of the data in both dimensions. Questions of computational cost and ease of decision making, however, should also play a large role.

As a last point, it should be noted that DST can be represented as a single operator in this measurement-operation framework. This theory consists of a single

intersection operator that produces a cross product expansion of the evidential-space and then maps it back onto itself (i.e., this corresponds to the uncorrelated addition of evidence to the same evidential-space). Such an operator is most useful in sequence-of-events or independent information combination applications which have a natural combinatorial structure. If the evidential base of support is correlated (i.e., dependent), however, then DST can not be applied. In such cases, it gives rise to combinatorial explosions and paradoxical results where they are unwarranted.

In general, most applications require consideration of the correlated nature of the evidence being combined. Without an experimental base, these considerations are lost. Each of the theories (OT, DST or FST) remains too abstract, to the detriment of the applications that require their generality.

4. SAMPLE PROBLEMS

4.1. COMBINATORIAL ANALYSIS

Several illustrative examples will now be discussed to put the ideas presented in the last section into concrete terms. A classic urn problem was chosen for these analyses. It clearly illustrates the roles of probabilities and possibilities in a framework where results from OT, FST and PT can all be compared. Despite its simplicity, these problems contain concrete realizations of a number of the important concepts which find use in most practical applications of uncertainty analysis – measurement, quantification of masses and memberships, logical inferencing, and verification.

As an example then, consider the following problem:

> "Given an urn containing two balls, each of which can be white w or black b. Draw balls from the urn, replacing them after noting their color. Infer from the results the contents of the urn."

This statement, with no further amplification (i.e., no mention of randomness or ensembles of urns) constitutes the full description of the problem. Only logical inferencing is to be used in its solution.

The relevant possibility set for this problem is the set of possible contents of the urn, which are

$$\Theta = \{x_1 , x_2 , x_3 , x_4\} , \tag{15}$$

where

$$x_1 = \{w, w\} , \quad x_2 = \{w, b\} , \quad x_3 = \{b, w\} \quad \text{and} \quad x_4 = \{b, b\} . \tag{16}$$

The experimental measurement which serves to completely specify the evidence that can be obtained for inferencing is the selection of a ball from the urn. This procedure has two possible operational results from which evidential support can be obtained – selection of the first ball s_1 or the second s_2. In tabular form, the evidence that can be drawn from these two experimental operations can be defined for each possibility as follows:

s_1 , $m_1 = \frac{1}{2}$	w	w	b	b
s_2 , $m_2 = \frac{1}{2}$	w	b	w	b
	x_1	x_2	x_3	x_4

Figure 2. The general experimental multivalued mapping for selecting balls from the urn.

Here, the number and corresponding colors of each ball has been explicitly taken into account. These data and the inferences which can be drawn from the two selection operations completely describe the experimental system and its support sets.

It is instructive at this point to use a novel method to explain the source of the probabilistic weights assigned to the two experimental operations in the figure. This will be done in order to remain faithful to the original statement of the problem which purposefully avoids mention of randomness or statistical ensembles (concepts which are quite useful, however, in most applications). The idea here is to use the group-theoretical methods proposed by Perey[12] to assign these probability measures. This approach is quite general and applicable to all such deductive inference problems.

To be specific, this approach uses the symmetries present in the definition of an experimental measurement operation to identify a possibility-generating group for the results of the measurement. The invariant measures of this group then uniquely define the probabilities for the results. Since we are dealing with finite, discrete possibility-spaces in this paper, the measures are all uniform. They are simply related to the order of the permutation group which abstractly represents the actual discrete measurement group which arises from the experimental system. More general measures result from experimental systems characterized by continuous groups.

For the problem at hand, the symmetries involved stem from the fact that the balls were considered to be labelled by number and color. While the inferencing results are dependent on these labels, the measurement operation is only able to discern one of these labels – the color. The remaining label (i.e., the number) always

remains unknown. The results of any measurement must thus be invariant under permutation of this number label.

As a result of this experimental symmetry, a cyclic group of $O(2)$ [isomorphic to a permutation group of $O(2)$] characterizes the system at hand. The invariant measure of this group is $\frac{1}{2}$. A probabilistic measure of $\frac{1}{2}$ is, therefore, assigned to each of the selection events s_1 and s_2. Although this result could also have been derived using arguments about randomness or Laplace's principle of indifference (as is usually done), the group-theoretical approach has much broader scope in artificial intelligence applications. It's justification is derived from deductive logic alone.

A complete analysis of the inferencing for this problem could also be made using this group-theoretical methodology, but its further use will not be explored here. OT and FST will be employed instead. The quantification of the number-label possibilities by group methods was our only concern.

Using the above assignments allows us now to proceed with an analysis of specific inferencing results. Suppose then that the first selection results in a white ball. The measurement diagram equivalent to Fig. 1 for this case can then be defined as

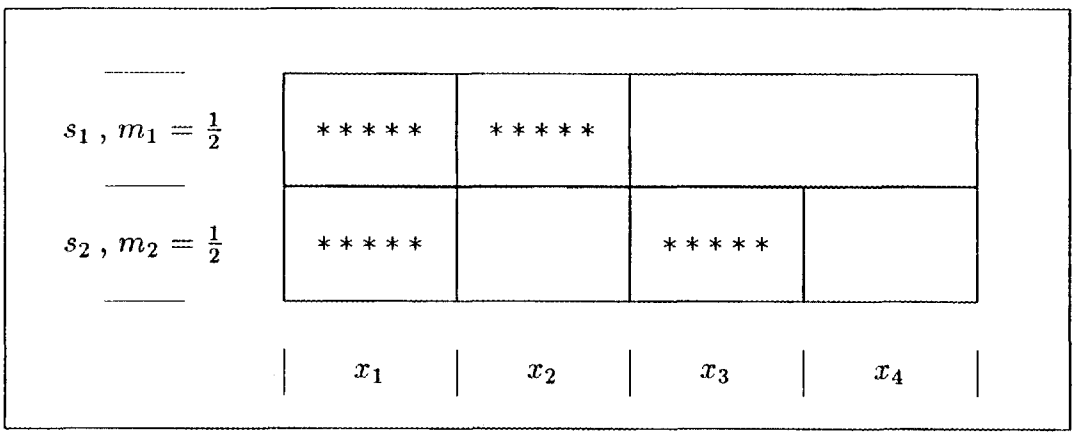

Figure 3. The experimental multivalued mapping for selecting a white ball from the urn.

Converting these results into their respective OT and FST representations using the methods described in the last section, gives the OT mass distribution **A**

$$\mathbf{A}_w = \{\overset{\frac{1}{2}}{(x_1, x_2)}, \overset{\frac{1}{2}}{(x_1, x_3)}\}, \tag{17}$$

and the corresponding fuzzy set $\mathcal{A}$

$$\mathcal{A}_w = \{\overset{1}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{0}{x_4}\}. \tag{18}$$

Here, the w subscript denotes the result after choosing a white ball.

If the first ball chosen were black on the other hand, the diagram would look like

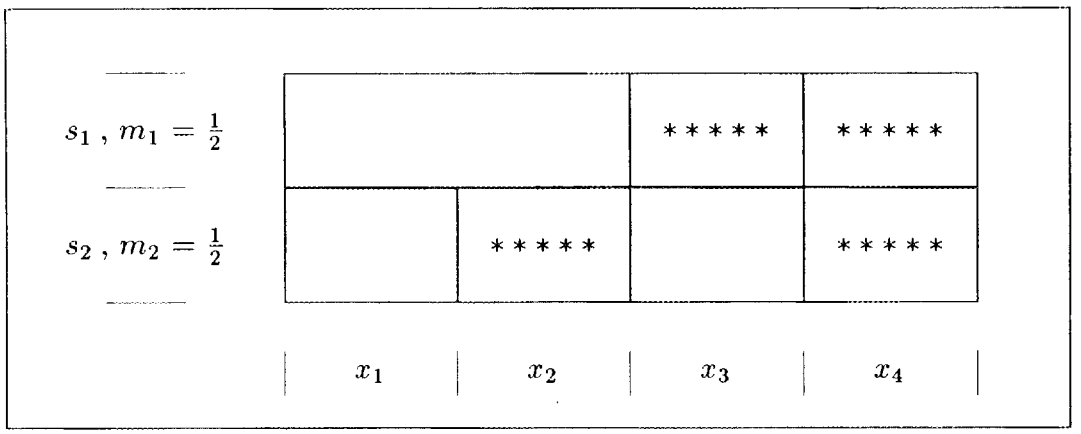

Figure 4. The experimental multivalued mapping for selecting a black ball from the urn.

The OT result would now be

$$\mathbf{A}_b = \{(\overset{\frac{1}{2}}{x_2, x_4}), (\overset{\frac{1}{2}}{x_3, x_4})\}, \tag{19}$$

and the FST result is

$$\mathcal{A}_b = \{\overset{0}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{1}{x_4}\}. \tag{20}$$

We see immediately from the respective definitions of the complements in each theory that the operation of choosing a ball has two complementary states – choosing white is quantitatively equivalent to not choosing black and vice versa. Also clear is the fact that either choice eliminates one possibility from all future deductions (i.e., one of the resulting masses or membership functions is zero).

Suppose we now consider the case of choosing two white balls in succession. If we are to make deductive inferences based on the results of the first and the second selections, it is clear that this case is described by the conjunction of two individual selection operations. But which operators are needed in each theory? If we naively used the MIN operator for a FST analysis, the results would be highly non-intuitive (and simply wrong!). This would be an example of using an abstract algebra without carefully analyzing the problem. The strength of FST is that a number of algebras are available to compensate for the suppression of evidential information. One of these algebras can correctly be applied to the analysis of any

given problem. The correct choice here is a probabilistic algebra of possibilities[10], with membership products used for the intersection operation.

In OT, the choice is made on the basis of classical PT considerations. Fundamentally, we are just compounding evidential-space information represented by a probability mapping. The dependencies present in the measurement process (or lack thereof) are the major factors here. In this case, the measurements are independent and uncorrelated. A combinatorial expansion of the evidential-space is the desired result and an uncorrelated intersection rule is therefore needed. This is precisely the intersection operation defined in the original OT algebra[1]. In this context, the problem at hand is also a natural choice for DST. The combinatorial conjunction of two selection operations is precisely described by Dempster's rule and this operator is also appropriate here.

Applying the OT intersection operator then, gives the following results for the selection of two white balls in succession:

$$\mathbf{A}_{ww} = \mathbf{A}_w \circledwedge \mathbf{A}_w = \{(x_1, x_2)^{\frac{1}{4}}, (x_1, x_3)^{\frac{1}{4}}, (x_1)^{\frac{1}{2}}\}\,. \tag{21}$$

A detailed analysis of these results shows their logical soundness. First, we see that this inference result can be broken into two parts. If the same ball were drawn both times, no new information is generated beyond that available after one selection alone. The first two terms in Eq.(21) convey this fact (i.e., they give the same inferences arrived at after selecting only one white ball). If, however, two different balls were chosen, then a precise inference can be made that the urn contained two white balls. This result is given by the third term. The mass assigned to these two alternatives reflects the fact that there is still an invariance in the problem. That is, the labels on the balls selected are unknown. Either combinatorial analysis or group theory can be used to substantiate these quantitative results.

If we take the OT results and convert them into an equivalent fuzzy set by summing maximum probabilities for each individual possibility, we get

$$\mathcal{A}_{ww} = \{x_1^{1}, x_2^{\frac{1}{4}}, x_3^{\frac{1}{4}}, x_4^{0}\}\,. \tag{22}$$

This result is clearly identical to that obtained directly from FST using a membership product intersection rule

$$\mu_{ww}(x_i) = \mu_w(x_i)\mu_w(x_i) \qquad \forall x_i \in \Theta\,, \tag{23}$$

together with the results of the white selection operation given in Eq.(18).

Here, we see that all the evidence still supports the x_1 possibility while the remaining two are decreased by a factor of two. The possibility that the urn contains two white balls is unity from these FST results. The corresponding probability lies

between $\frac{1}{2}$ and 1 as given by the OT results in Eq.(21). Both results are consistent with the intuitive conception of possibilities as being upper bounds to probabilities.

It interesting to note here how natural an area combinatorial analysis is for the possibility interpretation of FST. In this case, the result and its complement completely describe the possibility-space in a probabilistic sense. That is, a normalized probability distribution for these results, consistent with a PT analysis of the same problem, can be defined by simply scaling the FST results to reflect the total non-zero possibilities that remain. Conditional probabilities as well can be defined by eliminating certain possibilities and rescaling.

Looking at this problem from another viewpoint, we might ask what inferences could be drawn using the evidence derived from selecting the first ball *or* the second. In this case, the union operators in both theories are of interest.

The OT result in the case of selecting two white balls is

$$\mathbf{A}_{ww} = \mathbf{A}_w \textcircled{\vee} \mathbf{A}_w = \{\overset{\frac{1}{4}}{(x_1, x_2)}, \overset{\frac{1}{4}}{(x_1, x_3)}, \overset{\frac{1}{2}}{(x_1, x_2, x_3)}\}\,. \tag{24}$$

Using the probabilistic FST union operator defined as

$$\mu_{ww}(x_i) = \mu_w(x_i) + \mu_w(x_i) - \mu_w(x_i)\mu_w(x_i) \qquad \forall x_i \in \Theta\,, \tag{25}$$

gives

$$\mathcal{A}_{ww} = \{\overset{1}{x_1}, \overset{\frac{3}{4}}{x_2}, \overset{\frac{3}{4}}{x_3}, \overset{0}{x_4}\}\,. \tag{26}$$

Converting the OT result given in Eq.(24) into a fuzzy set as before, we see that OT gives a result that is equivalent to the FST result shown in Eq.(26). The two theories again, in this case, are thus seen to have identical algebraic properties as far as inferencing is concerned.

An analysis of the results of additional selection operations in this problem further strengthens this conclusion. For example, consider the case of continued selection of white balls from the urn. It is clear here, that the repeated application of the OT intersection operator will lead eventually to results that converge to the certain inference that the urn contains two white balls. This conclusion is the limiting possibility obtained using the FST intersection operator as well. The repeated operations in both cases square the probabilities (possibilities) associated with selecting the same ball over and over again, thereby making this event one of increasingly smaller probability (possibility). The remaining inference in both cases is then the one corresponding to selecting different balls, thus leading to the conclusion that the urn contained two white balls.

4.2. CONDITIONAL PROBABILITIES

A reanalysis of some of the results obtained in the last section will be made at this point to give a concrete example of the roles played by conflict and universal evidential support in both theories. For the urn problem just analyzed, these concepts

can be introduced by adding new information into the statement of the problem. In this case, we will say that it is known already that one ball is white. This case is thus equivalent to the original problem after the first white ball was selected. It is, therefore, a conditional probability reanalysis of what was done above. However, we will treat this additional knowledge as the starting point of a new analysis instead.

For these stated conditions then, the possibility set is reduced to considering only

$$\Theta = \{x_1 \, , x_2\} \, , \tag{27}$$

with

$$x_1 = \{w, w\} \, , \quad \text{and} \quad x_2 = \{w, b\} \, . \tag{28}$$

The selection operation is now represented by the following diagram:

$s_1 \, , \, m_1 = \frac{1}{2}$	w	w
$s_2 \, , \, m_2 = \frac{1}{2}$	w	b
	x_1	x_2

Figure 5. The conditional experimental multivalued mapping for selecting a ball from the urn.

Let us again assume that a white ball is chosen. The result of this operation is represented in OT as

$$\mathbf{A}_w = \{(\overset{\frac{1}{2}}{x_1}) \, , \overset{\frac{1}{2}}{\Theta}\} \, . \tag{29}$$

Here we see specifically how universal support appears in the context of the measurement process. The choice of a white ball conveys no new information *if* it was the one defined to be white. It simply supports all the inferencing possibilities (i.e., it gives universal support). Since the balls are not labelled, however, this choice is indistinguishable from the choice of the ball whose color is unknown, where we could gain some knowledge. This symmetry is again the source of the introduction of probabilities into the problem, as given in the diagram.

Taking the complement of this representation, we see immediately how null support enters into the picture. The complement operation result is

$$\mathbf{A}_b = \widetilde{\mathbf{A}}_w = \{\overset{\frac{1}{2}}{\phi}\,, \overset{\frac{1}{2}}{(x_2)}\}\,, \tag{30}$$

and universal support is changed into null support.

This event is also inescapable null support, arising in an experimental context. The null operation for selecting a ball is not measurable as such, but the definition of the unary operation of complementation forces this resulting support from logical considerations. In simple terms, the complement operation here, produces results which are identical to those derived from choosing a black ball. This means that if a white ball were chosen by an operation which could only select a black ball, the event would be considered to represent conflict. Since this event is a logical possibility, it must be recorded as such in a possibility analysis.

If we now convert the above two selection operations into fuzzy sets, we see clearly how they appear in a FST framework. The resulting membership functions are

$$\mathcal{A}_w = \{\overset{1}{x_1}\,, \overset{\frac{1}{2}}{x_2}\}\,, \tag{31}$$

for the white ball and

$$\mathcal{A}_b = \widetilde{\mathcal{A}}_w = \{\overset{0}{x_1}\,, \overset{\frac{1}{2}}{x_2}\}\,, \tag{32}$$

for the black ball.

Here, universal support again appears as a non-zero, uniform possibilistic measure over the entire possibility set and null support gives rise to an unnormalized possibility distribution. The deficit from unity in this normalization is the full measure of the null support in the evidential-space.

A reanalysis of the results obtained in the last subsection for the case where two white balls are selected is also instructive. Despite the differences in the possibility sets, the logic is identical and similar results should be expected. For this case, the OT representation operated on by an uncorrelated intersection operator yields the general result

$$\mathbf{A}_{ww} = \mathbf{A}_w \textcircled{\wedge} \mathbf{A}_w = \{\overset{\frac{3}{4}}{(x_1)}\,, \overset{\frac{1}{4}}{\Theta}\}\,. \tag{33}$$

The equivalent FST representation and a probabilistic intersection rule gives

$$\mathcal{A}_{ww} = \{\overset{1}{x_1}\,, \overset{\frac{1}{4}}{x_2}\}\,. \tag{34}$$

These results are seen to be identical to those obtained previously when the indistinguishability of the balls was taken into account. Logical soundness is what

underlies both analyses. Logically consistent inferences are the result. In the OT framework evidential support is retained, while in the FST context, possibilities alone are emphasized – these are the only differences in the results.

4.3. CORRELATED COMBINATIONS

At this point, it is most instructive to consider the possibility of selecting balls in the original urn problem in some correlated manner, for example, full correlated selection. In this latter procedure, the current experimental operation is identical to some other operation which was performed in the past. That is, whatever ball is currently chosen is known identically to be the one chosen in a previous selection operation. Such an eventuality, and all its generalizations, are handled in OT by using operators that are correlated by selection operation. These operations break down the assumption of independence of the successive selection operations and combinatorial operators are no longer applicable.

For a fully correlated problem then, the original combination rules of OT must be used in fully correlated form. That is, intersection or union operations must be performed on each individual support element and not on a cross product of all the terms (as in the independent, uncorrelated version of the definition of the algebra). The need for such correlated operations, in general, results from having some information about the sources of individual evidential support in any combination procedure. This knowledge results in operators which are no longer independent of each other, thereby requiring correlated treatment.

For example, assume that we select two white balls in succession and then make a third selection of a white ball that exactly duplicates the selection operation for the first ball. That is, if the first (or second) ball were selected in the first operation then the first (or second) ball is also selected in the third operation. The fully correlated OT intersection operation for this case is then given as

$$\begin{aligned}
\mathbf{A}_{ww\cap_c w} &= \mathbf{A}_{ww} \circledwedge_c \mathbf{A}_w \\
&= \{\overset{\frac{1}{4}}{(x_1,x_2)},\overset{\frac{1}{4}}{(x_1)},\overset{\frac{1}{4}}{(x_1,x_3)},\overset{\frac{1}{4}}{(x_1)}\} \circledwedge_c \{\overset{\frac{1}{2}}{(x_1,x_2)},\overset{\frac{1}{2}}{(x_1,x_3)}\} \qquad (35)\\
&= \{\overset{\frac{1}{4}}{(x_1,x_2)},\overset{\frac{1}{4}}{(x_1,x_3)},\overset{\frac{1}{2}}{(x_1)}\}.
\end{aligned}$$

where correlated operations are denoted by a subscript c.

This operation has been written out explicitly here to show how the correlations have been handled. In this case, the first two terms of $\mathbf{A}_{ww}$ are correlated to the first term in $\mathbf{A}_w$ and the second two in $\mathbf{A}_{ww}$ are correlated to the second in $\mathbf{A}_w$. Only these two sets of intersection operations are then performed. The resulting masses remain unchanged by the operation due to the correlation effect and no mass

products are needed. After collecting masses into common subsets, the final results are as given.

These same results are also seen to arise if the full correlation had been between the third and the second selection operations. Here, the first and the fourth terms of $\mathbf{A}_{ww}$ are correlated to the first term in $\mathbf{A}_w$ and the second and third in $\mathbf{A}_{ww}$ are correlated to the second in $\mathbf{A}_w$. Performing these correlated intersections again gives the results in Eq.(35).

In both the cases above, the underlying reason the results are identical (and the same as $\mathbf{A}_{ww}$), is that the repetition of identical operations adds no new information to the inferencing procedure. The inference results must, therefore, remain unchanged. Fully correlated OT operations bear this fact out.

For completeness, the result of using a fully correlated union operation in this analysis are also given here. Repeating the correlated operations given in Eq.(35), but with the OT union operator this time gives

$$\begin{aligned}
\mathbf{A}_{ww\cup_c w} &= \mathbf{A}_{ww} \textcircled{\vee}_c \mathbf{A}_w \\
&= \{\overset{\frac{1}{4}}{(x_1, x_2)}, \overset{\frac{1}{4}}{(x_1)}, \overset{\frac{1}{4}}{(x_1, x_3)}, \overset{\frac{1}{4}}{(x_1)}\} \textcircled{\vee}_c \{\overset{\frac{1}{2}}{(x_1, x_2)}, \overset{\frac{1}{2}}{(x_1, x_3)}\} \\
&= \{\overset{\frac{1}{2}}{(x_1, x_2)}, \overset{\frac{1}{2}}{(x_1, x_3)}\}\,.
\end{aligned} \tag{36}$$

This is the same result which also would have been obtained if the second and the third experimental operations were correlated.

The whole point of investigating these correlated operators becomes clear when we look at the equivalent results in FST. Obviously the probabilistic FST algebra used in the last subsections will not reproduce these results. In this case we must turn to the FST counterpart of the correlated OT operators, these being the classic MAX and MIN operators.

Repeating the analysis given above with FST using the MAX and MIN operators is seen to give

$$\begin{aligned}
\mathcal{A}_{ww\cap w} &= \mathcal{A}_{ww} \cap \mathcal{A}_w \\
&= \{\overset{1}{x_1}, \overset{\frac{1}{4}}{x_2}, \overset{\frac{1}{4}}{x_3}, \overset{0}{x_4}\} \cap \{\overset{1}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{0}{x_4}\} \\
&= \{\overset{1}{x_1}, \overset{\frac{1}{4}}{x_2}, \overset{\frac{1}{4}}{x_3}, \overset{0}{x_4}\}\,.
\end{aligned} \tag{37}$$

for the intersection operation and

$$\begin{aligned}\mathcal{A}_{ww\cup w} &= \mathcal{A}_{ww} \cup \mathcal{A}_{w} \\ &= \{x_1^{1}, x_2^{\frac{1}{4}}, x_3^{\frac{1}{4}}, x_4^{0}\} \cup \{x_1^{1}, x_2^{\frac{1}{2}}, x_3^{\frac{1}{2}}, x_4^{0}\} \\ &= \{x_1^{1}, x_2^{\frac{1}{2}}, x_3^{\frac{1}{2}}, x_4^{0}\},\end{aligned} \tag{38}$$

for the union operation.

If we now convert the original correlated OT results into their fuzzy set representations we also get for the intersection result

$$\mathcal{A}_{ww\cap_c w} = \{x_1^{1}, x_2^{\frac{1}{4}}, x_3^{\frac{1}{4}}, x_4^{0}\}, \tag{39}$$

and for the union result

$$\mathcal{A}_{ww\cup_c w} = \{x_1^{1}, x_2^{\frac{1}{2}}, x_3^{\frac{1}{2}}, x_4^{0}\}. \tag{40}$$

Clearly these results are identical. The fully correlated OT union and intersection operators are equivalent counterparts to the respective FST operators MAX and MIN in this case. This connection, holds in even more general terms for classes of problems characterized by consonant information. This fact has been observed before[14,15,16], but was stated in terms of comparing plausibility and possibility measures. The source of the connection here is clearly related to the fact that the MAX and MIN operators are simply set union and intersection operations performed in the possibility domain, correlated by possibility-set element. The OT counterparts are union and intersection operations performed on possibility sets correlated by evidential-set element. The relationship is one-to-one when the evidential information is consonant, as is the case here.

Going back to the analysis of the original urn problem, we can discover under what circumstances this one-to-one relationship breaks down. The differences are clearly illustrated if we were to select a black ball at any point in a succession of white choices. Despite the simplicity of this operation, more careful analysis is required than first might be thought. In this instance, we must deal with a mass distribution and its complement which, even in combinatorial analysis, are correlated in evidential-space. This experimental correlation must be taken into account in the analysis.

Noting then, that if in any two successive selection operations a white ball and then a black ball are chosen, this could only have been done if the first ball were chosen in one operation and the second ball in the other. There is full anti-correlation in these two operations. Once this is noted, there is now certainty that

the urn contains one white and one black ball. No further selections can change this conclusion as well – correlations assure this.

In OT with correlations, the above results can be illustrated simply by looking at the anti-correlated selection of a white ball and a black ball. For this intersection operation we get

$$\begin{aligned}\mathbf{A}_{wb} = \mathbf{A}_w \circledwedge_c \mathbf{A}_b &= \{\overset{\frac{1}{2}}{(x_1, x_2)}, \overset{\frac{1}{2}}{(x_1, x_3)}\} \circledwedge_c \{\overset{\frac{1}{2}}{(x_2, x_4)}, \overset{\frac{1}{2}}{(x_3, x_4)}\} \\ &= \{\overset{\frac{1}{2}}{(x_2)}, \overset{\frac{1}{2}}{(x_3)}\}.\end{aligned} \tag{41}$$

Here, the first term in $\mathbf{A}_w$ is correlated to the first term in $\mathbf{A}_b$ and the second term in $\mathbf{A}_w$ is correlated to the second term in $\mathbf{A}_b$. Also, note that if the full correlation alluded to in the beginning of this section were present (as opposed this anti-correlation), we see immediately that the correlated OT algebra is idempotent [i.e, the results given in Eq.(13) would be obtained].

If we use FST to analyze the problem, we see that we get similar results. That is, using the MIN on the white selection operation and its complement we get

$$\begin{aligned}\mathcal{A}_{wb} = \mathcal{A}_w \cap \mathcal{A}_b &= \{\overset{1}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{0}{x_4}\} \cap \{\overset{0}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{1}{x_4}\} \\ &= \{\overset{0}{x_1}, \overset{\frac{1}{2}}{x_2}, \overset{\frac{1}{2}}{x_3}, \overset{0}{x_4}\}.\end{aligned} \tag{42}$$

and clearly the conclusion is the same.

For this case, DST can easily be shown to arrive at the same conclusion as well. The DST results are derived not by using correlated operations, however (such operations are not considered in DST), but by using the renormalization condition already built into Dempster's rule. The choice of a black ball after choosing a white one introduces mass into the null set, indicating conflict, which is then removed by renormalization. A conditional reinterpretation of the results is thus required to remain consistent with the correct inferencing result. The use of DST for the correlated analyses that gave rise to the results in Eqs.(35) and (36), however, would lead to incorrect results even considering renormalizations of conflict. Such dependent combinations are simply not allowed in DST.

These results again highlight the fact that abstract algebras need to be tied to an experimental measurement basis in order to avoid inconsistent or paradoxical results. A full knowledge of the range of applicability of the operators chosen for analysis is important in any given application. Strict use of the full evidential base is sometimes essential in order to resolve such problems.

5. VERIFICATION

The final area that needs to be discussed is the experimental verification of uncertain inferencing. Without verification procedures, the results of either OT or FST remain fundamentally abstract. While the measurement procedure discussed in the previous sections reduces this abstractness with regard to quantification of the two theories, it does not eliminate it with respect to verifiability. In general, the only consistency that is clearly present in both theories is that they both use deductive logic to derive inferences from a common experimental base. While this is of great value, and possibly sufficient justification for use of either theory, conditions under which the probabilistic nature of the inferences can be tested also need to be explored.

Since the inference procedures in OT are fundamentally logical and deductive, testing the results constitutes a verification of the completeness of the possibility set and the invariance of the probabilistic measures which are being used. If the theoretical results are found to be lacking, then these are the areas of concern. The completeness of the possibility set is a modelling problem and the correctness of the measures is a probabilistic quantification problem. Both contribute to the accuracy of the inferences and both are problem dependent. We will concentrate here on the quantification aspects of this problem, since modelling verification is more theoretical in nature.

To understand the circumstances under which the quantitative aspects of probabilistic inferencing with OT can be verified, we need to refer back to Fig. 1. This diagram was constructed by assigning a measure to each element in the evidential-space and noting the subset of the possibility set it supported. The evidential-space measures are primarily statistical or group-theoretical in nature and, as such, imply that a frequency interpretation of the probabilistic results can be used for testing. Noting this fact, it is clear that repeated inferencing within the same problem framework is sufficient to generate frequencies with which to test such probabilistic results.

Looking at the urn problem as an example, it is clear that if the ball selection operations were repeated often enough, the probabilistic inferences could be tested statistically. Repeated selections from the same urn, however, increase the information available, so this is not what is needed. Separate, independent sets of selections from the same urn are more relevant (e.g., two balls selected from the urn in separate verification tests). The results of all these tests forms a statistical ensemble in which limiting frequencies can be used to verify the probabilistic inferences. As long as all the evidential elements are eventually used in proportion to their measure, the probabilities will be verified. It should be mentioned here, that it does not matter what the composition of the urn is, the testing will be done on

the inferences drawn from selection operations with that urn. Verification can be made with any urn having *any* of the possible compositions.

While the above procedure constitutes a rigorous test of this probabilistic methodology, it is often the case in practice that a number of different elements of the possibility set are to be encountered in inferencing. Probabilistic testing over the range of these possibilities might then be the most important aspect of the problem for decision making. In particular, it might be important to maintain consistent, verifiable inferencing over a distribution of the possibilities tested. The verification of inferencing in such problems highlights the link between OT and classical PT.

In such classes of problems, the fact that the OT inference procedure is indifferent to the actual possibility set member being studied leads quite naturally to the introduction of measures in the full possibility-space. Under these circumstances, the multivalued nature of the evidential mapping reduces to a one-to-one mapping which can be handled by classical probability methods (e.g., Bayesian inference theory). All the verification procedures available for statistical hypothesis testing can be used here.

In general, the measures introduced into the full possibility-space can be arbitrary and the results can still be verified. Specific priors are required in a Bayesian analysis, while its OT counterpart uses a specific member of the family of distributions implied by the multivalued mapping. The results in both cases will be identical, since they are both founded on probability theory. Classical statistical methods can be used again to verify the inferencing results.

We can illustrate the meaning of this verification procedure for the urn problems just discussed. There, only a single urn containing two specific balls was under investigation. The probabilities introduced were related to measures in the evidential-space. If we were to apply the inferencing logic to any of the other possible combinations of balls which could have been in the urn, it is clear that the logic would remain the same. The inferencing is, therefore, indifferent to the actual contents of the urn. This is a necessary logical condition for deductive inferencing in which the contents of the urn are unknown.

It is this latter invariance that implicitly creates a measure in the full possibility-space. For example, the space of ball number and color both create the inferencing domain. Group invariance principles can be used on this full possibility-space to yield measures for each of its members. The inferencing problem is then reduced to classical terms with measures (uniform in this case) on all possibility set members. Further analysis can be carried out with continued use of group-theoretical principles or classical Bayesian techniques.

In the Bayesian case, the inferencing problem for the urn is identical to one carried out using a non-informative prior. This induced measure implies that the deductive process will yield the same results had the problem been stated in statistical terms. That is, the problem could be stated in terms of four equiprobable urns, each containing one of the four different possible combinations of balls, and

each being sampled and used to derive inferences as before. The limiting frequencies of such a statistical ensemble of tests would yield the probabilistic results derived from the OT logic, provided the selection of the urns and balls was done in an experimental manner consistent with the stated invariances. In simple terms, this means that the urns and the balls must be chosen randomly to give the uniform measures explicitly derived from invariance arguments.

In general then, we see that the conditions under which the probable inferences can be verified are those in which repeated inferencing over a distribution of possibilities can be made. The limiting frequencies of the statistical ensemble of results which are generated in these tests can then be used to verify all logical conclusions which could be drawn in theory. This testing procedure is the same as that used in physics for theories that are inherently probabilistic in nature (e.g., quantum mechanics).

The results of either a PT or OT analysis differ, therefore, only in representational form in this regard. The OT representation retains the multivalued character of the evidential measurements. PT uses additional invariance arguments (with an eye towards verification) to reduce the evidential support subsets to evidential support elements in Θ. The multivalued nature of the OT results is consequently lost, but the inferences derived in PT are more specific. The relative merits of either theory in this regard must be left to the decision maker, to whom the merits of these two opposing representations are most meaningful.

As a final point, it is interesting to note that by virtue of the comments above, many of the inferencing operations defined in OT (and, therefore, in FST if given a probabilistic base) have counterparts which can be defined in terms of random variables. The connection between OT, FST and random set theory already noted in the literature[13] bears this out. This opens up interesting possibilities when it comes time to develop algorithms to do logical inferencing under uncertainty. While exact results will not be generated using random variables in place of analytic operators, sufficiently good statistics can probably be obtained without accounting for all the possibilities needed to be completely rigorous. Low probability inferences will suffer the most in such procedures, but major conclusions might be obtained far more efficiently. Random variable procedures are also ideally suited for treating the correlations which ofttimes characterize the real inferencing problems encountered in practice.

6. CONCLUSIONS

The probabilistic measurement basis proposed for both OT and FST is seen to provide a number of important foundations for both theories. The quantification of masses and membership functions is tied to a common measurement procedure. Logical inferencing in both theories based on the same experimental data are seen to be, in general, consistent. In this same probabilistic framework many of the operators needed for inferencing can be shown to have equivalent counterparts in both theories. The measurement basis also removes a large measure of abstractness from the theories making it necessary to carefully analyze any problem before choosing inference operators. These considerations should eliminate many sources of inconsistency and paradoxical behavior in practice.

The proposed probabilistic base also provides options for applying a wealth of probabilistic techniques to both the collection of experimental data and the verification process. These methods and their underlying theoretical foundations should make it easier to provide a sound scientific basis for inferencing under uncertainty (outside of the simple logical basis already built into both OT an FST). These methods offer efficient means for data collection, representation, and verification. They also provide algorithms which might be useful in approximating inferencing results, where such approximations are useful.

The sample problems considered display a full range of representations for dealing with uncertainties. The extensions to PT embodied in Dempster's conception of multivalued probabilistic mappings are seen to play a key role in opening up these new representational forms. The success of FST in treating problems outside the bounds of classical probability theory is seen to be closely tied to such mappings. The fact that they can be interpreted in terms of families of probability distributions makes it also clear that probability theory is still at the foundation of all of this work.

While much research remains to be done, it is clear from the results presented that the competing uncertainty theories (DST, FST, OT, and PT) have much in common, despite diverse axiomatic foundations and representational forms. A unification of all the theories should in principle be possible, if they are to be logically and experimentally sound, for these are the two foundations of the scientific method. Probability theory is seen to offer some strong points in moving toward this goal.

REFERENCES

1. E. M. Oblow, "O-Theory - A Hybrid Uncertainty Theory", *Int. J. Gen. Sys.*, **13**, 95, 1987.

2. E. M. Oblow, "A Probabilistic-Propositional Framework for the O-Theory Intersection Rule", *Int. J. Gen. Sys.*, **13**(4), 1987; see also, *Foundations of O-Theory I: The Intersection Rule*, Report ORNL/TM-9983, Oak Ridge National Laboratory, February 1986.

3. E. M. Oblow, *Extension of O-Theory to Problems of Logical Inferencing*, Report ORNL/TM-10107, Oak Ridge National Laboratory, March 1987.

4. E. M. Oblow, "O-Theory: A Probabilistic Alternative to Fuzzy Set Theory", In *Uncertainty in Knowledge-Based Systems*, eds. R. Yager and B. Bouchon, Springer Verlag, Berlin, 1987.

5. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 2rd ed., (1968), Vol. II, 3rd ed., (1971), John Wiley, N. Y.

6. T. Fine, *Theories of Probability*, Academic Press, N. Y., 1973.

7. A. Dempster, "Upper and Lower Probabilities Induced by a Multivalued Mapping", *Annal of Math. Statistics*, **38**, 325, 1967.

8. G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, N. J., 1976.

9. L.Zadeh, "Fuzzy Sets", *Information and Control*, **8**, 338, 1965.

10. A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets*, Vol. I, Academic Press, N. Y., 1975.

11. G. E. P. Box, W. Hunter and J. Hunter, *Statistics for Experimenters: An Introduction to Design, Data Analysis and Model Building*, John Wiley, N. Y., 1978.

12. F. G. Perey, *Application of Group Theory to Data Reduction*, Report ORNL-5908, Oak Ridge National Laboratory, September 1982

13. I. R. Goodman and H. T. Nguyen, (1985), *Uncertainty Models for Knowledge-Based Systems*, North-Holland Press, N. Y., 1985.

14. H. Prade, "A Computational Approach to Approximate and Plausible Reasoning with Applications to Expert Systems", *IEEE Trans. Pattern Anal. Mach. Intel.*, **PAMI-7**, May 1983.

15. D. Dubois and H. Prade, "On Several Representations of an Uncertain Body of Evidence", In *Fuzzy Information and Decision Processes*, M. M. Gupta and E. Sanchez, Eds., North-Holland, 1982.

16. J. Kampé de Fériet, "Interpretation of Membership Functions of Fuzzy Sets in Terms of Plausibility and Belief", In *Fuzzy Information and Decision Processes*, M. M. Gupta and E. Sanchez, Eds., North-Holland, 1982.

INTERNAL DISTRIBUTION

1. S. M. Babcock
2–6. J. Barhen
7. M. Beckerman
8. G. de Saussure
9. J. R. Einstein
10. E. Halbert
11. W. R. Hamel
12. J. K. Ingersoll
13. J. P. Jones
14. C. C. Jorgensen
15. G. Liepins
16–20. F. C. Maienschein
21. R. C. Mann
22–26. E. M. Oblow
27. F. G. Perey
28–32. F. G. Pin
33. V. Protopopescu
34. V. R. Uppuluri
35–39. C. R. Weisbin
40. B. A. Worley
41. A. Zucker
42. P. W. Dickson, Jr. (Consultant)
43. G. H. Golub (Consultant)
44. R. M. Haralick (Consultant)
45. D. Steiner (Consultant)
46. EPMD Reports Office
47–48. Laboratory Records Department
49. Laboratory Records, ORNL-RC
50. Document Reference Section
51–52. Central Research Library
53. ORNL Patent Office

EXTERNAL DISTRIBUTION

54. Office of Assistant Manager, Energy Research and Development, DOE-ORO, Oak Ridge, TN 37831
55. David Abraham, Analytic Disciplines, Inc., Second Floor, 1370 Piccard Drive, Rockville, MD 20850
56. Bruce Abramson, University of California, Los Angeles, CA 90024
57. Malcolm Bauer, Bell Communications Research, Morristown, NY 07974
58. Carlos Berenstein, University of Maryland, College Park, MD 20742
59. Raj K. Bhatnagar, University of Maryland, College Park, MD 20742
60. Piero P. Bonissone, General Electric Co., Corporate Research and Development, Schenectady, NY 12301
61. Wayne Book, Department of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332
62. Jack Breese, Stanford University, Stanford, CA 94305
63. B. G. Buchanan, Computer Science Department, Stanford University
64. B. Chandrasekaran, Ohio State University, Columbus, OH 43210
65. P. Cheeseman, NASA Ames Research Center, Moffett Field, CA 94035
66. Chee-Yee Chong, Advanced Decision Systems, Mountain View, CA 94040

67. Marvin S. Cohen, Decision Science Consortium, Inc., Falls Church, VA 22043
68. N. C. Dalkey, University of California, Los Angeles, CA 90024
69. Keith S. Decker, General Electric Corporate Research and Development, Schenectady, NY 12301
70. D. Dubois, Universite Paul Sabatier, Laboratoire Langages et Systemes Informatiques, 118 route de Narbonne, 31062 TOULOUSE CEDEX, FRANCE
71. John Fox, Imperial Cancer Research Fund Laboratories, London WC2A 3PX
72. Robert M. Fung, Advanced Decision Systems, Mountain View, CA 94040
73. R. R. Gawronski, University of West Florida, Department of Systems Sciences, Pensacola, FL 32514-003
74. T. D. Garvey, SRI International, 333 Ravenswood Ave., Menlo Park, CA 94025
75. Matthew L. Ginsberg, Stanford University, Stanford, CA 94305
76. I. R. Goodman, Naval Ocean Systems Center, San Diego, CA 92152
77. Benjamin N. Grosof, Stanford University, Stanford, CA 94305
78. Henry Hamburger, George Mason University, Fairfax, VA 22030
79. Stephen Jose Hanson, Bell Communications Research, Morristown, NJ 07974
80. David Heckerman, Stanford University, Stanford, CA 94305
81. Ewald Heer, 5329 Crown Avenue, LaCanada, CA 91011
82. Max Henrion, Carnegie-Mellon University, Pittsburgh, PA 15213
83. Peter D. Holden, McDonnell Douglas Corporation, St. Louis, MO 63166
84. F. Holley, Human Engineering Laboratory, Attn: ANXHE-FH (Holley), Aberdeen Proving Grounds, MD 21005
85. Samuel Holtzman, Strategic Decisions Group, Menlo Park, CA 94025
86. Eric Horvitz, Stanford University, Stanford, CA 94305
87. Daniel Hunter, TRW/DSG, Redondo Beach, CA 90278
88. Ramesh Jain, The University of Michigan, Department of Electrical Engineering and Computer Science, Ann Arbor, MI 48109-1109
89. Rodney W. Johnson, Naval Research Laboratory, Washington, DC 20375
90. M. H. Kalos, AEC Computing and Applied Mathematics Center, Courant Institute of Math Science, New York University, 251 Mercer Street, New York, NY 10012
91. Laveen N. Kanal, University of Maryland, College Park, MD 20742
92. Karl G. Kempf, McDonnell Douglas Corporation, St. Louis, MO 63166
93. David Lavine, L.N.K. Corporation, College Park, MD 20742
94. John F. Lemmer, Knowledge Systems Concepts, Inc., Rome, NY 13440
95. Todd S. Levitt, Advanced Decisions Systems, Mountain View, CA 94040
96. Gerald Shao-Hung Liu, Schlumberger/Sentry, San Jose, CA 95115
97. Ronald P. Loui, University of Rochester, Rochester, NY 14627

98. O. P. Manley, Division of Engineering, Mathematical and Geosciences, Office of Basic Energy Sciences, U.S. Department of Energy, Washington, DC 20545
99. Alexander Meystel, Drexel University, ECE, Philadelphia, PA 19104
100. Dana Nau, University of Maryland, College Park, MD 20742
101. J. F. Palmer, NCUBE Corporation, 915 E. La Vieve Lane, Tempe, AZ 85284
102. J. Pearl, Computer Science Department, University of California, 405 Hilgard Ave., Los Angeles, CA 90024
103. A. P. Pentland, Artificial Intelligence Center, SRI International, 333 Ravenswood Ave., Menlo Park, CA 94025
104. Bruce M. Perrin, McDonnell Douglas Corporation, St. Louis, MO 63166
105. H. E. Pople, Decision Systems Laboratory, University of Pittsburgh School of Medicine, Pittsburgh, PA 15261
106. H. Prade, Universite Paul Sabatier, Laboratorie Langages et Systems Informatiques, 118 route de Narbonne, 31062 TOULOUSE CEDEX, FRANCE
107. Paul Purdom, Indiana University, Bloomington, IN 47405
108. M. Rabins, Department of Mechanical Engineering, Wayne State University, Detroit, MI 48202
109. Z. Ras, Computer Science Department, University of Tennessee, 8 Ayres Hall, Knoxville, TN 37946
110. Karl N. Reid, Oklahoma State University, Stillwater, OK 74074
111. Larry Rendell, University of Illinois, Urbana-Champaign, IL 61801
112. E. H. Ruspini, AI Center, SRI International, 333 Ravenswood Avenue, Menlo Park, CA 94025
113. E. Sandewall, Department of Computer and Information Science, Linkoping University, Linkoping, SWEDEN
114. G. Saridis, Electrical, Computer, and Systems Engineering Department, Rensselaer Polytechnic Institute, 15th Street, Troy, NY 12180
115. Ross D. Shachter, Stanford University, Stanford, CA 94305
116. Glenn Shafer, School of Business, University of Kansas, Lawrence, KS 66045
117. John E. Shore, Naval Research Laboratory, Washington, DC 20375
118. P. Smets, Laboratory for Medical Statistics, School of Public Health, Brussels University (ULB), 808 route de Lennick, 1070 Brussels, BELGIUM
119. R. Sobeck, Laboratoire d'Automatique et d'Analyse des Systemes du C.N.R.S. 7, Avenue de Colonel-Roche, F-31077 Toulouse CEDEX, FRANCE
120. Ray Solomonoff, Oxbridge Research, Cambridge, MA 02238
121. David J. Spiegelhalter, MRC Biostatistics Unit, Cambridge, ENGLAND
122. Michael C. Tanner, Ohio State University, Columbus, OH 43210

123. T. R. Thompson, Artificial Intelligence Project, System Planning Corporation, 1500 Wilson Boulevard, Arlington, VA 22209
124. Chun-Hung Tzeng, Ball State University, Muncie, IN 47306
125. David S. Vaughn, McDonnell Douglas Corporation, St. Louis, MO 63166
126. P. H. Winston, Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139
127. Ben P. Wise, Carnegie-Mellon University, Pittsburgh, PA 15213
128. Robert M. Yadrick, McDonnell Douglas Corporation, St. Louis, MO 63166
129. Ronald Yager, Machine Intelligence Institute, Iona College, New Rochelle, NY 10801
130. Lotfi A. Zadeh, Computer Science Division, University of California, Berkeley, CA 94720
131. M. Zemankova, Computer Science Department, University of Tennessee, 8 Ayres Hall, Knoxville, TN 37946
132. Alf C. Zimmer, University of Regensburg, Federal Republic of Germany

133–162. Technical Information Center (TIC)