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Extension of O-Theory to Problems of Logical Inferencing

E. M. Oblow

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ABSTRACT

This paper extends Operator-Uncertainty Theory (OT) to the problem of uncertainty propagation in logical inferencing systems. The OT algebra and propositional interpretations presented in previous papers are applied here to derive operators for logical inferencing in the presence of conflict and undecidability. Operators for propagating uncertainties through the logical operations of disjunction and conjunction are defined. In addition, new OT operators for implication, modus ponens and modus tollens are also proposed.

The operators derived using the OT methodology are found to give rise to a four-valued logic similar to that used in computer circuit design. This framework allows uncertainty in inferencing to be represented in the form of rules convenient for use in expert systems as well as logical networks. The theory is general enough to deal with questions of conflict and undecidability, and to propagate their effects through the most widely used inference operations.

EXTENSION OF O-THEORY TO PROBLEMS OF LOGICAL INFERENCING

1. INTRODUCTION

This paper considers the problem of propagating uncertainty through logical inference operations. The uncertainty structures to be considered find application in rule-based expert systems and logical inference networks. The analysis employs Operator-Uncertainty Theory (OT)[1,2], a hybrid uncertainty theory based on the probabilistic concepts of Dempster-Shafer Theory (DST)[3] and the set-theoretic operations of Fuzzy Set Theory (FST)[4]. This new theory defines set representations for conflict and undecidability, thereby generalizing probability theory in a set-theoretic framework similar to that proposed in random set theory[5].

In two previous papers[1,2], the set-theoretic and propositional foundations of OT were presented. The applicability of this hybrid theory to uncertainty propagation in logical inferencing systems was discussed but never completely demonstrated. We propose in this paper, to show how this theory can be applied to propagate uncertainties through logical inference operations.

This work differs from other uncertainty approaches to logical inferencing and expert systems[6,7], in that OT will be seen to give rise, in general, to a multivalued-logic solution to these problems. This logic allows both conflict and undecidability in information sources to be represented and for such knowledge structures to be propagated through the inferencing process. The undecidability representation used is closely related to Shafer's concept of uncommitted belief[3]. The conflict representation, on the other hand, is distinctly an OT concept, related to the assignment of a probability-like measure to the null set.

2. BACKGROUND FOR OT

OT is a probabilistic uncertainty theory based on DST, developed for representing, propagating and combining certain non-standard but frequently occurring forms of uncertain information. In particular, it was developed to handle sources of information characterized by measurable forms of undecidability and conflict. It was formulated in a set-theoretic framework, analogous to FST, to generalize the use of the set operations of union, intersection, and complement in probabilistic uncertainty problems.

The theory deals with uncertainties, much like probability theory, by starting with a conventional possibility set of elementary events $\Theta = \{x_1, x_2, \ldots, x_n\}$. However, instead of defining probabilities for these elementary events and using a calculus to derive probabilities over a σ-field of these events, a mapping of probabilities is defined directly over the entire σ-field. Thus, in OT, a mapping $m: 2^{\Theta} \rightarrow [0,1]$ is defined which takes all the subsets of Θ and maps them into real numbers in the interval [0,1]. The subsets of Θ, which form a power set denoted in the mapping definition by 2^{Θ}, are the σ-field for uncertainty problems OT is designed to handle.

The masses m, which constitute the OT mapping, are normalized to unity like the probabilities of disjoint elementary events. In this case, however, they sum to unity over all the elements of 2^{Θ} and the mass on the null-set is not by definition set to zero. In probability terms then, we can say that the mapping m defines a probability for every member of the σ-field consisting of all subsets of Θ including the null-set ϕ. The masses are thus analogous to probability densities defined over the elements of 2^{Θ}, rather than just over Θ.

For the purposes of this paper, the possibility sets used will be restricted to those that are partitions of a finite universal possibility set denoted by $\Theta=\{x_i;i=1,N\}$. The partitions will appear only in two forms: 1) those composed of set-theoretic propositions partitioning the set Θ or 2) those composed of truth-values of such propositions. An example of the former case is the partition $\Theta = \{A,A^c\}$, where A and A^c denote the proposition A and its complement A^c. An example of the latter, is $\Theta = \{T,F\}$ where T and F represent the truth-values true and false respectively.

A mass mapping defined over either a power set of propositions or truth-values, will be used to represent the evidence directly supporting the propositions or truth-values denoted by each power set element. Direct support will mean that the evidence bears directly on only one particular power set element and is not resolvable into any of its subsets. The evidence will be represented probabilistically by the use of unit normalized mass distributions defined over the elements of 2^Θ. This extended probabilistic interpretation of evidential support is much the same as that proposed by Shafer[3] for DST.

As an example, consider the possibility set $\Theta = \{A,A^c\}$, representing a partition of the universal set $\Theta=\{x_i;i=1,N\}$ by the two propositions, "x_i is in A" or "x_i is in A^c". The mass distribution defined by the mapping $m:2^\Theta\rightarrow[0,1]$ for this A-partition of Θ, will be denoted by $\underline{A}$ and defined as

$$\underline{A} = [\ x,m_A(x)\ ;\ x\epsilon 2^\Theta\ ] \quad . \tag{1}$$

This distribution represents the direct evidence supporting the truth of the compound propositions in 2^Θ derived from the A-partition of Θ, which for this case are given by

$$2^\Theta = \{\ A\cap A^c\ ,\ A\ ,\ A^c\ ,\ A\cup A^c\ \} \quad . \tag{2}$$

Here, for instance, if A and A^c are complementary propositions in a propositional lattice[8] of subsets of Θ, then $A \cap A^c$ defines the statement that "x_i is in A and x_i is in A^c" and is the representation for conflict in this partition. Likewise in this context, $A \cup A^c$ represents the statement "x_i is in A or x_i is in A^c" and is the representation for undecidability. For notational simplicity, the $\cap$ connective will be dropped for the rest of the paper and terms like $A \cap A^c$ will be denoted simply as AA^c.

In extending OT to inference problems, the and connective $\cap$ used between any two complementary propositions (e.g. A and A^c) will always represent conflicting evidence. In addition, all complementary propositions will be assumed to be complements relative to a particular partition of Θ (i.e. relative to a particular sub-lattice of propositions in 2^Θ). The compound proposition representing conflict (e.g. AA^c) will, in general then, always have the set-theoretic interpretation of being a non-null element of the power set 2^Θ. In some of the applications to be discussed, however, such conflict will be represented by the null-element ϕ of 2^Θ. For these latter cases, this particular form of conflict will be referred to as "absolute". This will be the case, for instance, in the treating logical inferencing with implication rules. In general, however, such conflict will be viewed as relative to a partition, and represented by a non-null element of 2^Θ.

The or connective $\cup$, in this context, will always be used to represent evidence which has an undecidable character of the exclusive variety. That is, for example, $A \cup A^c$ specifies that "x_i is in A or A^c but not both". This is the same usage commonly applied to such a proposition in DST. In set-theoretic terms again, $A \cup A^c$ is meant to be a distinct element of the power set 2^Θ. For the logical inferencing problems to be discussed, this element will be defined to be Θ itself, in order to restrict the resulting inferences to compatible frames of reference.

In this paper, the OT intersection operator will be used extensively to combine mass distributions representing independent assessments of evidence supporting the truth of any propositions in 2^{Θ}. This operator represents the essence of deductive inference for the problems to be discussed. That is, the information contained in all independent assessments of evidence will be used conjunctively for further inferencing. In examining these problems from the standpoint of truth-values, however, both the OT intersection and union operators will be used to show the underlying unity of the results developed.

In combining independent evidence, the sets of mutually complementary propositions to be combined will be assumed to represent different independent partitions of some common underlying space. In particular, $\Theta=\{x_i;i=1,N\}$ will used to represent the common framework for all partitions. The combination of independent partitions by the OT intersection operator will thus result in a new partition of Θ. The mass assignments in this new partition will then represent the probabilistic evidence supporting the truth of the compound propositions formed by combining the elemental propositions from each partition.

Notationally, the intersection combination procedure will be represented, for example, by

$$\underline{S}_I = \underline{S}_A \otimes \underline{S}_B \quad , \tag{3}$$

where $\underline{S}_A$ and $\underline{S}_B$ are two mass distributions derived from the possibility set partitions of Θ given by $S_A=\{A,A^c\}$ and $S_B=\{B,B^c\}$. The distributions $\underline{S}_A$ and $\underline{S}_B$ will be assumed to be derived from independent sources. $\underline{S}_I$ then, is a mass distribution defined over the power set 2^{Θ} composed of compound propositions formed using A, A^c, B and B^c conjunctively.

In Eq. (3), $\otimes$ is the OT intersection operator which generates the mass assignments of the compound propositions $s_{I_k} \in 2^{\Theta}$. These masses $m(s_{I_k})$, are defined by

$$m(s_{I_k}) = \sum_{s_{I_k}=s_{A_i} \cap s_{B_j}} m(s_{A_i})m(s_{B_j}) \quad . \tag{4}$$

where in this notation, the sum is over all i and j subject to the constraint $s_{I_k}=s_{A_i} \cap s_{B_j}$. Here, s_{A_i} and s_{B_j} are the propositions which are elemental members of the power sets of the partitions of S_A and S_B. Also, if the $m(s_{A_i})$ and $m(s_{B_j})$ individually sum to unity, then it is clear that the $m(s_{I_k})$ do also.

3. SIMPLE PROPOSITIONAL EXPRESSIONS AND TRUTH VALUES

Using the definitions of OT concepts just given, we will now investigate some simple propositional expressions which can be evaluated using the information derived by combining evidence from two independent sources. Assume then, that source S_A assesses the evidence supporting propositions defined for an A-partition of Θ (i.e. propositions involving A and A^c) and source S_B, makes a similar determination but with regard to a B-partition. Both represent their evidence in terms of mass distributions for their respective partitions, these distributions being denoted by $\underline{S}_A$ and $\underline{S}_B$, respectively.

For the sake of generality, it will be assumed that both sources encounter evidence which requires the use of the full representational structure of an OT mass distribution. That is, some evidence is found to be conflicting, some undecidable in nature, some supportive of a particular proposition being investigated and some supportive of its complement. For clarity, these four categories will be represented by the symbols: C_A, U_A, A, and A^c for S_A and C_B, U_B, B, and B^c for S_B, respectively. Here, C ≡ conflict, U ≡ undecidable and superscript c ≡ complement).

In this case then, the following two mass distributions are assumed to arise from the assessments described:

$$\underline{S}_A = \left[\overset{m(C_A)}{C_A}, \quad \overset{m(A)}{A}, \quad \overset{m(A^c)}{A^c}, \quad \overset{m(U_A)}{U_A} \right],$$

$$\underline{S}_B = \left[\overset{m(C_B)}{C_B}, \quad \overset{m(B)}{B}, \quad \overset{m(B^c)}{B^c}, \quad \overset{m(U_B)}{U_B} \right]. \qquad (5)$$

The mass assigned to each proposition in 2^Θ appears, in this notation, above the respective proposition and the following definitions are implicitly assumed:

$$C_A \equiv AA^c \ , \ U_A \equiv AUA^c \ , \ C_B \equiv BB^c \ , \ U_B \equiv BUB^c \quad . \qquad (6)$$

Note here, that while mass has been assigned directly to the statements C_A and C_B (representing conflict), this is not intended to imply that this is the predominant way in which conflict enters OT. To the contrary, such assignments are the exception to the rule. They arise much more frequently as a result of combining information from independent sources. The assignments were made here, however, to highlight the use of this representational form. The operations performed on and with these conflict elements are of primary importance in this case.

With the mass distributions thus given, we can now use the OT intersection operator to combine this information. This operator is used, because we want to combine and use the information conjunctively. That is, we intend to do logical inferencing using the combined information in

$\underline{S}_A$ and $\underline{S}_B$. The result of applying this operator, as defined in Eq. (4), are

$$\begin{aligned}\underline{S}_I = [\; & \overset{m(C_A)m(C_B)}{C_A C_B} \;,\; \overset{m(C_A)m(B)}{C_A B} \;,\; \overset{m(C_A)m(B^c)}{C_A B^c} \;,\; \overset{m(C_A)m(U_B)}{C_A U_B} \;, \\ & \overset{m(A)m(C_B)}{A C_B} \;,\; \overset{m(A)m(B)}{AB} \;,\; \overset{m(A)m(B^c)}{AB^c} \;,\; \overset{m(A)m(U_B)}{A U_B} \;, \\ & \overset{m(A^c)m(C_B)}{A^c C_B} \;,\; \overset{m(A^c)m(B)}{A^c B} \;,\; \overset{m(A^c)m(B^c)}{A^c B^c} \;,\; \overset{m(A^c)m(U_B)}{A^c U_B} \;, \\ & \overset{m(U_A)m(C_B)}{U_A C_B} \;,\; \overset{m(U_A)m(B)}{U_A B} \;,\; \overset{m(U_A)m(B^c)}{U_A B^c} \;,\; \overset{m(U_A)m(U_B)}{U_A U_B} \;] \; . \qquad (7)\end{aligned}$$

Substituting the definitions of C and U into this expression and simplifying the results, we can also write this as

$$\begin{aligned}\underline{S}_I = [\; & \overset{m(C_A)m(C_B)}{AA^cBB^c} \;,\; \overset{m(C_A)m(B)}{AA^cB} \;,\; \overset{m(C_A)m(B^c)}{AA^cB^c} \;,\; \overset{m(C_A)m(U_B)}{AA^c(BUB^c)} \;, \\ & \overset{m(A)m(C_B)}{ABB^c} \;,\; \overset{m(A)m(B)}{AB} \;,\; \overset{m(A)m(B^c)}{AB^c} \;,\; \overset{m(A)m(U_B)}{A(BUB^c)} \;, \\ & \overset{m(A^c)m(C_B)}{A^cBB^c} \;,\; \overset{m(A^c)m(B)}{A^cB} \;,\; \overset{m(A^c)m(B^c)}{A^cB^c} \;,\; \overset{m(A^c)m(U_B)}{A^c(BUB^c)} \;, \\ & \overset{m(U_A)m(C_B)}{(AUA^c)(BB^c)} \;,\; \overset{m(U_A)m(B)}{(AUA^c)B} \;,\; \overset{m(U_A)m(B^c)}{(AUA^c)B^c} \;,\; \overset{m(U_A)m(U_B)}{(AUA^c)(BUB^c)} \;] \; . \qquad (8)\end{aligned}$$

The sixteen compound propositions in this form are seen to arise from the conjunctive combination of all the elementary propositions in the mass distributions $\underline{S}_A$ and $\underline{S}_B$. The masses of these compound propositions are derived similarly from mass product operations on each of these elementary terms. The sum of the sixteen masses so derived, are readily seen to preserve unit normalization.

A Venn diagram of the universal set Θ for the combination of the sets S_A and S_B for this case, is shown in Figure 1. This diagram will be used extensively to establish the correspondence between the set-theoretic and propositional notations used in the rest of this paper.

The basic components of this diagram are the elements of the underlying set $\Theta=\{x_i;i=1,N\}$. These elements are used to construct all compound propositions made up of the propositions A, A^C, B and B^C and, therefore, are a representation of an (A,B)-space. A list of the most basic of these constructions are as follows:

$$
\begin{aligned}
A &= \{ x_1, x_4, x_5, x_6, x_8, x_9, x_{10} \} \quad , \\
A^C &= \{ x_2, x_3, x_6, x_7, x_8, x_9, x_{12} \} \quad , \\
B &= \{ x_1, x_2, x_5, x_6, x_7, x_9, x_{11} \} \quad , \\
B^C &= \{ x_3, x_4, x_5, x_7, x_8, x_9, x_{13} \} \quad , \\
AB &= \{ x_1, x_5, x_6, x_9 \} \quad , \\
AB^C &= \{ x_4, x_5, x_8, x_9 \} \quad , \\
A^CB &= \{ x_2, x_6, x_7, x_9 \} \quad , \\
A^CB^C &= \{ x_3, x_7, x_8, x_9 \} \quad , \\
ABB^C &= \{ x_5 \} \; , \\
AA^CB &= \{ x_6 \} \; , \\
A^CBB^C &= \{ x_7 \} \; , \\
AA^CB^C &= \{ x_8 \} \; , \\
AA^CBB^C &= \{ x_9 \} \quad .
\end{aligned}
\tag{9}
$$

For the present case, we will make some further simplifications to restrict the analysis to logical inference problems. For such cases, we will assume that

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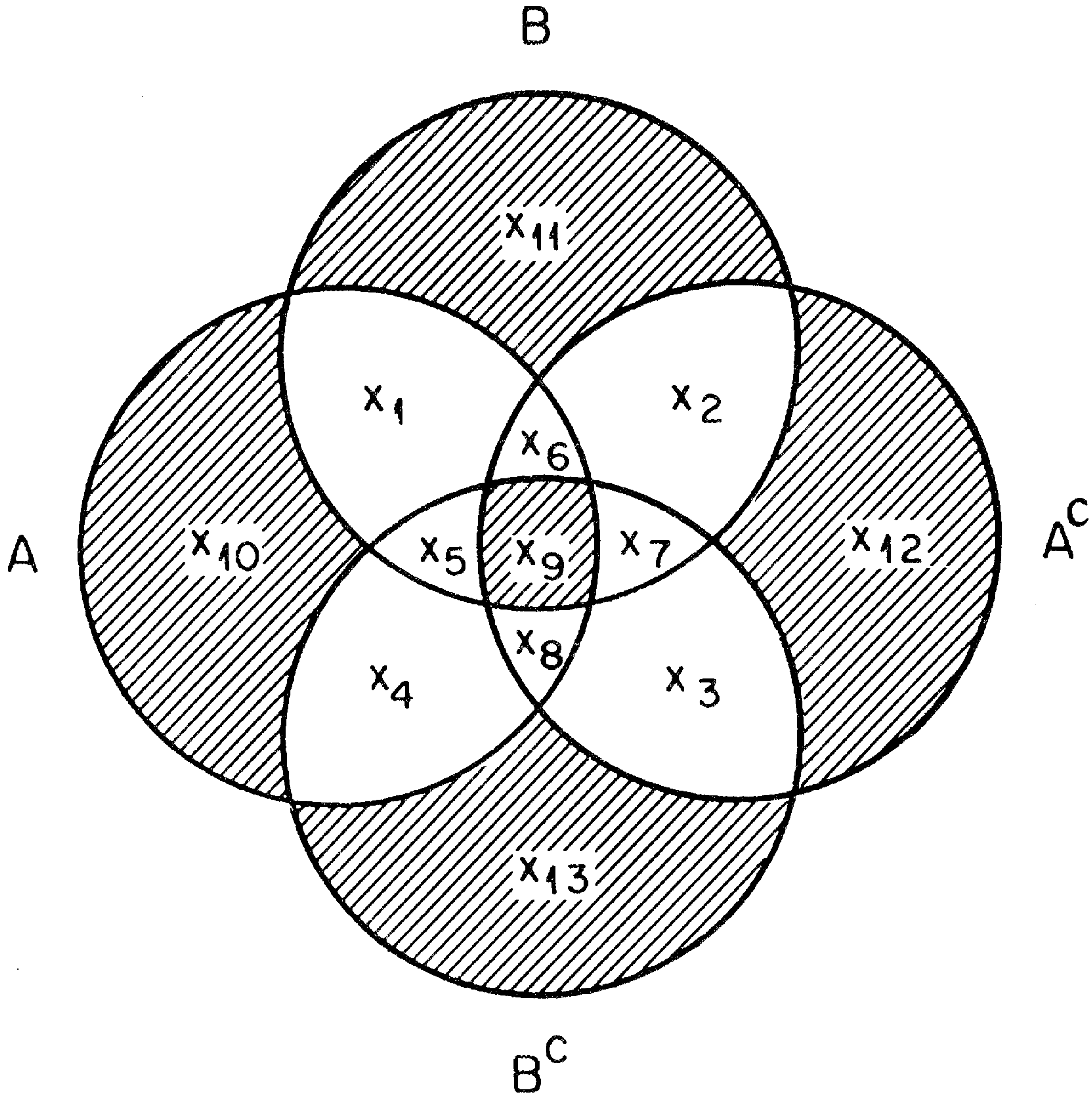

Figure 1. Venn diagram for possibility set Θ.

S_A and S_B refer to the same frame of reference Θ, that is, S_A and S_B are compatible partitions. We then have

$$\{ x_{10}, x_{11}, x_{12}, x_{13} \} \equiv \{\phi\} \quad , \tag{10}$$

and

$$\begin{aligned} \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \} &= \{\Theta\} \\ &\equiv AUA^C = BUB^C \quad . \end{aligned} \tag{11}$$

Furthermore, since $AA^CBB^C = \{x_9\}$ is a subset of all the power set elements constructed from propositions A and B, we can treat it without loss of generality as if it were a null-set element for this problem. In essence, we are assuming here that all compound propositions in the (A,B)-space form a propositional lattice with sup=$\{\Theta\}$ and inf=$\{x_9\}$. For this case then we assume that AA^CBB^C represents absolute conflict and therefore,

$$\{ x_9 \} \equiv \{\phi\} \quad . \tag{12}$$

With these assumptions, the complete underlying universal set Θ has been defined for this problem and OT can now be applied directly to solve inferencing problems. Eq. (8) represents the combined information from which such inferences will be made. Although the number of terms in this mass distribution is greater than either of the mass distributions used to construct it, the process of inferencing will reduce this complexity to a more manageable form by projecting the results onto different inference partitions.

We propose to illustrate the inference process by asking several specific inference questions. Each question will define a projection (i.e. an inference partition) of the general results. Specifically, we first would like to know what Eq. (8) allows us to conclude about the two elemental propositions AB and AUB, and their representation in terms of truth-values. The answer to these questions will define simple logical operators which then can be used directly in solving more complex problems by recursive application.

3A. The AB Case

To determine what bearing the combined information in Eq. (8) has on the proposition AB, we first have to partition Θ using AB and its propositional complement $(AB)^C$ and then find the partition masses. We will do this by defining the partition power set for this case as follows:

$$2^\Theta = \{ C_{AB} , AB , (AB)^C , U_{AB} \} \quad . \tag{13}$$

This representation provides the definitions of the compound propositions for which partition masses must be determined.

To construct a projection of Eq. (8) for the AB-partition, we first express its elemental propositions in terms of power set elements of 2^Θ as follows:

$$AB \equiv \{AB\} = \{ x_1, x_5, x_6 \} \quad , \tag{14a}$$

$$(AB)^C \equiv \{AB^C, A^CB, A^CB^C\} = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} \quad . \tag{14b}$$

From this definition, the conflict and undecidable partition elements are found to be

$$C_{AB} \equiv (AB)\cap(AB)^C = \{AA^CB, ABB^C, AA^CBB^C\} = \{ x_5, x_6 \} \quad , \tag{15a}$$

$$U_{AB} \equiv (AB)\cup(AB)^C = \{AB, AB^C, A^CB, A^CB^C\} = \{\Theta\} = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} \quad . \tag{15b}$$

Note here, that for this partition, the propositions AB, AB^C, A^CB, and A^CB^C can be considered to be the elementary propositions of the (A,B)-space. In this light, all compound propositions are composed of disjunctions or conjunctions of these four basic propositions. This observation holds true for all the partitions of the (A,B)-space to be discussed. The four elementary propositions represent a level in the propositional lattice about which undecidability and conflict can be treated symmetrically in OT.

The mass assignments for the AB-partition can now be made by projecting the sixteen propositions in Eq. (8) onto the four given in Eqs. (14) and (15). Masses for each of the four partition propositions are defined by summing the projected masses of those compound propositions in Eq. (8) that directly support each of the AB-partition propositions. In this process, the projection and the masses supporting each proposition are defined using the inherent order in the propositional lattice. This order is based on set-inclusion as defined by the compound propositions given in Eq. (9) and Eqs. (14) and (15).

Applying this procedure gives rise to the following projection mapping:

$$\{AA^CB, ABB^C, AA^CBB^C\} \rightarrow C_{AB}$$

$$\{AB\} \rightarrow AB$$

$$\{AB^C, A^CB, A^CB^C, AA^CB^C, A^CBB^C, (A \cup A^C)BB^C, AA^C(B \cup B^C), A^C(B \cup B^C), (A \cup A^C)B^C\} \rightarrow [AB]^C$$

$$\{A(B \cup B^C), (A \cup A^C)B, (A \cup A^C)(B \cup B^C)\} \rightarrow U_{AB} \qquad (16)$$

The partition masses for the AB-partition elements here, are

$$m(C_{AB}) = m(AA^C)m(BB^C)+m(AA^C)m(B)+ m(A)m(BB^C) \quad , \tag{17a}$$

$$m(AB) = m(AB) \quad , \tag{17b}$$

$$m([AB]^C) = m(AA^C)m(B^C)+m(AA^C)m(BUB^C)+ m(A)m(B^C)+m(A^C)+ m(AUA^C)m(BB^C)+m(AUA^C)m(B^C) \quad , \tag{17c}$$

$$m(U_{AB}) = m(A)m(BUB^C)+m(AUA^C)m(B)+ m(AUA^C)m(BUB^C) \quad . \tag{17d}$$

This partitioning process is most easily summarized in the form of a table which relates the final partition results to the initial partition propositions which were combined with the OT intersection rule. The table for this AB-partition is found to be

			B		
	AB	BB^C	B	B^C	BUB^C
	AA^C	C_{AB}	C_{AB}	$(AB)^C$	$(AB)^C$
A	A	C_{AB}	AB	$(AB)^C$	U_{AB}
	A^C	$(AB)^C$	$(AB)^C$	$(AB)^C$	$(AB)^C$
	AUA^C	$(AB)^C$	U_{AB}	$(AB)^C$	UAB

Table 1. Summary table for AB-partition.

The masses for the elements of this table are those already given in Eq. (17).

3B. AUB Case

The same procedure illustrated above can be used to derive results for a partition based on the proposition AUB. For this case, the partition power set is

$$2^{\Theta} = \{ C_{AUB} , AUB , (AUB)^{C} , U_{AUB} \} , \qquad (18)$$

and the propositional and set-theoretic definitions of these terms are as follows:

$$AUB \equiv \{AB, A^{C}B, AB^{C}\} = \{ x_1, x_2, x_4, x_5, x_6, x_7, x_8 \} , \qquad (19a)$$

$$(AUB)^{C} \equiv \{A^{C}B^{C}\} = \{ x_3, x_7, x_8 \} , \qquad (19b)$$

$$C_{AUB} \equiv (AUB)\cap(AUB)^{C} = \{AA^{C}B^{C}, A^{C}BB^{C}, AA^{C}BB^{C}\} = \{ x_7, x_8 \} , \qquad (19c)$$

$$U_{AUB} \equiv (AUB)U(AUB)^{C} = \{AB, AB^{C}, A^{C}B, A^{C}B^{C}\} = \{\Theta\} = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} . \qquad (19d)$$

As in the AB case before, if we collect the appropriate propositional terms from Eq. (8) and form a projection mapping, we can define such a mapping for the AUB-partition as follows:

$$\{AA^{C}B^{C}, A^{C}BB^{C}, AA^{C}BB^{C}\} \rightarrow C_{AUB}$$

$$\{A^{C}B^{C}\} \rightarrow (AUB)^{C}$$

$$\{AB, A^{C}B, AB^{C}, AA^{C}B, ABB^{C}, A(BUB^{C}), (AUA^{C})B, (AUA^{C})BB^{C}, AA^{C}(BUB^{C})\} \rightarrow AUB$$

$$\{A^{C}(BUB^{C}), (AUA^{C})B^{C}, (AUA^{C})(BUB^{C})\} \rightarrow U_{AUB} \qquad (20)$$

Rewriting these results again in table form, we get the following summary table for the AUB partition:

		B			
	AUB	BB^C	B	B^C	BUB^C
	AA^C	C_{AUB}	AUB	C_{AUB}	AUB
A	A	AUB	AUB	AUB	AUB
	A^C	C_{AUB}	AUB	$(AUB)^C$	U_{AUB}
	AUA^C	AUB	AUB	U_{AUB}	U_{AUB}

Table 2. Summary table for AUB-partition.

with masses

$$m(C_{AUB}) = m(AA^C)m(BB^C)+m(AA^C)m(B^C)+m(A^C)m(BB^C) \quad , \tag{21a}$$

$$m((AUB)^C) = m(A^CB^C) \quad , \tag{21b}$$

$$\begin{aligned} m(AUB) = {} & m(AA^C)m(B)+m(AA^C)m(BUB^C)+ \\ & m(A)+m(A^C)m(B)+ \\ & m(AUA^C)m(BB^C)+m(AUA^C)m(B) \quad , \end{aligned} \tag{21c}$$

$$\begin{aligned} m(U_{AUB}) = {} & m(A^C)m(BUB^C)+m(AUA^C)m(B^C)+ \\ & m(AUA^C)m(BUB^C) \quad . \end{aligned} \tag{21d}$$

These results, together with those given in the last subsection, can be considered now to constitute the basic conjunction and disjunction operators of an OT propositional inferencing calculus. It should be noted that each of the summary tables displays a similar projection mapping pattern. That is, the number of terms in each of the four types of partition elements (i.e. C, U, etc.) is the same. In addition, the central four elements of the table, which are key elements of a classical propositional calculus, have the same operational properties as the column and row propositions. That is, they can be conjunctively and disjunctively combined with each other to generate the other twelve elements of the non-classical calculus. This pattern is useful in understanding the derivation of an OT truth-operator representation of these results which will now be developed.

4. TRUTH-VALUE OPERATORS

The propositional operators developed in the last section can be put into another from, if we define a truth-propositional (i.e. logical) interpretation for the mass distributions $\underline{S}_A$, $\underline{S}_B$, and $\underline{S}_I$. To do this, we will assume that the set-theoretical forms previously discussed can be treated alternately as truth-propositions (i.e. statements with definite truth-values). The propositional masses corresponding to these statements will then represent the direct evidence supporting these truth-propositions. In this interpretation, we will make use of the patterns present in both the AB and AUB summary tables to define a correspondence between elementary set-theoretic propositions (e.g. A, A^c, AUA^c and AA^c) and truth-values for these propositions (e.g. T, F, TUF and TF). The transformation thus defined, yields a four-valued logic that serves as an alternate interpretation of the simple propositional algebra just developed.

To begin, we first require the truth-propositional transformation to include conventional Boolean algebra as a subset. For this to be the case, the relationships between set-theoretic propositions and truth-propositions are defined as follows:

$$A \equiv T \quad , \quad A^c \equiv F \quad , \quad AUA^c \equiv TUF \quad , \quad AA^c \equiv TF \quad , \qquad (22a)$$

$$B \equiv T \quad , \quad B^c \equiv F \quad , \quad BUB^c \equiv TUF \quad , \quad BB^c \equiv TF \quad . \qquad (22b)$$

Using these expressions, we can now construct logical truth-tables of the same form as Tables 1 and 2. These tables define OT truth-operators representing the basic propositional combination procedures developed in the last section. In particular, the table definitions of the conjunction operator ∧ and disjunction operator ∨, are seen to be

	∧	B: TF	T	F	TUF
	TF	TF	TF	F	F
A	T	TF	T	F	TUF
	F	F	F	F	F
	TUF	F	TUF	F	TUF

Table 3. Truth-table for the OT conjunction operator.

and

	∨	B: TF	T	F	TUF
	TF	TF	T	TF	T
A	T	T	T	T	T
	F	TF	T	F	TUF
	TUF	T	T	TUF	TUF

Table 4. Truth-table for the OT disjunction operator.

These results can readily be seen to form a four-valued truth algebra (i.e. logic) with the central four elements of each table reproducing the results normally defined in a two-valued Boolean algebra (i.e. the central elements form the Boolean truth tables for the conjunction and disjunction operators). The rest of each table constitutes a four-valued OT truth-algebra which is constructed from the basic Boolean results by using the definitions of conflict and undecidability given in Eqs. (6) and (22). The four-valued algebra so derived is most easily summarized by the following simple Boolean lattice:

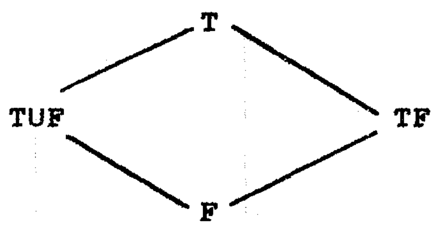

Figure 2. Lattice structure for OT four-valued truth algebra.

This lattice has been used in the past by others[9,10] as a basis for the development of a four-valued logic suitable for computer circuit design and other related applications. It is seen here to arise as a natural consequence of partitioning the results derived from the basic OT intersection operation. Moreover, it provides a useful viewpoint from which to rederiving these operators from the set-theoretic representation of the full OT algebra, as will be seen next.

5. DIRECT DERIVATION OF FOUR-VALUED LOGIC

As an alternative way of looking at the truth algebra (i.e. logic) just presented, we will explore a derivation of truth-operators directly from the set-theoretic definitions of the general OT algebra. If we again assume that we want to develop a four-valued truth algebra that includes Boolean operations as a subset, we can start with a two-element set $\Theta=\{x,\tilde{x}\}$ having a power set 2^{Θ} of the form

$$2^{\Theta} = \{ \phi , x , \tilde{x} , \Theta \} . \tag{23}$$

If we are to preserve the core Boolean algebra we desire, the symbols x and $\tilde{x}$ used here, can be interpreted in truth-value form in the following manner:

$$\phi = F , \Theta = T , x = T \cup F , \tilde{x} = TF . \tag{24}$$

In this framework, it is clear that the conventional Boolean truth operations arise in this set-theoretic framework from applying the standard set union, intersection and complement operations to the ϕ and Θ elements. In an extended four-valued algebra, we see that x and $\tilde{x}$ are also truth-propositions which have a conventional set-theoretic interpretation. That is, they obey the standard set intersection, union and complement rules

$$x \cap \tilde{x} = \phi , x \cup \tilde{x} = \Theta , \tilde{\tilde{x}} = x . \tag{25}$$

In this truth-propositional context, x and $\tilde{x}$ can be seen to be propositions with intermediate truth-values lying between T and F. These intermediate truth-propositions are consistent with the intended OT interpretation of x and $\tilde{x}$ as undecidability and conflict, respectively.

The assumptions above can now used together with the previously published OT set-theoretic definitions of union, intersection, and complement[1] to define four-valued OT logic operators. The masses in this formalism will represent the evidence directly supporting the particular truth-value of each proposition. The OT union operation $\oplus$ and its underlying set operations given in Eqs. (24) and (25), in this context is defined by

$$\underline{S}_U = \underline{S}_A \oplus \underline{S}_B \ , \tag{26a}$$

$$m(s_{U_k}) = \sum_{s_{A_i} \cup s_{B_j} = s_{U_k}} m(s_{A_i}) m(s_{B_j}) \ , \tag{26b}$$

where for the elements $s \in 2^{\Theta}$ we have only the four truth-values

$$s_1 = TF \ , \ s_2 = T \ , \ s_3 = F \ , \ s_4 = TUF \ . \tag{26c}$$

Using this OT union operator, we get the following truth-table for the OT logical disjunction operator:

		S_B			
	∨	TF	T	F	TUF
	TF	TF	T	TF	T
S_A	T	T	T	T	T
	F	TF	T	F	TUF
	TUF	T	T	TUF	TUF

Table 5. Truth-table for the set-theoretic OT disjunction operator.

Here, the masses assigned to each element of this table are, by virtue of the OT union combination rule, the same product mass assignments given in Eq. (26b).

Similarly, using the OT intersection rule given before [see Eqs. (3) and (4)] as

$$\underline{S}_I = \underline{S}_A \otimes \underline{S}_B \ , \tag{27a}$$

$$m(s_{I_k}) = \sum_{s_{A_i} \cap s_{B_j} = s_{I_k}} m(s_{A_i}) m(s_{B_j}) \ , \tag{27b}$$

with

$$s_1 = TF \; , \; s_2 = T \; , \; s_3 = F \; , \; s_4 = TUF \quad , \qquad (27c)$$

gives the following truth-table for the OT logical conjunction operator:

		s_B			
	$\wedge$	TF	T	F	TUF
	TF	TF	TF	F	F
s_A	T	TF	T	F	TUF
	F	F	F	F	F
	TUF	F	TUF	F	TUF

Table 6. Truth-table for the set-theoretic OT conjunction operator.

The masses here are those given in Eq. (27b).

As can be seen from these results, they correspond precisely to those of the four-valued algebra derived previously in Section 3. The Boolean truth-tables for the disjunction and conjunction operations are again a subset of this four-valued logic. The lattice given in Fig. 2, also summarizes these results. The rederivation is, therefore, seen to be consistent with the results given in Refs. [9] and [10] mentioned previously, which were derived directly using lattice theory only. The specific interpretation of the extended truth-propositions used here as conflict and undecidability, are the essence of the extension of OT over conventional probabilistic methodologies.

Much of what has been presented so far also has immediate interpretation in DST, in that the undecidable uncertainty representation has the same operational characteristics of Shafer's uncommitted belief. This aspect of the theory is sufficient to formulate a three-valued logic for interpreting the results presented. This logic is clearly a subset of the OT formalism. The major difference between DST and OT, in this case, resides in the handling of conflict which always occurs in DST theory as a result of the conjunctive combination of information. In OT this conflict is represented explicitly and propagated as such. In DST it is eliminated by renormalization.

6. LOGICAL INFERENCE AND IMPLICATION RULES

The developments presented in the last two sections are intended for applications which require evaluations of truth-values in logical or Boolean networks. Expert system applications can be also be addressed, if inferencing with rules can be represented in this same OT framework. We will try to add such rules by looking at rule-based inference from a logical operator point of view. In this context, the OT formalism will be used to define the inferencing operations of implication, modus ponens and modus tollens. These operations will be defined in both set-theoretic and truth-value forms suitable for use in computer inferencing algorithms. For the purposes of this paper, rules will be interpreted in a strictly logical fashion. Non-logical (i.e. heuristic) alternatives, suitable for more general use in expert systems, however, will also be briefly mentioned.

6A. A Partition for Logical Implication

The methods outlined in the previous sections can be used to handle logical implication if we treat the classical material implication $A \rightarrow B$ set-theoretically. That is, assume that material implication is equivalent to the compound proposition $A^c \cup B$. A mass distribution representing this implication rule can then be written in the following form:

$$\underline{S}_R = \left[\begin{array}{cccc} m(C_R) & m(R) & m(Rc) & m(U_R) \\ C_R\ , & R\ , & R_C\ , & U_R \end{array} \right] . \qquad (28)$$

Here the components of the rule are defined to be

$$R \equiv A^C \cup B = \{AB, A^C B, A^C B^C\} = \{\ x_1, x_2, x_3, x_5, x_6, x_7, x_8\ \} \ , \qquad (29a)$$

$$R^C \equiv AB^C = \{AB^C\} = \{\ x_4, x_5, x_6\ \} \ , \qquad (29b)$$

$$C_R \equiv RR^C = \{AB, A^C B, A^C B^C\} \cap \{AB^C\} = \{ABB^C, AA^C B^C, AA^C BB^C\} = \{\ x_5, x_6\ \} \ , \qquad (29c)$$

$$U_R \equiv R \cup R^C = \{AB, A^C B, A^C B^C\} \cup \{AB^C\} = \{AB, A^C B, AB^C, A^C B^C\} = \{\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\ \} \ . \qquad (29d)$$

In this context, it is clear that material implication is being considered as a proposition for there is evidence to measure its validity and negation as well as its internal conflict and undecidability. This propositional form should be quite easy to use in expert system applications and is consistent with the classical use of logical implication in two-valued logic in the limit of non-conflicting and fully resolved information. Its use in rule based-inferencing can now be illustrated by developing suitable OT inference operators.

6B. A Modus Ponens Operator

The logical inference made when a modus ponens operation is invoked is the use of a premise and a rule to deduce a conclusion. Specifically, for the context of this paper, the truth of proposition A, together with an implication rule R, of the form $A \rightarrow B$, will be used to deduce the truth of proposition B. In this case, a mass distribution for proposition A will be conjunctively combined with a mass distribution for the rule R (for which proposition A acts as a premise) to deduce the mass distribution for proposition B.

This procedure can be put into the OT formalism by assuming that the mass distributions for the premise and the rule are both assessed independently. The inferences about B will be drawn from the conjunctive combination of these two information sources. This combination will be performed using the same methods illustrated in the previous sections. That is, the OT intersection rule and the four-valued logic developed for the conjunction and disjunction operators will be applied.

To begin, we treat the premise of an implication rule as a OT mass distribution which is represented in the following partition of 2^{Θ}:

$$\underline{S}_A = \left[\begin{array}{cccc} m(C_A) & m(A) & m(A^c) & m(U_A) \\ C_A\ , & A\ , & A^c\ , & U_A \end{array} \right] . \tag{30}$$

Applying the OT intersection operator to the premise and implication rule mass distributions, we see that the resulting mass distribution can be written as follows:

$$\begin{array}{l} \underline{S}_{AR} = \Big[\begin{array}{cccc} m(C_A)m(C_R) & m(C_A)m(R) & m(C_A)m(R^c) & m(C_A)m(U_R) \\ AA^cRR^c\ , & AA^cR\ , & AA^cR^c\ , & AA^c(RUR^c)\ , \end{array} \\[2ex] \quad \begin{array}{cccc} m(A)m(C_R) & m(A)m(R) & m(A)m(R^c) & m(A)m(U_R) \\ ARR^c\ , & AR\ , & AR^c\ , & A(RUR^c)\ , \end{array} \\[2ex] \quad \begin{array}{cccc} m(A^c)m(C_R) & m(A^c)m(R) & m(A^c)m(R^c) & m(A^c)m(U_R) \\ A^cRR^c\ , & A^cR\ , & A^cR^c\ , & A^c(RUR^c)\ , \end{array} \\[2ex] \quad \begin{array}{cccc} m(U_A)m(C_R) & m(U_A)m(R) & m(U_A)m(R^c) & m(U_A)m(U_R) \\ (AUA^c)(RR^c)\ , & (AUA^c)R\ , & (AUA^c)R^c\ , & (AUA^c)(RUR^c) \end{array} \Big] . \end{array} \tag{31}$$

If the compound propositions in this representation are rewritten in terms of the basic propositions of an (A,B)-space defined in Fig.1 and Eq. (9), the propositional terms appearing in these results can be put into the following summary table:

		R			
	AR	RR^C	R	R^C	RUR^C
	AA^C	AA^CB^C	AA^C	AA^CB^C	AA^C
A	A	$ABB^CUAA^CB^C$	$ABUAA^CB^C$	AB^C	A
	A^C	AA^CB^C	A^C	AA^CB^C	A^C
	AUA^C	$ABB^CUAA^CB^C$	A^CUB	AB^C	Θ

Table 7. Summary table for the conjunction of AR.

As before, this summary table representation can be used to answer specific inference questions by partitioning the results by the specific proposition being investigated. In this case inferences about B are of interest.

Using the B-partition, represented as

$$S_B = \{ BB^C , B , B^C , BUB^C \} , \qquad (32)$$

we can now collect the terms in Table 7 which bear directly on the propositions in this B-partition to determine the inferences which can be drawn about B. In this projection process, various assumptions can be made about the allowable mass distributions which can be represented in (A,B)-space. Each of these assumptions will lead to different inferencing results, some of which will be strictly logical and others purely heuristic in use.

The most important of these assumptions relates to the handling of the propositional term AA^CB^C which appears in a number of positions in the summary table. For instance, it is entirely possible to treat this term as being equivalent to $C_A B^C$ and thus conclude that any inferences which yield this term imply that B^C is true (i.e. we can treat the conflict represented in AA^CB^C as being conflict in an A-partition with no conflicting effect on inferences about B). This approach is a heuristic possibility, one that has definite applicability in most practical expert systems. It is one which does not follow, however, from strictly logical considerations, as we will see shortly.

If instead, we are to treat rules from a strictly logical point of view, then the central four propositional results in Table 7 must yield the inferences expected from classical two-valued logic. In this case the term representing the intersection of A and R must yield the result AB and, therefore, the inference B in the modus ponens case under discussion. This implies that the AA^CB^C term must either have no mass assigned to it by definition, or it must be treated as a null-element in this analysis. To be consistent with the other OT logical inference operations which can also be analogously defined from these combined mass distribution results (i.e. $A^C \rightarrow B$, $A \rightarrow B^C$, etc.), all such conflicting propositional forms must be treated similarly.

To comprehensively treat all these cases then, the null-space approach will be used in the analysis which follows. In set-theoretic form, this assumption for all rule-based inferencing will thus be

$$\{ x_5 , x_6 , x_7 , x_8 \} = \{\phi\} \quad . \tag{33}$$

This strictly logical interpretation of the modus ponens operation then leads to the following table, which summarizes the mapping of the results in Table 7 into a B-partition:

	B		R		
		RR^C	R	R^C	RUR^C
	AA^C	C_B	C_B	C_B	C_B
A	A	C_B	B	B^C	U_B
	A^C	C_B	U_B	C_B	U_B
	AUA^C	C_B	U_B	B^C	U_B

Table 8. Summary table for B-partition of the conjunction of AR.

If we now use the same truth-value mapping given in Eq. (22), we can convert these results into a truth-table representation for a four-valued OT modus ponens operator. The resulting table is as follows:

	MP		R		
		TF	T	F	TUF
	TF	TF	TF	TF	TF
A	T	TF	T	F	TUF
	F	TF	TUF	TF	TUF
	TUF	TF	TUF	F	TUF

Table 9. Truth-table for the OT modus ponens operator.

The masses corresponding to the four terms in this table are

$$m(TF) = m(C_A)+m(A)m(C_R)+m(A^c)m(C_R)+ m(A^c)m(R^c)+m(U_A)m(C_R) \quad , \tag{34a}$$

$$m(T) = m(A)m(R) \quad , \tag{34b}$$

$$m(F) = m(A)m(R^c)+m(U_A)m(R^c) \quad , \tag{34c}$$

$$m(TUF) = m(A)m(U_R)+m(A^c)m(R)+m(A^c)m(U_R)+ m(U_A)m(R)+m(U_A)m(U_R) \quad . \tag{34d}$$

It is clear from Tables 8 and 9, that the result of applying a classic two-valued logic to the modus ponens operation are preserved in the operations defined in the central four elements of these tables. That is, if A is true and the rule R is true, then B is true. If A is false and R is true, then the truth-value of B is undecidable, in that it can be true or false. The other terms in these tables, however, represent a generalization of these results which can be used to propagate both conflict and undecidability through a rule-based inferencing system.

An important point to note here, is that the central portion of Table 9 (i.e. the Boolean logic), is not conflict free. Even if mass is not assigned directly to a conflicting proposition it will appear as an inference resulting from mass being assigned to the propositions A=F and R^c=F. This is a direct result of interpreting the inference as $AR^c=\{\phi\}$ (i.e. absolute conflict) in a B-partition of the combined results.

6C. A Modus Tollens Operator

The same procedure above can be used by analogy to generate an OT modus tollens operator. This operator will represent the deductions that can be made about A given information, in the form of mass distributions, about B and the implication rule R.

First, the mass distributions for the partitions of the rule R and modus tollens premise B are combined using the OT intersection operation to give

$$\underline{S}_{BR} = [\ \overset{m(C_B)m(C_R)}{BB^cRR^c}\ ,\ \overset{m(C_B)m(R)}{BB^cR}\ ,\ \overset{m(C_B)m(R^c)}{BB^cR^c}\ ,\ \overset{m(C_B)m(U_R)}{BB^c(RUR^c)}\ ,$$

$$\overset{m(B)m(C_R)}{BRR^c}\ ,\ \overset{m(B)m(R)}{BR}\ ,\ \overset{m(B)m(R^c)}{BR^c}\ ,\ \overset{m(B)m(U_R)}{B(RUR^c)}\ ,$$

$$\overset{m(B^c)m(C_R)}{B^cRR^c}\ ,\ \overset{m(B^c)m(R)}{B^cR}\ ,\ \overset{m(B^c)m(R^c)}{B^cR^c}\ ,\ \overset{m(B^c)m(UR)}{B^c(RUR^c)}\ ,$$

$$\overset{m(U_B)m(C_R)}{(BUB^c)(RR^c)}\ ,\ \overset{m(U_B)m(R)}{(BUB^c)R}\ ,\ \overset{m(U_B)m(R^c)}{(BUB^c)R^c}\ ,\ \overset{m(U_B)m(U_R)}{(BUB^c)(RUR^c)}\ ]\ . \quad (35)$$

Substituting the definitions of the propositions in the R-partition [i.e. Eq. (29)] into these results, we get the following summary table:

			R		
	BR	RR^c	R	R^c	RUR^c
	BB^c	ABB^c	BB^c	ABB^c	BB^c
B	B	ABB^c	B	ABB^c	B
	B^c	$ABB^cUAA^cB^c$	$A^cB^cUABB^c$	AB^c	B^c
	BUB^c	$ABB^cUAA^cB^c$	A^cUB	AB^c	Θ

Table 10. Summary table for the conjunction of BR.

These results can now be used to form an A-partition. We use an A-partition here since we are trying to make inferences about A from the combined information in proposition B and the rule R. Again, as in the case of the modus ponens operator, many definitions of this operator are possible. To maintain a strictly logical interpretation of the modus tollens operation, however, the problematical ABB^C proposition in this case will be assumed to be a null-space element (i.e. an absolute conflict term). For general logical consistency then, we use Eq. (33) again to give the following summary table:

		R			
	A	RR^C	R	R^C	RUR^C
B	BB^C	C_A	C_A	C_A	C_A
	B	C_A	U_A	C_A	U_A
	B^C	C_A	A^C	A	U_A
	BUB^C	C_A	U_A	A	U_A

Table 11. Summary table for the A-partition of the conjunction of BR.

If we now use the same proposition to truth-value mapping given in Eq. (22), we can convert these results into the truth table representation of an OT modus tollens operator. This table is as follows:

		R			
	MT	TF	T	F	TUF
B	TF	TF	TF	TF	TF
	T	TF	TUF	TF	TUF
	F	TF	F	T	TUF
	TUF	TF	TUF	T	TUF

Table 12. Truth-table for the OT modus tollens operator.

with corresponding masses

$$m(TF) = m(C_B)+m(B)m(C_R)+m(B)m(R^c)+ m(B^c)m(C_R)+m(U_B)m(C_R) \quad , \tag{36a}$$

$$m(T) = m(B^c)m(R^c)+m(U_B)m(R^c) \quad , \tag{36b}$$

$$m(F) = m(B^c)m(R) \quad , \tag{36c}$$

$$m(TUF) = m(B)m(R)+m(B)m(U_R)+m(B^c)m(U_R)+ m(U_B)m(R)+m(U_B)m(U_R) \quad . \tag{36d}$$

These results again clearly show that classical two-valued modus tollens deductions are preserved in the central portions of these tables. That is, if B is false and the rule R is true, the we deduce that A is false. If B is true and R is true, then the truth-value of A is undecidable, in that A can be true or false. The added feature of the four-valued OT logic table definition of this operation is that conflict and undecidability can also be represented and propagated through inferencing schemes.

6D. A Logical Implication Rule

As a final point, we will look at the manner in which material implication can be derived as the result of an OT combination operation. This approach allows implication rules to be constructed from information about the premise and conclusion of a specific implication.

Using the classical definition of material implication again, the rule R, denoting $A \rightarrow B$, can be written as $A^c \cup B$. This implication rule can be viewed as an R-partition of the information contained in mass distributions for A and B.

Using the combined information given in Eq. (8) as the underlying basis for this partition, we see that the rule itself can be represented by the following table:

	R	B: BB^c	B: B	B: B^c	B: BUB^c
	AA^c	C_R	R	C_R	R
A	A	C_R	R	R^c	U_R
	A^c	R	R	R	R
	AUA^c	R	R	U_R	U_R

Table 13. Summary table for the R-partition of the conjunction of AB.

The masses for each term in this table are easily seen to be

$$m(C_R) = m(AA^c)m(BB^c)+m(AA^c)m(B^c)+m(A)m(BB^c) \quad , \qquad (37a)$$

$$m(R) = m(AA^c)m(BB^c)+m(AA^c)m(B^c)+m(A)m(B)+ m(A^c)+m(AUA^c)m(BB^c)+m(AUA^c)m(B) \quad , \qquad (37b)$$

$$m(R^c) = m(A)m(B^c) \quad , \qquad (37c)$$

$$m(U_R) = m(A)m(BUB^c)+m(AUA^c)m(B^c)+ m(AUA^c)m(BUB^c) \quad . \qquad (37d)$$

If we again use the truth-value mapping given in Eq. (22), we can convert the summary table results for R into truth-table form. This resulting table is as follows:

		B			
	R	TF	T	F	TUF
	TF	TF	T	TF	T
A	T	TF	T	F	TUF
	F	T	T	T	T
	TUT	T	T	TUF	TUF

Table 14. Truth-table for the OT logical implication rule.

The *corresponding* truth-propositional masses are then

$$m(TF) = m(C_A)m(C_B)+m(C_A)m(B^c)+m(A)m(C_B) \tag{38a}$$

$$m(T) = m(C_A)m(C_B)+m(C_A)m(B^c)+m(A)m(B)+ m(A^c)+m(U_A)m(C_B)+m(U_A)m(B) \quad , \tag{38b}$$

$$m(F) = m(A)m(B^c) \quad , \tag{38c}$$

$$m(TUF) = m(A)m(U_B)+m(U_A)m(B^c)+m(U_A)m(U_B) \quad . \tag{38d}$$

This construction is suitable now for evaluating logical rules from independent premise and conclusion information sources. It is this form which lends itself most easily to expert system applications. Logical rules can be elicited from experts based on the observed relationships between premises and conclusions and mass distribution data.

6E. Summary of OT Operators

In this section a series of operators were defined to handle logical implication in inferencing systems. We have seen here clearly, that the modus ponens and modus tollens operators as well as the definition of the rule R as A^CUB, all represent material implication in classical logic. That is, the assumptions made in deriving these OT operators are all logically consistent with this form of implication. In this sense, if A implies B, then B^C implies A^C and A^CUB can logically be defined as material implication. Furthermore, we see that a two-valued Boolean logical representation of implication is a subset of these OT four-valued operators as well. The basis for all this consistency is again the assumption that the conflict terms, represented by $\{x_5, x_6, x_7, x_8\}$, are all defined as null-space elements.

The logical results so derived, can also be viewed in set-theoretic terms by defining a power set based on the four propositions AB, A^CB, AB^C, and A^CB^C, which now have null intersections. This interpretation is, therefore, also consistent with DST, in that no conflict terms, other than ϕ, appear in the final results. The point of departure of the OT results with DST in this case, however, is that mass is assigned to ϕ in OT and can be propagated through further inferences.

In addition, the fact that conflict appears directly in the Boolean core of the OT logic operators for modus ponens and modus tollens, further differentiates the OT results from those based probability theory or Lukasiewicz' three-valued logic[11]. They more closely resemble the proposed rules for general Post algebras, particularly Belnap's four-valued subset of this algebra[12].

7. CONCLUSIONS

The results derived in this paper extend Operator-Uncertainty Theory (OT) to problems of uncertainty propagation in truth-value and rule-based inferencing systems. The OT algebra has been used here to derive operators for the propagation of logical uncertainties which allow for representations of conflict and undecidability in propositional forms. Propagation operators for the propositional and logical operations of disjunction and conjunction were defined, and in addition, logical operators for implication, modus ponens and modus tollens were also proposed.

The operators derived using the OT methodology were found to give rise to a four-valued logic similar to that used in computer circuit design, analogous to Belnap's four-valued algebra. This framework allows uncertainty in inference rules to be represented in a very convenient form for use in expert systems. The theory is general enough to deal with questions of conflict and undecidability, and to propagate their effects through all the basic logical inference operations.

For many expert system problems, the possibility of assigning masses (i.e. probabilistic evidence) to both premises and rules is an important extension of existing inference schemes. The OT formalism provides a convenient theoretical basis which accommodates probabilistic mass assignments and their propagation through rules. The applicability of the OT methodology in any particular expert system will depend on the definitions or other constraints applied by the researcher to the problem being modelled. In some cases, rules will be logically defined and the full methodology developed in this paper can be applied. In other cases, a more heuristic basis for defining rules might be more appropriate, requiring other interpretations or partitions of the OT results presented.

In any event, for those rule-based systems which are amenable to either probabilistic or frequency interpretations of their rules and premises, OT represents a comprehensive framework for handling inferencing under uncertainty. The OT formalism should, in this case, be applicable to either logical or heuristic rule-based expert systems.

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