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Foundations of 0-Theory I: The Intersection Rule

E. M. Oblow

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FOUNDATIONS OF O-THEORY I: THE INTERSECTION RULE*

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ABSTRACT

In a previous paper, O-theory (OT), a hybrid uncertainty theory was proposed for dealing with problems of uncertainty in logical inference. The foundations of one of the concepts introduced, the OT intersection operator, are explored in this paper. The developments rely solely on set-theoretic and probability theory notions.

The original OT intersection rule had as its basis Dempsters' rule of combination. A more fundamental basis for the OT rule will be shown to be classical probability theory. Mass assignments in the Dempster-Shafer theory (DST) formalism are first reevaluated yielding a more basic relationship between masses and probabilities. These results are then used to show that the OT intersection rule can be derived from first principles in probability theory. Furthermore, without resort to conditional probabilities, a simpler axiomatic basis can be used to establish this rule. It can, therefore, be used as alternative to Bayes' theorem for combining probabilistic belief consistently. Dempsters' rule will be shown to be a special case of the OT result. It too will be derived from probability theory axioms.

The formal connection between mass and probability presented also makes distinctions between DST and probability theory less consequential. DST is still seen to be a generalization of the concept of probability, but it now fits within the original probability framework. The DST conception of uncommitted belief will also be shown to be compatible with probability theory.

FOUNDATIONS OF O-THEORY I:
THE INTERSECTION RULE

E. M. OBLOW

1) Introduction

In a previous paper[1], O-theory[1,2] (OT), a hybrid uncertainty theory was proposed for dealing with problems of uncertainty in logical inference. The theory was explained in a language which combined concepts from Dempster-Shafer theory[3,4] (DST) and fuzzy set theory[5] (FST). Since publication of that paper, other papers have also introduced similar notions[6,7]. Additional research has revealed a rich foundation in both set theory and classical probability theory for the concepts introduced. This paper will explore the foundations of one of these concepts, the OT intersection operator[1]. The developments to be presented will reveal its fundamental importance in combining belief sets. This rule will be used as a means of introducing a detailed foundation for OT in general.

In its original formulation, the OT intersection rule had as its basis Dempsters' rule of combination[3] as defined in DST[4]. In this paper we will demonstrate that a fundamental basis for the OT intersection rule can be found in classical probability theory[8,9]. To accomplish this, some fundamental notions about mass assignments in the DST formalism must first be reevaluated. A more basic relationship between masses and probabilities will grow out of this reexamination. These results will then be used to show that the OT intersection rule can indeed be derived from first principles in probability theory. Furthermore, without resort to conditional probabilities, a simpler axiomatic basis can be used to establish this rule. In this context it can be used as an alternative to Bayes' theorem for combining probabilistic belief consistently. Dempsters' rule will be shown to be a special case of the OT result and it too will be derived from probability theory axioms.

The particular approach used in this paper is to reduce certain DST concepts to a probability basis. This more fundamental view offers a firmer foundation for OT. It also holds promise for reinterpreting the computational algorithms used in the applications of DST. The OT probabilistic framework will be shown to be more comprehensive. Using this approach, some of the problems that have plagued DST in the past[10,11] can be recast into a more favorable light.

As in the previous OT paper, only finite sets will be dealt with. This simplifies the developments considerably, but the choice is merely due to the current lack of proofs needed to extend the theory to continuous problems. For the intersection rule to be discussed, however, it is believed that this restriction can be eliminated without too much additional effort. The results presented will be based on purely probabilistic concepts for which continuous analogies already exist.

2) Intersection Rule in OT

A brief notational review of some basic concepts in OT will be needed before the foundation for the intersection operator can be presented. The starting point for the developments which follow is, as always[1], a possibility set $\Theta=\{x_1,x_2,\ldots x_n\}$ whose elements x_i are disjoint and completely span Θ. A mapping, m, of the power set 2^Θ into real numbers in [0,1] (i.e. $m:2^\Theta\rightarrow[0,1]$), gives rise to the masses which are assigned to each element of 2^Θ. These masses sum to unity over all the power set elements, and together with the power set, they form a belief set $\underline{\Theta}$ in the OT formalism.

Given a group of these belief sets $\underline{A}$, $\underline{B}$, $\underline{C}$,....., which are presumed to be independent, the problem is to devise a rule for combining these sets in a consistent and useful manner. All the information contained in the initial belief sets individually should be represented faithfully in their combination.

In OT, the combination of belief sets by the intersection rule is denoted by

$$\underset{\sim}{S} = \underset{\sim}{A} \otimes \underset{\sim}{B} \otimes \underset{\sim}{C} \ldots\ldots \quad , \tag{1}$$

where $\otimes$ is the OT intersection operator. The developments to follow will derive the previously given definition of this operator from first principles. This will be accomplished after first devising a new probabilistic representation for OT belief sets. Further developments will specify the field in which $\underset{\sim}{S}$ resides and show that it is a subset of the one used in classical probability theory.

3) Probability Representation

To begin the developments, a reformulation of the DST-based structure of OT belief sets is proposed. This reformulation allows probability theory to be applied directly to such a set. In addition to its fundamental importance in OT, this reformulation can also serve as a computational tool which might be useful in some applications.

The basic premise used in reformulating the belief sets, is that all mass in the belief set $\underset{\sim}{\Theta}$ which is not assigned to an elemental member of Θ can be redistributed to such elemental members without loss of generality or computational rigor. This redistribution must be done in parametric form to retain this generality.

For example, start with masses m, a possibility set $A=\{x_1^A, x_2^A, \ldots, x_n^A\}$, the set of integers $\{1,\ldots,n\}$ denoted by N and the belief set $\underset{\sim}{A}$, given by

$$\underset{\sim}{A} = [\ \overset{m_1^A}{x_1^A},\ \overset{m_2^A}{x_2^A}, \ldots, \overset{m_{12}^A}{(x_1^A, x_2^A)}, \ldots, \overset{m_{12\cdots n}^A}{(x_1^A, x_2^A, \ldots, x_n^A)}\ ]\ ,$$

$$\equiv \{ \; (x^A_\alpha, m^A_\alpha) \;|\; \alpha \subset N, \; x^A_\alpha \in 2^A \; \} \quad . \tag{2}$$

We can redistribute the mass of any typical power set members of 2^A (i.e. m^A_α) to elemental members of A in the following manner:

$$\text{for,} \qquad x^A_\alpha \equiv \{ \; x^A_{\alpha_i} \;|\; i \in \alpha, \; x^A_{\alpha_i} \in A \; \} \quad , \tag{3a}$$

$$\text{let,} \qquad m(x^A_{\alpha_i}) \equiv a^i_\alpha \quad , \tag{3b}$$

$$\text{where,} \quad m^A_\alpha \equiv \sum_{i \in \alpha} a^i_\alpha \quad , \quad a^i_\alpha \geq 0 \quad , \quad \forall i \in \alpha \quad . \tag{3c}$$

Here, the a^i_α's are unspecified parameters except for single element subsets (i.e. where $x^A_\alpha = x^A_i$), in which case $a^i_i = m^A_i$.

If we redistribute the masses of all the members of 2^A as prescribed in Eq.(3), the result after collecting terms will be

$$\underset{\sim}{A} = \left\{ \begin{matrix} p^A_1 \\ x^A_1 \end{matrix} \; , \; \begin{matrix} p^A_2 \\ x^A_2 \end{matrix} \; , \; \ldots\ldots \; , \; \begin{matrix} p^A_n \\ x^A_n \end{matrix} \right\} \quad , \tag{4}$$

where the p^A_i's are now probabilities defined as

$$p^A_i \equiv p_A(x^A_i) = \sum_{\alpha \cap i} a^i_\alpha \quad , \qquad i \in N, \; \alpha \subset N \quad . \tag{5}$$

Note here for future use, that

$$\sum_i \sum_{\alpha \cap i} a^i_\alpha = \sum_i p^A_i = \sum_\alpha m^A_\alpha = \sum_\alpha \sum_{i \in \alpha} a^i_\alpha \tag{6}$$

In the form given in Eq.(4), $\underset{\sim}{A}$ is now a belief set which can be used in classical probability theory. Masses assigned to members of the power set 2^A are replaced by probabilities associated only with elemental members of the set A. Since parameters are used in the definitions of the individual probabilities, however, this representation is clearly an extension of the classical probability concept. The notions of upper and lower probabilities for elemental members of A, defined in DST, are also fundamentally part of this reformulation and must be used to interpret all results. In the current developments they simply represent evaluations of the elemental probabilities using the upper and lower limits of the a_α^i's in each expression.

Several important points should be noted here before continuing the developments. First, this construct can easily be converted back to its original DST form by redistributing the probabilities back into the power set elements from which they were derived (Eq.(6) is useful in this regard). This ability to go back and forth between probabilities in A and masses in 2^A will be uses extensively in OT. It can be viewed as one of the basic foundations needed to derive the OT intersection rule.

Second, it should be noted that to be rigorous in both DST and classical probability theory, we must assume that both $m(\phi)=0$ and $p(\phi)=0$. The $m(\phi)=0$ condition is not a stated premise of OT, but since it is used inconsequentially in the discussions to follow, it will be assumed and no further comments are needed here. The $p(\phi)=0$ condition is fundamental to probability theory, and this assumption will be used strictly. The distinctions between the two are related to normalization rules which will explained later.

Finally, note that when viewed in a probability framework, the masses in non-elemental subsets of A in DST now appear as parameters, which explicitly represent a correlated relationships between the elemental probabilities. If actual values were used for the a_α^i's this interpretation would be lost. The DST notion of mass in non-elemental sets can, therefore, be reinterpreted in a much more powerful way in probability theory than simply uncommitted belief.

To complete these initial developments, one further representational change needs to be made. This change will be to convert the set notation used to this point into a functional notation. Here, belief sets will be explicitly represented as the union of their elemental members. This notational change, as simple as it may seem, is another fundamental point, as will be seen shortly.

In this light, let

$$A \equiv (x_1^A \cup x_2^A \cup \ldots\ldots \cup x_n^A) = \bigcup_i x_i^A \quad . \tag{7}$$

Immediately, we see that probability theory can be used to evaluate probabilities for functions of this form. For instance, the probability of A itself can now be established by noting that the elemental x_i's are disjoint, as assumed before. That is, if we apply the union rule for disjoint sets in probability theory to the expression in Eq.(7), we get

$$p(A) = p_A(\bigcup_i x_i^A) = \sum_i p_A(x_i^A) = \sum_i p_i^A \ . \tag{8}$$

This result can likewise be repeated for the other sets B, C,... to establish their probabilities.

4) Derivation of Intersection Rule

Using the concepts introduced in the previous section, we can now derive the definition proposed previously for the OT intersection operator. In the same vein, we can show under what simplifying assumptions Dempsters' rule of combination can also be derived as a subset of this definition.

If we denote the OT intersection of belief sets again as

$$\underline{S} = \underline{A} \otimes \underline{B} \otimes \underline{C} \ldots\ldots\ldots \quad , \tag{9}$$

but now assume that the probability representation is to be used, we propose to show that the previously defined $\otimes$ operator can simply be interpreted as $\cap$, the conventional intersection operator in set theory.

Using the form given in Eq.(7) for belief sets, we first observe that using $\cap$ in place of $\otimes$ in Eq.(9), gives

$$S = (A \cap B \cap C \cap \ldots) \quad . \tag{10}$$

Breaking the sets A, B, C,... down into their elemental components, using Eq.(7) as a guide, yields

$$S = [(\bigcup_i x_i^A)\cap(\bigcup_j x_j^B)\cap(\bigcup_k x_k^C)\ldots] \quad , \; i,j,k,..=1,..,n \; , \tag{11}$$

which can be rearranged into equivalent disjunctive normal form as follows:

$$S = \bigcup_{ijk..} (x_i^A \cap x_j^B \cap x_k^C \cap \ldots) \quad , \quad i,j,k,..=1,..,n \quad . \tag{12}$$

Defining $s_{ijk\ldots}$ for later use as

$$s_{ijk\ldots} = (x_i^A \cap x_j^B \cap x_k^C \cap \ldots) \quad , \tag{13}$$

we finally get

$$S = \bigcup_{ijk..} s_{ijk...} \quad , \quad i,j,k,..=1,..,n \quad . \tag{14}$$

To make it clear that S is a belief set in probabilistic form (albeit with more members), the probabilities defined for its elemental members $s_{ijk...}$ must be evaluated. This is accomplished by simply using the classical probability intersection rule for sets that are independent. Since $\underline{A}$, $\underline{B}$, $\underline{C}$,... were all assumed to be independent, their elements are also and we can write for any general $s_{ijk...}$,

$$\begin{aligned} p_S(s_{ijk...}) &= p_S(x_i^A)p_S(x_j^B)p_S(x_k^C)... \\ &= p_i^A \; p_j^B \; p_k^C \; ... \quad . \end{aligned} \tag{15}$$

Using Eqs.(7) and (8) and the fact that the x's are disjoint, we also see that

$$\begin{aligned} p(S) &= p_S(\bigcup_{ijk..} s_{ijk...}) = \sum_{ijk..} p_S(s_{ijk...}) \\ &= \sum_i p_S(x_i^A) \sum_j p_S(x_j^B) \sum_k p_S(x_k^C) \; ... \\ &= p(A)p(B)p(C)... \quad , \\ & \quad i,j,k,..=1,..,n \quad . \end{aligned} \tag{16}$$

Although it is not apparent yet, Eqs.(14) to (16) represent the most fundamental form of the OT intersection rule. Much can be gained from analyzing results in this simple framework.

To continue, however, we must first establish the connection between the general results given in Eq.(14) and those published previously[1]. This is accomplished by converting them to mass and power set form. Using the definitions of the individual probabilities in terms of masses (i.e. Eqs.(3) and (5)), this first step can easily be performed. Basically, this consists of collecting terms with masses in parametric form and identifying them with terms representing specified masses only. This collection process results in a mass mapping for the power set of S.

The key to eliminating the parametric masses from Eq.(15), is to note that products of these masses are derived fundamentally from the intersection of power set elements in the original mass representations of $\underline{A}$, $\underline{B}$, $\underline{C}$,... Because of this observation, it is just as easy to show the desired results by deriving Eqs.(14) and (15) starting from the original power set forms of $\underline{A}$, $\underline{B}$, $\underline{C}$, ... This indeed will be done later.

For the moment, note that each individual x_i^A in probability form has parametric masses derived from many power set elements. This fact is another fundamental OT precept which can be written in functional form as

$$x_i^A \equiv \bigcup_{\alpha \cap i} x^A_{\alpha\, i} \quad , \tag{17a}$$

$$p_A(x_i^A) = \sum_{\alpha \cap i} p_A(x^A_{\alpha\, i}) = \sum_{\alpha \cap i} a^i_\alpha \quad . \tag{17b}$$

Using this expression as a guide, Eq.(14) can be rewritten as

$$S = \bigcup_{ijk..} [(\bigcup_{\alpha \cap i} x^A_{\alpha\, i}) \cap (\bigcup_{\beta \cap j} x^B_{\beta\, j}) \cap (\bigcup_{\gamma \cap k} x^C_{\gamma\, k}) \; \dots \;]$$

$$= \bigcup_{ijk..} \left[\bigcup_{\alpha \cap i} \bigcup_{\beta \cap j} \bigcup_{\gamma \cap k} \ldots (x^A_{\alpha_i} \cap x^B_{\beta_j} \cap x^C_{\gamma_k} \ldots) \right] . \qquad (18)$$

Switching the order of the union operations then gives

$$S = \bigcup_{\alpha\beta\gamma..} \left[\bigcup_{i \in \alpha} \bigcup_{j \in \beta} \bigcup_{k \in \gamma} \ldots (x^A_{\alpha_i} \cap x^B_{\beta_j} \cap x^C_{\gamma_k} \ldots) \right]$$

$$= \bigcup_{\alpha\beta\gamma..} \left[(\bigcup_{i \in \alpha} x^A_{\alpha_i}) \cap (\bigcup_{j \in \beta} x^B_{\beta_j}) \cap (\bigcup_{k \in \gamma} x^C_{\gamma_k}) \ \ldots \right] \qquad (19)$$

The terms in parenthesis here are easily seen from Eq.(3a) to be power set elements, so that we can finally write,

$$S = \bigcup_{\alpha\beta\gamma..} (x^A_\alpha \cap x^B_\beta \cap x^C_\gamma \cap \ \ldots) = \bigcup_{\alpha\beta\gamma..} s_{\alpha\beta\gamma..} \quad . \qquad (20)$$

This is the final power set representation of the intersection rule with no parametric masses present. To see this more clearly, the masses of each element must be evaluated. As before, using the probability intersection rule for independent sets gives

$$m^S_{\alpha\beta\gamma..} = p_S(s_{\alpha\beta\gamma..}) = p_S(x^A_\alpha \cap x^B_\beta \cap x^C_\gamma \cap ..)$$

$$= m^A_\alpha \; m^B_\beta \; m^C_\gamma \ \ldots \quad . \qquad (21)$$

These results can easily be related to the published DST form of the OT intersection rule[1], by simply assuming

$$x^A_i = x^B_i = x^C_i = \ \ldots \ = x^S_i \quad , \quad \forall i \in N \quad . \qquad (22)$$

This reduces S back to the same form and dimensionality as the original power sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, ... and gives them all a common notation and meaning as belief sets.

As indicated previously, it is also possible to arrive at the results given in Eqs.(18) to (21) by using the same approach presented above but starting with the power set form of $\underline{A}$, $\underline{B}$, $\underline{C}$,... (i.e. Eq.(2)). To see this, we start again with the functional notation in Eq.(10). Since we now typically have,

$$A = [\ x_1^A \cup x_2^A \cup \ldots \cup (x_1^A, x_2^A) \cup \ldots \cup (x_1^A, x_2^A, \ldots, x_n^A)]$$

$$= \bigcup_{\alpha} x_{\alpha}^A \quad , \quad x_{\alpha}^A \in 2^A \ , \ \alpha \subset N \quad , \tag{23}$$

we can expand S out in terms like those above, to get

$$S = [\ (\bigcup_{\alpha} x_{\alpha}^A) \cap (\bigcup_{\beta} x_{\beta}^B) \cap (\bigcup_{\gamma} x_{\gamma}^C) \ \ldots\ ]$$

$$= \bigcup_{\alpha\beta\gamma..} (\ x_{\alpha}^A \cap x_{\beta}^B \cap x_{\gamma}^C \cap \ldots\ ) \quad ,$$

$$= \bigcup_{\alpha\beta\gamma..} s_{\alpha\beta\gamma..} \quad , \tag{24}$$

$$\alpha, \beta, \gamma, \ldots \subset N \quad , \quad x_{\alpha}^A \in 2^A, \ x_{\beta}^B \in 2^B, \ x_{\gamma}^C \in 2^C, \ \ldots \quad ,$$

where a typical mass term is again given by Eq.(21).

Since each term like x_{α}^A is itself composed of the union of its elements, each $s_{\alpha\beta\gamma..}$ can be expanded to give

$$S = \bigcup_{\alpha\beta\gamma..} [(\bigcup_{i\in\alpha} x_{\alpha_i}^A) \cap (\bigcup_{j\in\beta} x_{\beta_j}^B) \cap (\bigcup_{k\in\gamma} x_{\gamma_k}^C) \ \ldots\ ]$$

$$= \bigcup_{\alpha\beta\gamma..} [\bigcup_{i\in\alpha} \bigcup_{j\in\beta} \bigcup_{k\in\gamma} \ldots (x_{\alpha_i}^A \cap x_{\beta_j}^B \cap x_{\gamma_k}^C \ldots)\ ] \quad . \tag{25}$$

After switching the order of the union operations and using the results in Eq.(17), we finally get

$$S = \bigcup_{ijk..} [\bigcup_{\alpha \cap i} \bigcup_{\beta \cap j} \bigcup_{\gamma \cap k} [(x^A_{\alpha_i}) \cap (x^B_{\beta_j}) \cap (x^C_{\gamma_k}) \ldots]$$

$$= \bigcup_{ijk..} [(\bigcup_{\alpha \cap i} x^A_{\alpha_i}) \cap (\bigcup_{\beta \cap j} x^B_{\beta_j}) \cap (\bigcup_{\gamma \cap k} x^C_{\gamma_k}) \ \ldots]$$

$$= \bigcup_{ijk..} (x^A_i \cap x^B_j \cap x^C_k \cap \ldots) = \bigcup_{ijk..} s_{ijk..} \ . \qquad (26)$$

These results, when combined with the simplifying assumptions given in Eq.(22), are precisely the definition of the intersection rule in OT published previously[1]. If the mass in the null set elements in this latter case is eliminated by renormalization, then this result is also seen to be that given by Dempster's rule of combination in DST.

The fact that both the probabilistic form, given in Eq.(14), and the power set form, given in Eq.(20), are equivalent and can be reduced to Dempster's rule is most interesting. The fact that this derivation was carried out straightforwardly in both a probabilistic and mass framework using only probability and set theory concepts is remarkable.

Comparison of Eq.(20) with the more compact form published previously[1], makes it clear that, mass assigned directly to the null set and mass appearing in the null set as a result of OT intersection operations, are both computational artifacts. In the former case, $m(\phi)$ can be ignored in any intersection operation. This mass can eventually be accounted for by subtracting the masses in all other elements of S (see for example, Eqs.(8) and (16)). In the latter case, the more fundamental interpretation presented here reveals that such mass is identified with the non-null elements $s_{ijk\ldots}$ (i.e. for which $i \neq j \neq k \neq \ldots$). These elements represent the OT 'null set' in the broader definition of set S just derived. The interpretation of these results in DST are somewhat different and these differences will be explored shortly. For now, the role of the belief set S in classical probability theory needs to investigated in more detail.

5) Interpretation of Results

To make the results presented in the last section clearer and to demonstrate their use in more concrete terms, a simple example will be explored. While this example is easier to deal with, it looses none of the generality of the complete derivation. It simply allows a systematic interpretation of the results to be illustrated more graphically.

As an example, then, take

$$\underline{S} = \underline{A} \otimes \underline{B} \quad , \tag{27}$$

with

$$\underline{A} = \{ \overset{m_1^A}{x_1^A} , \overset{m_2^A}{x_2^A} , \overset{m_{12}^A}{(x_1^A , x_2^A)} \} \quad , \tag{28}$$

$$\underline{B} = \{ \overset{m_1^B}{x_1^B} , \overset{m_2^B}{x_2^B} , \overset{m_{12}^B}{(x_1^B , x_2^B)} \} \quad , \tag{29}$$

and

$$m_1^A + m_2^A + m_{12}^A = 1 \quad , \quad m_1^B + m_2^B + m_{12}^B = 1 \quad . \tag{30}$$

Following the reformulation procedure given in Eqs.(3) to (5) and Eq.(7), $\underline{A}$ and $\underline{B}$ can be converted equivalently to

$$A = \left({x_1^A}^{p_1^A} \cup {x_2^A}^{p_2^A} \right) \quad , \tag{31}$$

$$B = \left({x_1^B}^{p_1^B} \cup {x_2^B}^{p_2^B} \right) \quad , \tag{32}$$

with

$$p_1^A = m_1^A + a_{12}^1 \quad , \qquad p_2^A = m_2^A + a_{12}^2 \quad , \tag{33a}$$

$$p_1^B = m_1^B + b_{12}^1 \quad , \qquad p_2^B = m_2^B + b_{12}^2 \quad , \tag{33b}$$

and

$$a_{12}^1 + a_{12}^2 = m_{12}^A \quad , \tag{34a}$$

$$b_{12}^1 + b_{12}^2 = m_{12}^B \quad . \tag{34b}$$

The OT intersection operation can now be rewritten using the set intersection symbol to give

$$S = [(x_1^A \cup x_2^A) \cap (x_1^B \cup x_2^B)] \quad , \tag{35}$$

which in disjunctive normal form is

$$S = [s_{11} \cup s_{12} \cup s_{21} \cup s_{22}] = \bigcup_{ij} s_{ij} \quad . \tag{36}$$

Here, the s_{ij}'s and their probabilities are given by

$$s_{11} = x_1^A \cap x_1^B \quad , \qquad p_{11}^S = p_1^A \, p_1^B \quad , \tag{37a}$$

$$s_{12} = x_1^A \cap x_2^B \quad , \qquad p_{12}^S = p_1^A \, p_2^B \quad , \tag{37b}$$

$$s_{21} = x_2^A \cap x_1^B \quad , \qquad p_{21}^S = p_2^A \, p_1^B \quad , \tag{37c}$$

$$s_{22} = x_2^A \cap x_2^B \quad , \qquad p_{22}^S = p_2^A \, p_2^B \quad . \tag{37d}$$

These results can be converted into belief set notation by expanding the probabilities in terms of masses as

$$p_{11}^S = (m_1^A + a_{12}^1)(m_1^B + b_{12}^1) = m_1^A m_1^B + a_{12}^1 m_1^B + b_{12}^1 m_1^A + a_{12}^1 b_{12}^1 \quad , \tag{38a}$$

$$p_{12}^S = (m_1^A + a_{12}^1)(m_2^B + b_{12}^2) = m_1^A m_2^B + a_{12}^1 m_2^B + b_{12}^2 m_1^A + a_{12}^1 b_{12}^2 \quad , \tag{38b}$$

$$p_{21}^S = (m_2^A + a_{12}^2)(m_1^B + b_{12}^1) = m_2^A m_1^B + a_{12}^2 m_1^B + b_{12}^1 m_2^A + a_{12}^2 b_{12}^1 \quad , \tag{38c}$$

$$p_{22}^S = (m_2^A + a_{12}^2)(m_2^B + b_{12}^2) = m_2^A m_2^B + a_{12}^2 m_2^B + b_{12}^2 m_2^A + a_{12}^2 b_{12}^2 \quad . \tag{38d}$$

After collecting terms with parametric masses which add up to the given masses m_{12}^A and m_{12}^B, we finally get the power set form of S given by

$$\begin{aligned} S = [\; & \overset{m_1^A \, m_1^B}{s_{11}} , \; \overset{m_1^A \, m_2^B}{s_{12}} , \; \overset{m_2^A \, m_1^B}{s_{21}} , \; \overset{m_2^A \, m_2^B}{s_{22}} , \\ & \overset{m_1^A \, m_{12}^B}{(s_{11}, s_{12})} , \overset{m_2^A \, m_{12}^B}{(s_{21}, s_{22})} , \overset{m_{12}^A \, m_1^B}{(s_{11}, s_{21})} , \overset{m_{12}^A \, m_2^B}{(s_{12}, s_{22})} \; , \\ & \overset{m_{12}^A \, m_{12}^B}{(s_{11}, s_{12}, s_{21}, s_{22})} \;] \quad . \end{aligned} \tag{39}$$

A simple Venn diagram can be drawn for these results to interpret them more easily. This diagram is shown in Fig.1.

Using the results above, together with their diagramatic representation, allows some fundamental notions about the OT intersection operation to be established. The first observation which can be made, is that A and B have a common area of agreement in S in the region in which they intersect (i.e. the shaded area in Fig.1). All the probability originally in A and B separately is now assigned to this intersection region. This amounts to the same thing as saying, the combined results lie in the field of all intersecting elements of A and B - a subfield of the classical σ-field of probability theory. Moreover, the probabilities (masses) assigned to these intersection elements are products of probabilities (masses) in A and B, a result that can be derived from the intersection rule for independent sets in probability theory. The axiomatic basis for OT is, therefore, a subset of that used to derive probability theory (i.e. no notion of conditioning is needed to get a combination rule similar to Bayes' theorem).

The fact that the OT intersection of the two belief sets A and B, produces a consistent picture in S, is another remarkable fact. Only Bayes' theorem is commonly held to produce such results using probability theory as a basis[12]. This consistency is clearly seen in Fig.1, when it is noted that, as was the case before combining information, the circles enclosing x_1^A, x_2^A, x_1^B, and x_2^B, still contain their original probabilities.

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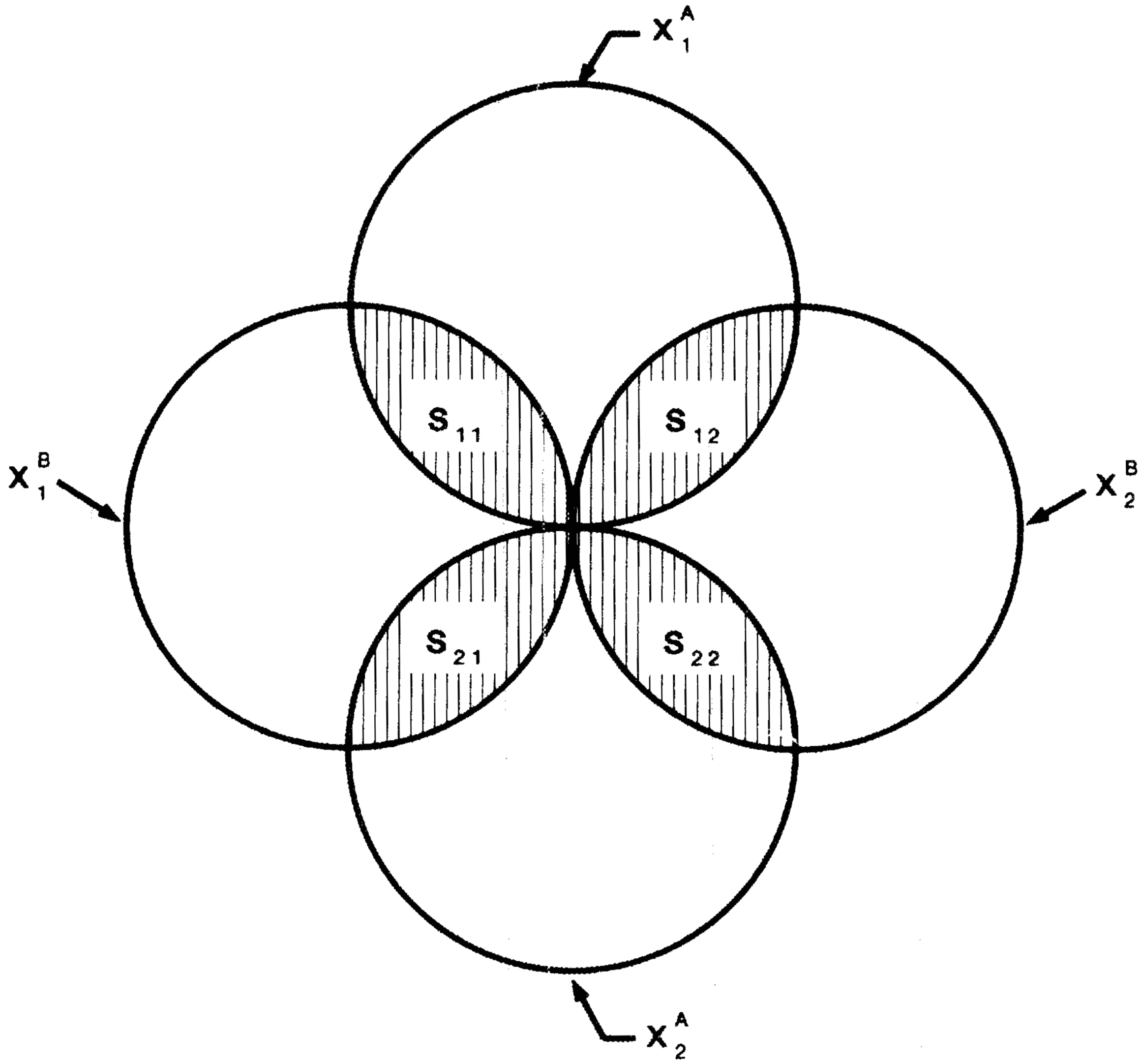

FIG 1. VENN DIAGRAM REPRESENTING BELIEF SET S

That is, in Fig.1, the circles representing A and B can be seen from Eq.(39) to enclose regions with probabilities given by

$$p(x_1^A) = p(s_{11})+p(s_{12}) = p_1^A \quad , \tag{40a}$$

$$p(x_2^A) = p(s_{21})+p(s_{22}) = p_2^A \quad , \tag{40b}$$

$$p(x_1^B) = p(s_{11})+p(s_{21}) = p_1^B \quad , \tag{40c}$$

$$p(x_2^B) = p(s_{12})+p(s_{22}) = p_2^B \quad . \tag{40d}$$

These are the same values they had before combination. In this picture, then, each belief set retains its original character in combination, yet S is a consistent representation of all the information present.

The usefulness and interpretation of each of the individual intersection elements s_{ij} that span S can only be made in the context of the particular application under investigation (i.e. as is the case in probability theory, where A and B can be used to represent sequences of events, propositions, functional expressions, etc.).

6) DST - A Special Case

For the purposes of this example, we can consider the possible interpretation of the results just derived in the context of DST. This formalism constitutes a special case of the general OT results. That is, the elements s_{11} and s_{22} can be considered to result from the combination of consonant information and s_{12} and s_{21} can be considered to result from combining dissonant information. In DST, this dissonant component is disallowed and must be redistributed by renormalization. In OT, it remains explicitly, and in most of its interpretations it can be dealt with as such.

To formally recreate the results of combining A and B in DST using Dempsters' rule, all one needs to do is make the simplifying assumption that

$$x_1^A = x_1^B = x_1^S \quad \text{and} \quad x_2^A = x_2^B = x_2^S \quad , \qquad (41)$$

yielding the Venn diagram in Fig. 2. From this diagram and the DST definition of the null set, it is immediately apparent that

$$s_{12} = \phi, \; s_{21} = \phi, \; s_{11} = x_1^S, \; s_{22} = x_2^S \; . \qquad (42)$$

Since s_{12} and s_{21} are assumed to be null elements in DST, any mass (probability) associated with these elements is renormalized proportionately into x_1^S, x_2^S and (x_1^S, x_2^S). In OT, even with the assumption made in Eq.(41), this renormalization is not required. Both s_{12} and s_{21} can be considered to be a legitimate elements of S in their own right. These elements are designated as ϕ only for the purposes of comparison with and application of DST. It can be demonstrated, for instance, that the interpretation and use of s_{12} and s_{21} as they are actually defined is more appropriate in sequence-of-events or set function applications. In these latter cases, the individual s_{ij}'s have direct meaning in terms of possible events and/or functions. The concepts of renormalization and the null space, are more or less meaningless here.

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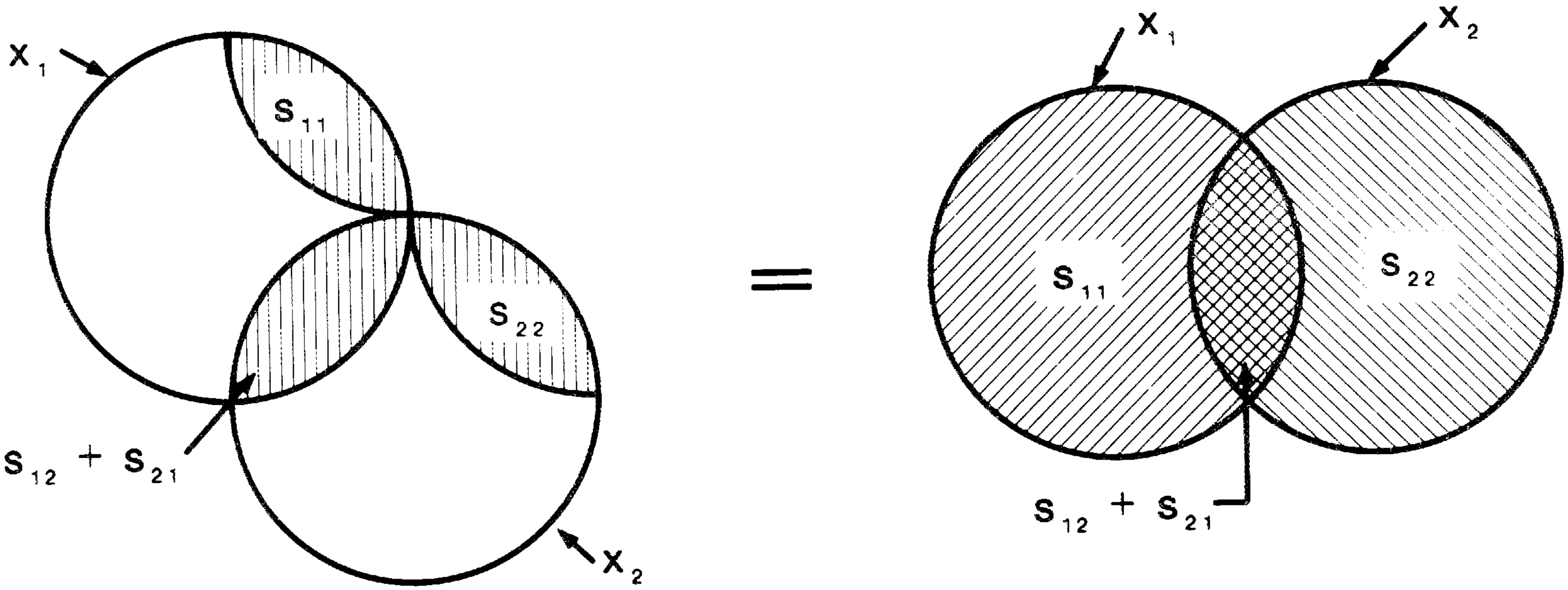

FIG 2. VENN DIAGRAMS REPRESENTING DST INTERPRETATION OF BELIEF SET S

In any event, the transformation back to DST theory can be completed by using Eqs.(39) and (42) to give

$$\begin{array}{ccccc} & m_S(\phi) & m_S(x_1^S) & m_S(x_2^S) & m_S(x_1^S,x_2^S) \\ \underline{S} = [& \phi \ , & x_1^S \ , & x_2^S \ , & (x_1^S,x_2^S) \ ] \ , \end{array} \qquad (43)$$

with masses

$$m_S(\phi) \quad = \quad m_1^A \, m_2^B + m_2^A \, m_1^B \quad , \qquad (44a)$$

$$m_S(x_1^S) \quad = \quad m_1^A \, m_1^B + m_1^A \, m_{12}^B + m_1^B \, m_{12}^A \quad , \qquad (44b)$$

$$m_S(x_2^S) \quad = \quad m_2^A \, m_2^B + m_2^A \, m_{12}^B + m_2^B \, m_{12}^A \quad , \qquad (44c)$$

$$m_S(x_1^S,x_2^S) = m_{12}^A \, m_{12}^B \quad . \qquad (44d)$$

In the DST formalism, $m(\phi)\neq 0$ is not allowed and this mass must be redistributed proportionally among the other elements of S. After renormalization of $m_S(\phi)$ in DST fashion we get the final result that,

$$\begin{array}{cccc} & m_1^S & m_2^S & m_{12}^S \\ S = [& x_1^S \ , & x_2^S \ , & (x_1^S,x_2^S) \ ] \ , \end{array} \qquad (45)$$

with masses

$$m_1^S \ = m_S(x_1^S)/[1-m_S(\phi)] \quad , \qquad (46a)$$

$$m_2^S \ = m_S(x_2^S)/[1-m_S(\phi)] \quad , \qquad (46b)$$

$$m_{12}^S = m_S(x_1^S,x_2^S)/[1-m_S(\phi)] \quad . \qquad (46c)$$

Eqs.(45) and (46) represent the classical DST result for this problem[4]. This result can clearly be seen to arise from OT using the simplifying assumptions given in Eqs.(41).

In OT these assumptions need not be made (although they can be useful in analyzing some problems). The result in Eq.(39) can instead be used directly.

For instance, in viewing OT results in a DST framework, the mass in (s_{11}, s_{12}) is seen to represent both the consonant and dissonant components of the combination of the original DST power set elements x_1^A and (x_1^B, x_2^B). Dempsters' rule assigns the mass in this combination directly and completely to x_1^S (i.e. s_{11}). This intersection ignores the possibility, originally assumed, that some mass in (x_1^B, x_2^B) might have been in x_2^B. This possibility is not forgotten in OT. A distinction is therefore made between components s_{11} and (s_{11}, s_{12}) even when s_{12} is viewed as being in the OT null set. The renormalization problem in DST and its paradoxes are more easily dealt with in OT specifically because this distinction can be made.

In a future paper on OT foundations[13], the distinctions made above between OT and DST will be used to avoid some of the basic conflicts between DST and probability theory. These conflicts must be resolved in light of the results presented in this paper, which prove Dempsters' rule can be derived directly from probability theory. The general formulation of OT presented here will prove to be most useful in this regard.

7) Summary and Conclusions

A deeper investigation of the foundations of the OT intersection operator has provided a purely probabilistic and set-theoretic interpretation of its origins. The intersection operation was found to combine belief from independent sources into a new belief set, in a manner previously thought only to be possible in a probability framework with the use of Bayes' theorem. Dempsters' rule of combination was found to be a special case of this OT intersection operation.

The computational aspects of the developments presented make it clear that masses and probabilities can be used interchangeably in the OT framework. This flexibility should open up new areas of application for OT and the DST framework in general. These new areas are those that were solely in the domain of probability theory before.

The formal connection between mass and probability presented here also makes the distinctions between DST and probability theory far less consequential. DST indeed is a generalization of the probability concept, but it now fits within the original probability framework. The DST conception of uncommitted belief is, therefore, a notion that is compatible with classical probability theory. In addition, Dempsters' rule can now also be justified as a combination operation which can be used on an equal basis with Bayes' theorem. The fact that Dempsters' rule and its generalization in OT does not rely on conditional probabilities is a strong point in its favor as well.

The concept of prior knowledge, if still essential to any analyses, can also conceivably be handled in an OT framework by specifying a separate prior belief set. This new set can be combined with other belief information using the OT intersection rule. Uncommitted belief can still be used effectively in this framework to represent the lack of knowledge, as in DST.

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