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STRESS INDICES AND FLEXIBILITY FACTORS FOR MOMENT LOADINGS ON ELBOWS AND CURVED PIPE

W. G. Dodge
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Errata
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FOREWORD

The work reported here was done at Oak Ridge National Laboratory (ORNL) as part of the ORNL Piping Program — Design Criteria for Piping, Pumps, and Valves, under the direction of W. L. Greenstreet, Associate Head, Solid Mechanics Department, and S. E. Moore, Program Coordinator. The ORNL Piping Program is funded by the U.S. Atomic Energy Commission (USAEC) under the Nuclear Safety Research and Development Program as the AEC supported portion of an AEC-industry cooperative effort for the development of design criteria for nuclear power plant piping components, pumps, and valves. The AEC-industry cooperative effort is coordinated by the Pressure Vessel Research Committee (PVRC) of the Welding Research Council, under the Subcommittee to Develop Stress Indices for Piping, Pumps, and Valves. J. L. Mershon of the AEC Division of Reactor Development and Technology is the USAEC cognizant engineer.

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NOMENCLATURE

C	Stress index
E	Modulus of elasticity
J	Polar moment of inertia of the curved pipe
k_p	Flexibility factor
$\vec{M}$	Applied moment vector
M_i	In-plane bending moment component
M_o	Out-of-plane bending moment component
M_t	Torsional moment component
P	Internal pressure
r	Mean cross-sectional radius of the curved pipe
R	Bend radius of the centerline of the curved pipe
S	Stress intensity
t	Wall thickness of the curved pipe
w	Displacement of the pipe wall relative to the centerline of the bend
w_r, w_t	Radial and circumferential components of w, respectively
Z	Section modulus of the curved pipe
α	Arc angle of pipe bend
ϕ	Circumferential angle
ν	Poisson's ratio
β, η	Position angles of the moment vector $\vec{M}$
γ	R/r, radius ratio parameter
λ	$tR/r^2\sqrt{1-\nu^2}$, bend characteristic parameter
ψ	PR^2/Ert, internal pressure loading parameter
σ, τ	Normal and shear stresses, respectively

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ABSTRACT

Flexibility factors and stress indices for elbows and curved pipe loaded with an arbitrary combination of in-plane, out-of-plane, and torsional bending moments are developed for use with the simplified analyses procedures of present-day design codes and standards. An existing analytical method was modified for use in calculating these factors, the equations were programmed for the IBM-360 computer, and computed results were compared with experimental data to establish the adequacy of the modified method. Parametric studies were then performed to obtain desired information. The results are presented in both tabular and graphical form. Approximate equations of best fit, developed from the tabulated values, are presented in a form which can be used directly in the codes and standards. The present equations are slightly more conservative than the ones in current use. However, experimental and analytical studies now in progress may indicate further modifications in the stress indices and flexibility factors for elbows.

Keywords: stress indices, pipe elbows, curved pipe, stress analysis, piping code, ANSI B31.7, pressure vessel code, ASME BPV Section III.

INTRODUCTION

The stress analysis of elbows and curved pipe has been the subject of numerous theoretical and experimental studies since Bantlin,[1*] in 1910, first demonstrated experimentally that a curved pipe responds differently under load than predicted by simple beam theory; and von Karman,[2] in 1911, presented the first rational explanation of this discrepancy. The purpose of the present study is to select, from among the existing

*Superscript numbers refer to similarly numbered references at the end of this report.

theoretical methods, the most appropriate analysis for use in developing stress indices and flexibility factors; and to develop these factors in a form suitable for use in codes and standards. This study is one of several being conducted under the Oak Ridge National Laboratory's Piping Program[3] as part of a joint USAEC-industry program for the refinement, extension, and development of methods of design for nuclear reactor plant piping, pumps, and valves.

The general approach taken here is that of the nuclear power section,[4] B31.7, of the American National Standards Institute (ANSI) Standard Code for Pressure Piping, in which stress indices are used in conjunction with simplified design formulas for calculating design stresses for class I piping. In this code the maximum shear stress theory of failure (Tresca condition) is used as the design criterion; and stress indices are defined in terms of the ratio of the "stress intensity" to a nominal stress, where the stress intensity is defined as twice the absolute value of the maximum shear stress at a point in the component for a given loading condition.

In the simplified analysis method of division 1-705 of ANSI B31.7, the piping code[4] uses three types of stress indices, B_i, C_i, and K_i, corresponding roughly to the three categories of allowable stresses: primary, secondary, and peak, respectively. The B_i indices are used with Eq. (9) of division 1-705 to assure against catastrophic membrane failure, and are thus related to the primary stresses. The C_i indices are used with Eq. (10) to insure that shakedown to elastic behavior will occur after application of a few load cycles, and are thus related to the sum of primary plus secondary stresses. The K_i indices are generally elastic stress concentration factors and are used along with the C_i indices in Eq. (11) to insure against fatigue failure.

For a combination of loads applied simultaneously to a piping component, the simplified design formulas of the code are based on the assertion that the maximum stress intensity existing anywhere in the component is equal to or less than the sum of the maximum stress intensities due to the loads taken individually. It is thus appropriate to develop stress indices for moment loadings without considering other loads which may exist on the component in application. The present study deals with stress indices for elbows and curved pipe of the primary plus secondary category for externally applied moment loadings ["C_2" as used in Eq. (10) of the piping code] and the corresponding flexibility factors for use in piping system analyses.

The first task was a rather extensive literature review of currently available theoretical stress analyses for elbows, and selection of the most appropriate methods for developing the stress indices and flexibility factors. Since there is some advantage in treating an arbitrary combination of in-plane, out-of-plane, and torsional moment loadings, one of the criteria for selecting the appropriate analysis was that consistent solutions for each of these loadings should be available, or easily developed. The analysis should also be accurate as verified with experimental data.

The remaining sections of the report deal with the flexibility factors and the mathematical development of stress indices for combined loads. Numerical results, obtained from a parameter study, are presented in graphical and tabular form, and as approximate formulas that are conservative and easy to use in design analyses. The last section is a discussion of the present results and recommendations for further study.

A summary discussion of the analytical method, which was selected from the published literature and modified for use in this study, is given in Appendix A; improved equations for use with this method are developed in Appendix B. The equations in Appendix B account for membrane force components that were neglected in the original development. A list of symbols used in the text is given in the Nomenclature.

SURVEY AND EVALUATION OF ANALYTICAL METHODS

Published closed-form theoretical stress analyses for curved pipe and elbows with bending moment loads fall into one of three categories depending on the approach taken in defining the mathematical problem and solving the equations. These are the minimum potential energy approach first used by von Karman[2] in 1911, the mechanics-of-materials approach first used by Turner and Ford[5] in 1957, and the thin-shell theory approach first used by Tueda[6] in 1936. All of these early solutions were for an ideal torus loaded with an in-plane bending moment, and in each case the problem was simplified by neglecting the stress variations along the length of the elbow. Various extensions and modifications to these analyses have since been published and numerous experimental studies have been conducted to verify the theoretical results.

Minimum Potential Energy Analyses

The first rational stress analysis for elbows was published by von Karman[2] in 1911 for the problem of in-plane bending of curved pipe. Later, in 1943, Vigness[7] generalized von Karman's analysis to include out-of-plane bending. They reasoned that due to the curvature of the pipe bend, the longitudinal stresses in the tube wall tend to distort the shape of the cross section, which in turn produces a greater flexibility and a different stress distribution than predicted by simple beam theory. The forms of distortion of the cross section under the different types of moment loadings are shown in Fig. 1. This flattening of the cross section is due to the longitudinal stresses, which in a curved tube produce component forces acting toward and away from the center of the tube.

Both von Karman and Vigness obtained solutions by representing the displacements of the wall with trigonometric series, and determined the coefficients of these series by minimizing the total potential energy. For torsional moment loading, they assumed that a curved pipe would respond in the same manner as a straight pipe, and that the cross section would remain circular.

In order to simplify the analysis, three major assumptions were made:

1. The ratio of the pipe radius to the bend radius was neglected relative to 1.

2. The circumferential membrane strain was assumed to be zero, so that the length of any segment of the circumference of the tube wall would remain constant.

3. Plane sections transverse to the centerline of the tube were assumed to remain plane and perpendicular to the deformed centerline.

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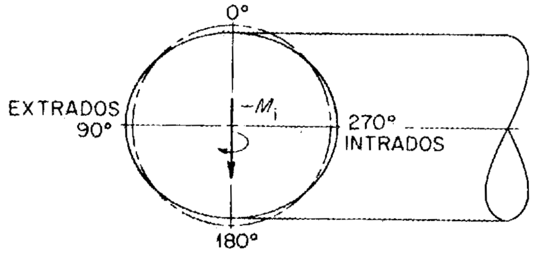
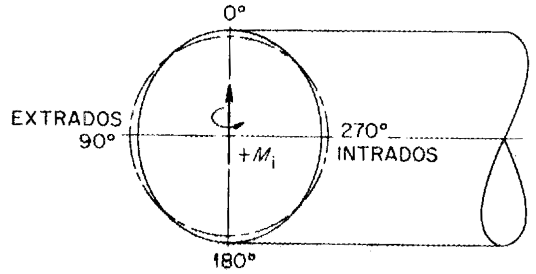
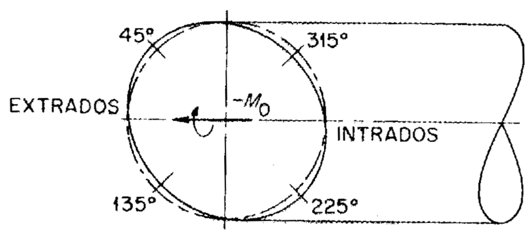


(a) IN-PLANE BENDING, TENDING TO OPEN THE ELBOW



(b) IN-PLANE BENDING, TENDING TO CLOSE THE ELBOW



(c) OUT-OF-PLANE BENDING

Fig. 1. Cross section distortions of elbows loaded with in-plane and out-of-plane bending moments.

For both in-plane and out-of-plane bending, the analytical results indicated that the angular rotation of the plane sections and the maximum stresses were greater for curved pipe than for straight pipe, thereby agreeing with the experimental results of Bantlin The rotation and the maximum stresses for elbows relative to the same quantities for straight pipe, that is, the flexibility factor and the stress-intensification factor, were found to be greater than 1.0 and to depend only on the dimensionless parameter lambda*

$$\lambda = \frac{tR}{r^2 \sqrt{(1 - \nu^2)}} . \tag{1}$$

In 1955, Kafka and Dunn[8] incorporated the influence of internal pressure into von Karman's in-plane bending analysis. This was done by including in the potential energy expression the elastic work done on the tube by the pressure during deformation of the cross section. The influence of internal pressure is to reduce the distortion caused by the bending load; hence, internal pressure tends to reduce both the bending stresses and the flexibility of the elbows.

The following year, in 1956, Rodabaugh and George[9] rederived and generalized the von Karman-Vigness analyses including the effects of in-plane and out-of-plane bending as well as the influence of internal pressure. When internal pressure is included, the flexibility factor and the stress-intensification factor depend on a second dimensionless parameter psi:

$$\psi = \frac{PR^2}{Ert} . \tag{2}$$

The more complete analysis of Rodabaugh and George was written in terms of infinite series which they evaluated using one, two, and three terms to obtain explicit expressions for the flexibility factor and the in-plane and out-of-plane stress intensity factors. These expressions form the basis for the stress indices currently in use in the nuclear power

*Since both von Karman and Vigness used uniaxial stress-strain relations in their analyses, neither author obtained the term $\sqrt{(1 - \nu^2)}$.

piping code (see USAS B31.7-1969, Appendices D and F). Appendix A of the present report contains a brief summary of the results obtained by Rodabaugh and George.

There is one other analytical development based on the minimum potential energy formulation which is of interest in the current study. In 1952, Gross[10] pointed out that von Karman's analysis results in a logical inconsistency which may be significant for elbows with a small bend radius. In his analysis, von Karman had reasoned that the deformation of the cross section of a curved tube is caused entirely by longitudinal membrane stress components acting toward and away from the center of curvature of the bend. For simplicity he assumed that the circumferential membrane strain could be set equal to zero.

In order to maintain static equilibrium in the tube wall, the inward force resultant of the longitudinal membrane stresses must be balanced by transverse shear forces and a circumferential membrane force. For in-plane bending the circumferential force should have a maximum absolute value at the side of the elbow along the neutral bending axis, ($\phi = 0$); it should be symmetric about $\phi = 0$; and equal to zero at $\phi = \pm \pi/2$. The experimental results of Gross[10] on elbows with pipe-radius to bend-radius ratios in the neighborhood of 1/3 and others indicate that this conclusion is correct. However, because of the zero circumferential membrane assumption, the von Karman analysis yields a circumferential membrane force which is skew-symmetric about $\phi = 0$, and hence zero at $\phi = 0$ [see, for example, Appendix A, Eq. (A-14)].

In his analysis, Gross determined the circumferential membrane force required to maintain static equilibrium based on the assumption that the von Karman analysis gives the correct longitudinal membrane force; and applied this as an additive ... "correction for transverse compression." Actually this analysis determines a more accurate approximation of the circumferential membrane force and should replace the corresponding force determined by the von Karman analysis. A generalization of this correction is developed in Appendix B for use in conjunction with the equations developed by Rodabaugh and George (see Appendix A). Use of the modified equations requires the introduction of a third dimensionless parameter, gamma, defined by

$$\gamma = R/r \ . \tag{3}$$

Mechanics-of-Materials Analyses

A second type of analysis, different from that developed by von Karman, was used by Turner and Ford[5] to analyze the in-plane bending of curved tubes without internal pressure. Their analysis, published in 1957, used a mechanics-of-materials approach which was more complex than the strain-energy approach. Two of the simplifying assumptions used by von Karman, however, were not utilized by Turner and Ford. The parameter r/R was not neglected relative to 1, nor was the circumferential strain on the midwall surface assumed to be zero. On the basis of their results, they concluded that the peak stresses and flexibilities determined by the minimum potential energy analyses are, by a combination of circumstances, unlikely to be in error by more than 5 to 10%. In 1966, Smith[11] generalized the analysis of Turner and Ford to include out-of-plane bending.

Thin-Shell Theory Analyses

A third approach has been followed by several authors for the analysis of elbows and curved tubes loaded with in-plane bending moments without internal pressure. Using thin-shell theory, Tueda[6] in 1936 reduced the problem to two coupled ordinary differential equations which he integrated by means of a power series. In 1951, Clark and Reissner,[12] using the thin-shell theory developed earlier by Reissner,[13] also obtained two coupled ordinary differential equations with variable coefficients. In order to solve these equations, approximations were made which were essentially equivalent to the assumptions made by von Karman in his analyses.

In 1968, Cheng and Thailer[14] published an analysis for in-plane bending which was based on the two differential equations of Clark and Reissner[12] but without their simplifying assumptions. Their solution was in the form of a series expansion which identically satisfied the equilibrium equation, but satisfied the compatibility equation only for small values of the pipe-radius to bend-radius ratio, that is, $r/R \ll 1.0$. Rather than impose a restriction on this ratio, the coefficients were determined so as to minimize the complimentary energy, and hence approximately satisfy the compatibility equation. The resulting series converges quite rapidly, requiring only four terms for elbows with bend parameter values around $\lambda = 0.05$ and only six terms for $\lambda = 0.01$. They also investigated the convergence characteristics of a number of previously published analyses and found that, with the exception of the von Karman-type solutions, convergence was very slow, requiring on the order of 40 to 50 terms. Cheng and Thailer reported, on the other hand, that six terms were found to be adequate for the minimum potential energy formulation of Symonds and Pardue,[15] which is a slight extension of the von Karman-Vigness analysis to include the radius ratio parameter r/R in the elastic energy equations; however, they still retained the assumption of zero circumferential membrane strain.

Most Appropriate Solution for Design Use

Since Cheng and Thailer's thin-shell theory analysis retains the radius ratio $\gamma = R/r$ as a parameter and is not restricted by the assumption of an inextensible circumferential midsurface, it should be more accurate than the von Karman-type minimum potential energy solutions. The rapid convergence characteristics tend to favor its use over the mechanics-of-materials analysis of Turner and Ford,[5] which appears to give about the same accuracy. Thus, if Cheng and Thailer's method could be generalized to include out-of-plane bending and the effects of internal pressure, it would probably be the most satisfactory to use as a basis for developing stress indices and flexibility factors. However, since such an analysis is not available at this time, the minimum potential energy solution of Rodabaugh and George as corrected by the modifications developed in Appendix B was selected as the basis for the C_2 stress-index development. The modified analysis was programmed for the ORNL computer and used in the remainder of the study.

The computer program ELBØW* was written to evaluate as many terms in the series solution as needed to guarantee convergence. Previous authors had shown that the number of terms required for convergence increased with decreasing values of the elbow bend parameter λ, usually requiring between three and six terms for λ greater than about 0.1. The computer program was therefore written to compute up to 20 terms if necessary. For λ values on the order of $\lambda = 0.01$, 15 or 16 terms were sufficient to achieve six-digit accuracy in the calculated stresses, whereas only five or six terms were required for $\lambda \approx 0.1$. This is an important factor in the cost of doing large-scale parameter studies where several hundred cases may be analyzed.

COMPARISON OF ANALYTICAL AND EXPERIMENTAL RESULTS

Analytical results obtained with the computer program discussed above were compared with experimental results published in the literature. Since the analysis is capable of considering bending loads, both with and without internal pressure acting at the same time, the experimental data obtained by Rodabaugh and George[9] in 1957 for the case of in-plane bending was used as one test problem. Their elbow was a 30-in.-OD long-radius ($R = 45$ in.) 90° welding end elbow with a 0.5-in. nominal wall thickness. The average wall thickness was 0.515 in., with a variation of +0.058 in. to −0.094 in.; and the average outside diameter was 29.973 in., with a variation of +0.160 in. to −0.172 in. These dimensions give a nominal bend parameter value of $\lambda = 0.1118$ [see Eq. (1)]. The elbow was instrumented with SR-4 strain gages located every 15° around the circumference at the midpoint of the bend (45°) on both the inside and outside surfaces. The elbow was tested with an in-plane bending moment and internal pressures of 0, 400, 800, and 1100 psi; the corresponding internal pressure parameter values from Eq. (2) were $\psi = 0$, 0.0037, 0.0074, and 0.0102, respectively.

*A listing of the computer program is given in the report, "ELBØW: A Fortran Program for the Calculation of Stresses, Stress Indices, and Flexibility Factors for Elbows and Curved Pipe" (to be published). Copies may be obtained from the authors, Dodge and Moore.

Using the nomenclature of Fig. 2, comparisons between the normalized experimental stresses, $\sigma_i/(M/Z)$, in the circumferential and longitudinal directions and the present analytical results as a function of angular position ϕ are shown in Figs. 3, 4, and 5 for internal pressure values of 0, 400 psi, and 800 psi, respectively. The analytical results are shown as dotted (...) and solid (——) lines for the inside and outside surfaces, respectively; and the experimental results are shown as open triangles and circles. As these figures show, the overall agreement is good for both the inside and outside surfaces, and especially good for the maximum values. Although not shown here, the present analytical results agree somewhat better with the experimental results than the unmodified analysis used earlier by Rodabaugh and George.

A second example test problem was also compared with experimental results for both an in-plane and an out-of-plane bending moment applied to the elbow. These data were obtained from the paper by Smith and Ford[16] published in 1967. Smith and Ford reported in-plane bending results from three elbows, two of which were also tested with an out-of-plane bending moment applied through the connecting pipes. The first of these, their model no. 1, had an outside diameter, D_o, of 6.556 in., a wall thickness, t, of 0.244 in., a mean radius, r, of 3.156 in., and a bend radius, R, of 18.23 in. The corresponding dimensionless parameters were $\gamma = 5.776$, $\lambda = 0.4846$, and $\psi = 0$. This model was instrumented with electrical resistance strain gages at the center of the bend around the outside surface. Although the gage lengths were not reported, since the tests were conducted within the last few years (around 1966), it is presumed that reasonably small gages were used and that modern techniques were used in the experimental stress analysis. The published experimental results for this model should therefore provide a fairly good test case for the present analysis, even though the bend parameter ($\lambda \approx 0.5$) is somewhat larger than desirable. Smith and Ford's other model, their no. 3, was considered to be too thick for general theory validation; the parameter values for this model were $\gamma = 3.045$ and $\lambda = 0.8903$.

Comparisons between the normalized experimental stresses from Smith and Ford's model no. 1 and the present analytical results are shown in Figs. 6 and 7. Figure 6 is for the case of in-plane bending and Fig. 7

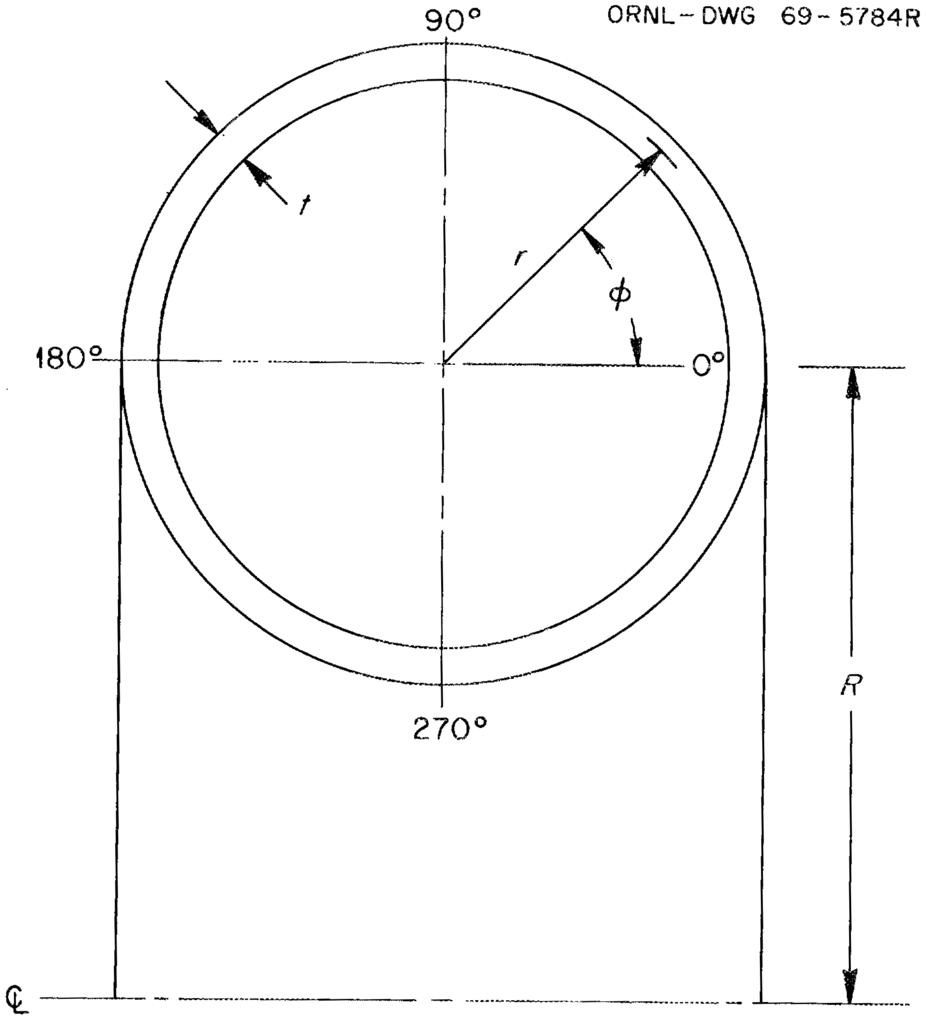


Fig. 2. Coordinate system and dimensions for analytical model of elbows.

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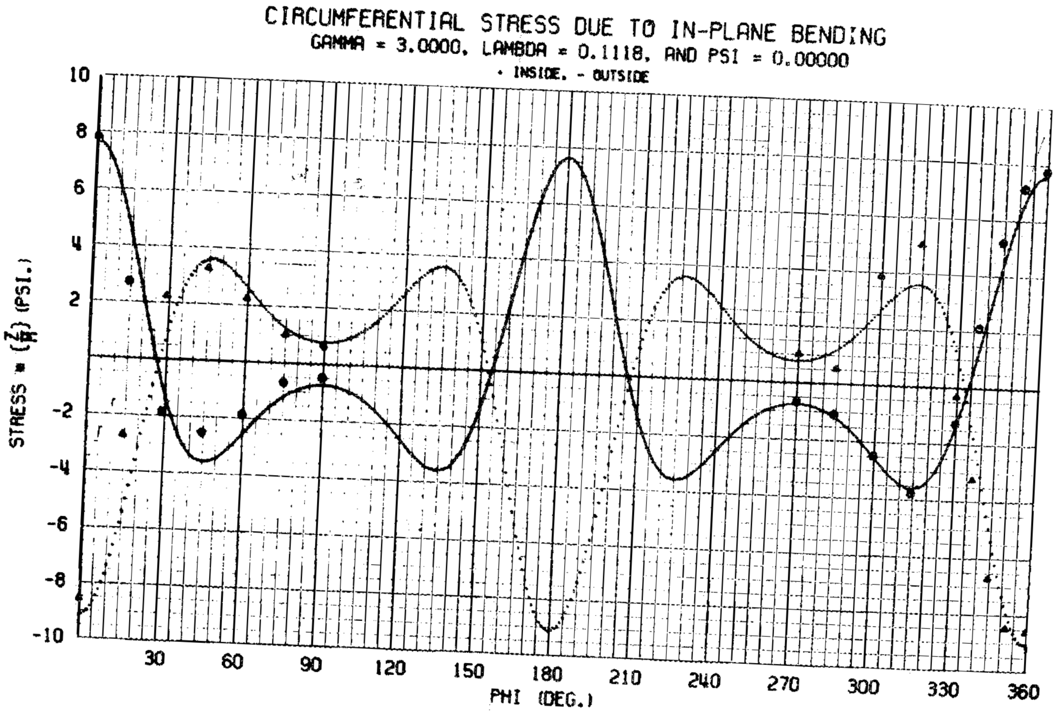

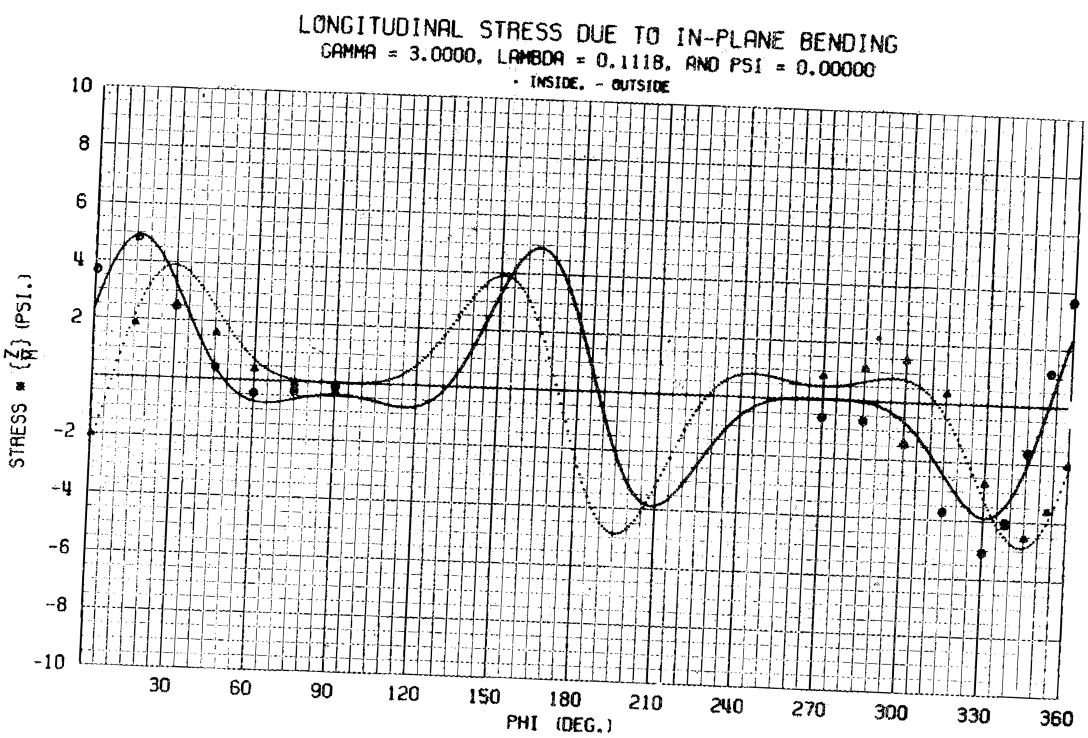

Fig. 3. Comparison of calculated stresses and experimental data from ref. 9 for a 30-in. elbow loaded with an in-plane bending moment.

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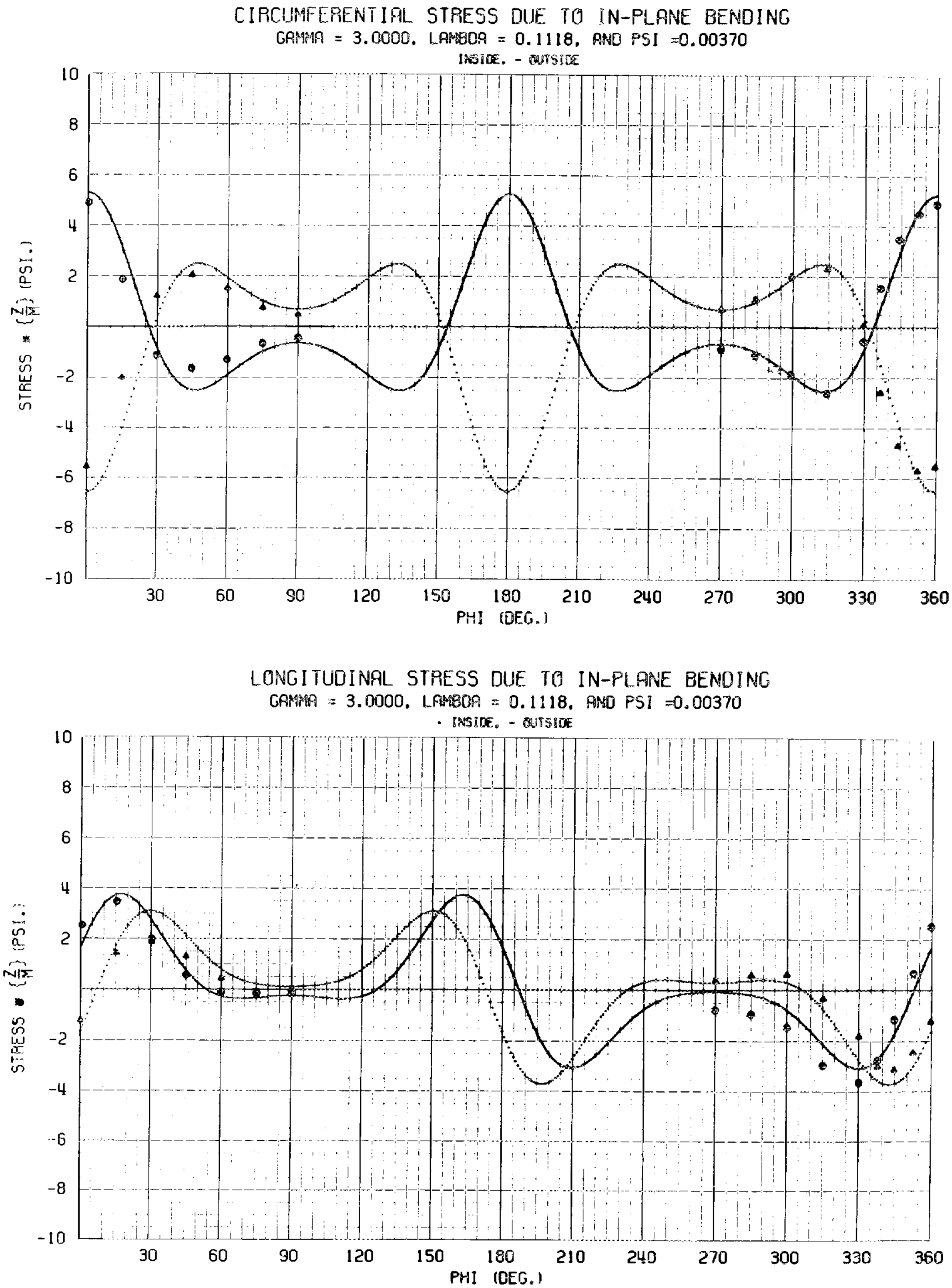

Fig. 4. Comparison of calculated stresses and experimental data from ref. 9 for a 30-in. elbow loaded with an in-plane bending moment and an internal pressure of 400 psi.

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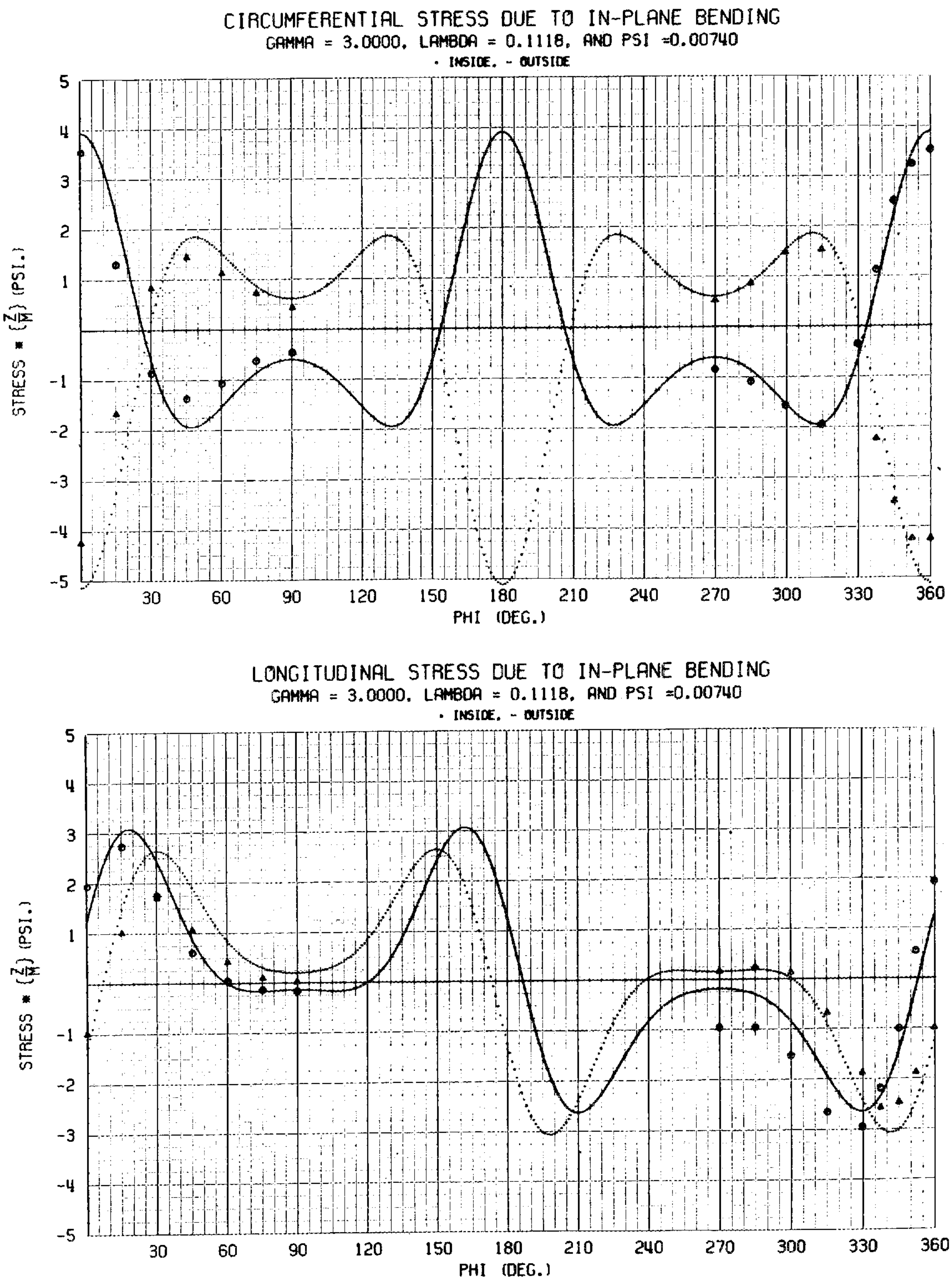

Fig. 5. Comparison of calculated stresses and experimental data from ref. 9 for a 30-in. elbow loaded with an in-plane bending moment and an internal pressure of 800 psi.

ORNL DWG. 72-222

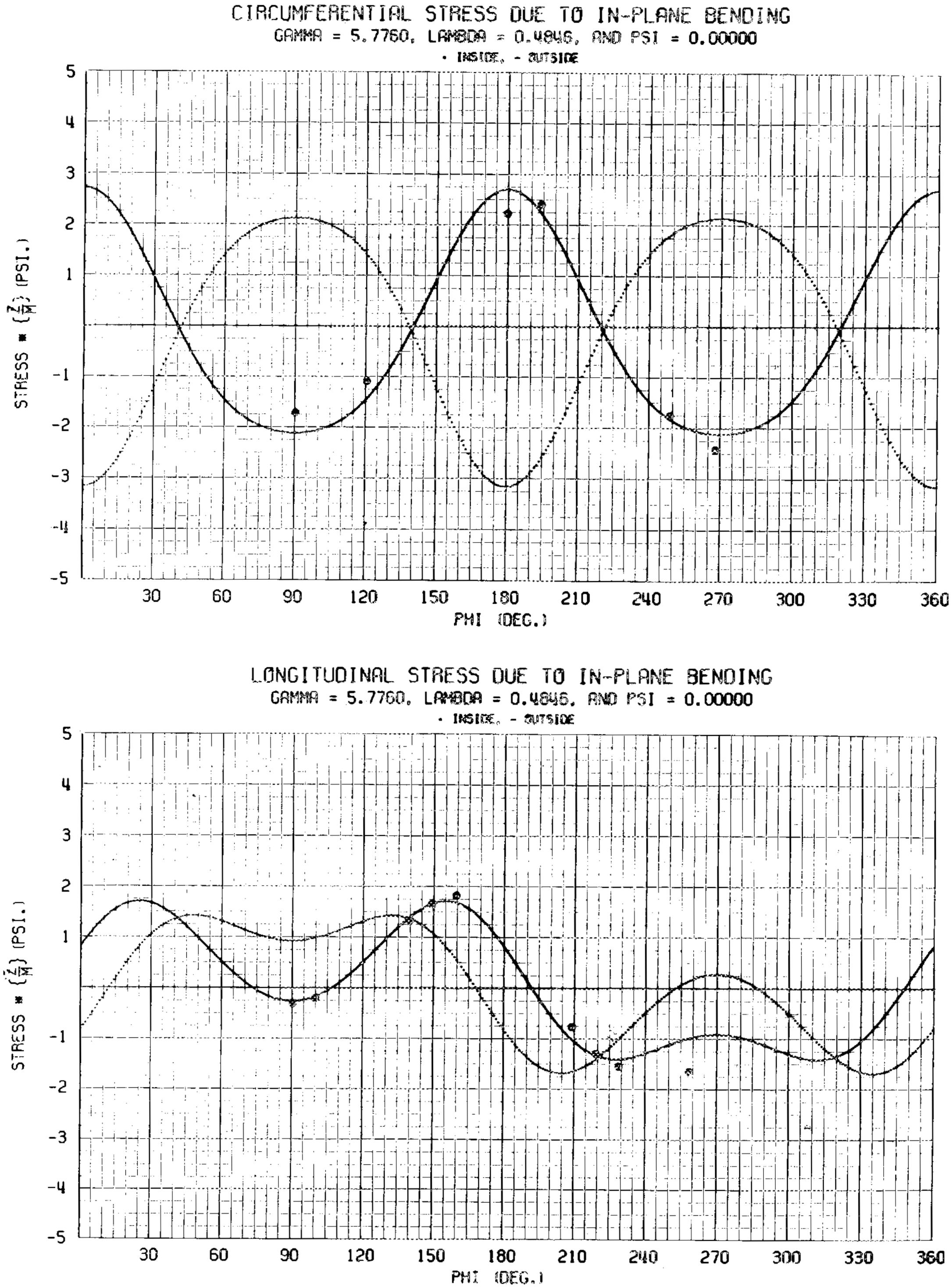

Fig. 6. Comparison of calculated stresses and experimental data from ref. 16 for a 6-in. elbow loaded with an in-plane bending moment.

ORNL DWG. 72-223

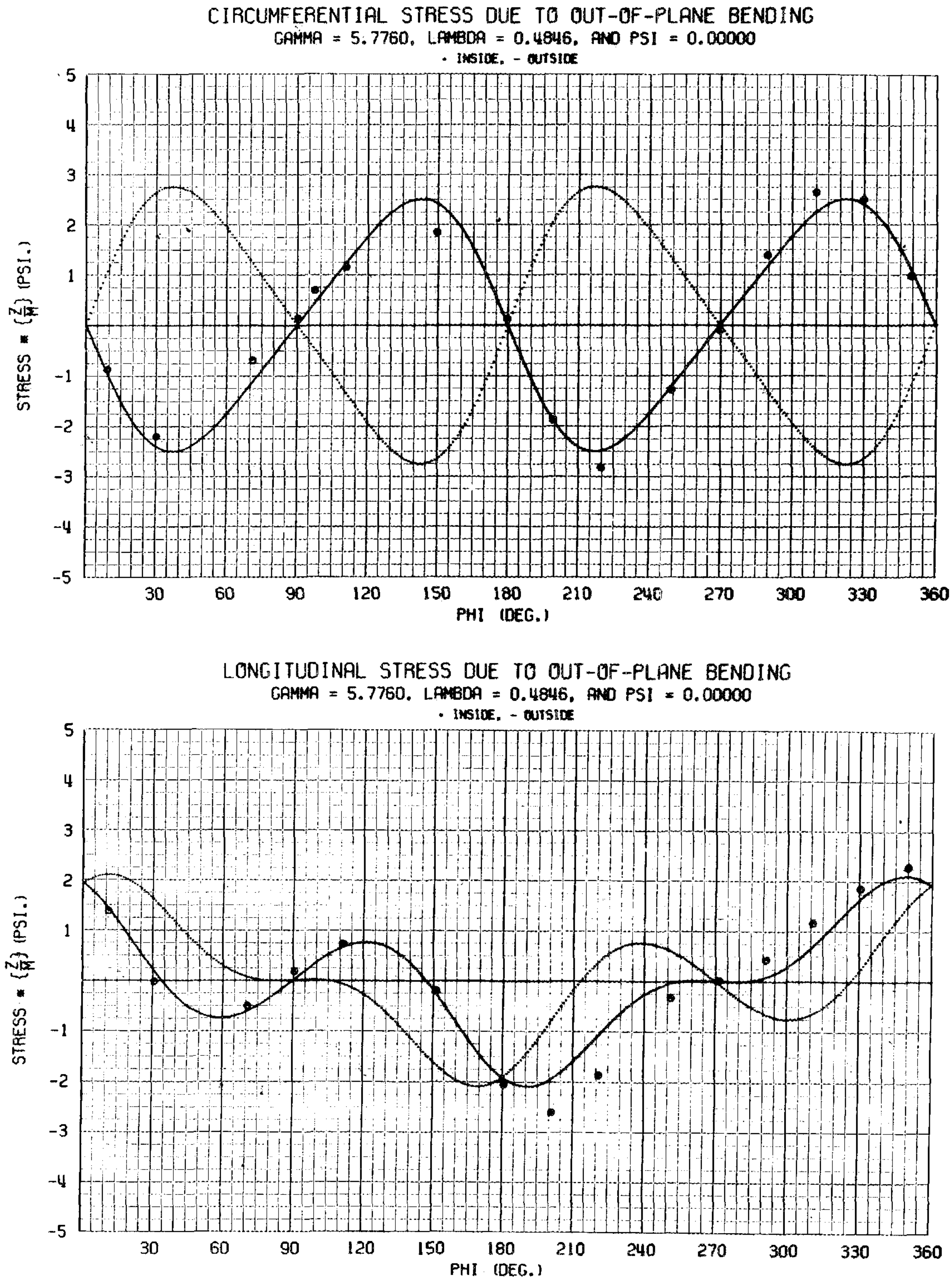

Fig. 7. Comparison of calculated stresses and experimental data from ref. 16 for a 6-in. elbow loaded with an out-of-plane bending moment.

is for out-of-plane bending. The angular position ϕ is relative to the same position used above (see Fig. 3). For the in-plane bending case the agreement between the experimental and analytical results is quite good and is about the same as obtained in the previous example. For the out-of-plane bending case the agreement between experimental and analytical results is also good for the circumferential stresses and reasonably good for the longitudinal stresses, as shown in Fig. 7. When these figures are compared with Smith and Ford's paper, where they compared the experimental results with analytical results obtained using the mechanics-of-materials analysis of Smith,[11] it appears that both analyses give about the same relative comparison with the experimental data.

These two examples thus tend to reinforce our conclusion that the modified minimum potential energy analysis developed in Appendix B from the work of Rodabaugh and George is the most suitable analytical method for developing stress indices and flexibility factors for elbows and curved pipe. It would perhaps be more conclusive if the analysis could be compared with experimental data from elbows with very small λ-parameter values, say on the order of $\lambda = 0.01$ to 0.1, which were loaded with in-plane and out-of-plane bending moments in the presence of internal pressure. After the present study was completed, two reports by Jacobs and Surosky[17,18] containing recent experimental data for nine elbows ($\lambda \approx 0.2$) were brought to the attention of the authors. These data will be compared with the present analyses at a later date. For additional information and references, the interested reader is referred to the survey report[19] by Rodabaugh and Pickett published in 1970.

DESIGN FORMULAS

For general design purposes, simple, reasonably accurate, and conservative stress analysis formulas are often more desirable than the complete set of equations, which in the present case requires the use of a high-speed computer. This is the approach taken by most design codes and standards in an effort to minimize the combined cost of the piping system components and the design stress analysis of the system. For nuclear piping system design, Section III of the ASME Boiler and Pressure

Vessel Code[20] and Section B31.7 of the ANSI Standard Code for Pressure Piping[4] use simplified stress analysis formulas in terms of stress indices and flexibility factors. Flexibility factors are used in the analysis of the overall flexibility of the system, and stress indices are used to evaluate the adequacy of a specific component in the piping system under specific loading conditions determined from the system flexibility analysis. The detailed analysis procedures of these codes also allow the use of the complete solutions described earlier if the conservative stress-index analysis should prove too restrictive.

Flexibility Factors

The flexibility factor k may be defined as the ratio of a relative rotation to a nominal rotation:

$$k = \frac{\theta_{ab}}{\theta_{nom}} , \tag{4}$$

where

θ_{ab} = rotation of end "a" of a piping component with respect to end "b" of that component due to a moment loading M, and in the direction of the moment M,

θ_{nom} = nominal rotation due to the moment loading M.

For curved pipe and elbows, it is convenient to use the ratio of the relative rotation of two transverse planes in the component which are an infinitesimal distance apart to the relative rotation of corresponding planes in similar straight pipe under the same loading conditions. Using this definition the analysis of Rodabaugh and George predicts equal flexibility factors for both in-plane and out-of-plane bending moment loads. When the effects of internal pressure are also included, the flexibility factor k_p is given as a function of the two parameters λ and ψ:

$$k_p = k_p(\lambda,\psi) . \tag{5}$$

Numerical values for $k_p(\lambda,\psi)$ obtained from Eq. (A-12), Appendix A, are given in Table 1 for the parameter range $0.01 \le \lambda \le 5.0$ and $0 \le \psi \le 0.512$; a graphical description is shown in Fig. 8.

When the internal pressure parameter ψ is zero, $k = k_p(\lambda,0)$ is essentially a logarithmic straight line for values of $\lambda < 1.0$. This suggests a simple formula of the type

$$k = A\lambda^n ; \qquad \lambda \le 1.0 \tag{6}$$

where the constants A and n may be determined from the numerical values given in the table. The equation given by Rodabaugh and George [see Ref. 9, Eq. (33)] fits the tabulated values with an error of less than 5% if $A = 1.66$ and $n = -1.0$, that is,

$$k = k_p(\lambda,0) = \frac{1.66}{\lambda} ; \qquad \lambda \le 1.0 . \tag{7}$$

For internal pressure values greater than zero, they suggest using an equation of the following form:

$$k_p(\lambda,\psi) = \frac{k}{1 + f(\lambda,\psi)} , \tag{8}$$

where $f(\lambda,\psi)$ is a correction factor for the influence of internal pressure.

Using Eq. (8) and the tabulated values for $k_p(\lambda,\psi)$ listed in Table 1, the following approximating expression for f was developed in the present study:

$$f = 1.75\,\lambda^{-4/3} \exp(-1.15\,\psi^{-1/4}) ; \qquad \begin{array}{l} 0.05 \le \lambda \le 1.0 \\ 0 \le \psi \le 0.1 \end{array} \tag{9}$$

where exp (...) is the Napierian number e raised to the power indicated in parenthesis. This expression results in flexibility factors which are slightly smaller than the tabulated values (less than 10%). The accuracy is as good, however, as the expression given by Rodabaugh and George in Ref. 9 and appears to fit a wider range of nondimensional parameters.

Table 1. Flexibility factors for elbows loaded with in-plane or out-of-plane bending moments

Flexibility Factors for Elbows

LAMBDA	PSI									
	0.0	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.512
0.010	173.206	55.377	35.398	21.927	13.460	8.304	5.205	3.363	2.288	1.352
0.015	115.470	50.447	33.859	21.526	13.366	8.282	5.200	3.362	2.288	1.352
0.020	86.602	45.719	32.124	21.020	13.239	8.252	5.193	3.360	2.288	1.352
0.030	57.734	37.792	28.619	19.827	12.907	8.170	5.174	3.355	2.286	1.352
0.040	43.300	31.832	25.458	18.547	12.502	8.062	5.147	3.349	2.285	1.352
0.050	34.639	27.341	22.756	17.292	12.055	7.933	5.114	3.340	2.283	1.352
0.060	28.865	23.886	20.482	16.115	11.589	7.787	5.075	3.330	2.280	1.352
0.070	24.741	21.167	18.568	15.036	11.122	7.630	5.030	3.319	2.277	1.352
0.080	21.647	18.980	16.949	14.056	10.663	7.464	4.981	3.305	2.273	1.351
0.100	17.315	15.698	14.384	12.376	9.797	7.120	4.871	3.274	2.265	1.351
0.150	11.555	10.919	10.357	9.410	8.006	6.273	4.556	3.176	2.237	1.349
0.200	8.675	8.352	8.055	7.529	6.684	5.523	4.222	3.056	2.200	1.346
0.300	5.728	5.609	5.496	5.285	4.919	4.347	3.593	2.790	2.107	1.338
0.400	4.204	4.149	4.097	3.997	3.815	3.511	3.065	2.524	2.001	1.328
0.500	3.293	3.265	3.238	3.185	3.088	2.916	2.646	2.284	1.892	1.315
0.600	2.709	2.693	2.678	2.649	2.593	2.491	2.322	2.078	1.788	1.301
0.800	2.041	2.035	2.029	2.018	1.996	1.956	1.884	1.768	1.608	1.271
1.000	1.693	1.691	1.688	1.683	1.673	1.655	1.620	1.561	1.471	1.240
1.500	1.321	1.321	1.320	1.319	1.317	1.313	1.305	1.290	1.264	1.171
2.000	1.184	1.183	1.183	1.183	1.182	1.181	1.178	1.173	1.163	1.122
3.000	1.083	1.082	1.082	1.082	1.082	1.082	1.081	1.080	1.078	1.067
5.000	1.030	1.030	1.030	1.030	1.030	1.030	1.030	1.030	1.029	1.028

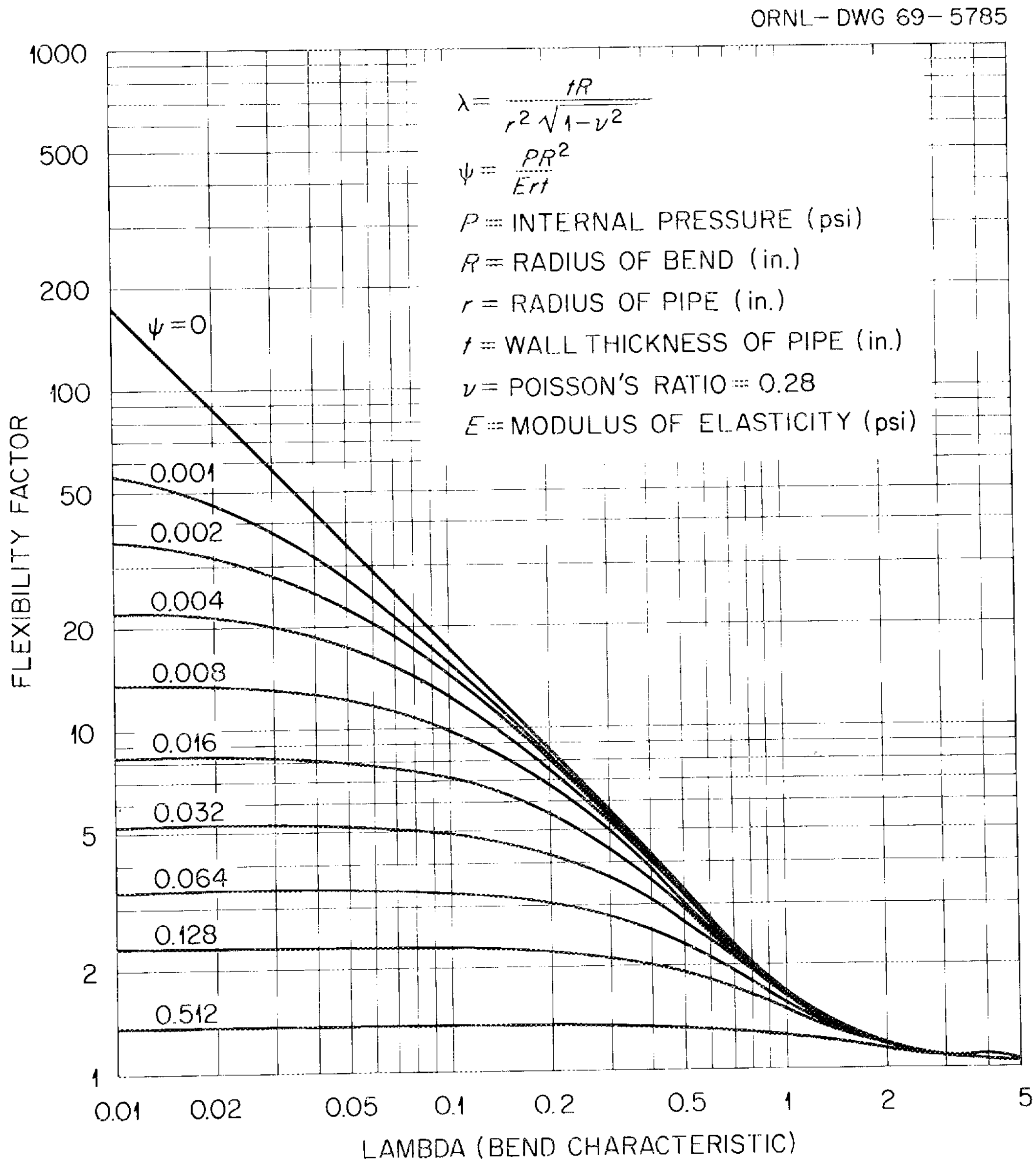

Fig. 8. Flexibility factors for elbows as a function of the bend characteristic λ and the internal pressure parameter ψ.

Stress-Index Development

In general, the moment loadings which are applied to an elbow in a piping system consist of in-plane bending moments M_i, out-of-plane bending moments M_o, and torsional moments M_t. For each of these loading conditions the distribution of stresses will be different. In particular, the maximum values will have different magnitudes and will be located at different positions, as can easily be verified by examining the equations in Appendix A. It can also be verified that for combined loadings the maximum stresses may be located at still different positions with maximum values which will in general be less than the sum of the individual maximums. For design purposes it will therefore be advantageous to determine the maximum stresses for any arbitrary combination of moment loadings, and to represent these values in a convenient form. Since both Section III and ANSI B31.7 use the maximum shear stress criterion (Tresca condition) for specifying design allowable stresses, this criterion is used as the basis for developing stress indices for general design use.

According to the Tresca condition, multiaxial stress-state yielding will occur when twice the absolute value of the maximum shear stress, or the stress intensity S as defined in the piping codes, exceeds the yield stress of a simple uniaxial tensile specimen. At some position in the component the stress intensity will have a maximum value S_{max}, which in the present case will depend on the dimensional parameters of the elbow λ, ψ, and γ. Furthermore, if the total bending moment load is represented as a vector $\vec{M}$, with individual vector components $\vec{M}_i$, $\vec{M}_o$, and $\vec{M}_t$ as shown in Fig. 9, then for a given vector magnitude $|\vec{M}|$ there will be some angular position of the vector $\vec{M}(\eta,\beta)$ which will give the largest value for the maximum stress intensities. Thus for combined moment loadings, S_{max} will depend on the additional parameters η and β.

Therefore let $\bar{\bar{S}}$ be the largest value of the stress intensities S_{max} corresponding to all moment vectors with fixed magnitude:

$$\bar{\bar{S}}\,(\lambda,\psi,\gamma) = S_{max}\,(\lambda,\psi,\gamma,\eta,\beta) \quad . \tag{10}$$

In other words, for a given elbow which is loaded with any combination of in-plane, out-of-plane, or torsional moments whose vector sum $|\vec{M}|$ is fixed,

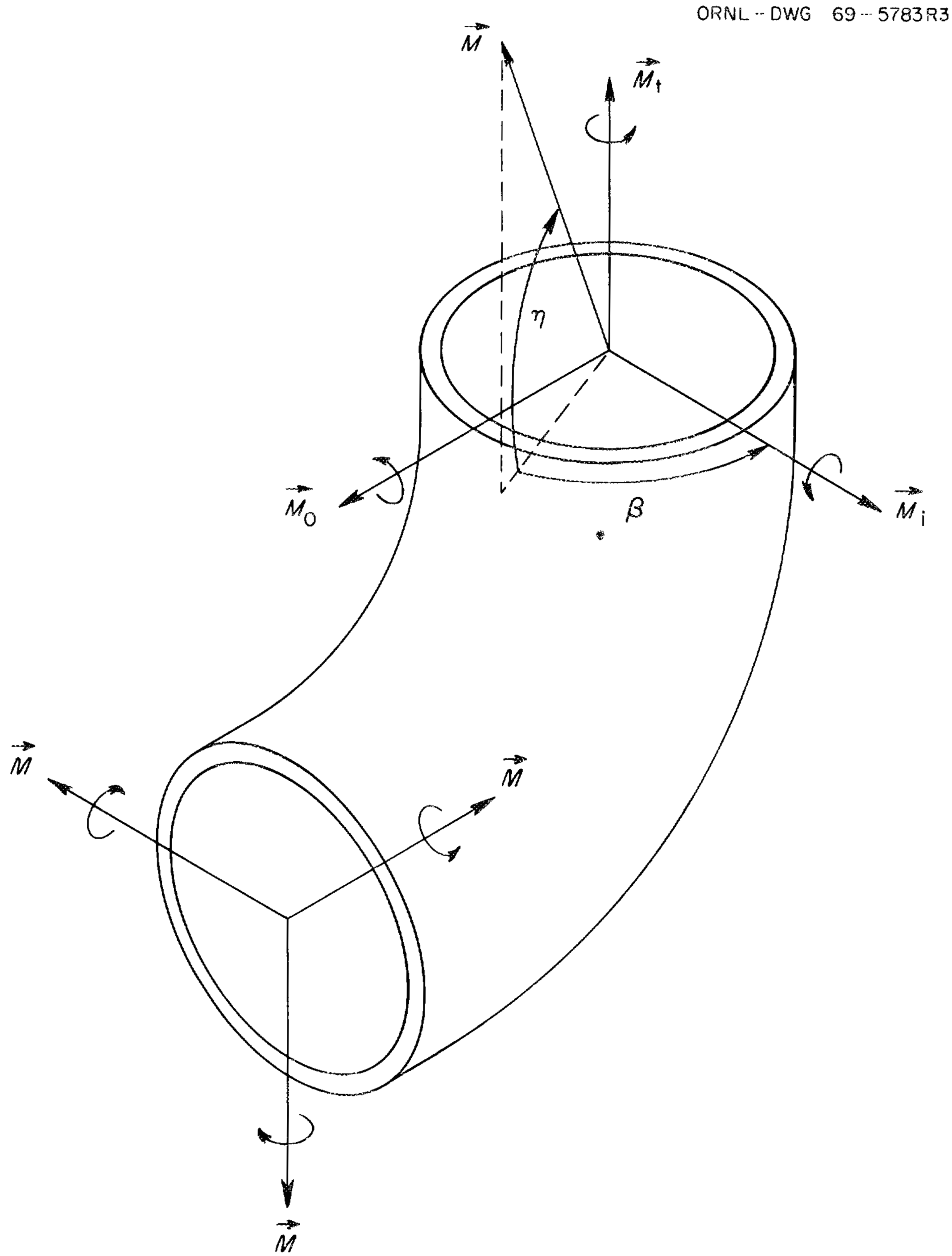

Fig. 9. Moment vector loadings on an elbow or curved pipe.

$\overline{S}$ is the maximum value of the stress intensity. The quantity $\overline{S}$ thus provides a convenient and conservative variable for use in developing stress indices. In keeping with the format of the piping code and the desired simplicity, we define the stress index as the ratio of $\overline{S}\ (\lambda,\psi,\gamma)$ to the maximum bending stress in a corresponding straight pipe:

$$C\ (\lambda,\psi,\gamma) = \frac{\overline{S}\ (\lambda,\psi,\gamma)}{M/Z} ,$$

where Z is the section modulus, $M = |\vec{M}|$ is the magnitude of the resultant moment vector,

$$M = \sqrt{M_i^2 + M_o^2 + M_t^2} , \tag{11}$$

and M_i, M_o, and M_t are the magnitudes of the individual vector components, respectively. In the following development the magnitude M is set equal to the section modulus so that

$$M/Z \equiv 1 , \tag{12}$$

$$C\ (\lambda,\psi,\gamma) = \overline{S}\ (\lambda,\psi,\gamma) , \tag{13}$$

and the magnitude of the individual vector components are:

$$\begin{aligned} M_i &= Z \cos \eta \cos \beta , \\ M_o &= Z \cos \eta \sin \beta , \\ M_t &= Z \sin \eta . \end{aligned} \tag{14}$$

The stresses which are produced in an arbitrary element of the elbow by this combination of loadings, as shown in Fig. 10, are

$$\begin{aligned} \sigma_\ell &= \frac{M_i}{Z} \sigma_\ell^i + \frac{M_o}{Z} \sigma_\ell^o , \\ \sigma_c &= \frac{M_i}{Z} \sigma_c^i + \frac{M_o}{Z} \sigma_c^o , \\ \tau &= \frac{M_t r}{J} = \frac{M_t}{2Z} , \end{aligned} \tag{15}$$

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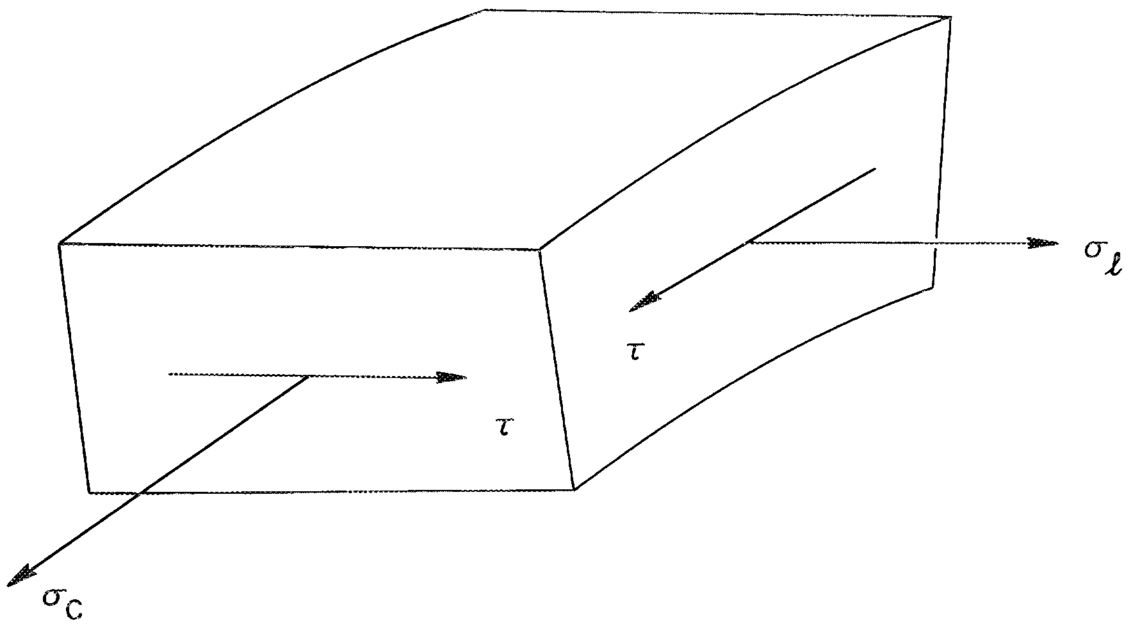

Fig. 10. State-of-stress in a differential element of a curved tube.

where the subscripts ℓ and c denote the longitudinal and circumferential directions, respectively, τ is the shear stress, and σ_ℓ^i, σ_c^i, σ_ℓ^o, and σ_c^o are the normalized stresses in the elbow due to in-plane bending and out-of-plane bending, respectively. The normalizing factor M/Z is the maximum bending stress due to a moment loading M in a piece of straight pipe with the same section modulus Z as the elbow [see Eqs. (A-13) to (A-16), Appendix A].

The principal normal stresses for the state-of-stress indicated in Fig. 10 are:

$$\sigma_1 = \frac{1}{2}(\sigma_\ell + \sigma_c) + \sqrt{\frac{1}{4}(\sigma_\ell - \sigma_c)^2 + \tau^2} ,$$

$$\sigma_2 = \frac{1}{2}(\sigma_\ell + \sigma_c) - \sqrt{\frac{1}{4}(\sigma_\ell - \sigma_c)^2 + \tau^2} , \qquad (16)$$

$$\sigma_3 = 0 .$$

Since the stress intensity S is twice the maximum shear stress, S will be the maximum of the three quantities:

$$S_1 = |\sigma_1 - \sigma_2| = \sqrt{(\sigma_\ell - \sigma_c)^2 + 4\tau^2} ,$$

$$S_2 = |\sigma_1 - \sigma_3| = \frac{1}{2}\left|(\sigma_\ell + \sigma_c) + S_1\right| , \qquad (17)$$

$$S_3 = |\sigma_2 - \sigma_3| = \frac{1}{2}\left|(\sigma_\ell + \sigma_c) - S_1\right| .$$

The problem for a given elbow, that is, fixed λ, ψ, γ, thus reduces to determining the maximum of S_1, S_2, and S_3 for all angular positions of the moment vector $\vec{M}(\eta, \beta)$.

Consider first the equation for S_1. After substituting Eqs. (14) and (15) into the first of Eqs. (17):

$$S_1 = \sqrt{(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta} , \qquad (18)$$

where

$$A = \sigma_\ell^i - \sigma_c^i \,,$$
$$B = \sigma_\ell^o - \sigma_c^o \,. \qquad (19)$$

Stationary values of β and η are found by setting the partial derivatives of S_1^2 with respect to β and η, respectively, equal to zero. The resulting equations are:

$$(A \sin\beta - B\cos\beta)(A\cos\beta + B\sin\beta)\ \cos^2\eta = 0 \,,$$
$$[1 - (A\cos\beta + B\sin\beta)^2]\ \sin\eta\ \cos\eta = 0 \,. \qquad (20)$$

The solutions for Eqs. (20) yield either a maximum at $\eta = 90°$ of

$$S_1 = 1 \,, \qquad (21)$$

or a maximum at $\eta = 0$ and $\beta = \arctan(B/A)$ of

$$S_1 = \sqrt{A^2 + B^2} \,. \qquad (22)$$

Determining maximum values for S_2 and S_3 is somewhat more complicated. After substituting Eqs. (14) and (15) into both the second and third of Eqs. (17):

$$S_2 = \frac{1}{2}\left|(C\cos\beta + D\sin\beta)\ \cos\eta + S_1\right| \qquad (23)$$

and

$$S_3 = \frac{1}{2}\left|(C\cos\beta + D\sin\beta)\ \cos\eta - S_1\right| \,, \qquad (24)$$

where

$$C = \sigma_\ell^i + \sigma_c^i \,,$$
$$D = \sigma_\ell^o + \sigma_c^o \,, \qquad (25)$$

and as before, Eq. (18),

$$S_1 = \sqrt{(A\cos\beta + B\sin\beta)^2\ \cos^2\eta + \sin^2\eta} \,.$$

For stationary values of S_2, the following two equations obtained from Eq. (23) must be solved:

$$\frac{\partial S_2}{\partial \beta} = 0 = (-C \sin \beta + D \cos \beta) \cos \eta$$
$$+ [(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta]^{-1/2}$$
$$\times [(-A \sin \beta + B \cos \beta)(A \cos \beta + B \sin \beta) \cos^2 \eta] , \quad (26)$$

and

$$\frac{\partial S_2}{\partial \eta} = 0 = -(C \cos \beta + D \sin \beta) \sin \eta$$
$$+ [(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta]^{-1/2}$$
$$\times [1 - (A \cos \beta + B \sin \beta)^2] \sin \eta \cos \eta . \quad (27)$$

A similar set of equations is also obtained for S_3, differing only in the sign ($\pm$) of the second terms. Because of the complexity of these equations, explicit solutions for β and η were not determined. However, the angle η can be expressed as a function of the angle β from Eq. (27). Rearranging this equation gives the three-term product:

$$0 = \{\sin \eta\}\{[(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta]^{-1/2}\}$$
$$\{(C \cos \beta + D \sin \beta)[(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta]^{1/2}$$
$$- [1 - (A \cos \beta + B \sin \beta)^2 \cos \eta]\} . \quad (28)$$

Since the second term in braces $\{\ldots\}$ cannot vanish for any value of η or β, it may be discarded immediately leaving the first and third terms, which may be set equal to zero independently. If we consider the first term, that is, $\sin \eta = 0$, then a maximum stress intensity occurs at $\eta = 0$. Then using Eqs. (18) and (23),

$$S_2 = \frac{1}{2} |(C + A) \cos \beta + (D + B) \sin \beta| \quad (29)$$

or, in terms of stress components from Eqs. (19) and (25),

$$S_2 = |\sigma_\ell^i \cos \beta + \sigma_\ell^o \sin \beta| \ . \tag{30}$$

Thus by setting $\partial S_2/\partial\beta = 0$ from Eq. (30),

$$\beta = \text{arc tan } (\sigma_\ell^o/\sigma_\ell^i) \ , \tag{31}$$

and a maximum value for S_2 is given by

$$S_2 = \sqrt{(\sigma_\ell^i)^2 + (\sigma_\ell^o)^2} \ . \tag{32}$$

A similar result is obtained for S_3. At $\eta = 0$,

$$S_3 = |\sigma_c^i \cos \beta + \sigma_c^i \sin \beta| \ , \tag{33}$$

and from setting $\partial S_3/\partial\beta = 0$,

$$\beta = \text{arc tan } (\sigma_c^o/\sigma_c^i) \ , \tag{34}$$

giving a maximum value at $\eta = 0$ of

$$S_3 = \sqrt{(\sigma_c^i)^2 + (\sigma_c^o)^2} \ . \tag{35}$$

When the third term in Eq. (28) is set equal to zero, we obtain the following:

$$(C \cos \beta + D \sin \beta) \sqrt{(A \cos \beta + B \sin \beta)^2 \cos^2 \eta + \sin^2 \eta}$$
$$= [1 - (A \cos \beta + B \sin \beta)^2] \cos \eta \ . \tag{36}$$

Squaring both sides and collecting terms gives

$$\tan \eta = \frac{\pm 1}{C \cos \beta + D \sin \beta} \{1 + (A \cos \beta + B \sin \beta)^2 \ [(A \cos \beta$$
$$+ B \sin \beta)^2 - (C \cos \beta + D \sin \beta)^2 - 2]\}^{1/2} \ . \tag{37}$$

The same result is obtained when the corresponding equations for S_3, Eqs. (24), etc., are used. Hence, Eq. (37) gives the value for η which makes both S_2 and S_3 stationary for each value of β.

Parameter Study and Results

The stress index C_2 (λ,ψ,γ) for a given elbow was determined numerically by first solving the equations of Appendices A and B for the individual stress components as a function of angular position ϕ around the circumference of the elbow for each of the applied loads M_i, M_o, and M_t. The maximum stress intensity $\overline{S}$ (λ,ψ,γ) and consequently the stress index C_2 [see Eq. (13)] was then determined by combining the stresses as indicated above and choosing the largest of the values obtained by evaluating Eqs. (21), (22), (32), and (35) and those obtained from Eqs. (23) and (24) when β was scanned with η given by Eq. (37). A complete parameter study was conducted using a Fortran program written for the ORNL IBM-360 computer for values of λ which ranged between 0.01 and 5.0, for values of ψ between 0 and 0.512, and for four values of the radius ratio parameter γ = 2, 3, 5, and 10. The resulting stress indices are plotted in Figs. 11 through 14, and tabular values are given in Tables 2, 3, 4, and 5. As one might expect from the work of previous authors, the stress indices are not strongly dependent on the radius ratio for $\gamma = R/r > 2.0$, which can be verified by comparing the numbers in the different tables.

A conservative approximation for stress indices, which slightly overestimates the tabulated values, is given by

$$C_2 = \frac{2.25\ \lambda^{-2/3}}{1 + \lambda^{-4/3}\ \exp\left(-\psi^{-1/4}\right)} \ ; \qquad \begin{array}{l} 0.05 \le \lambda \le 1.0 \\ 0 \le \psi \le 0.1 \ . \end{array} \tag{38}$$

In order to take advantage of the slight dependence of C_2 on the parameter γ, one might use the better approximation

$$C_2 = \frac{2\lambda^{-2/3}\ (1 + 0.25\ \gamma^{-1})}{1 + \lambda^{-4/3}\ \exp\left(-\psi^{-1/4}\right)} \ ; \qquad \begin{array}{l} 0.05 \le \lambda \le 1.0 \\ 0 \le \psi \le 0.1 \ . \end{array} \tag{39}$$

For $\psi = 0$, Eq. (39) is similar to an approximation proposed by Cheng and Thailer.[21] For zero internal pressure it gives stress indices which are about 12% higher than the code values (Table NB-3683.2-1, Section III, Ref. 20) for $\gamma = 2$, and values which are equal to the code indices for $\gamma = \infty$.

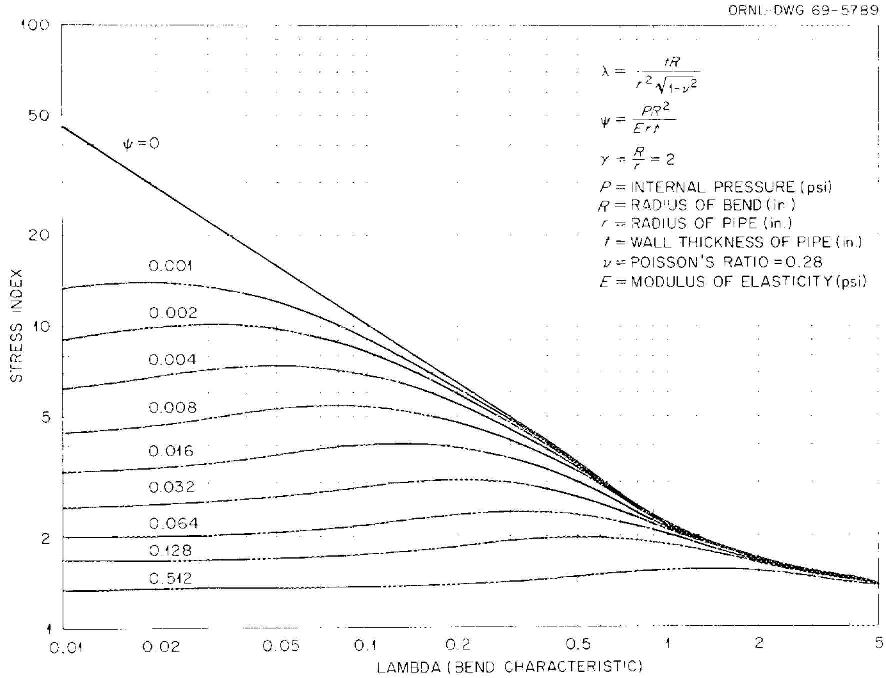

Fig. 11. Stress indices for an arbitrary combination of in-plane and out-of-plane bending or torsional moments as a function of λ and ψ for the radius ratio parameter γ = 2.

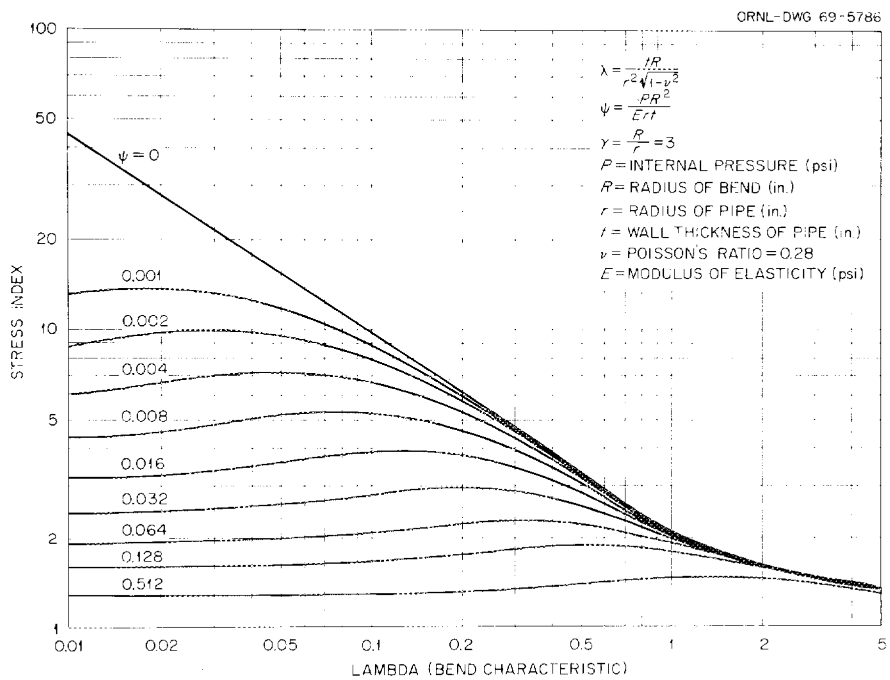

Fig. 12. Stress indices for an arbitrary combination of in-plane and out-of-plane bending or torsional moments as a function of λ and ψ for the radius ratio parameter γ = 3.

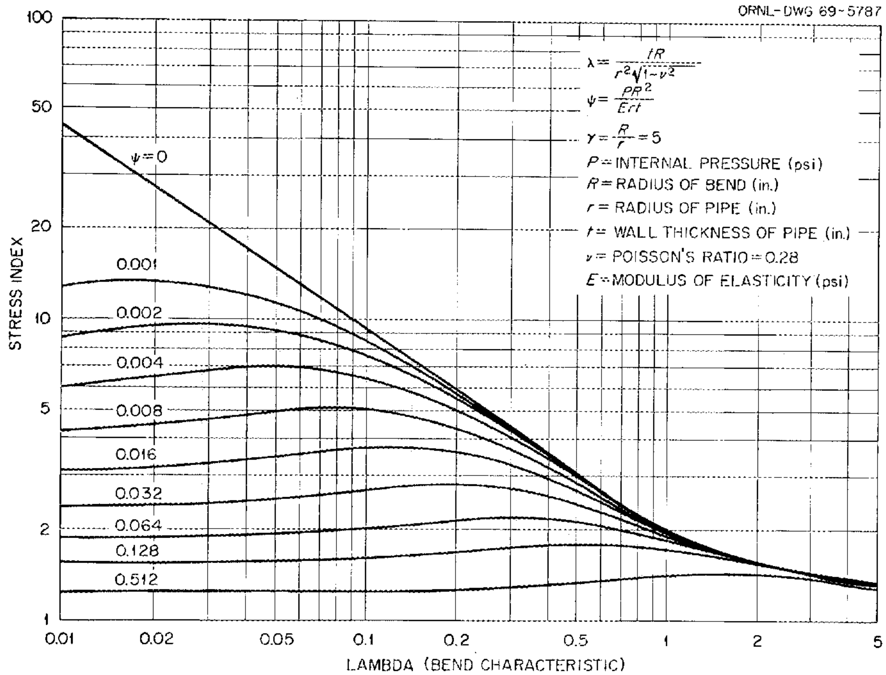

Fig. 13. Stress indices for an arbitrary combination of in-plane and out-of-plane bending or torsional moments as a function of λ and ψ for the radius ratio parameter $\gamma = 5$.

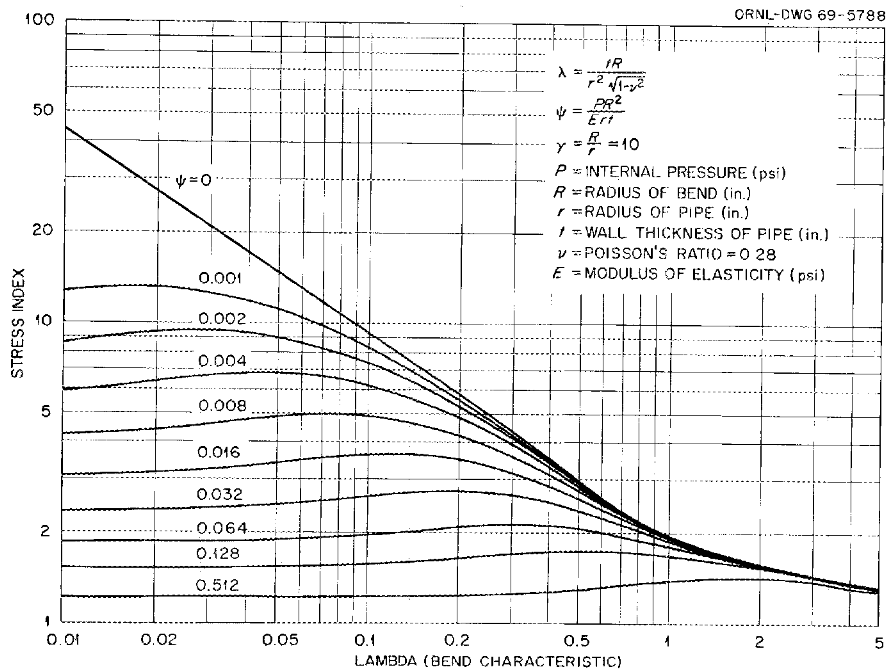

Fig. 14. Stress indices for an arbitrary combination of in-plane and out-of-plane bending or torsional moments as a function of λ and ψ for the radius ratio parameter $\gamma = 10$.

Table 2. C_2 stress indices for elbows with radius ratio $\gamma = 2.0$

STRESS INDICES FOR ELBOWS
WITH
GAMMA = 2.0

LAMBDA	PSI									
	0.0	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.512
0.010	45.669	13.394	9.037	6.223	4.438	3.284	2.510	1.993	1.659	1.340
0.015	35.054	13.917	9.573	6.519	4.568	3.331	2.530	2.001	1.663	1.341
0.020	29.062	13.948	9.911	6.782	4.699	3.384	2.551	2.010	1.667	1.342
0.030	22.323	13.523	10.123	7.156	4.945	3.502	2.597	2.030	1.676	1.344
0.040	18.515	12.906	9.971	7.339	5.141	3.615	2.649	2.051	1.685	1.347
0.050	16.020	12.155	9.768	7.382	5.282	3.718	2.701	2.073	1.695	1.349
0.060	14.233	11.420	9.522	7.331	5.372	3.807	2.751	2.097	1.705	1.351
0.070	12.877	10.745	9.211	7.225	5.420	3.882	2.799	2.120	1.715	1.353
0.080	11.810	10.142	8.883	7.113	5.434	3.942	2.843	2.144	1.726	1.356
0.100	10.218	9.123	8.240	6.903	5.391	4.021	2.923	2.188	1.748	1.361
0.150	7.856	7.360	6.924	6.192	5.120	4.043	3.047	2.287	1.802	1.374
0.200	6.526	6.247	5.992	5.542	4.824	3.932	3.084	2.356	1.851	1.387
0.300	5.017	4.897	4.783	4.571	4.203	3.631	3.012	2.413	1.925	1.414
0.400	4.137	4.074	4.012	3.895	3.683	3.328	2.863	2.401	1.966	1.440
0.500	3.542	3.504	3.468	3.397	3.266	3.035	2.699	2.350	1.981	1.465
0.600	3.109	3.086	3.062	3.017	2.931	2.775	2.546	2.284	1.977	1.486
0.800	2.524	2.513	2.502	2.480	2.442	2.389	2.294	2.143	1.938	1.517
1.000	2.223	2.219	2.215	2.207	2.192	2.163	2.109	2.018	1.880	1.536
1.500	1.859	1.858	1.857	1.854	1.850	1.840	1.823	1.790	1.734	1.538
2.000	1.688	1.688	1.687	1.686	1.684	1.680	1.672	1.657	1.630	1.515
3.000	1.516	1.516	1.516	1.515	1.515	1.513	1.511	1.506	1.497	1.453
5.000	1.379	1.379	1.379	1.379	1.379	1.379	1.378	1.377	1.375	1.364

Table 3. C_2 stress indices for elbows with radius ratio $\gamma = 3.0$

STRESS INDICES FOR ELBOWS
WITH
GAMMA = 3.0

LAMBDA	PSI									
	0.0	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.512
0.010	44.850	13.097	8.825	6.076	4.332	3.189	2.427	1.916	1.586	1.270
0.015	34.338	13.589	9.325	6.341	4.441	3.232	2.442	1.923	1.589	1.271
0.020	28.412	13.607	9.642	6.587	4.556	3.278	2.459	1.930	1.592	1.272
0.030	21.755	13.062	9.834	6.939	4.782	3.378	2.500	1.944	1.598	1.273
0.040	17.999	12.462	9.675	7.107	4.967	3.479	2.543	1.962	1.605	1.275
0.050	15.540	11.728	9.374	7.140	5.098	3.575	2.587	1.981	1.613	1.276
0.060	13.781	11.008	9.138	7.084	5.180	3.658	2.633	2.000	1.621	1.278
0.070	12.448	10.347	8.836	6.976	5.222	3.727	2.677	2.020	1.630	1.280
0.080	11.399	9.756	8.516	6.836	5.232	3.781	2.718	2.041	1.639	1.282
0.100	9.836	8.759	7.889	6.574	5.184	3.850	2.788	2.081	1.657	1.286
0.150	7.522	7.034	6.606	5.886	4.875	3.862	2.897	2.168	1.703	1.296
0.200	6.222	5.948	5.698	5.255	4.550	3.748	2.929	2.229	1.747	1.307
0.300	4.752	4.634	4.522	4.314	3.952	3.412	2.855	2.279	1.813	1.331
0.400	3.897	3.834	3.773	3.658	3.450	3.101	2.707	2.264	1.850	1.355
0.500	3.317	3.280	3.244	3.175	3.045	2.818	2.550	2.216	1.864	1.377
0.600	2.895	2.872	2.849	2.804	2.719	2.579	2.403	2.153	1.862	1.397
0.800	2.363	2.356	2.349	2.335	2.307	2.256	2.165	2.020	1.824	1.425
1.000	2.096	2.092	2.088	2.081	2.066	2.038	1.988	1.902	1.773	1.448
1.500	1.771	1.770	1.769	1.767	1.762	1.753	1.736	1.705	1.651	1.461
2.000	1.612	1.612	1.611	1.610	1.608	1.605	1.597	1.582	1.556	1.443
3.000	1.451	1.450	1.450	1.450	1.449	1.448	1.446	1.441	1.432	1.388
5.000	1.317	1.317	1.317	1.317	1.316	1.316	1.316	1.315	1.313	1.301

Table 4. C_2 stress indices for elbows with radius ratio $\gamma = 5.0$

STRESS INDICES FOR ELBOWS
WITH
GAMMA = 5.0

LAMBDA	PSI									
	0.0	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.512
0.010	44.195	12.880	8.683	5.972	4.261	3.137	2.380	1.873	1.546	1.233
0.015	33.766	13.334	9.164	6.228	4.353	3.170	2.393	1.878	1.547	1.233
0.020	27.891	13.340	9.460	6.460	4.464	3.208	2.407	1.883	1.549	1.233
0.030	21.300	12.720	9.625	6.786	4.673	3.299	2.438	1.896	1.555	1.234
0.040	17.585	12.107	9.454	6.939	4.842	3.391	2.476	1.909	1.560	1.235
0.050	15.156	11.386	9.137	6.963	4.962	3.477	2.516	1.924	1.566	1.237
0.060	13.420	10.678	8.830	6.902	5.036	3.551	2.556	1.941	1.572	1.238
0.070	12.104	10.028	8.536	6.791	5.072	3.612	2.594	1.959	1.579	1.239
0.080	11.070	9.447	8.223	6.650	5.078	3.661	2.630	1.976	1.587	1.240
0.100	9.531	8.467	7.609	6.328	5.025	3.723	2.694	2.011	1.602	1.243
0.150	7.254	6.774	6.351	5.641	4.714	3.727	2.793	2.088	1.642	1.252
0.200	5.979	5.709	5.462	5.026	4.345	3.612	2.817	2.142	1.680	1.261
0.300	4.540	4.424	4.313	4.108	3.751	3.280	2.738	2.185	1.740	1.282
0.400	3.704	3.642	3.583	3.469	3.263	2.956	2.594	2.168	1.773	1.304
0.500	3.137	3.101	3.065	2.997	2.868	2.686	2.442	2.121	1.786	1.324
0.600	2.724	2.701	2.678	2.634	2.573	2.470	2.301	2.061	1.781	1.343
0.800	2.262	2.255	2.248	2.234	2.208	2.158	2.070	1.935	1.753	1.379
1.000	2.024	2.021	2.017	2.010	1.996	1.970	1.921	1.838	1.713	1.401
1.500	1.726	1.725	1.724	1.722	1.717	1.709	1.692	1.662	1.609	1.425
2.000	1.588	1.587	1.587	1.586	1.584	1.580	1.572	1.558	1.532	1.420
3.000	1.441	1.440	1.440	1.440	1.439	1.438	1.436	1.431	1.422	1.375
5.000	1.307	1.307	1.307	1.307	1.307	1.307	1.306	1.305	1.303	1.291

Table 5. C_2 stress indices for elbows with radius ratio γ = 10.0

STRESS INDICES FOR ELBOWS
WITH
GAMMA = 10.0

LAMBDA	PSI									
	0.0	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.512
0.010	43.704	12.745	8.596	5.917	4.217	3.108	2.358	1.854	1.528	1.216
0.015	33.336	13.174	9.051	6.155	4.308	3.133	2.367	1.857	1.529	1.216
0.020	27.501	13.158	9.331	6.371	4.408	3.170	2.377	1.861	1.530	1.216
0.030	20.959	12.522	9.478	6.676	4.599	3.251	2.406	1.869	1.533	1.217
0.040	17.275	11.841	9.300	6.817	4.755	3.333	2.438	1.881	1.537	1.218
0.050	14.868	11.130	8.980	6.835	4.865	3.410	2.472	1.894	1.542	1.218
0.060	13.149	10.431	8.611	6.770	4.933	3.480	2.508	1.908	1.547	1.219
0.070	11.847	9.790	8.311	6.657	4.964	3.538	2.544	1.923	1.553	1.220
0.080	10.823	9.216	8.003	6.515	4.967	3.584	2.577	1.939	1.559	1.221
0.100	9.302	8.248	7.398	6.192	4.910	3.641	2.635	1.970	1.572	1.223
0.150	7.054	6.578	6.159	5.458	4.598	3.635	2.722	2.039	1.607	1.230
0.200	5.796	5.529	5.285	4.853	4.233	3.517	2.742	2.087	1.641	1.238
0.300	4.381	4.266	4.156	3.953	3.600	3.185	2.660	2.124	1.694	1.258
0.400	3.560	3.498	3.439	3.327	3.123	2.868	2.517	2.105	1.725	1.277
0.500	3.003	2.966	2.931	2.863	2.758	2.605	2.368	2.058	1.734	1.298
0.600	2.611	2.595	2.580	2.551	2.496	2.396	2.232	1.998	1.729	1.319
0.800	2.199	2.192	2.186	2.173	2.149	2.102	2.021	1.891	1.713	1.352
1.000	1.976	1.973	1.969	1.962	1.949	1.923	1.875	1.794	1.677	1.381
1.500	1.711	1.710	1.709	1.707	1.703	1.695	1.679	1.649	1.599	1.419
2.000	1.587	1.587	1.586	1.585	1.583	1.579	1.572	1.558	1.531	1.420
3.000	1.441	1.440	1.440	1.440	1.439	1.438	1.436	1.431	1.422	1.375
5.000	1.307	1.307	1.307	1.307	1.307	1.307	1.306	1.305	1.303	1.291

SUMMARY AND DISCUSSION

During the past 60 years many analytical and experimental studies have been conducted to determine the stresses in and the flexibility of pipe elbows and curved tubes under moment loadings. Minimum potential energy methods, mechanics-of-materials, and thin-shell theory have all been used to formulate equations and to derive solutions. Of these, the minimum potential energy approach has been most widely studied, and at present is the most well developed, although thin-shell theory appears to have a good deal of undeveloped potential. The minimum potential energy approach was first used by von Karman[2] in 1911 to obtain an analytical solution for the case of in-plane bending. His solution has been extended by Vigness,[7] Symonds and Pardue,[15] Kafka and Dunn,[8] Rodabaugh and George,[9] Gross,[10] and in the present study to improve the accuracy and to include other loading conditions.

Recent advances in the development of safety codes and standards[4,20] for the nuclear power industry, where structural integrity is of the utmost importance, have emphasized the need for practicable as well as conservative design formulas. The present study was therefore undertaken to develop conservative stress indices and flexibility factors for elbows loaded with an arbitrary combination of in-plane and out-of-plane bending and torsional moments using the best available analytical solutions. For various reasons, including theoretical considerations and comparisons with experimental data, the paper by Rodabaugh and George was selected as the basis from which to proceed.

The Rodabaugh and George solution was modified to provide a more accurate approximation for the circumferential membrane stress, and equations were derived for calculating the stress indices and flexibility factors. These equations were programmed for the ORNL IBM-360 digital computer and a parameter study was conducted for a range of dimensionless variables which includes those of current interest. Tabulated values and graphical plots of the resulting indices and flexibility factors are given in the body of the report. The tabulated values were used to develop approximating formulas, which are slightly conservative, for a restricted range of the variables in a form which can be used directly in the nuclear piping codes and standards.

The flexibility factor is given by

$$k_p(\lambda,\psi) = \frac{1.66\ \lambda^{-1}}{1 + 1.75\ \lambda^{-4/3}\ \exp\left(-1.15\ \psi^{-1/4}\right)}\ ; \qquad \begin{matrix} 0.05 \le \lambda \le 1.0 \\ 0 \le \psi \le 0.1 \end{matrix} \qquad (40)$$

where*

$$\lambda = \frac{tR}{r^2\sqrt{1-\nu^2}}$$

is the bend characteristic parameter of the elbow or curved pipe, and

$$\psi = \frac{PR^2}{Ert}$$

is the nondimensional internal pressure parameter.

A conservative approximation for the stress index (which is simply multiplied by the maximum stress M/Z in straight pipe due to bending to calculate the maximum stress intensity in the elbow) is given by

$$C_2 = \frac{2\lambda^{-2/3}\left(1 + 0.25\ \gamma^{-1}\right)}{1 + \lambda^{-4/3}\ \exp\left(-\psi^{-1/4}\right)}\ ; \qquad \begin{matrix} 0.05 \le \lambda \le 1.0 \\ 0 \le \psi \le 0.1 \end{matrix} \qquad (41)$$

where $\gamma = R/r$ is the ratio of the bend radius to pipe radius of the elbow.

The influence of internal pressure is to reduce both the flexibility factor and the stress index. The major portion of the bending loads in a piping system results from restraining the thermal expansions. Thus, the effect of internal pressure is to increase the bending loads due to the reduction in flexibility and simultaneously to reduce the effectiveness of the bending loads in producing stresses. A conservative approach to piping system analysis is to determine the bending load using a flexibility analysis which includes the effect of internal pressure but to

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*The upper limit of $\lambda \ge 1.0$ on the applicable range may be replaced by the more conservative condition that $k_p(\lambda,\psi) \ge 1.0$ with very little loss in accuracy in order to conform with present code practice.

neglect the effect of internal pressure in determining the stress intensity due to these loads.

Two potentially important problems have not been discussed in the present study. These are end effects and geometric deviations from the ideal torus used as the basic analytical model. Several publications report results of experimental investigations to determine the effect of the attached structures at the ends of pipe elbows. Pardue and Vigness[22] and Vissat and del Buono[23] verify that the attached structures have a significant effect on the stresses and flexibilities. At present there does not appear to be any "in depth" theoretical investigation into the problem of end effects. Such an investigation is needed. Kalnins[24] has developed a theoretical analysis which utilizes thin-shell theory to treat this problem. If his analysis proves to be satisfactory from a computational viewpoint, it could be used as a basis for further study. The desirable results of such a study would be simple correction factors, which could be applied to the results given in this report, to account for the effects of end conditions. As the flexibility factor is related to the degree to which the cross section becomes oval during deformation, the use of the flexibility factors developed in the present study will lead to an overestimation of the overall flexibility of an elbow. Also, the stress indices may be more conservative than necessary because of the effects of end conditions.

Additional work is also needed on the influence of geometrical deviations of commercial elbows from the idealized perfect toroidal section, particularly out-of-roundness and nonuniform wall thickness. All published experimental studies have used commercial grade elbows where these geometric deviations are present to at least some degree; however, it is not possible to determine from these data what portion of the stresses is due to geometric deviations and what portion is due to other effects. There are, however, two theoretical papers which give some insight into the problem. These are the paper by Clark, Gilroy, and Reissner[25] published in 1952, and a recent paper by Findley and Spence,[26] both of which treat bends with out-of-roundness.

An experimental study currently under way at Oak Ridge National Laboratory (see Ref. 3, Task 4) will shed further light on both the above questions. As a result of that study, it is expected that modifications to the stress indices and flexibility factors developed in the present study will be made.

REFERENCES

1. A. Bantlin, "Formanderung and Beanspruchung federnder Ausgliechrohre," Z., V.D.I., 54: 43-49 (1910).

2. Th. von Karman, "Über die Formänderung dünnwandieger Rohre, insbersondere federnder Ausgliechrohre," Z., V.D.I., 55: 1889-1895 (1911).

3. S. E. Moore and W. L. Greenstreet, "Program for the ORNL Piping Program — Design Criteria for Piping, Pumps, and Valves" (1970 unpublished).

4. USA Standard, Code for Pressure Piping, Nuclear Power Piping, USAS B31.7-1969, American Society of Mechanical Engineers, New York (1969).

5. C. E. Turner and H. Ford, "Examination of the Theories for Calculating the Stresses in Pipe Bends Subjected to In-Plane Bending," Proc. Inst. Mech. Engrs., 171: 513-515 (1957).

6. M. Tueda, "Bourdon Tubes and Bending of Thin Curved Tubes," Kyoto Imperial University, Memoirs, July 1934, February 1936.

7. I. Vigness, "Elastic Properties of Curved Tubes," Trans. ASME, 65: 105-120 (1943).

8. P. G. Kafka and M. B. Dunn, "Stiffness of Curved Circular Tubes With Internal Pressure," J. Appl. Mech., 23: 247-254 (1956).

9. E. C. Rodabaugh and H. H. George, "Effect of Internal Pressure on Flexibility and Stress-Intensification Factors of Curved Pipe or Welding Elbows," Trans. ASME, 79: 939-948 (1957).

10. N. Gross, "Experiments on Short-Radius Pipe Bends," Proc. Inst. Mech. Engrs., 1(B): 465-479 (1952-53).

11. R. T. Smith, "Theoretical Analysis of the Stresses in Pipe Bends Subjected to Out-of-Plane Bending," J. Mech. Eng. Sci., 9(2)(1967).

12. R. A. Clark and E. Reissner, "Bending of Curved Tubes," Advances in Applied Mechanics, Academic Press, New York, Vol. 2, pp. 93-122, 1951.

13. E. Reissner, "On Bending of Curved Thin-Walled Tubes," Proc. Nat. Acad. Sci. U.S., 35: 204-208 (1949).

14. D. H. Cheng and H. J. Thailer, "In-Plane Bending of Curved Circular Tubes," J. Eng. Ind., Trans., ASME Paper No. 68-PVP-12, September 1968.

15. P. S. Symonds and T. E. Pardue, "Characteristics of Short Radius Tube Bends," second partial report, Naval Research Laboratory Report 0-2761, 1946.

16. R. T. Smith and H. Ford, "Experiments on Pipe Lines and Pipe Bends Subjected to Three-Dimensional Loadings," J. Mech. Eng. Sci., 9(2): 124-137 (1967).

17. R. L. Jacobs and H. Surosky, "Experimental Investigation of the Effects of End Conditions and Bend Angle on the Stress Distributions of Welding Elbows, Phase I," Naval Ship Research and Development Center, Annapolis Division Report No. 2835, November 1968.

18. R. L. Jacobs and H. Surosky, "Experimental Investigation of the Effects of End Conditions and Bend Angle on the Stress Distributions of Welding Elbows, Phase II," Naval Ship Research and Development Center, Annapolis Division Report No. 3044, September 1969.

19. E. C. Rodabaugh and A. G. Pickett, "Survey Report on Structural Design of Piping Systems and Components," USAEC Report TID-25553, Division of Technical Information, December 1970.

20. ASME Boiler and Pressure Vessel Code, Section III. Nuclear Power Plant Components, American Society of Mechanical Engineers, New York, 1971.

21. D. H. Cheng and H. J. Thailer, "On Bending of Curved Circular Tubes," J. Eng. Ind., Trans. ASME, 62-66 (February 1970).

22. T. E. Pardue and I. Vigness, "Properties of Thin-Walled Curved Tubes of Short-Bend Radius," Trans. ASME, 73: 77-81 (1951).

23. P. L. Vissat and A. J. del Buono, "In-Plane Bending Properties of Welding Elbows," Trans. ASME, 77: 161-175 (1955).

24. A. Kalnins, "Stress Analysis of Curved Tubes," Pressure Vessel Technology, Part I — Design and Analysis, Proceedings of the First International Conference on Pressure Vessel Technology, Delft, The Netherlands, American Society of Mechanical Engineers, New York, 1969.

25. R. A. Clark, T. I. Gilroy, and E. Reissner, "Stresses and Deformations of Toroidal Shells of Elliptical Cross Section," J. Appl. Mech., 19: 37-48 (1952).

26. G. E. Findley and J. Spence, "Bending of Pipe Bends with Elliptic Cross Sections," Welding Research Council Bulletin 164, Welding Research Council, August 1971.

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APPENDIX A

THE ANALYSIS OF RODABAUGH AND GEORGE

In 1956, Rodabaugh and George[9] published an analysis of the stresses and deformations for pipe elbows subjected to external bending loads including the effect of internal pressure on the bending stresses. A brief summary of that analysis and a restatement of the pertinent equations are given below.

Following the analyses of von Karman[2] and Vigness,[7] the strains were assumed to be due to two distinct forms of deformation — the warping of the tube cross section, and the beam-type bending of the tube centerline. The warping of the cross section was assumed to be independent of position along the tube centerline, the ratio r/R of the tube radius to bend radius was neglected in comparison to unity, and the circumferential membrane strain was assumed to be zero. The strain-displacement equations of thin-shell theory, incorporating the above assumptions, were used to obtain the strain due to warping of the cross section. A strength-of-materials analysis was then used to determine the strain due to beam-type bending of the tube centerline. The resulting longitudinal strains, given by the following equations, are constant across the tube wall.

<u>In-Plane Bending</u>:

$$\epsilon_{\ell} = \frac{1}{R}\left(\frac{\Delta\alpha}{\alpha} r \sin\phi + w_t \cos\phi - \frac{dw_t}{d\phi}\sin\phi\right); \qquad \text{(A-1)}$$

<u>Out-of-Plane Bending</u>:

$$\epsilon_{\ell} = \frac{1}{R}\left(\frac{R}{\rho} r \cos\phi + w_t \cos\phi - \frac{dw_t}{d\phi}\sin\phi\right); \qquad \text{(A-2)}$$

where $\Delta\alpha/\alpha$ is the change in curvature of the tube centerline due to in-plane bending, ρ is the radius of curvature of the deformed tube centerline in the plane perpendicular to the plane containing the pipe bend, and w_t is the circumferential displacement of the tube wall relative to the centerline. From the assumption that the circumferential membrane strain is zero it follows that the radial displacement (positive outward) of the tube wall relative to the centerline is

$$w_r = -\frac{dw_t}{d\phi} \ . \tag{A-3}$$

The warping of the cross section was assumed, for out-of-plane bending, to be given by

$$w_t = \sum_{n=1}^{\infty} C_n \cos 2n\phi \tag{A-4}$$

and, for in-plane bending, by

$$w_t = \sum_{n=1}^{\infty} C_n \sin 2n\phi \ , \tag{A-5}$$

where the coefficients C_n are to be determined.

Using these expressions for the deformation, the total strain energy due to deformation is given by:

$$U = \frac{\pi r t E}{2R^2} \left\{ r^2 \xi^2 + 3r\xi C_1 + \frac{9}{4} C_1^2 \right.$$

$$+ \frac{1}{4} \sum \left[C_n^2 (1-2n)^2 - 2C_n C_{n+1} (2n-1)(2n+3) \right.$$

$$\left. \left. + C_{n+1}^2 (2n+3)^2 \right] + \frac{\lambda^2}{12} \sum C_n^2 (8n^3 - 2n)^2 \right\} , \tag{A-6}$$

where $\xi = \Delta\alpha/\alpha$ for in-plane bending and $\xi = R/r$ for out-of-plane bending. The change in potential energy of the internal pressure per unit length of the pipe, due to the warping of the tube wall, is given by

$$V_p = 2\pi P \sum n^2 (4n^2 - 1) C_n^2 \ . \tag{A-7}$$

The principle of least work requires that the coefficients, C_n, satisfy the set of linear equations:

$$\frac{\partial}{\partial C_1}(U + V_p) = 0 = 3 + (5 + 6\lambda^2 + 24\psi)d_1 - \frac{5}{2}d_2 \ ,$$

$$\frac{\partial}{\partial C_2}(U + V_p) = 0 = -\frac{5}{2}d_1 + (17 + 600\lambda^2 + 480\psi)d_2 - \frac{21}{2}d_3 \ ,$$

$$\vdots \tag{A-8}$$

$$\frac{\partial}{\partial C_n}(U + V_p) = 0 = -\frac{1}{2}(2n-3)(2n+1)\ d_{n-1}$$

$$+\left[(4n^2+1) + (8n^3 - 2n)^2\frac{\lambda^2}{6} + 8n^2\ (4n^2-1)\psi\right]d_n$$

$$-\frac{1}{2}(2n-1)(2n+3)\ d_{n+1} \ ,$$

where

$$d_n = \frac{C_n}{r\xi} \ . \tag{A-9}$$

Truncating the series after N terms, a set of N linear equations for N+1 unknown values of d_n is obtained. With $d_{N+1} = 0$, the values of d_1, d_2, ..., d_N are the solution to this set of equations.

These values of d_1, d_2, ..., d_N yield a minimum for the energy $U + V_p$. The value of ξ is determined by equating this minimum energy to the work done by the bending moment, that is,

$$(U + V_p)_{min} = \frac{1}{2}\frac{M}{R}\xi \ . \tag{A-10}$$

The values of C_1, C_2, ..., C_N are obtained from Eq. (A-9).

The flexibility factor, k_p, is given by

$$k_p = \frac{EI}{RM}\xi \ . \tag{A-11}$$

After simplification, k_p may be expressed as

$$k_p = \frac{1}{1 + \frac{3}{2} d_1} \ . \tag{A-12}$$

Utilizing the strain-displacement relations (A-1) and (A-2) and the appropriate stress-strain relations, the nondimensional stresses, obtained by dividing the calculated stresses by M/Z are, for in-plane bending:

$$\sigma_\ell^i = F\left[D \sin\phi + \frac{1}{2}\sum_{n=1}^{N} A_n \sin(2n+1)\phi \pm \frac{\nu\lambda}{2}\sum_{n=1}^{N} B_n \cos 2n\phi\right], \tag{A-13}$$

$$\sigma_c^i = F\left[\nu D \sin\phi + \frac{\nu}{2}\sum_{n=1}^{N} A_n \sin(2n+1)\phi \pm \frac{\lambda}{2}\sum_{n=1}^{N} B_n \cos 2n\phi\right], \tag{A-14}$$

and for out-of-plane bending:

$$\sigma_\ell^o = F\left[D \cos\phi + \frac{1}{2}\sum_{n=1}^{N} A_n \cos(2n+1)\phi \pm \frac{\nu\lambda}{2}\sum_{n=1}^{N} -B_n \sin 2n\phi\right], \tag{A-15}$$

$$\sigma_c^o = F\left[\nu D \cos\phi + \frac{\nu}{2}\sum_{n=1}^{N} A_n \cos(2n+1)\phi \pm \frac{\lambda}{2}\sum_{n=1}^{N} -B_n \sin 2n\phi\right], \tag{A-16}$$

where the terms with positive and negative signs correspond to the outside and inside surface bending stresses respectively, and

$$\begin{aligned} F &= \frac{k_p}{1-\nu^2} \ , \\ D &= 1 + \frac{3}{2} d_1 \ , \\ A_n &= (1-2n)d_n + (2n+3)d_{n+1} \ , \\ B_n &= (2n - 8n^3)d_n \ . \end{aligned} \tag{A-17}$$

APPENDIX B

CIRCUMFERENTIAL MEMBRANE FORCE CORRECTION

In comparing his experimental results with the analytical results obtained from the von Karman theory, Gross[10] noticed that this theory results in a logical inconsistency. In his analysis, von Karman had reasoned that, due to the curvature of the pipe bend, the longitudinal membrane forces yield components which deform the tube cross section, and logically, produce a compressive circumferential membrane force. For in-plane bending, this force should be symmetric about ϕ equals zero, be zero at ϕ equals $\pm\pi/2$, and have its maximum absolute value at ϕ equals zero. The experimental results of Gross and others indicate that this conclusion is correct. The von Karman analysis, however, yields a circumferential membrane force which is skew-symmetric about ϕ equals zero and hence zero at ϕ equals zero; see, for example, Eq. (A-14).

By assuming that the von Karman analysis correctly determines the longitudinal membrane force and the circumferential shell bending moment, Gross determined the circumferential membrane force necessary to satisfy the static equilibrium conditions. The following constitutes a generalization of his analysis.

Consider the two free bodies, A and B, shown in Fig. 15. These free bodies are obtained from a segment of curved tube of arc length α by passing longitudinal planes with fixed values of ϕ. The longitudinal membrane force is denoted as n_L, m_C is the circumferential shell bending moment, and Q denotes the transverse shearing force.

For in-plane bending, the longitudinal membrane force obtained from Eq. (A-13) is:

$$n_L(\phi) = \frac{tFM_i}{Z}\left[D \sin\phi + \frac{1}{2}\sum_{n=1}^{N} A_n \sin(2n+1)\phi\right] . \qquad \text{(B-1)}$$

The transverse shearing force Q, obtained from Eq. (A-14), is

$$Q(\phi) = \frac{dm_C}{rd\phi} = -\frac{\lambda t^2}{6r}\sum_{n=1}^{N} nB_n \sin 2n\phi .$$

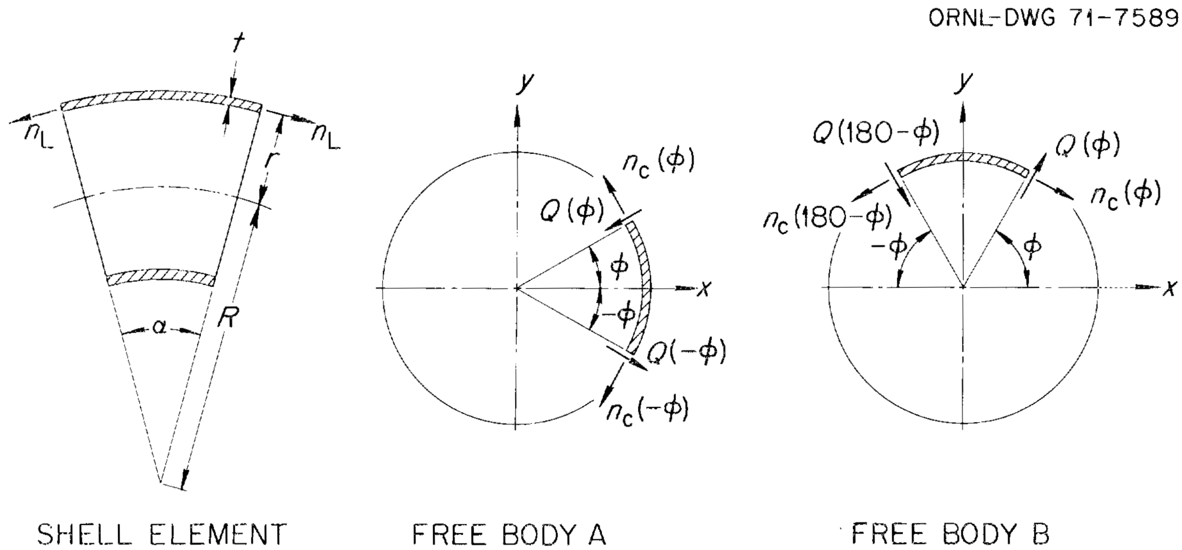

Fig. 15. Free body diagrams for the determination of circumferential membrane forces.

Thus,

$$Q(180 - \phi) = -\frac{\lambda t^2}{6r} \sum nB_n \sin 2n\,(180 - \phi)$$

$$= \frac{\lambda t^2}{6r} \sum nB_n \sin 2n\phi$$

$$= -Q(\phi) \; . \qquad \text{(B-2)}$$

Also,

$$Q(-\phi) = -\frac{\lambda t^2}{6r} \sum nB_n \sin 2n\,(-\phi)$$

$$= \frac{\lambda t^2}{6r} \sum nB_n \sin 2n\phi$$

$$= -Q(\phi) \; . \qquad \text{(B-3)}$$

On free body A, from statics the sum of the forces in the y direction is zero:

$$\Sigma F_y = 0 = R\alpha \{-[Q(\phi) + Q(-\phi)] \sin\phi + [n_C(\phi) - n_C(-\phi)] \cos\phi\} \; ,$$

but from Eq. (B-2),

$$Q(\phi) = -Q(-\phi) \; ;$$

hence,

$$n_C(\phi) = n_C(-\phi) \; .$$

Also from statics, the sum of forces in the x direction is zero:

$$\Sigma F_x = 0 = R\alpha \{[Q(-\phi) - Q(\phi)] \cos\phi - [n_C(\phi) + n_C(\phi)] \sin\phi\} \; .$$

Thus,

$$Q(\phi) = -n_C(\phi) \frac{\sin\phi}{\cos\phi} \; .$$

Similarly for free body B:

$$\Sigma F_x = 0 = R\alpha \{[Q(\phi) + Q(180 - \phi)] \cos \phi$$

$$+ [n_C(\phi) - n_C(180 - \phi)] \sin \phi\} ;$$

thus,

$$n_C(\phi) = n_C(180 - \phi) .$$

Also,

$$\Sigma F_y = 0 = - 2\sin \frac{\alpha}{2} \int_{\phi}^{180-\phi} n_L \, r d\xi$$

$$+ R\alpha [Q(\phi) - Q(180 - \phi)] \sin \phi$$

$$- R\alpha [n_C(\phi) + n_C(180 - \phi)] \cos \phi ;$$

$$2r \frac{\alpha}{2} \int_{\phi}^{\pi/2} n_L \, d\xi = R\alpha [Q(\phi) \sin \phi - n_C(\phi) \cos \phi]$$

$$= - R\alpha \, n_C(\phi) [\sin^2 \phi + \cos^2 \phi]/\cos \phi ;$$

or

$$n_C(\phi) = - \frac{r \cos \phi}{R} \int_{\phi}^{\pi/2} n_L \, d\xi . \qquad \text{(B-4)}$$

Substituting the expression for n_L given in Eq. (B-1) and performing the integration indicated in Eq. (B-4) we obtain for the membrane force

$$n_C(\phi) = - k_p \frac{M_i t}{\gamma Z(1 - \nu^2)} \left\{ D \cos \phi + \frac{1}{2} \sum_{n=1}^{N} \frac{A_n}{(2n + 1)} \cos (2n + 1)\phi \right\} \cos \phi , \qquad \text{(B-5)}$$

where $\gamma = R/r$ is the radius ratio parameter. Performing a similar analysis for out-of-plane bending gives the membrane force

$$n_c(\phi) = k_p \frac{M_o t}{\gamma Z(1 - \nu^2)} \left\{ D \sin \phi + \frac{1}{2} \sum_{n=1}^{N} \frac{A_n}{(2n + 1)} \sin (2n + 1)\phi \right\} \cos \phi . \qquad \text{(B-6)}$$

The normalized circumferential stresses, to be used in place of Eqs. (A-14) and (A-15) for in-plane and out-of-plane bending obtained from Eqs. (B-5) and (B-6), respectively are then:

<u>In-Plane Bending</u>:

$$\sigma_c^i = F \left\{ - \frac{1}{\gamma} \left[D \cos \phi + \frac{1}{2} \sum_{n=1}^{N} \frac{A_n}{(2n + 1)} \cos (2n + 1)\phi \right] \cos \phi \pm \frac{\lambda}{2} \sum_{n=1}^{N} B_n \cos 2n\phi \right\} ; \qquad \text{(B-7)}$$

<u>Out-of-Plane Bending</u>:

$$\sigma_c^o = F \left\{ \frac{1}{\gamma} \left[D \sin \phi + \frac{1}{2} \sum_{n=1}^{N} \frac{A_n}{(2n + 1)} \sin (2n + 1)\phi \right] \cos \phi \pm \frac{\lambda}{2} \sum_{n=1}^{N} - B_n \sin 2n\phi \right\} ; \qquad \text{(B-8)}$$

where F, D, A_n, and B_n are defined in Appendix A, and the radius ratio parameter $\gamma = R/r$ has been introduced into the analysis. It might be noted that although Eqs. (B-7) and (B-8) are approximate, they do satisfy the conditions of static equilibrium and as a consequence the circumferential membrane strain is not zero as originally assumed by von Karman. These equations should be as accurate as any of the analyses discussed in the body of the text.