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CLASSICAL PLASMA PHENOMENA FROM A QUANTUM MECHANICAL VIEWPOINT

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ABSTRACT

Non-linear phenomena in a classical plasma are treated in this paper. The plasma is regarded as a collection of particles (electrons and ions) and quasi-particles (plasmons, phonons, photons, etc.). The interaction between particles and quasi-particles is described by an interaction Hamiltonian which is written in terms of creation and annihilation operators. The transition probability for any process may be calculated from the usual formulas of quantum mechanical perturbation theory. These quantum mechanical methods provide a straightforward method for deriving quasi-linear equations and the corrections to them due to wave-wave interactions and wave-particle scattering. The derivation of conservation laws and H-theorems is particularly simple.

Kinetic equations for a plasma are derived which in various limits reduce to the Wyld-Pines, Balescu-Lenard, Boltzmann and quasi-linear equations.

A variety of non-linear interactions of plasma waves is discussed.

CONTENTS

CHAPTER 1. INTRODUCTION

It is natural and sometimes necessary to use quantum mechanics to discuss the phenomena of solid state plasma physics. However, the subject matter of this work is that of gaseous plasmas where quantum effects are negligible and classical physics is quite adequate. In such circumstances it may seem eccentric to employ quantum mechanics and then to discard the quantum corrections by taking the classical limit. Indeed, this is the more difficult approach to the simple problems of plasma physics, but when the more difficult problems involving non-linear interactions are considered a quantum mechanical viewpoint has certain advantages. It is useful to view the waves in a plasma as being composed of quasi-particles, the quanta of the waves. These quasi-particles interact with the particles of the plasma and with each other. These interactions are described in terms of an "interaction Hamiltonian," a "vertex function" or a "matrix element" for the interaction. This language is useful even when all calculations are made classically. In some cases the quantum mechanical calculations are more straightforward and less difficult than the corresponding classical calculations.

The subject matter of this work has been discussed from a classical viewpoint in the reviews of Kadomtsev,[1] Vedenov[2] and Tsytovich.[3] That is, we shall be concerned with non-linear effects in a plasma. We shall suppose that the non-linear effects are small, so that the linear theory is a good first approximation. We shall not discuss strong turbulence although progress has recently been made in this field.[4]

There is a large body of literature in which Green's function techniques of quantum statistical mechanics are applied to plasma physics.[5,6,7,8] These techniques seem to be applicable only to plasmas in thermal equilibrium. For that reason they are not very useful in the study of gaseous plasmas where the plasmas of most interest depart appreciably from thermal equilibrium.

The work in this paper is more closely related to that of Pines and Schrieffer[9] who gave a quantum mechanical derivation of the quasi-linear equations which were originally derived classically by Drummond and Pines[10] and by Vedenov, Velikhov and Sagdeev.[11] Pines and Schrieffer found interaction Hamiltonians for particles and plasmons (the quanta of plasma oscillations) and particles and phonons (the quanta of ion sound waves). They used these together with the Fermi Golden Rule to write equations for the rate of change of the particle and quasi-particle distribution functions. In the classical limit these equations reduced to the classical quasi-linear equations. It is clear from the derivation of these equations that Landau damping or growth can be described as the competition between absorption and stimulated emission of quasi-particles by particles. Another useful feature of this derivation is that terms due to spontaneous emission, which are often overlooked in classical derivations, appear quite naturally.

Wyld and Pines[12] used the Fermi Golden Rule to write equations for the rate of change of the particle distribution functions due to collisions. They assumed that the matrix element for a coulomb collision, $4\pi e^2/\vec{q}^{\,2}$, must be modified by the factor $\epsilon^{-1}(\vec{q},\omega)$ where ϵ is the dielectric function of the plasma, $\hbar\vec{q}$ is the momentum transfer and $\hbar\omega$

is the energy transfer in the collision. When the classical limit is taken, the equation of Wyld and Pines reduces to the Balescu[13]-Lenard[14] equation or the Boltzmann equation depending on just how the limiting process was carried out. The Balescu-Lenard equation had previously been derived by more tedious arguments than the one just described.

These papers by Pines and Schrieffer and by Wyld and Pines were important because they showed the ease with which equations describing classical plasmas could be derived by the perturbation theory formulas of quantum mechanics. Also, they gave insight into the physics behind the equations.

A quantum mechanical theory of non-linear phenomena in a very strong magnetic field was given by Walters and Harris.[15,16] The field was assumed to be so strong that the electron motion was essentially one-dimensional. The electrons were described by the fluid equations for a cold plasma. A Hamiltonian was found which gave the fluid equations as the Hamiltonian equations of motion. The Hamiltonian was written in terms of creation and annihilation operators for plasmons. The non-linear terms in the fluid equations gave rise to terms in the Hamiltonian describing three plasmon interactions; that is terms containing products of three creation or annihilation operators. An ion-plasmon interaction Hamiltonian was also derived. Using the Fermi Golden Rule, equations were derived for the rate of change of the ion and plasmon distribution functions. The three-plasmon interaction of Walters and Harris had previously been derived classically by Aamodt and Drummond.[17]

Quantum mechanical calculations of the interaction of three quasi-particles have been made for the case of a plasma with no magnetic field by Krishan[18] and by Krishan and Selim.[19] Calculations of three and four plasmon interactions in an unmagnetized plasma have been made by Zakharov.[20]

In a recent paper Ross[21] has considered wave-particle and wave-wave interactions from a quantum mechanical viewpoint. His starting point is the shielded coulomb matrix element discussed above. By examination of this in the neighborhood of frequencies which satisfy $\epsilon(\vec{q},\omega) = 0$ he is able to interpret the matrix element in terms of a quasi-particle propagator and a particle-quasi-particle vertex. Having obtained the particle-quasi-particle vertex, he adopted the point of view that the interaction of plasma waves is mediated by the particles and calculated the matrix elements for three wave interactions and wave-particle scattering. He limited the calculations to the quasi-one dimensional problem of a plasma in a very strong magnetic field. He also used the formalism of temperature dependent Green's functions which seem to be applicable to plasmas in thermal equilibrium only. However, when he reached the point in the calculations at which results were expressed in terms of particle distribution functions, these distribution functions were allowed to be arbitrary and the equations were assumed to be still valid. The point of view of Ross is very close to that of the writer and our results are in agreement.

Attention should also be directed to the paper of Gailitis et al.[22] Although the calculations are classical, the language is quantum mechanical. The paper deals with the interaction of plasmons, phonons and photons in an isotropic plasma.

The plan of this work is the following: In the second chapter we introduce the dielectric tensor of the plasma. The dielectric tensor, which is familiar to classical plasma physicists, is derived quantum mechanically in such a way as to emphasize the similarity in the classical and quantum mechanical derivations. The second quantization formalism is introduced in this chapter. A knowledge of the second quantization formalism which is sufficient for understanding this work may be obtained from Davidov.[23] We also discuss the linear theory of propagation of waves in a plasma, the damping and growth of these waves and the concepts of positive and negative wave energies.

In Chapter 3 we quantize the electromagnetic field in the plasma as has previously been done by Alekseev and Nikitin[24] and by Kihara, Aono and Dodo.[25] Our formulation is somewhat better suited to the purposes of this work. We also derive the particle-quasi-particle interaction Hamiltonian. The picture of the plasma which we have at the end of this chapter is that the plasma consists of a collection of particles and quasi-particles which interact only weakly. This is the picture developed in the early papers of Bohm and Pines.[6] We should point out that the picture is not completely consistent since motion of the particles is involved in the quasi-particle excitations as well. This lack of consistency is not likely to cause problems in the weakly non-linear systems considered here. The particles can be divided into the two classes: The "resonant particles" and the "non-resonant particles." The resonant particles are those which can emit and absorb quasi-particles and are a small minority in a weakly turbulent plasma. The non-resonant particles oscillate under the influence of the fields of the wave. They contribute to the energy and momentum of the wave, but after the wave has passed they return to their original state.

These results are used in Chapter 4 to derive the quasi-linear equations. This derivation is essentially the same as that of Pines and Schrieffer.[9] We discuss the conservation theorems and the H-theorem for these equations. The one-dimensional quasi-linear equations are sufficiently simple that their consequences can be examined in some detail and we do this, generally following Drummond and Pines[10] and Vedenov, Velikhov and Sagdeev.[11] The quasi-linear equations are not adequate to explain the phenomenon of oscillatory damping observed by Malmberg and Wharton.[26] We give a qualitative explanation of this phenomenon. Quantitative theories have been published by O'Neil[27] and by Al'Tshul' and Karpman.[28]

In Chapter 5 we look more critically at the interaction of a particle with a plasma. The particle, of course, interacts with the fluctuating fields. These may be calculated by the dressed test particle method of Rostoker[29] and Ichamaru.[30] The relation between the scattering by fluctuating fields, the two-particle scattering via a screened coulomb potential and the emission and absorption of quasi-particles is clarified. The scattering of photons by a plasma is quite similar to the scattering of particles, so it is convenient to consider the two processes together. The enhanced scattering of electromagnetic waves by a plasma as the plasma approaches a state of instability[31] is explained in terms of stimulated emission of plasmons.

In Chapter 6 we review the previously mentioned Wyld-Pines equation.[12] Building upon the insight gained in Chapters 4 and 5 we see what must be done to this equation to make it applicable to weakly unstable plasmas. This leads us to what we believe are equations which preserve the essential features of the Wyld-Pines equation and the quasi-linear equations. The derivation is in the same spirit as the derivation of the Wyld-Pines equation and elementary derivations of the Boltzmann equation. Kinetic theory being the slippery subject that it is, we would not be so presumptuous as to claim that our equations are logically unassailable. They do seem to be as plausible and probably more transparent than other equations which have been proposed. Only the first order interaction between particles and fluctuating fields is included in the equations discussed in this chapter. Higher order interactions are discussed in the next chapter.

In Chapter 7 we return to the point of view that the fundamental interactions are the particle-quasi-particle interactions and proceed to build from these the three-wave and wave-particle scattering interactions by the formulas of second and third order perturbation theory. We feel that in this chapter the advantages of the quantum mechanical approach are most apparent.

In Chapter 8 we return to classical physics and consider the interaction of a few monochromatic waves. The Hamiltonian formulation derived in previous chapters is of considerable utility here. What had appeared in previous chapters as problems in quantum mechanical perturbation theory now appears as the problem of non-linearly coupled classical oscillators. This aspect of non-linear plasma theory is closely related to the problems of non-linear optics.[32]

CHAPTER 2. THE DIELECTRIC TENSOR AND WAVE PROPAGATION

For many purposes in plasma physics it is convenient and sufficiently accurate to consider the plasma as a dispersive medium characterized by a dielectric tensor $\overleftrightarrow{\epsilon}(\vec{q},\omega)$. In this chapter we shall give a quantum mechanical derivation of the dielectric tensor and discuss its use in the study of wave propagation.

2.1 Conductivity and Dielectric Tensors

Our starting point is the Hamiltonian for a particle of species s in an electromagnetic field.

$$\mathcal{H}_s = \frac{1}{2m_s}\left|\vec{p} - \frac{e_s}{c}\vec{A}(\vec{x})\right|^2 + e_s\,\phi(\vec{x}) \tag{2.1}$$

where $\vec{A}$ and ϕ are the vector and scalar potentials of the field. We write

$$\vec{A} = \vec{A}_o + \vec{A}_1 \tag{2.2}$$

$$\phi = \phi_o + \phi_1 \tag{2.3}$$

where $\vec{A}_o$ and ϕ_o are the potentials for the zero order electric and magnetic fields. We may also include in $e_s\phi$ any gravitational or fictitious fields which it is convenient to consider. The potentials $\vec{A}_1$ and ϕ_1 are regarded as small perturbations. We shall divide $\mathcal{H}_s$ into zero, first and second order parts.

$$\mathcal{H}_s = \mathcal{H}_{so} + \mathcal{H}_{s1} + \mathcal{H}_{s2} \tag{2.4}$$

$$\mathcal{H}_s = \frac{1}{2m_s}\left|\vec{p} - \frac{e_s}{c}\vec{A}_o\right|^2 + e_s\,\phi_o \tag{2.5}$$

$$\mathcal{H}_{s1} = -\frac{e_s}{m_s}\left(\vec{p} - \frac{e_s}{c}\vec{A}_o\right)\cdot\vec{A}_1 + e_s\,\phi_1 \tag{2.6}$$

$$\mathcal{H}_{s_2} = \frac{e_s^{\,2}}{2m_s\,c^2}A_1^2 \tag{2.7}$$

Since $\mathcal{H}_{s2}$ is quadratic in the small perturbation $\vec{A}_1$, it will be neglected in the following.

We shall usually work in the coulomb gauge, so

$$\nabla \cdot \vec{A} = 0 \tag{2.8}$$

We expand $\vec{A}_1(\vec{x},t)$ and $\phi(\vec{x},t)$ in a Fourier series in a box of volume V and assume the usual periodic boundary conditions. Thus

$$\vec{A}_1(\vec{x},t) = \sum_{\vec{q}} \vec{A}_1(\vec{q},t)\, e^{i\vec{q}\cdot\vec{x}} \tag{2.9}$$

and

$$\vec{A}_1(\vec{q},t) = \int \frac{d^3x}{V}\, e^{-i\vec{q}\cdot\vec{x}}\, A_1(\vec{x},t) \tag{2.10}$$

with similar equations for $\phi(\vec{x},t)$ and $\phi(\vec{q},t)$.

Now, let $\chi_{sa}(\vec{x})$ be a solution of

$$\mathcal{H}_{so}\, \chi_{sa}(\vec{x}) = E_{sa}(\vec{x})\, \chi_{sa}(\vec{x}) \tag{2.11}$$

The subscript a denotes the quantum numbers associated with the energy eigenvalue E_{sa} and eigenfunction χ_{sa}. We now go over to the second quantization formalism.[23] Let

$$\Psi_s(\vec{x},t) = \sum_a C_{sa}(t)\, \chi_{sa}(\vec{x}) \tag{2.12}$$

and interpret C_{sa} and C^+_{sa} as annihilation and creation operators for particles of species s in the state a. The Hamiltonian for particles of species s is

$$H_s = \int d^3x\, \Psi_s^{+}\, \mathcal{H}_s\, \Psi_s = H_{so} + H_{s1} + H_{s2} \tag{2.13}$$

where

$$H_{so} = \sum_a E_{sa}\, C^+_{sa}\, C_{sa} \tag{2.14}$$

$$H_{s1} = \sum_{\vec{q}} \sum_a \sum_{a'} C^+_{sa'}\, C_{sa} \Big\{ -\frac{e_s}{c} \vec{A}_1(\vec{q},t) < a' \Big| \vec{v}\, e^{iq\cdot x} \Big| a > + e_s\, \phi_1(\vec{q},t) < a' \Big| e^{i\vec{q}\cdot\vec{x}} \Big| a > \Big\} \tag{2.15}$$

where $\vec{v} = (\vec{p} - eA_o/c)/m_s$ is the velocity operator. H_{s2} comes from $\mathcal{H}_{s2}$ and is neglected.

We assume that the operators C_{sa} and C^+_{sa} obey the Fermion commutation relations

$$\left[C_{sa}, C_{s'a'}\right]_+ = \left[C^+_{sa}, C^+_{s'a'}\right]_+ = 0 \tag{2.16}$$

$$\left[C_{sa}, C^+_{s'a'}\right]_+ = \delta_{ss'}\, \delta_{aa'} \tag{2.17}$$

where $\left[A,B\right]_+ = AB - BA$. Of course, the ions may be Bosons rather than Fermions. However, when we take the classical limit, as we always will, it will not make any difference, so for convenience we shall always treat both electrons and ions as Fermions.

The change with time of any operator is given by the Heisenberg equations of motion.

$$\frac{\partial P}{\partial t} = \frac{i}{h}\left[H,P\right]_- = \frac{i}{\hbar}(HP - PH) \tag{2.18}$$

In particular let us consider the operator $C^+_{sb'}\, C_{sb}$. Using Eqs. (2.13), (2.14) and (2.15) for H and the commutation relations Eqs. (2.16) and (2.17) we find

$$\begin{aligned}
\frac{\partial}{\partial t} C^+_{sb'}\, C_{sb} &= \frac{i}{\hbar}\left[H_{so}, C^+_{sb'}\, C_{sb}\right] + \frac{i}{\hbar}\left[H_{s1}, C^+_{sb'}\, C_{sb}\right] \\
&= \frac{i}{\hbar}(E_{sb'} - E_{sb})\, C^+_{sb'}\, C_{sb} \\
&+ \frac{ie_s}{\hbar}\sum_{\vec{q}}\sum_{a}\Big\{C^+_{sa}\, C_{sb}\Big[-\frac{1}{c}\vec{A}_1(\vec{q},t)\cdot\langle a|\vec{v}\, e^{i\vec{q}\cdot\vec{x}}|b'\rangle \\
&+ \phi_1(\vec{q},t)\,\langle a|e^{i\vec{q}\cdot\vec{x}}|b'\rangle \\
&- C^+_{sb'}\, C_{sa}\Big[-\frac{1}{c}\vec{A}_1(\vec{q},t)\cdot\langle b|\vec{v}\, e^{i\vec{q}\cdot\vec{x}}|a\rangle \\
&+ \phi_1(\vec{q},t)\,\langle b|e^{i\vec{q}\cdot\vec{x}}|a\rangle\Big\}
\end{aligned} \tag{2.19}$$

Now, we shall define

$$F_s(b',b,t) = \sum_{\alpha} P_\alpha \langle\alpha|C^+_{sb'}\, C_{sb}|\alpha\rangle \tag{2.20}$$

where P_α is the probability that the system is in the state $|\alpha\rangle$.. Note that $F_s(b',b,t)$ is both a quantum mechanical and an ensemble average of the operator $C^+_{sb'}\, C_{sb}$. As will be made clear later $F_s(b',b,t)$ is closely related to the classical distribution function $f_s(\vec{x},\vec{v},t)$. By averaging Eq. (2.19) we obtain the equation obeyed by

$F_s(b',b,t)$. It is

$$\frac{\partial F_s}{\partial t}(b',b,t) = \frac{i}{\hbar}(E_{sb'} - E_{sb})\, F_s\,(b',b,t)$$
$$+ \frac{ie_s}{\hbar} \sum_{\vec{q}} \sum_a \Big\{ F_s\,(a,b,t) \Big[-\frac{1}{c}\, \vec{A}_1 \cdot < a \,|\, \vec{v}\, e^{i\vec{q}\cdot\vec{x}} \,|\, b' >$$
$$+ \phi_1 < a \,|\, e^{i\vec{q}\cdot\vec{x}} \,|\, b' > \Big]$$
$$- F_s\,(b',a,t) \Big[-\frac{1}{c}\, \vec{A}_1 \cdot < b \,|\, \vec{v}\, e^{i\vec{q}\cdot\vec{x}} \,|\, a >$$
$$+ \phi_1 < b \,|\, e^{i\vec{q}\cdot\vec{x}} \,|\, a > \Big] \Big\} \qquad (2.21)$$

This is the quantum mechanical analog of the Vlasov equation.

We now linearize Eq. (2.21) by writing

$$F_s\,(b,b',t) = F_{so}\,(b)\, \delta_{bb'} + F_{s1}\,(b'b,t) \qquad (2.22)$$

and treating F_{s1}, $\vec{A}_1$ and ϕ_1 as small perturbations. Note that since $C^+_{sb}\, C_{sb}$ is the number operator for particles ofspecies s and state b, $F_{so}(b)$ can be interpreted as the ensemble average number of particles of species s in state b. The linearized quantum Vlasov equation is

$$\frac{\partial F_{s1}}{\partial t}(b',b,t) = \frac{i}{\hbar}(E_{sb'} - E_{sb})\, F_{s1}\,(b',b,t)$$
$$+ \frac{ie_s}{\hbar} \sum_{\vec{q}} \Big[F_{so}\,(b) - F_{so}\,(b') \Big]$$
$$\Big[-\frac{1}{c}\, \vec{A}_1\,(\vec{q},t) \cdot < b \,|\, \vec{v}\, e^{i\vec{q}\cdot\vec{x}} \,|\, b' >$$
$$+ \phi_1\,(\vec{q},t) < b \,|\, e^{i\vec{q}\cdot\vec{x}} \,|\, b' > \Big] \qquad (2.23)$$

Next, we shall assume that the time dependence of F_{s1}, $\vec{A}_1$ and ϕ_1 is given by

$$F_{s1},\ \vec{A}_1,\ \phi_1 \sim e^{-i\omega t + \eta t} \qquad (2.24)$$

where the positive infinitesimal η has been introduced to make the perturbations vanish at $t = -\infty$. As will be seen it leads to the Landau prescription for avoiding singularities in integrals. Eq. (2.23) can now be solveu with the result

$$F_{s1}(b',b,\omega) = - e_s \frac{F_{so}(b) - F_{so}(b')}{\hbar\omega - (E_{sb} - E_{sb'}) + i\eta\hbar}$$

$$\sum_{\vec{q}} \Big[- \frac{1}{c} \vec{A}_1(\vec{q},\omega) < b | \vec{v} e^{i\vec{q}\cdot\vec{x}} | b' >$$

$$+ \phi_1(\vec{q},\omega) < b | e^{i\vec{q}\cdot\vec{x}} | b' > \Big] \tag{2.25}$$

This will be used to calculate the current in the plasma. A current density operator is defined by

$$\vec{J}(\vec{x},t) = \sum_s e_s \psi_s^+ (\vec{x},t) \frac{1}{2} (\overleftarrow{v} + \vec{v}) \psi_s (\vec{x},t)$$

$$= \sum_s \sum_b \sum_{b'} \frac{e_s}{2V} C^+_{sb'} C_{sb} X^*_{sb'} \frac{1}{2} (\overleftarrow{v} + \vec{v}) X_{sb} \tag{2.26}$$

where we have used $\overleftarrow{v}$ to denote a velocity operator which operates on the function to its left. The Fourier transform of $\vec{J}(\vec{x},t)$ is

$$J(q,t) = \int \frac{d^3x}{V} e^{-i\vec{q}\cdot\vec{x}} \vec{J}(x,t)$$

$$= \sum_s \sum_{b'} \sum_b \frac{e_s}{2V} C^+_{sb'} C_{sb} < b' | \vec{v} e^{-i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}} \vec{v} | b > \tag{2.27}$$

and its average is

$$< \vec{J}(\vec{q},t) > = \sum_s \sum_{b'} \sum_b \frac{e_s}{2V} F_s(b',b,t) < b' | \vec{v} e^{-i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}} \vec{v} | b > \tag{2.28}$$

where <. . . .> denotes the quantum mechanical and ensemble average of Eq. (2.20). The part of <J> proportional to the perturbing fields will be denoted by $<\vec{J}_1(\vec{q},)>$. It is obtained by replacing F_s in Eq. (1.28) by F_{s1} as given by Eq. (2.25). It is convenient to express $\vec{A}_1$ and ϕ_1 in terms of the electric field $\vec{E}_1$ by using

$$\vec{E}_1(\vec{q},\omega) = \frac{i\omega}{c} \vec{A}_1 - iq\phi_1 \tag{2.29}$$

from which

$$\phi_1(\vec{q},\omega) = \frac{i\vec{q}\cdot\vec{E}_1}{q^2} \tag{2.30}$$

$$\vec{A}_1(\vec{q},\omega) = \frac{c}{i\omega} (\overleftrightarrow{I} - \frac{\vec{q}\,\vec{q}}{q^2}) \cdot \vec{E}_1 \tag{2.31}$$

We may now write

$$\langle J_1(\vec{q},\omega)\rangle = \sum_{q'} \overleftrightarrow{\sigma}(\vec{q},\vec{q}',\omega) \cdot \vec{E}_1(\vec{q}',\omega) \tag{2.32}$$

where

$$\overleftrightarrow{\sigma}(\vec{q},\vec{q}',\omega) = -\sum_s \sum_b \sum_{b'} \frac{ie_s^2}{2V} \frac{F_{so(b)} - F_{so(b')}}{h\omega - (E_{sb} - E_{sb'}) + i\eta\hbar}$$
$$\langle b' \left| \vec{v}\, e^{i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}}\, \vec{v} \right| b \rangle \left\{ \frac{\vec{q}'}{q'^2} \langle b | e^{+i\vec{q}'\cdot\vec{x}} | b' \rangle \right.$$
$$\left. + \frac{1}{\omega} \langle b | \vec{v}\, e^{i\vec{q}'\cdot\vec{x}} | b' \rangle \cdot \left[\overleftrightarrow{I} - \frac{\vec{q}'\vec{q}'}{q'^2} \right] \right\} \tag{2.33}$$

This is quite general. We have made no assumptions about the homogeneity or isotropy of the plasma. All of the difficulties are concealed in the matrix elements. We shall now make simplifying assumptions which permit us to reduce Eq. (2.33) to a more useful form.

First, we shall assume that $\vec{A}_o$ and ϕ_o are zero. Then

$$\mathcal{H}_{so} = \frac{1}{2m_s} p^2 = -\frac{\hbar^2}{2m_s} \nabla^2 \tag{2.34}$$

$$\chi_{sb}(\vec{x}) = \chi_{s\vec{k}}(\vec{x}) = \langle \vec{x} | \vec{k} \rangle = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \tag{2.35}$$

$$\vec{v} = \frac{1}{m_s} \vec{p} = \frac{\hbar}{im_s} \frac{\partial}{\partial \vec{x}} \tag{2.36}$$

$$E_{sk} = \frac{\hbar^2 k^2}{2m_s} \tag{2.37}$$

$$\langle \vec{k} | e^{i\vec{q}'\cdot\vec{x}} | \vec{k}' \rangle = \delta_{\vec{k},\vec{k}'+\vec{q}'} \tag{2.38}$$

$$\langle \vec{k} | \vec{v}\, e^{i\vec{q}'\cdot\vec{x}} | \vec{k}' \rangle = \frac{\hbar}{m_s} (k' + q')\, \delta_{\vec{k},\vec{k}'+\vec{q}'} \tag{2.39}$$

$$\langle \vec{k}' | \vec{v} e^{-i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}}\vec{v} | \vec{k} \rangle = \frac{\hbar}{m_s} (2\vec{k} - \vec{q})\, \delta_{\vec{k},\vec{k}'+\vec{q}} \tag{2.40}$$

Using these results in Eq. (2.33) gives

$$\overleftrightarrow{\sigma}(\vec{q},\vec{q}',\omega) = \delta_{\vec{q}\,\vec{q}'}\, \overleftrightarrow{\sigma}(\vec{q},\omega) \tag{2.41}$$

where

$$\overleftrightarrow{\sigma}(\vec{q},\omega) = -\sum_s \sum_{\vec{k}} \frac{ie_s^{\,2}}{V} \frac{F_{so}(\vec{k}) - F_{so}(\vec{k}-\vec{q})}{h\omega - \frac{\hbar^2 k^2}{2m_s} + \frac{\hbar^2}{2m_s}\left|\vec{k}-\vec{q}\right|^2 + i\eta\hbar}$$

$$\frac{\hbar}{m_s}\left(\vec{k} - \frac{1}{2}\ \vec{q}\right)\left\{\frac{\hbar\vec{k}}{\omega m_s} + \frac{\vec{q}}{q^2}\left(1 - \frac{\hbar\vec{k}\cdot\vec{q}}{\omega m_s}\right)\right\} \tag{2.42}$$

Now, we shall replace $F_{so}(\vec{k})$ by the corresponding velocity distribution $f_{so}(\vec{v})$ where $\vec{v} = \hbar\vec{k}/m_s$. Also we shall let the volume of the system become infinite and use

$$\sum_{\vec{k}} \xrightarrow[V\to\infty]{} V\int d^3v \tag{2.43}$$

The conductivity tensor now takes the form

$$\overleftrightarrow{\sigma}(\vec{q},\omega) = -\sum_s \frac{ie_s^{\,2}}{\hbar\omega}\int d^3v\ \frac{f_{so}(\vec{v}) - f_{so}(\vec{v} - \hbar\vec{q}/m_s)}{\omega - \vec{q} - \vec{v} + \frac{\hbar q^2}{2m_s} + i\eta}$$

$$\left(\vec{v} - \frac{\hbar\vec{q}}{2m_s}\right)\left[\vec{v} + \frac{\vec{q}}{q^2}\left(\omega - \vec{q}\cdot\vec{v}\right)\right] \tag{2.44}$$

The dielectric tensor is related to the conductivity tensor by

$$\overleftrightarrow{\epsilon}(\vec{q},\omega) = \overleftrightarrow{I} + \frac{4\pi i}{\omega}\ \overleftrightarrow{\sigma}(q,\omega) \tag{2.45}$$

The dielectric constant (or more properly, dielectric function) is defined as

$$\epsilon(\vec{q},\omega) = \frac{1}{q^2}\ \vec{q}\cdot\overleftrightarrow{\epsilon}(\vec{q},\omega)\cdot\vec{q} \tag{2.46}$$

Using Eqs. (2.44) and (2.45) we obtain

$$\epsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^{\,2}}{\hbar\omega q^2}\int d^3v\ \frac{f_{so}(\vec{v}) - f_{so}\left(v - \frac{\hbar\vec{q}}{m_s}\right)}{\omega - \vec{q}\cdot\vec{v} + \frac{\hbar q^2}{2m_s} + i\eta}$$

$$\left(\vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}\right) \tag{2.47}$$

which may also be put into the form

$$\epsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{hq^2} \int d^3v \, \frac{f_{so}(\vec{v}) - f_{so}(\vec{v} - h\vec{q}/m_s)}{\omega - \vec{q}\cdot\vec{v} + \frac{\hbar q^2}{2m_s} + i\eta} \tag{2.48}$$

The classical dielectric function is obtained by taking the $\hbar \to 0$ limit. It is

$$\epsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{m_s q^2} \int d^3v \, \frac{q \cdot \partial f_{so}/\partial \vec{v}}{\omega - \vec{q}\cdot\vec{v} + i\eta} \tag{2.49}$$

The matrix elements in Eq. (2.33) can also be evaluated for the case of a plasma in a uniform magnetic field. We take the vector potential to be

$$\vec{A}_o = - \vec{e}_x B_y \tag{2.50}$$

This gives a uniform magnetic field in the z-direction. Then

$$\mathcal{H}_{so} = \frac{1}{2m_s} \left| \vec{p} - \frac{e_s}{d} \vec{A}_o \right|^2 \tag{2.51}$$

$$\chi_{sb} = \chi_{sn}k_xk_z = < \vec{X} | n, k_x, k_y >$$

$$= \frac{1}{L} G_n (y - y_o) \, e^{i(k_x x + k_z z)}$$

where $L^3 = V$ is the volume of the box in which the system is quantized. $G_n(y - y_o)$ is a harmonic oscillator wave function with quantum number n centered about

$$y_o = \frac{\hbar k_x}{m_s \ \ cs} \tag{2.53}$$

where $\omega_{cs} = e_s B/m_s c$ is the cyclotron frequency. The components of the velocity operator are

$$v_x = \frac{1}{m_s}\left(p_x - \frac{e}{c} A_o\right) = \omega_{cs}(y - y_o) \tag{2.54}$$

$$v_y = \frac{1}{m_s} p_y \tag{2.55}$$

$$v_z = \frac{1}{m_s} p_z \tag{2.56}$$

The energy eigenvalues are

$$E_{n,k_x,k_z} = \hbar\,\omega_{cs}\,(n + 1/2) + \frac{\hbar^2 k_z{}^2}{2m_s} \tag{2.57}$$

The necessary matrix elements have been evaluated by Walters.[15] In this paper we shall only need the matrix elements in the classical limit (i.e. in the limit of large quantum numbers and $\hbar \to 0$. These limiting forms are

$$< n,k_x,k_z \left| e^{i\vec{q}\cdot\vec{x}} \right| n',\, k_x',\, k_z' > \tag{2.58}$$

$$= \delta_{k_x',k_x,q_x}\;\delta_{k_z',k_z-q_z}\; J_{n-n'}\; e^{i(n-n')\phi_q + iq_y y_o}$$

$$< n,\, k_x,\, k_z \left| e^{i\vec{q}\cdot\vec{x}}\, v_x \right| n',\, k_x',\, k_z' >$$

$$= \delta_{k_x',k_x - q_x}\;\delta_{k_z',k_z - q_z}\sqrt{\frac{\hbar\,\omega_{cs}}{2m_x}}\; e^{iq_y y_o} \tag{2.59}$$

$$\left[\sqrt{n'}\; J_{n-n'\,\eta\,1}\; e^{i(n-n'+1)\phi_q} + \sqrt{n'+1}\; J_{n-n'-1}\; e^{i(n-n'-1)\phi_q}\right]$$

$$< n,\, k_x,\, k_z \left| e^{i\vec{q}\cdot\vec{x}}\, v_y \right| n',\, k_x',\, k_z' >$$

$$= \delta_{k_x',\, k_x - q_x}\;\delta_{k_z',\, k_z - q_z}\sqrt{\frac{\hbar\,\omega_{cs}}{2m_s}}\; e^{iq_y y_o} \tag{2.60}$$

$$\left[\sqrt{n'}\; J_{n-n'+1}\; e^{i(n-n'+1)\phi_q} - \sqrt{n'+1}\; J_{n-n'-1}\; e^{i(n-n'-1)\phi_q}\right]$$

$$< n,\, k_x,\, k_z \left| e^{i\vec{q}\cdot\vec{x}}\, v_z \right| n',\, k_x',\, k_z' >$$

$$= \delta_{k_x',\, k_x - q_x}\;\delta_{k_z',\, k_z - q_z}\;\frac{\hbar\, k_z}{m_s}\; J_{n-n'}\; e^{i(n-n')\phi_q + iq_y y_o} \tag{2.61}$$

In the above equations J_n is the Bessel function of order n. Its argument is $q_\perp v/\omega_{cs}$ where the $v_\perp$ is related to the quantum number n by

$$\frac{1}{2} m_s v^2 = \hbar \omega_{cs} n \tag{2.62}$$

In the classical limit n and n' become infinite but their difference n - n' remains finite. The angle ϕ_q is defined by

$$q_x + i q_y = q_\perp e^{i\phi_q} \tag{2.63}$$

The matrix elements given here may be used in Eq. (2.33) to obtain the conductivity tensor. We shall omit the rather tedious details. If the plasma is uniform in space, then $f_{so}(n,k_x,k_z)$ must be independent of k_x, and it may be shown that Eq. (2.41) holds. In the classical limit the conductivity tensor is given by[33]

$$\overleftrightarrow{\sigma}(\vec{q},\omega) = -\sum_s \frac{ie_s^2}{m_s\omega} \sum_{n=-\infty}^{+\infty} \int d^3v \frac{\overleftrightarrow{S}_s}{\omega - k_z v_z - n\omega_{cs} + i\eta} \tag{2.64}$$

where

$$\overleftrightarrow{S}_s = \begin{bmatrix} v_\perp U \left(\frac{n J_n}{\lambda_s}\right)^2 & -i v_\perp U \frac{n}{\lambda_s} J_n J_n' & v_\perp W \frac{n}{\lambda_s} J_n^2 \\ i v_\perp U \frac{n}{\lambda_s} J_n J_n & v_\perp U (J_n')^2 & i v_\perp W J_n J_n' \\ v_z U \frac{n}{\lambda_s} J_n^2 & -i v_z U J_n J_n' & v_z W J_n^2 \end{bmatrix} \tag{2.65}$$

and

$$\lambda_s = \frac{q_\perp v}{\omega_{cs}} \tag{2.66}$$

is the argument of J_n, J_n' denotes the derivative of J_n with respect to its argument and

$$U = (\omega - q_z v_z) \frac{\partial f_{so}}{\partial v_\perp} + k_z v_\perp \frac{\partial f_{so}}{\partial v_z} \tag{2.67}$$

$$W = \frac{n \omega_{cs} v_z}{v_\perp} \frac{\partial f_{so}}{v_\perp} + (\omega - n\omega_{cs}) \frac{\partial f_{so}}{\partial v_z} \tag{2.68}$$

The dielectric function is given by

$$\epsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{\hbar q^2} \sum_{\ell=-\infty}^{+\infty} \int d^3v \; \frac{f_{so}(v_\perp^2, v_z) - f_{so}(v^2 - \frac{2\hbar\omega_c}{m_s}\ell, v_z)}{\omega - \ell\omega_{cs} - q_z v_z + \frac{\hbar}{2m_s} q_z^2 + i\eta}$$

$$J_\ell^2 \left(\frac{q_\perp v_\perp}{\omega_{cs}}\right) \tag{2.69}$$

In the classical limit this becomes

$$\epsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{m_s q^2} \sum_{\ell=-\infty}^{+\infty} \int d^3v \; \frac{J_\ell^2 (q_\perp v_\perp/\omega_{cs})}{\omega - \ell\omega_{cs} - q_z v_z + i\eta}$$

$$\left[\frac{\ell\omega_{cs}}{v_\perp} \frac{\partial f_{so}}{\partial v_\perp} + q_z \frac{\partial f_{so}}{\partial v_z}\right] \tag{2.70}$$

2.2 The Relation Between the Quantum Mechanical Distribution Function and the Density Matrix

Let us consider a plasma without zero order electric or magnetic fields. Then the free particle eigenfunctions are given by Eq. (2.35). The expansion Eq. (2.12) can be written as

$$\psi_s(\vec{x},t) = \sum_{\vec{p}} C_{s\vec{p}}(t) \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{V}} \tag{2.71}$$

The expectation value of the number density of particles of species s can be written as

$$\langle n_s(\vec{x},t)\rangle = \langle \psi_s^+(\vec{x},t)\, \psi_s(\vec{x},t) \rangle$$

$$= \sum_{\vec{p}} \sum_{\vec{q}} \langle C^+_{s\vec{p}}(t)\, C_{s\vec{p}+\vec{q}}(t)\rangle \frac{e^{i\vec{q}\cdot\vec{x}}}{V} \tag{2.72}$$

where <. . . .> denotes the quantum mechanical and ensemble average of Eq. (2.20). This suggests that we define a distribution function by

$$F_s(\vec{x},\vec{p},t) = \sum_{\vec{q}} \langle C^+_{s\vec{p}}(t)\, C_{s\vec{p}+\vec{q}}(t)\rangle \frac{e^{i\vec{q}\cdot\vec{x}}}{V} \tag{2.73}$$

for then

$$\langle n_s(\vec{x},t)\rangle = \sum_{\vec{p}} F_s(\vec{x},\vec{p},t) \tag{2.74}$$

Furthermore, the momentum distribution function is given by

$$\langle n_s(\vec{p},t)\rangle = \int d^3x\, F_s(\vec{x},\vec{p},t) = \langle C^+_{s\vec{p}}(t)\; C_{s\vec{p}}(t)\rangle \tag{2.75}$$

Eqs. (1.74) and (1.75) are relations which any distribution function is expected to satisfy.

Returning to Eq. (2.73), we write it as

$$F_s(\vec{x},\vec{p},t) = \langle C^+_{s\vec{p}} \sum_{\vec{q}} C_{s\vec{p}+\vec{q}} \frac{e^{i\vec{q}\cdot\vec{x}}}{V}\rangle$$

$$= \langle C^+_{sp}\; e^{-i\vec{p}\cdot\vec{x}} \sum_{\vec{p}+\vec{q}} C_{s\vec{p}+\vec{q}} \frac{e^{i(p+q)\cdot\vec{x}}}{V}\rangle = \sqrt{V}\,\langle C^+_{sp}\; e^{-i\vec{p}\cdot\vec{x}}\; \psi_s(x,t)\rangle \tag{2.76}$$

Now using

$$C^+_{s\vec{p}}(t) = \int \frac{d^3x'}{V}\, e^{+i\vec{p}\cdot\vec{x}'}\; \psi^+_s(x',t) \tag{2.77}$$

We can write

$$F_s(\vec{x},\vec{p},t) = \int \frac{d^3x'}{\sqrt{V}}\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \langle \psi^+_s(\vec{x}',t)\; \psi_s(\vec{x},t)\rangle \tag{2.78}$$

The function

$$\rho_s(\vec{x}',\vec{x},t) = \langle \psi^+_s(\vec{x}',t)\; \psi_s(\vec{x},t)\rangle \tag{2.79}$$

is the density matrix. Our distribution function $F_s(\vec{x},p,t)$ is a Fourier transform of the density matrix. It has previously been used by Von Roos.[34] It is similar but not identical to the well known distribution function of Wigner.[35]

The Fourier transform of $F_s(\vec{x},\vec{p},t)$ is found from Eq. (2.73) to be

$$F_s(\vec{q},\vec{p},t) = \frac{1}{V}\langle C^+_{s\vec{p}}(t)\; C_{s\vec{p}+\vec{q}}(t)\rangle \tag{2.80}$$

We have previously defined

$$F_s(b',b,t) = \langle C^+_{sb'}(t)\; C_{sb}(t)\rangle \tag{2.81}$$

As the distribution function in quantum number space. It is a generalization of the better known distribution functions.

2.3 Wave Propagation

We shall now use the conductivity and dielectric tensors derived in the last section to discuss wave propagation in plasmas. The electric and magnetic fields are given in terms of the potentials by

$$\vec{B}_1 = \nabla \times \vec{A}_1 \tag{2.82}$$

$$\vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \phi_1 \tag{2.83}$$

As before the subscript 1 denotes a first order perturbation. In the usual way it is found from Maxwell's equations that the potentials obey the equations

$$\nabla^2 \vec{A}_1 - \nabla(\nabla \cdot \vec{A}_1) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}_1 = -\frac{4\pi}{c}\vec{J}_1 + \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi_1 \tag{2.84}$$

$$\nabla^2 \phi_1 = -4\pi\rho_1 - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{A}_1 \tag{2.85}$$

Now we look for solutions in the form of a plane wave. Thus

$$\vec{A}_1,\ \phi_1,\ J_1,\ \rho_1 \sim e^{i(\vec{q}\cdot\vec{x} - \omega t)} \tag{2.86}$$

and use the results of the last section to write

$$\vec{J}(\vec{q},\omega) = \overleftrightarrow{\sigma}(\vec{q},\omega)\cdot\vec{E}_1(\vec{q},\omega) \tag{2.87}$$

Also, the equation of continuity

$$\vec{q}\cdot\vec{J}_1(\vec{q},\omega) - \omega\rho_1(\vec{q},\omega) = 0 \tag{2.88}$$

can be used to eliminate ρ. When Eq. (2.45) is used it is found that Eqs. (2.84) and (2.85) can be written as

$$\left[-q^2\overleftrightarrow{1} + \vec{q}\,\vec{q} + \frac{\omega^2}{c^2}\overleftrightarrow{\epsilon}(\vec{q},\omega)\right]\cdot\vec{A}_1 = \frac{\omega}{c}\overleftrightarrow{\epsilon}(\vec{q},\omega)\cdot\vec{q}\,\phi \tag{2.89}$$

$$\vec{q}\cdot\overleftrightarrow{\epsilon}(\vec{q},\omega)\cdot\vec{q}\,\phi_1 = \frac{\omega}{c}\vec{q}\cdot\overleftrightarrow{\epsilon}(\vec{q},\omega)\cdot\vec{A}_1 \tag{2.90}$$

We shall find it convenient to work in the Coulomb gauge which we define by

$$\vec{q} \cdot \overleftrightarrow{\epsilon}(\vec{q},\omega) \cdot \vec{A}_1 = 0 \tag{2.91}$$

(This is the gauge condition which replaces Eq. (2.8) in a dispersive medium.) Then, Eq. (2.90) reduces to

$$\vec{q} \cdot \overleftrightarrow{\epsilon}(\vec{q},\omega) \cdot \vec{q}\,\phi_1 = 0 \tag{2.92}$$

In general it is not possible to separate the waves in a plasma into longitudinal and transverse waves; there is a coupling between them. However, for any isotropic plasma this separation is possible, for then the only vector available for the construction of the tensor $\overleftrightarrow{\epsilon}(\vec{q},\omega)$ is $\vec{q}$, hence $\overleftrightarrow{\epsilon}$ must have the form

$$\overleftrightarrow{\epsilon}(\vec{q},\omega) = (\overleftrightarrow{I} - \frac{\vec{q}\,\vec{q}}{q^2}\,\epsilon_T(\vec{q},\omega) + \frac{\vec{q}\,\vec{q}}{q^2}\,\epsilon_L(\vec{q},\omega) \tag{2.93}$$

where ϵ_L and ϵ_T are called the longitudinal and transverse dielectric functions. Note that $\epsilon_L(\vec{q},\omega)$ is just the function defined in the last section by Eq. (2.46) and called simply the dielectric function. With this form for $\overleftrightarrow{\epsilon}$ Eqs. (2.92), (2.89) and (2.91) become

$$q^2 \epsilon_L(\vec{q},\quad)\,\phi_1 = 0 \tag{2.94}$$

$$\left[-q^2\,\overleftrightarrow{I} + \vec{q}\,\vec{q} + \frac{\omega^2}{c^2}\,\overleftrightarrow{\epsilon}(\vec{q},\omega)\right] \cdot \vec{A}_1 = \frac{\omega}{c}\,\vec{q}\,\epsilon_L(\vec{q},\omega)\,\phi_1 \tag{2.95}$$

$$\vec{q} \cdot \vec{A}_1\,\epsilon_L(\vec{q},\omega) = 0 \tag{2.96}$$

Eq. (2.94) is satisfied by either $\phi_1 = 0$ or $\epsilon_L = 0$. In either case the right hand side of Eq. (2.95) vanishes, so the equations for ϕ_1 and $\vec{A}_1$ are uncoupled. The equation

$$\epsilon_L(\vec{q},\omega) = 0 \tag{2.97}$$

is the dispersion relation for longitudinal waves. For a given $\vec{q}$ it will have one or more solutions $\omega_{\vec{q}\sigma}$. These are the frequencies of longitudinal waves in the plasma. In general such frequencies are indicating exponential damping or growth of the corresponding wave.

If $\epsilon_{\perp} \neq 0$ then $\phi_1 = 0$. Eq. (2.96) becomes

$$\vec{q} \cdot \vec{A}_1 = 0 \tag{2.98}$$

indicating that these waves are transverse. Eq. (2.95) gives

$$\left[- q^2 + \frac{\omega^2}{c^2} \epsilon_T (q, \omega)\right] \vec{A}_1 = 0 \tag{2.99}$$

The relation

$$- q^2 + \frac{\omega^2}{c^2} \epsilon_T (q, \omega) = 0 \tag{2.100}$$

is the dispersion relation for transverse waves. It has solutions $\omega_{\vec{q}\sigma}$. For a given $\vec{q}$ there will be two independent polarization directions (in the plane perpendicular to $\vec{q}$), which have the same frequency.

It is common practice in plasma physics to assume that an approximate decoupling of longitudinal and transverse waves can be made even when the plasma is not isotropic. For longitudinal waves $\vec{A}_1$ is taken to be zero and the dispersion relation is obtained from Eq. (2.92). For transverse waves ϕ_1 is taken to be zero and $\vec{A}_1$ is taken to satisfy

$$\left[- q^2 \overleftrightarrow{1} + \vec{q}\,\vec{q} + \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon} (\vec{q}, \omega)\right] \cdot \vec{A}_1 = 0 \tag{2.101}$$

(If this equation is satisfied then so is Eq. (2.91)). The dispersion relation for transverse waves is obtained from the condition that Eq. (2.101) have a non-trivial solution; namely

$$\mathrm{DET} \left[- q^2 \overleftrightarrow{1} + \vec{q}\,\vec{q} + \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon} (\vec{q}, \omega)\right] = 0 \tag{2.102}$$

It may be shown that this approximate decoupling of longitudinal and transverse waves is a good approximation when the plasma pressure is much less than the magnetic field pressure. In what follows we shall always assume that the separation into longitudinal and tranverse waves is possible, either because the plasma is isotropic or because the approximate separation is a good one. In principle this assumption can be avoided but at the cost of some additional mathematical complexity.

Now, let us return to the dispersion relation for longitudinal waves

$$\epsilon_L(\vec{q},\omega) = \frac{1}{q^2}\,\vec{q}\cdot\overleftrightarrow{\epsilon}(\vec{q},\omega)\cdot\vec{q} = 0 \tag{2.103}$$

The way ϵ was defined in the last section it may be complex even when $\vec{q}$ and ω are real, so for real $\vec{q}$ and ω we shall write

$$\epsilon_L(\vec{q},\omega) = \epsilon_{L1}(\vec{q},\omega) + i\,\epsilon_{L2}(\vec{q},\omega) \tag{2.104}$$

For complex ω $\epsilon_L(\vec{q},\omega)$ is to be interpreted as the analytic continuation of $\epsilon_L(\vec{q},\omega)$ from the real ω axis. Now, let us write the solution of Eq. (2.103) as

$$\omega_{\vec{q}\sigma} = \Omega_{\vec{q}\sigma} + i\gamma_{\vec{q}\sigma} \tag{2.105}$$

and assume that

$$|\gamma_{\vec{q}\sigma}| \ll |\Omega_{\vec{q}\sigma}| \tag{2.106}$$

and

$$|\epsilon_{L2}(\vec{q},\Omega_{q\sigma})| \ll |\epsilon_{L1}(\vec{q},\Omega_{q\sigma})| \tag{2.107}$$

Expanding $\epsilon_L = 0$ in a Taylor series and neglecting products of small terms gives

$$\epsilon_{L1}(\vec{q},\Omega_{q\sigma}) + i\gamma_{q\sigma}\left(\frac{\epsilon_{L1}}{\partial\omega}\right)_{\Omega_{q\sigma}} + i\,\epsilon_{L2}(\vec{q},\Omega_{\vec{q}\sigma}) = 0 \tag{2.108}$$

Equating separately the real and imaginary parts to zero gives

$$\epsilon_{L1}(\vec{q},\Omega_{q\sigma}) = 0 \tag{2.109}$$

$$\gamma_{q\sigma} = -\frac{\epsilon_{L2}(\vec{q},\Omega_{q\sigma})}{\left(\frac{\partial\epsilon_{L1}}{\partial\omega}\right)_{\Omega_{q\sigma}}} \tag{2.110}$$

The real part of the frequency is determined from Eq. (2.109), and then the imaginary part is given by Eq. (2.110).

The physical significance of Eq. (2.110) can be made clear by noting that

$$\vec{J} = \sigma_L \vec{E} \tag{2.111}$$

so

$$\vec{E}^* \cdot \vec{J} = \sigma_L \left|\vec{E}\right|^2$$

and

$$\mathcal{R}_e \, \vec{E}^* \cdot \vec{J} = \frac{\omega}{4\pi} \, \epsilon_{L2} \left|\vec{E}\right|^2 \tag{2.113}$$

where Eq. (2.45) has been used. Eq. (2.110) can be written

$$\gamma_{q\sigma} = - \left[\frac{\mathcal{R}_e \, E^* \cdot \vec{J}}{\frac{1}{4\pi} \left|\vec{E}\right|^2 \frac{\partial}{\partial \omega} (\omega \epsilon_{L1})} \right]_{\omega = \Omega_{q\sigma}} = - \frac{P}{2W} \tag{2.114}$$

The numerator P is just the energy dissipated by the electric field of the wave in driving the current $\vec{J}$. In the denominator

$$W = \frac{1}{8\pi} \left|\vec{E}\right|^2 \left(\frac{\partial}{\partial \omega} \omega \epsilon_1 \right)_{\Omega_{q\sigma}} \tag{2.115}$$

has the following interpretation. The energy in the electric field is $\left|\vec{E}\right|^2/8\pi$. This must be modified by a factor which corrects for the kinetic energy of the oscillating particles in order to get the total energy of the wave. The correction factor may be shown to be the second factor in Eq. (2.115). (Reference 33, Chapter 1.)

If the energy of the wave, W, is positive then a wave will be damped if P is positive indicating that the energy of the wave is dissipated in driving the current. On the other hand P may be negative indicating a flow of energy from the plasma particles to the wave. In this case a positive energy wave will grow.

It is possible for ϵ_1 to be such that the energy of the wave is negative. If this is the case a wave will grow if P is positive, for then the plasma particles absorb energy from the wave making W even more negative and consequently increasing the amplitude of the wave. If P is negative the wave is damped. As an example of a distribution function which leads to negative energy waves consider the two-stream distribution function

$$f_o(\vec{v}) = n_1 \delta(\vec{v}) + n_2 \delta(\vec{v} - \vec{V}) \tag{2.116}$$

Using this Eq. (2.49) gives

$$\epsilon(\vec{q},\omega) = 1 - \frac{\omega_{P1}^2}{\omega^2} - \frac{\omega_{P2}^2}{(\omega - \vec{q}\cdot\vec{V})^2} \tag{2.117}$$

(We have assumed a single species, integrated by parts and let $\omega_{P1}^2 = 4\pi n_1 e^2/m$ and $\omega_{P2}^2 = 4\pi n_2 e^2/m$.) In Fig. 2.1 we have sketched $\epsilon(\vec{q},\omega)$ as a function of ω.

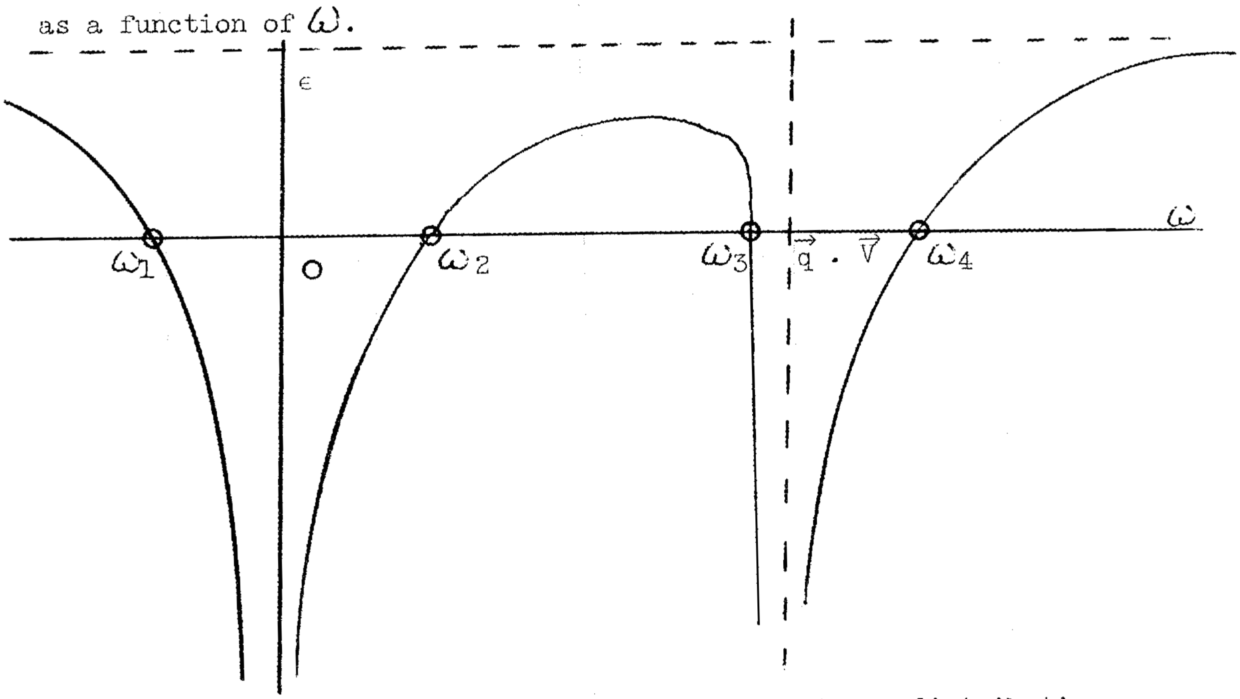


Fig. 2.1 Dielectric function for the two-stream distribution function.

The equation $\epsilon = 0$ is a fourth degree equation for ω. It has four roots. If ϵ is as drawn in Fig. 2.1 all four roots are real. Three of the frequencies ω_1, ω_2 and ω_4 have $\partial \omega \epsilon_1 / \partial \omega$ positive and so correspond to positive energy waves. The other root ω_3 corresponds to a negative energy wave. All of these waves are stable and undamped. If now $\vec{q} \cdot \vec{V}$ is decreased the two roots ω_2 and ω_3 will approach each other, become equal and then move off into the complex plane. Since all of the quantities in Eq. (2.117) are real, ω_2 and ω_3 are complex conjugates of each other. One of the roots corresponds to an exponentially growing wave. This instability, called the two-stream instability, may be thought of as due to the coupling of a positive and a negative energy wave. If energy is transferred from the negative energy wave to the positive energy wave then the amplitude of both waves will grow.

It should be noted that Eqs. (2.109) and (2.110) are not valid for the two-stream instability. In this example ϵ_2 was identically zero, and complex rather than real solutions of Eq. (2.109) were found.

The same sort of analysis that led to Eqs. (2.109) and (2.110) can be applied to the dispersion relation for transverse waves, Eq. (2.100). The result is

$$- q^2 + \frac{\Omega_{q\sigma}^2}{c^2} \epsilon_{T1} (\vec{q}, \Omega_{q\sigma}) = 0 \tag{2.118}$$

and

$$\gamma_{q\sigma} = \left[\frac{-\omega^2 \epsilon_{T2} (\vec{q},\omega)}{\frac{\partial}{\partial \omega} \omega^2 \epsilon_{T1} (\vec{q},\omega)} \right]_{\omega = \Omega_{q,\sigma}} \tag{2.119}$$

This can be written as

$$\gamma_{q\sigma} = - \frac{P}{2W} = - \left[\frac{\mathcal{R}_e \, \vec{E}^* \cdot \vec{J}}{\frac{1}{4\pi} \left|\vec{E}\right|^2 \frac{1}{\omega} \frac{\partial}{\partial \omega} \omega^2 \, \epsilon_{T1} (\vec{q},\omega)} \right]_{\omega = \Omega_{q,\sigma}} \tag{2.120}$$

In this case the factor necessary to obtain the total energy from the electric field energy is $\omega^{-1} \partial \omega^2 \, \epsilon_{T1} / \partial \omega$. It may be obtained from[33]

$$W = \frac{\left|\vec{E}\right|^2}{8\pi} \frac{\partial}{\partial \omega} \omega \, \epsilon_{T1} + \frac{\left|\vec{B}\right|^2}{8\pi} \tag{2.121}$$

when

$$\vec{K} \times \vec{B} = - \frac{\omega}{c} \overleftrightarrow{\epsilon} \cdot \vec{E} \tag{2.122}$$

is used.

CHAPTER 3. QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN A DISPERSIVE MEDIUM

In this chapter we shall begin by considering the plasma as a dispersive medium characterized by a dielectric tensor $\overleftrightarrow{\epsilon}\,(\vec{q},\omega)$ which is the real part of the dielectric tensor introduced in the last chapter. Waves with real frequencies can propagate in this medium. The electromagnetic field in the plasma will be expanded in these waves, a Hamiltonian for the system will be found and the system quantized by the usual prescription. This leads us to a description of plasma excitations in terms of quasi-particles (plasmons, phonons, photons, etc.). This quantization of the field in a plasma has been treated previously by Kihara, Aono and Dodo[25] by Alekseev and Nikitin[24] and others (see Reference 24 for other references).

We then derive a particle-quasi-particle interaction Hamiltonian. The interaction of quasi-particles with particles leads to a growth or decay of the number of quasi-particles (or equivalently, the intensities of the fields of the waves) which in more conventional treatments is determined from the imaginary part of $\overleftrightarrow{\epsilon}$.

3.1 Quantization of the Electromagnetic Field

We now expand the potentials $\phi\,(\vec{x},t)$ and $\vec{A}\,(\vec{x},t)$ in Fourier series in a large box of volume V. (These potentials are ϕ_1 and $\vec{A}_1$ of the last chapter but the subscript has been dropped.) The usual periodic boundary conditions are assumed.

$$\phi\,(\vec{x},t) = \sum_{\vec{k},\sigma}\left[\frac{4\pi\,\hbar\,\Omega_{\vec{k},\sigma}}{Vk^2\left[\frac{\partial}{\partial\omega}\,\omega\,\epsilon_{1L}\right]_{\Omega_{k\sigma}}}\right]^{1/2}$$

$$\left\{B_{\vec{k}\sigma}\, e^{i(\vec{k}\cdot\vec{x}-\Omega_{\vec{k}\sigma}t)} + B^{+}_{\vec{k},\sigma}\, e^{-i(\vec{k}\cdot\vec{x}-\Omega_{\vec{k}\sigma}t)}\right\} \tag{3.1}$$

$$\vec{A}\,(\vec{x},t) = \sum_{\vec{k},\sigma}\left[\frac{4\pi\,\hbar c^2}{V\Omega_{\vec{k}\sigma}\left[\frac{1}{\omega}\,\frac{\partial}{\partial\omega}\,\omega^2\,\epsilon_{1T\sigma}\right]_{\Omega_{k\sigma}}}\right]^{1/2}$$

$$\vec{u}_{\vec{k}\sigma}\left\{A_{\vec{k}\sigma}\, e^{i(\vec{k}\cdot\vec{x}-\Omega_{k\sigma}t)} + A^{+}_{k}\, e^{-i(\vec{k}\cdot\vec{x}-\Omega_{k\sigma}t)}\right\} \tag{3.2}$$

In the above $B_{\vec{k}\sigma}$ and $A_{\vec{k}\sigma}$ are Fourier coefficients. The reason for the factors in square brackets will be made clear presently. In Eq. (3.1) the $\Omega_{\vec{k}\sigma}$'s are the frequencies of longitudinal waves found by solving Eq. (2.109). In Eq. (3.2) the $\Omega_{\vec{k}\sigma}$'s are the frequencies of transverse waves found by solving Eq. (2.118). The polarization vectors $\vec{U}_{\vec{k}\sigma}$ are solutions of

$$\left[-k^2\,\overleftrightarrow{I} + \vec{k}\,\vec{k} + \frac{\Omega^2_{k\sigma}}{c^2}\,\overleftrightarrow{\epsilon}_1\,(\vec{k},\Omega_{\vec{k}\sigma})\right]\cdot\vec{u}_{\vec{k}\sigma} = 0 \tag{3.3}$$

They are normalized to unity. We define

$$\epsilon_{1T\sigma}\,(\vec{k},\omega) = \vec{u}_{\vec{k}\sigma}\cdot\overleftrightarrow{\epsilon}_1\,(\vec{k},\omega)\cdot\vec{u}_{\vec{k}\sigma} \tag{3.4}$$

It is this transverse dielectric function which must be used in Eq. (2.118) when solving for $\Omega_{\vec{k}\sigma}$.

We now calculate the time averaged energy in the electromagnetic field. For $\vec{A} = 0$ there is no magnetic field and

$$U_L = \frac{1}{8\pi}\int d^3x < E^2 > = \frac{1}{8\pi}\int d^3x < |\nabla\,\phi|^2 > \tag{3.5}$$

is the energy in the electric field of longitudinal waves. The angular brackets indicate a time average over a period which is long in comparison with the periods of oscillation. Using Eq. (3.1) in

Eq. (3.5) and carrying out the integration and time averaging gives

$$U_L = \sum_{\vec{k},\sigma} \frac{\hbar \Omega_{\vec{k},\sigma}}{\left| \frac{\partial}{\partial \omega} \omega \epsilon_{L1} \right|_{\Omega_{k,\sigma}}} B^{+}_{\vec{k},\sigma} B_{\vec{k},\sigma} \tag{3.6}$$

As was discussed in the last chapter, it is necessary to correct the electric field energy of each of the waves by the factor $\omega^{-1} \partial \omega \epsilon_{1L} / \partial \omega$ in order to get the total energy. Making this correction gives

$$H_L = \sum_{\vec{k},\sigma} S_{\vec{k},\sigma} \hbar \Omega_{\vec{k},\sigma} B^{+}_{\vec{k},\sigma} B_{k,\sigma} \tag{3.7}$$

for the total energy in longitudinal waves. Here

$$S_{\vec{k}\sigma} = \frac{\left(\frac{\partial}{\partial \omega} \omega \epsilon_{L1} \right)_{\Omega_{k\sigma}}}{\left| \frac{\partial}{\partial \omega} \omega \epsilon_{L1} \right|_{\Omega_{k\sigma}}} = \pm 1 \tag{3.8}$$

is the sign of the correction factor. It is positive for positive energy waves and negative for negative energy waves. Note that in this development $\Omega_{\vec{k}\sigma}$ is a positive frequency. The factor in the square bracket in Eq. (3.1) was chosen so that H_L would have the simple form of Eq. (3.7).

Next, we calculate the energy in the transverse fields. Letting $\phi = 0$, we write

$$\begin{aligned} U_T &= \frac{1}{8\pi} \int d^3x < E^2 + B^2 > \\ &= \frac{1}{8\pi} \int d^3x < \frac{1}{c^2} \left| \frac{\partial \vec{A}}{\partial t} \right|^2 + \left| \nabla \times \vec{A} \right|^2 > \end{aligned} \tag{3.9}$$

Substituting Eq. (3.2) into Eq. (3.9) gives

$$U_T = \sum_{\vec{k},\sigma} \frac{\hbar \Omega_{k\sigma}}{\left| \frac{1}{\omega} \frac{\partial}{\partial \omega} \omega^2 \epsilon_{1T\sigma} \right|_{\Omega_{k\sigma}}} A^{+}_{\vec{k}\sigma} A_{\vec{k}\sigma} \tag{3.10}$$

Applying the correction factor for transverse waves gives

$$H_L = \sum_{\vec{k},\sigma} S_{\vec{k}\sigma} \, \hbar \Omega_{\vec{k}\sigma} \, A^{+}_{\vec{k}\sigma} \, A_{k\sigma} \tag{3.11}$$

for the total energy in transverse waves. Again $S_{\vec{k}\sigma}$ is the sign of the energy of the wave.

So far in the discussion of this chapter $B_{\vec{k}\sigma}$ and $A_{\vec{k}\sigma}$ have been considered to be complex numbers and $B^{+}_{\vec{k}\sigma}$ and $A^{+}_{\vec{k}\sigma}$ have been taken to be their complex conjugates. The transition from classical to quantum mechanics is made by reinterpreting $B_{\vec{k}\sigma}$ and $A_{\vec{k}\sigma}$ as destruction operators for longitudinal and transverse quasi-particles of momentum $\hbar \vec{k}$, type σ and energy $S_{\vec{k}\sigma} \, \hbar \, \Omega_{\vec{k}\sigma}$; $B^{+}_{\vec{k}\sigma}$ and $A^{+}_{\vec{k}\sigma}$ are the corresponding creation operators. These operators are assumed to obey the commutation relations for Bosons

$$\left[B_{\vec{k}\sigma} , B_{\vec{k}', \sigma'}\right]_{-} = \left[B^{+}_{\vec{k}\sigma} , B^{+}_{\vec{k}', \sigma'}\right]_{-} = 0 \tag{3.12}$$

$$\left[B_{\vec{k}\sigma} , B^{+}_{\vec{k}', \sigma'}\right]_{-} = \delta_{\vec{k},\vec{k}'} \, \delta_{\sigma , \sigma'} \tag{3.13}$$

where $\left[A,B\right]_{-} = AB - BA$. The operators $A_{\vec{k}\sigma}$ and $A^{+}_{\vec{k}\sigma}$ obey similar relations. The operator $B^{+}_{\vec{k}\sigma} \, B_{\vec{k}\sigma}$ is the number operator for longitudinal quasi-particles of momentum $\hbar \vec{k}$ and type σ . It is a well known consequence[23] of the Boson commutation relations that its eigenvalues are $N_{\sigma}(\vec{k}) = 0, 1, 2, 3, \ldots \infty$. H_L and H_T are the Hamiltonians for longitudinal and transverse quasi-particles. The state of the system is specified when the number of quasi-particles of each type is given. That is, the state vectors are of the form

$$\left| \ldots N_{\sigma}(\vec{k}) \ldots N_{\sigma'}(\vec{k}') \ldots \right> \tag{3.14}$$

These are eigenvectors of H_L and H_T.

We shall need to know the effect of operatoring on these state vectors with $B_{\vec{k}\sigma}$ and $B^{+}_{\vec{k}\sigma}$. Again, it follows from the commutation relations that[23]

$$B_{\vec{k}\sigma}\Big| \ldots N_{\sigma}(\vec{k}) \ldots > = \sqrt{N_{\sigma}(\vec{k})}\,\Big| \ldots, N_{\sigma}(\vec{k}) - 1, \ldots > \tag{3.15}$$

$$B^{+}_{\vec{k}\sigma}\Big| \ldots N_{\sigma}(\vec{k}) \ldots > = \sqrt{N_{\sigma}(\vec{k}) + 1}\,\Big| \ldots, N_{\sigma}(\vec{k}) + 1, \ldots > \tag{3.16}$$

The operators $A_{\vec{k}\sigma}$ and $A^{+}_{\vec{k}\sigma}$ have similar effects.

This completes the quantum mechanical theory of plasma quasi-particles.

3.2 The Particle-Quasi-Particle Interaction

In Eq. (2.12) we expanded $\psi_s(\vec{x},t)$ in a set of eigenfunctions $\chi_{sa}(\vec{x})$ and interpreted the expansion coefficients C_{sa} and C^{+}_{sa} as destruction and creation operators for particles of species s in the state a. The zeroth order Hamiltonian for the particles was given by Eq. (2.14). The state vectors of the particle system have the form

$$\Big| \ldots N_s(a) \ldots > \tag{3.17}$$

where $N_s(a)$ is the number of particles of species s in the state a. Since the particles are all assumed to be Fermions the only possible values of $N_s(a)$ are zero and one. The $N_s(a)$ are eigenvalues of the number operator $C^{+}_{sa}\,C_{sa}$. It follows from the commutation relations that[23]

$$C_{sa}\Big| \ldots, N_s, (a) \ldots > = \pm\sqrt{N_s(a)}\,\Big| \ldots, 1 - N_s(a), \ldots > \tag{3.18}$$

$$C^{+}_{sa}\Big| \ldots, N_s(a), \ldots > = \pm\sqrt{1 - N_s(a)}\,\Big| \ldots, 1 - N_s(a), \ldots > \tag{3.19}$$

For our purposes the $\pm$ signs in the above equations can be ignored.

The state vectors of the system of non-interacting particles and quasi-particles is of the form

$$\left| \ldots . N_s(a) \ldots N_\sigma(\vec{k}) \ldots . \right> \tag{3.20}$$

(The index σ ranges over all types of quasi-particles including both longitudinal and transverse.)

From Eq. (2.6) we see that the term $e_s\phi$ is responsible for the interaction of particles of species s with longitudinal fields. We shall use this in the second quantization formalism to write the interaction Hamiltonian between particles and longitudinal quasi-particles as

$$H_{sL} = \int d^3x \; \psi_s^+ (\vec{x}) \; e_s\phi \; (\vec{x}) \; \psi_s \; (\vec{x}) \tag{3.21}$$

Using Eqs. (2.12) and (3.1) we can carry out the integrations and obtain

$$H_{sL} = \sum_a \sum_{a'} \sum_{\vec{k},\sigma} \Big\{ M_{sL\sigma}(\vec{k},a,a') \; C^+_{sa} \; C_{sa'} \; B_{\vec{k}\sigma}$$

$$+ M^*_{sL\sigma} (\vec{k},a,a') \; C_{sa} \; C^+_{sa'} \; B^+_{\vec{k}\sigma} \Big\} \tag{3.22}$$

where

$$M_{sL\sigma} (\vec{k},a,a') = \left[\frac{4\pi \; e_s^2 \; \hbar \; \Omega_{\vec{k}\sigma}}{V \, k^2 \left| \frac{\partial}{\partial \omega} \; \omega \, \epsilon_{1L} \right|_{\Omega_{k\sigma}}} \right]^{1/2} \langle a | e^{i\vec{k}\cdot\vec{x}} | \, a' > s \tag{3.23}$$

The interaction Hamiltonian between particles of species s and transverse quasi-particles will be divided into two parts $H^{(1)}_{sT}$ and $H^{(2)}_{sT}$ where

$$\overset{(1)}{H}_{sT} = - e_s \int d^3x \; \psi_s^+ (\vec{x}) \; \frac{1}{m_s} \; (\vec{p} - \frac{e_s}{c} \vec{A}_o) \cdot \vec{A}_1 \quad \psi_s (\vec{x})$$

$$= \sum_a \sum_{a'} \sum_{\vec{k},\sigma} \Big\{ \overset{(1)}{M}_{sT\sigma} (\vec{k},a,a') \; C^+_{sa} \; C_{sa'} \; A_{\vec{k}\sigma}$$

$$+ \overset{(1)*}{M}_{sT\sigma} (\vec{k},a,a') \; C_{sa} \; C^+_{sa'} \; A^+_{\vec{k}\sigma} \Big\} \tag{3.24}$$

and

$$\overset{(2)}{H}_{sT} = \frac{e_s^2}{2m_s \, c^2} \int d^3x \quad \psi_s^+ (x) \; A_1^2 \; (\vec{x}) \quad \psi_s (\vec{x})$$

$$= \sum_a \sum_{a'} \sum_{\vec{k},} \sum_{\vec{k}'\sigma'} \quad C^+_{sa} \; C_{sa'}$$

$$\Big\{ \overset{(2)}{M}_{sT\,\sigma\sigma'} (\vec{k},\vec{k}',a,a') \; A_{\vec{k}\sigma} \quad A_{\vec{k}'\,\sigma'}$$

$$+ \overset{(2)}{M}_{sT\,\sigma\sigma'} (\vec{k},-\vec{k}',a,a') \; A_{\vec{k}\sigma} \quad A^+_{\vec{k}'\,\sigma'}$$

$$+ \overset{(2)}{M}_{sT\,\sigma\sigma'} (-\vec{k},\vec{k}',a,a') \; A^+_{\vec{k}\sigma} \quad A_{\vec{k}'\,\sigma'}$$

$$+ \overset{(2)}{M}_{sT\,\sigma\sigma'} (-\vec{k},-\vec{k}',a,a') \; A^+_{\vec{k}\sigma} \quad A^+_{\vec{k}'\,\sigma'} \Big\} \tag{3.25}$$

In the above interaction Hamiltonians the vertex parts are given by

$$\overset{(1)}{M}_{sT\,\sigma} (\vec{k},a,a') = \left[\frac{4\pi e_s^2 \, \hbar}{V \, \Omega_{\vec{k}\sigma} \left| \frac{1}{\omega} \, \frac{\partial}{\partial \omega} \, \omega^2 \, \epsilon_{1T\sigma} \right|_{\Omega_{k\sigma}}} \right]^{1/2}$$

$$< a \,\Big|\, \vec{u}_{\vec{k}\sigma} \cdot \vec{v} \; e^{i\vec{k}\cdot\vec{x}} \,\Big|\, a' > s \tag{3.26}$$

and

$$\overset{(2)}{M}_{sT\,\sigma\sigma'} (\vec{k},\vec{k}',a,a') = \frac{e_s^2}{2m_s \, c^2} \left[\frac{4\pi \, \hbar \, c^2}{V} \right]$$

$$\frac{1}{\left[\frac{\partial}{\partial \omega} \, \omega^2 \, \epsilon_{It\sigma} \right]^{1/2}_{\Omega_{\vec{k}\sigma}} \left| \frac{\partial}{\partial \omega} \, \omega^2 \, \epsilon_{1T\,\sigma'} \right|^{1/2}_{\Omega_{k\,\sigma'}}}$$

$$< a \Big| \vec{u}_{\vec{k}\sigma} \cdot \vec{u}_{\vec{k}'\sigma'} \; e^{i(\vec{k} + \vec{k}') \cdot \vec{x}} \Big| \, a' > s \tag{3.27}$$

The necessary matrix elements for the evaluation of Eqs. (3.23), (3.26) and (3.27) are given in Eqs. (2.38) and (2.39) when the particles are free and in Eqs. (2.58), (2.59), (2.60) and (2.61) when the particles are in a uniform magnetic field.

It is convenient to represent each term in an interaction Hamiltonian by a Feynmann diagram. For instance the term containing $C^{+}_{sa}\, C_{sa'}\, B_{\vec{k}\sigma}$ in Eq. (3.22) describes the process in which a particle of species s in the state a' is destroyed, one is created in the state a, and a quasi-particle of type σ and momentum $\hbar\, \vec{k}$ is destroyed. The term containing $C_{sa}\, C^{+}_{sa'}\, B^{+}_{\vec{k},\sigma}$ describes the inverse process. These are represented by the diagrams in Fig. 3.1.

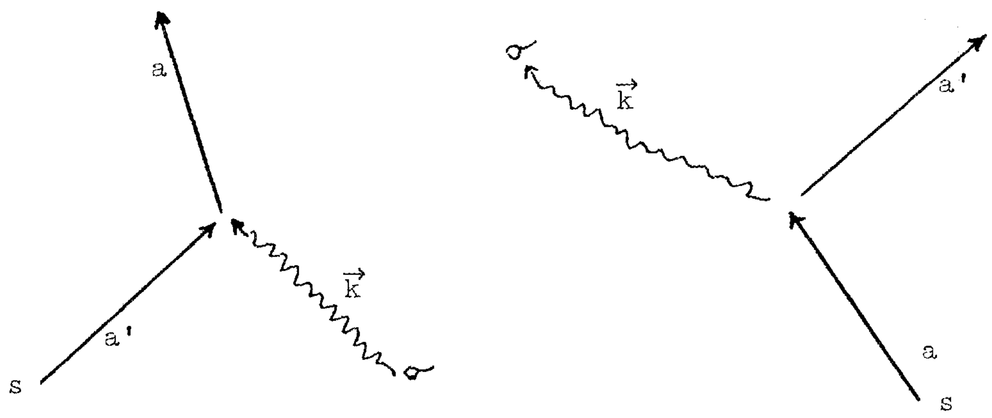


Fig. 3.1 Feynmann diagrams for the processes described by H_{sL} and $H^{(1)}_{sT}$.

The interaction Hamiltonian $H^{(1)}_{sT}$ contains the same type of terms but with transverse quasi-particles replacing longitudinal ones. The interaction Hamiltonian $H^{(2)}_{sT}$ contains terms describing processes in which two transverse quasi-particles are destroyed, two are created or one is created and one is destroyed.

Now that the interaction Hamiltonians are known the transition probability between initial state $|i>$ and final state $|f>$ may be calculated from the well known formula (the "Fermi Golden Rule"). Transition probability per unit time

$$= \frac{2\pi}{\hbar} |M|^2 \delta (E_i - E_f) \tag{3.28}$$

where E_i and E_f are the initial and final energies and M is the matrix element for the transition. It is given by

$$M = < f | H' | i > + \sum_I \frac{< F | H' | I > < I | H' | i >}{E_i - E_I + i\eta}$$

$$+ \sum_I \sum_{II} \frac{< f | H' | I > < I | H' | II > < II | H' | i >}{(E_i - E_I + i\eta)(E_i - E_{II} + i\eta)} + \ldots \tag{3.29}$$

Here, η is a positive infinitesimal, H' is the interaction Hamiltonian and the summations are over intermediate states.

We now have all of the necessary machinery to discuss quasi-linear theory and wave wave coupling.

CHAPTER 4. QUASI-LINEAR THEORY

As we shall use the term, quasi-linear theory deals with first order processes, that is, those transitions for which the matrix element is approximated by the first term in Eq. (3.29).

4.1 Quasi-Linear Theory of Longitudinal Waves with $B_o = 0$

We shall consider a plasma with no external magnetic field. The dielectric function is given by Eq. (2.48) or in the classical limit by Eq. (2.49). It may be divided into its real and imaginary parts by using the Plemelj formula

$$\frac{1}{X + i\eta} \xrightarrow[\eta \to 0+]{} P \frac{1}{X} - i\pi\,\delta(X) \tag{4.1}$$

where the P indicates that a principal value is to be taken in subsequent integrations. Using this in Eq. (2.48) we find

$$\epsilon(\vec{q},\omega) = \epsilon_1(\vec{q},\omega) + i\,\epsilon_2(\vec{q},\omega) \tag{4.2}$$

$$\epsilon_1(q,\omega) , 1 + \sum_s \frac{4\pi^2 e_s^2}{\hbar q^2} P \int d^3v \; \frac{f_{so}(\vec{v}) - f_{so}(\vec{v} - \hbar\vec{q}/m_s)}{\omega - \vec{q}\cdot\vec{v} + \frac{\hbar^2 q^2}{2m_s}} \tag{4.3}$$

$$\epsilon_2(\vec{q},\omega) = -\sum_s \frac{4\pi^2 e_s^2}{\hbar q^2} \int d^3v \; \left[f_{so}(\vec{v}) - f_{so}(\vec{v} - \hbar\vec{q}/m_s)\right]$$

$$\delta(\omega - \vec{q}\cdot\vec{v} + \hbar q^2/2m_s) \tag{4.4}$$

In the classical limit ϵ_1 and ϵ_2 are given by

$$\epsilon_1(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{m_s q^2} P \int d^3v \; \frac{\vec{q}\cdot\partial f_{so}/\partial\vec{v}}{\omega - \vec{q}\cdot\vec{v}} \tag{4.5}$$

and

$$\epsilon_2(\vec{q},\omega) = -\sum_s \frac{4\pi^2 e_s^2}{m_s q^2} \int d^3v \; \vec{q}\cdot\frac{\partial f_{so}}{\partial\vec{v}}\,\delta(\omega - \vec{q}\cdot\vec{v}) \tag{4.6}$$

The frequencies of longitudinal waves are to be found by solving $\epsilon_1(\vec{q},\omega) = 0$. For a plasma consisting of electrons and ions there are two solutions of this equation; one of these is the high frequency plasma oscillation and the other is the low frequency ion sound wave. To obtain the first of these we assume that the phase velocity of the wave is much greater than thermal velocities and expand $(\omega - \vec{q} \cdot \vec{v})^{-1}$ in Eq. (4.5) to obtain

$$\epsilon_1(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{m_s q^2 \omega} P\int d^3v\, \vec{q} \cdot \frac{\partial f_{so}}{\partial \vec{v}} \left[1 + \frac{\vec{q} \cdot \vec{v}}{\omega} + \frac{(\vec{q} \cdot \vec{v})^2}{\omega^2} + \ldots\right] \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left[1 + \frac{2}{\omega} < \vec{q} \cdot \vec{v} > + \frac{3}{\omega^2} <(\vec{q} \cdot \vec{v})^2> + \ldots\right] \qquad (4.7)$$

where we have neglected the ions and integrated by parts. Setting $\epsilon_1 = 0$ and solving for the frequency gives approximately

$$\Omega_{\vec{q}\lambda} \simeq \omega_{pe}\left(1 + \frac{2 <\vec{q} \cdot \vec{v}>}{\omega_{pe}} + \frac{3<(\vec{q} \cdot \vec{v})^2>}{\omega_{pe}^2}\right)^{1/2} \qquad (4.8)$$

The angular brackets denote averages with respect to the electron distribution function. We shall use the symbol λ to denote plasma oscillations (plasmons).

It is well known that ion sound waves are strongly damped unless the ion temperature is much less than the electron temperature. We shall assume that this is the case. To obtain the frequency of ion sound waves we shall assume that the phase velocity of the wave is much less than the electron thermal velocity but much greater than the ion thermal velocity. With these approximations Eq. (4.5) becomes

$$\epsilon_1(\vec{q},\omega) \simeq 1 + \frac{1}{q^2 L_e^2} - \frac{\omega_{pi}^2}{\omega^2} \qquad (4.9)$$

where $L_e = (T_e/4\pi ne^2)^{1/2}$ is the electron Debye length. Solving Eq. (4.9) for the frequency gives

$$\Omega_{\vec{q},\nu} \simeq \frac{\omega_{pi}\, qL_e}{\sqrt{1 + q^2 L_e^2}} = \frac{\sqrt{T_e/M_i}\; q}{\sqrt{1 + q^2 L_e^2}} \tag{4.10}$$

We shall use the symbol ν to denote ion sound waves (phonons).

We find

$$\left(\frac{\partial}{\partial \omega}\, \omega\, \epsilon_1\right)_{\Omega_{q\lambda}} \simeq 2 \tag{4.11}$$

and

$$\left(\frac{\partial}{\partial \omega}\, \omega\, \epsilon_1\right)_{\Omega_{q\nu}} \simeq 2 \left(1 + \frac{1}{q^2 L_e^2}\right) \tag{4.12}$$

Using these together with Eq. (2.38) in Eq. (3.23) gives

$$M_{s\lambda}\,(\vec{k},\vec{p},\vec{p}\,') = \left[\frac{2\pi e_s^2\, \hbar\, \Omega_{k\lambda}}{V\, k^2}\right]^{1/2} \delta_{\vec{p},\vec{p}\,' + \vec{k}} \tag{4.13}$$

and

$$M_{s\nu}\,(\vec{k},\vec{p},\vec{p}\,') = \left[\frac{2\pi e_s^2\, \hbar\, \Omega_{k\nu}}{V\, k^2 \left(1 + \frac{1}{k^2 L_e^2}\right)}\right]^{1/2} \delta_{\vec{p},\vec{p}\,' + \vec{k}} \tag{4.14}$$

for the vertex functions for the particle-plasmon and particle-phonon interactions. We have replaced the particle quantum numbers a and a' by the free particle wave vectors $\vec{p}$ and $\vec{p}\,'$. These vertex functions agree with those found by Pines and Schrieffer[9] and by Ross[21] using quite different methods.

To simplify the discussion which follows we shall consider only electrons and plasmons and neglect the ions and phonons. Eq. (4.13) may be used to write the electron-plasmon interaction Hamiltonian as

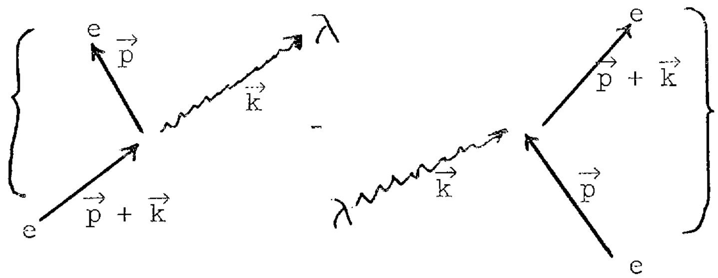
$$H_{e\lambda} = \sum_{\vec{p}} \sum_{k} \left[\frac{2\pi e^2\, \hbar\, \Omega_{k\lambda}}{V\, k^2}\right]^{1/2}$$

$$\left\{ C^{+}_{\vec{p}+\vec{k}}\; C_{\vec{p}}\; B_{\vec{k}} + C_{\vec{p}+\vec{k}}\; C^{+}_{p}\; B^{+}_{\vec{k}} \right\} \tag{4.15}$$

where we have used the Kronecker-Delta in Eq. (4.13) to eliminate one summation and have dropped some superfluous subscripts. Note that momentum is conserved at each vertex.

Now, we shall write equations for the rate of change of $N_e(\vec{p})$, the number of electrons of momentum $\hbar\,\vec{p}$ and $N_\lambda(\vec{k})$, the number of plasmons of momentum $\hbar\,\vec{k}$. Schematically we may write

$$\frac{\partial N_\lambda(\vec{k})}{\partial t} = \sum_{\vec{p}}$$



(4.16)

What we mean by this is that we add all of the processes in which a plasmon of momentum $\hbar\,\vec{k}$ is emitted and subtract all of those in which one is absorbed. This difference gives the increase in $N_\lambda(\vec{k})$. The schematic equation can be converted into a mathematical equation by replacing each diagram by the transition probability per unit time for the process. Using Eq. (3.28) and the first term in Eq. (3.29) gives

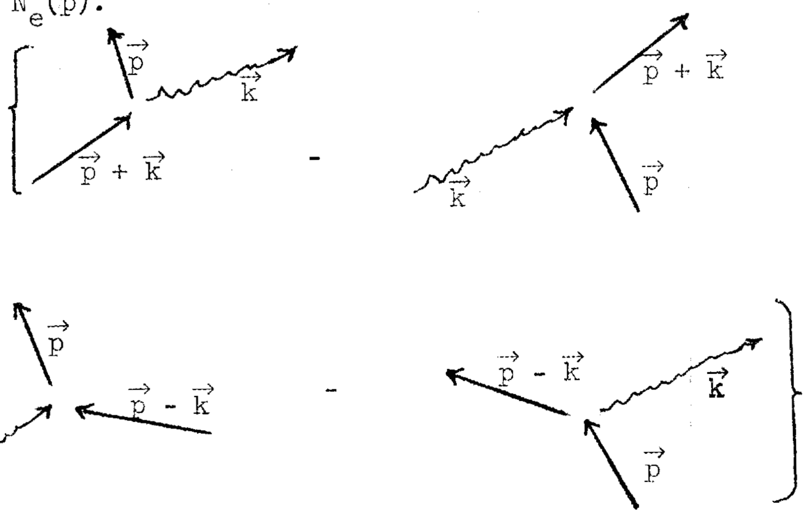
$$\frac{\partial N_\lambda(\vec{k})}{\partial t} = \sum_{\vec{p}} \frac{2\pi}{\hbar}\left[\frac{2\pi e^2\,\hbar\,\Omega_{k\lambda}}{V\,k^2}\right]\delta\left[\frac{\hbar^2}{2m}\left|\vec{p}+\vec{k}\right|^2 - \frac{\hbar^2}{2m}\,p^2\right.$$
$$\left. - \hbar\,\Omega_{\lambda k}\right]\left\{N_e(\vec{p}+\vec{k})\left[1 - N_e(\vec{p})\right]\left[N_\lambda(\vec{k}) + 1\right]\right.$$
$$\left. - \left[1 - N_e(\vec{p}+\vec{k})\right] N_e(\vec{p})\, N_\lambda(\vec{k})\right\} \tag{4.17}$$

Note that the square of the matrix element has two factors. One of these is just the square of the vertex part in Eq. (4.15). The other comes from the square of the matrix element of the creation and destruction operators. For instance, consider the term $C^{+}_{\vec{p}+\vec{k}}\, C_{\vec{p}}\, B_{\vec{k}}$ in Eq. (4.15) and the corresponding diagram in Eq. (4.16). From Eq.

(3.18) we see that the destruction of an electron of momentum $\hbar\vec{p}$ gives a factor $N_e(\vec{p})$. From Eq. (3.19) we see that creation of an electron of momentum $\hbar(\vec{p}+\vec{k})$ gives a factor $1 - N_e(\vec{p}+\vec{k})$. From Eq. (3.15) we see that destruction of a plasmon of momentum $\hbar\vec{k}$ gives a factor $N_\lambda(\vec{k})$. The product of these factors gives the last term in Eq. (4.17).

In a similar way we can write a schematic equation for the rate of change of $N_e(\vec{p})$.

$$\frac{\partial N_e(\vec{p})}{\partial t} = \sum_{\vec{k}}$$



(4.18)

The corresponding mathematical equation is

$$\frac{\partial N_e(\vec{p})}{\partial t} = \sum_{\vec{k}} \frac{2\pi}{\hbar}\left[\frac{2\pi e^2\,\hbar\,\Omega_{k\lambda}}{V\,k^2}\right]\Big\{\delta\Big[\frac{\hbar^2}{2m}|\vec{p}+\vec{k}|^2 - \frac{\hbar^2}{2m}p^2$$

$$-\hbar\,\Omega_{\lambda k}\Big]\Big[N_e(\vec{p}+\vec{k})\big[1-N_e(\vec{p})\big]\big[N_\lambda(\vec{k})+1\big] - \big[1-N_e(\vec{p}+\vec{k})\big]N_e(\vec{p})$$

$$N_\lambda(\vec{k})\Big] + \delta\Big[\frac{\hbar^2}{2m}|\vec{p}-\vec{k}|^2 + \hbar\,\Omega_{\lambda\vec{k}} - \frac{\hbar^2}{2m}p^2\Big]$$

$$\Big[N_e(\vec{p}-\vec{k})\big[1-N_e(\vec{p})\big]N_\lambda(\vec{k}) - \big[1-N_e(\vec{p}-\vec{k})\big]N_e(\vec{p})\big[N_\lambda(\vec{k})+1\big]\Big]\Big\}$$

(4.19)

Some consequences of Eqs. (4.17) and (4.19) may be seen immediately.

$$\frac{\partial}{\partial t} \sum_{\vec{p}} N_e(\vec{p}) = 0 \qquad (4.20)$$

$$\frac{\partial}{\partial t} \left\{ \sum_{\vec{p}} \hbar \vec{p} N_e(\vec{p}) + \sum_{\vec{k}} \hbar \vec{k} N_\lambda(\vec{k}) \right\} = 0 \qquad (4.21)$$

$$\frac{\partial}{\partial t} \left\{ \sum_{\vec{p}} \frac{\hbar^2 p^2}{2m} N_e(\vec{p}) + \sum_{\vec{k}} \hbar \Omega_{\lambda \vec{k}} N_\lambda(\vec{k}) \right\} = 0 \qquad (4.22)$$

These equations show that particles (but not quasi-particles), momentum and energy are conserved. This is not too surprising since they are conserved at each vertex.

We can define the entropy of the electron-plasmon system as

$$S = S_e + S_\lambda \qquad (4.23)$$

where

$$S_e = -K \sum_{\vec{p}} \left\{ \left[1 - N_e(p)\right] \log \left[1 - N_e(p)\right] + N_e(p) \log N_e(p) \right\} \qquad (4.24)$$

and

$$S_\lambda = K \sum_{\vec{k}} \left\{ \left[N_\lambda(\vec{k}) + 1\right] \log \left[N_\lambda(\vec{k}) + 1\right] - N_\lambda(\vec{k}) \log N_\lambda(\vec{k}) \right\} \qquad (4.25)$$

where K is Boltzmann's constant. S_e is the entropy of a system of Fermions and S_λ is the entropy of a system of Bosons.[36] By taking the time derivative of S and using Eqs. (4.17) and (4.19) it is not difficult to show that

$$\frac{dS}{dt} \geqslant 0 \qquad (4.26)$$

Furthermore, the equality holds when

$$N_e(\vec{p}) = \frac{1}{C \exp\left(\frac{\hbar^2 p^2}{2mT}\right) + 1} \qquad (4.27)$$

and

$$N_\lambda(\vec{k}) = \frac{1}{\exp\left(\frac{\hbar\Omega_{\lambda\vec{k}}}{T}\right) - 1} \tag{4.28}$$

since a brief calculation shows that these are equilibrium solutions of Eqs. (4.17) and (4.19). This seems to indicate that $N_e(\vec{p})$ approaches the Fermi-Dirac distribution and $N_\lambda(\vec{k})$ approaches the Planck distribution as time increases. What is missing from the proof is a proof that S has only one maximum.

We now pass to the classical limit by the prescription

$$\hbar \to 0 \tag{4.29a}$$

$$\hbar\,\vec{p} \to m\,\vec{v} \tag{4.29b}$$

$$N_e(\vec{p}) \ll 1 \tag{4.29c}$$

$$N_\lambda(\vec{k}) \to \infty \tag{4.29d}$$

$$\hbar\,\Omega_{\lambda k}\,N_\lambda(\vec{k}) = P_\lambda(\vec{k}) = \text{finite} \tag{4.29e}$$

$$\sum_{\vec{k}} \to V\int \frac{d^3k}{2\pi^3} \tag{4.29f}$$

$$\sum_{\vec{p}} N_e(\vec{p}) \to V\int d^3v\, f_e(\vec{v}) \tag{4.29g}$$

Eq. (4.29c) means that the electron gas is far from degeneracy. $P_\lambda(\vec{k})$ is the energy spectrum of plasma oscillations--a classically meaningful quantity. We let the volume of the box in which the system is quantized become infinite so that sums go over into integrals, hence Eqs. (4.29f) and (4.29g). In this limit Eqs. (4.17) and (4.19) become

$$\frac{\partial P_\lambda(\vec{k})}{\partial t} = 2\gamma_\lambda(\vec{k})\,P_\lambda(\vec{k}) + S_\lambda(\vec{k}) \tag{4.30}$$

$$\frac{\partial f_e(\vec{v})}{\partial t} = \frac{\partial}{\partial\vec{v}}\cdot\left(\overleftrightarrow{D}(\vec{v})\cdot\frac{\partial f_e(\vec{v})}{\partial\vec{v}}\right) + \frac{\partial}{\partial\vec{v}}\cdot\left(\vec{A}(\vec{v})f_e\right) \tag{4.31}$$

where

$$\gamma_\lambda(\vec{k}) = \frac{2\pi^2 e^2 \Omega_{\lambda k}}{mk^2} \int d^3v\, \vec{k} \cdot \frac{\partial f_e}{\partial \vec{v}}\, \delta(\vec{k} \cdot \vec{v} - \Omega_{\lambda \vec{k}}) \tag{4.32}$$

$$S_\lambda(\vec{k}) = \frac{4\pi^2 e^2 \Omega^2_{\lambda k}}{k^2} \int d^3v\, f_e(\vec{v})\, \delta(\vec{k} \cdot \vec{v} - \Omega_{\lambda \vec{k}}) \tag{4.33}$$

$$\overleftrightarrow{D}(\vec{v}) = \frac{4\pi^2 e^2}{m^2} \int \frac{d^3k}{(2\pi)^3} P_\lambda(\vec{k})\, \frac{\vec{k}\,\vec{k}}{k^2}\, \delta(\vec{k} \cdot \vec{v} - \Omega_{\lambda \vec{k}}) \tag{4.34}$$

$$\vec{A}(\vec{v}) = \frac{4\pi^2 e^2}{m} \int \frac{d^3k}{(2\pi)^3} \Omega_{\lambda \vec{k}} \frac{\vec{k}}{k^2}\, \delta(\vec{k} \cdot \vec{v} - \Omega_{\lambda \vec{k}}) \tag{4.35}$$

These are the classical quasi-linear equations.

The first term on the right hand side of Eq. (4.30) comes from the stimulated emission terms in Eq. (4.17). The second term comes from the spontaneous emission terms.

Eq. (4.32) agrees with the imaginary part of the frequency given by Eq. (2.110) when Eqs. (4.6) and (4.11) are used. In the zeroth order approximation where the particle-plasmon interaction is neglected the imaginary part of $\epsilon(\vec{k},\omega)$ is neglected and the plasmons have infinite lifetimes. The imaginary part of ϵ is a consequence of the processes of absorption and emission of plasmons by particles.

In the present derivation it is clear that Eq. (4.17) for the rate of change of $N_\lambda(\vec{k})$ must be accompanied by Eq. (4.19) for the rate of change of $N_e(\vec{p})$. Also, the terms due to spontaneous emission appear quite naturally in this derivation although they are often omitted in classical derivations.

It is easy to show that time-independent solutions of Eqs. (4.30) and (4.31) are

$$P_\lambda(\vec{k}) = T \quad \text{(Rayleigh-Jeans)} \tag{4.36}$$

$$f_e(\vec{v}) = C\, e^{-mv^2/2T} \quad \text{(Maxwell-Boltzmann)} \tag{4.37}$$

However, it is not possible to show that these solutions are approached asymptotically. Indeed, it is possible to show that there are initial $P_\lambda(\vec{k})$ and $f_e(\vec{v})$ which can never evolve into Eqs. (4.36) and (4.37). An example will be given presently.

Before discussing Eqs. (4.30) and (4.31) further it is convenient to write them in dimensionless units. To do so we define

$$\alpha = \sqrt{\frac{T}{m}}\ ,\ \vec{u} = \vec{v}/\alpha\ ,\ \tau = \omega_{pe} t$$

$$L_e = \alpha/\omega_{pe},\ \vec{q} = L_D \vec{k}$$

$$P(\vec{k}) = T\mathcal{P}(\vec{q})$$

$$f(v) = \frac{n}{\alpha^3} F(\vec{u}) \tag{4.38}$$

In dimensionless form Eqs. (4.30) and (4.31) are

$$\frac{\partial \mathcal{P}(\vec{q})}{\partial \tau} = \frac{\pi}{q^2} \int d^3u\, \delta(\vec{q}\cdot\vec{u} - 1) \left[\mathcal{P}(\vec{q})\ \vec{q}\cdot\frac{\partial F}{\partial \vec{u}} + F(\vec{u})\right] \tag{4.39}$$

$$\frac{\partial F(\vec{u})}{\partial \tau} = \pi\, \mathcal{E} \frac{\partial}{\partial \vec{u}} \cdot \int \frac{d^3q}{(2\pi)^3}\, \delta(\vec{q}\cdot\vec{u} - 1) \frac{\vec{q}}{q^2}$$
$$\left[\mathcal{P}(\vec{q})\ \vec{q}\cdot\frac{\partial F}{\partial \vec{u}} + F(\vec{u})\right] \tag{4.40}$$

where

$$\mathcal{E} = \frac{1}{n\, L_e^3} \tag{4.41}$$

is the so-called plasma parameter. In deriving these we have made the approximation $\Omega_{\lambda k} \simeq \omega_{pe}$. These equations take a particularly simple form in one dimension, namely

$$\frac{\partial \mathcal{P}(\vec{q})}{\partial \tau} = \frac{\pi}{q^2} \int du\, \delta(qu - 1) \left[\mathcal{P}(q)\ q\frac{\partial F}{\partial u} + F\right]$$
$$= \frac{\pi}{|q|^3} \left[q\,\mathcal{P}(q)\ F'\left(\frac{1}{q}\right) + F\left(\frac{1}{q}\right)\right] \tag{4.42}$$

$$\frac{\partial F(u)}{\partial \tau} = \pi \mathcal{E} \int \frac{dq}{2\pi} \, \delta \, (qu - 1) \, \frac{1}{|q|} \left[\mathcal{P}(q) \; q \, \frac{\partial F}{\partial u} + F(u) \right]$$

$$= \frac{1}{2} \mathcal{E} \, \frac{\partial}{\partial u} \left\{ \frac{u}{|u|} \left[\frac{\mathcal{P}(1/u)}{u} \, \frac{\partial F(u)}{\partial u} + F(u) \right\} \right. \tag{4.43}$$

where the prime denotes a derivative with respect to the argument. A further simplification can be made by making the change of variable $q = 1/u$ and writing $\mathcal{P}(u)$ instead of $\mathcal{P}(1/u)$. Then

$$\frac{\partial \mathcal{P}(u,\tau)}{\partial \tau} = \pi \, |u|^3 \left[\frac{1}{u} \, \mathcal{P}(u,\tau) \, \frac{\partial F(u,\tau)}{\partial u} + F(u,\tau) \right] \tag{4.44}$$

$$\frac{\partial F(u,\tau)}{\partial \tau} = \frac{1}{2} \mathcal{E} \, \frac{\partial}{\partial u} \left\{ \frac{u}{|u|} \left[\frac{\mathcal{P}(u,\tau)}{u} \, \frac{\partial F(u,\tau)}{\partial u} + F(u,\tau) \right] \right\} \tag{4.45}$$

We have now explicitly included τ as an argument of $\mathcal{P}(u,\tau)$ and $F(u,\tau)$. In dimensionless units the Rayleigh-Jeans and Maxwell-Boltzmann solutions are

$$\mathcal{P} = 1 \tag{4.46}$$

$$F = C \, e^{-1/2 \, u^2} \tag{4.47}$$

which clearly satisfy Eqs. (4.44) and (4.45).

An integral of Eqs. (4.44) and (4.45) can be obtained by combining them to give

$$\frac{\partial F}{\partial \tau} = \frac{1}{2} \mathcal{E} \, \frac{\partial}{\partial u} \left\{ \frac{u}{|u|} \, \frac{1}{\pi |u|^3} \, \frac{\partial \mathcal{P}}{\partial \tau} \right\} \tag{4.48}$$

from which

$$F(u,\tau) - \frac{1}{2\pi} \mathcal{E} \, \frac{\partial}{\partial u} \left[\frac{1}{u^3} \, \mathcal{P}(u,\tau) \right] = g(u) \tag{4.49}$$

where $g(u)$ is constant in time. Now, if at any time $\mathcal{P}$ and F are given by Eqs. (4.46) and (4.47) then

$$g(u) = C \, e^{-1/2 \, u^2} + \frac{\mathcal{E}}{2\pi} \, \frac{3}{u^4} \tag{4.50}$$

But clearly one can choose the initial conditions for F and $\mathcal{P}$ so that g(u) is some other function; with such a choice of initial conditions the equilibrium solutions, Eqs. (4.46) and (4.47) are never approached.*

As we have previously remarked, in classical derivations the terms due to spontaneous emission are often omitted. These are the terms S_λ and $\vec{A}\, f_e$ in Eqs. (4.30) and (4.31) and the corresponding terms in Eqs. (4.44) and (4.45). We shall now examine the quasi-linear equations with the spontaneous emission terms neglected. We write

$$\frac{\partial \mathcal{P}(u,\tau)}{\partial \tau} = \pi \frac{|u|^3}{u} \mathcal{P}(u,\tau) \frac{\partial F(u,\tau)}{\partial u} = 2\gamma(u,\tau)\,\mathcal{P}(u,\tau) \tag{4.51}$$

$$\begin{aligned} \frac{\partial F(u,\tau)}{\partial \tau} &= \frac{1}{2}\, \mathcal{E} \frac{\partial}{\partial u} \left[\frac{1}{|u|} \mathcal{P}(u,\tau) \frac{\partial F(u,\tau)}{\partial u} \right] \\ &= \frac{\partial}{\partial u} \left[D(u,\tau) \frac{\partial F(u,\tau)}{\partial u} \right] \end{aligned} \tag{4.52}$$

Again, it is found that g(u) is a constant of the motion. It may be used to write

$$F(u,\tau) - F(u,o) = \frac{\mathcal{E}}{2\pi} \frac{\partial}{\partial u} \frac{1}{u^3} \left[\mathcal{P}(u,\tau) - \mathcal{P}(u,o) \right] \tag{4.53}$$

from which

$$\mathcal{P}(u,\tau) - \mathcal{P}(u,o) = \frac{2\pi}{\mathcal{E}} u^3 \int du \left[F(u,\tau) - F(u,o) \right] \tag{4.54}$$

Now, to be specific let us assume that the initial distribution has the form of a Maxwellian with a bump on the tail as shown in Fig. 4.1.

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*I am indebted to Owen C. Eldridge for this observation.

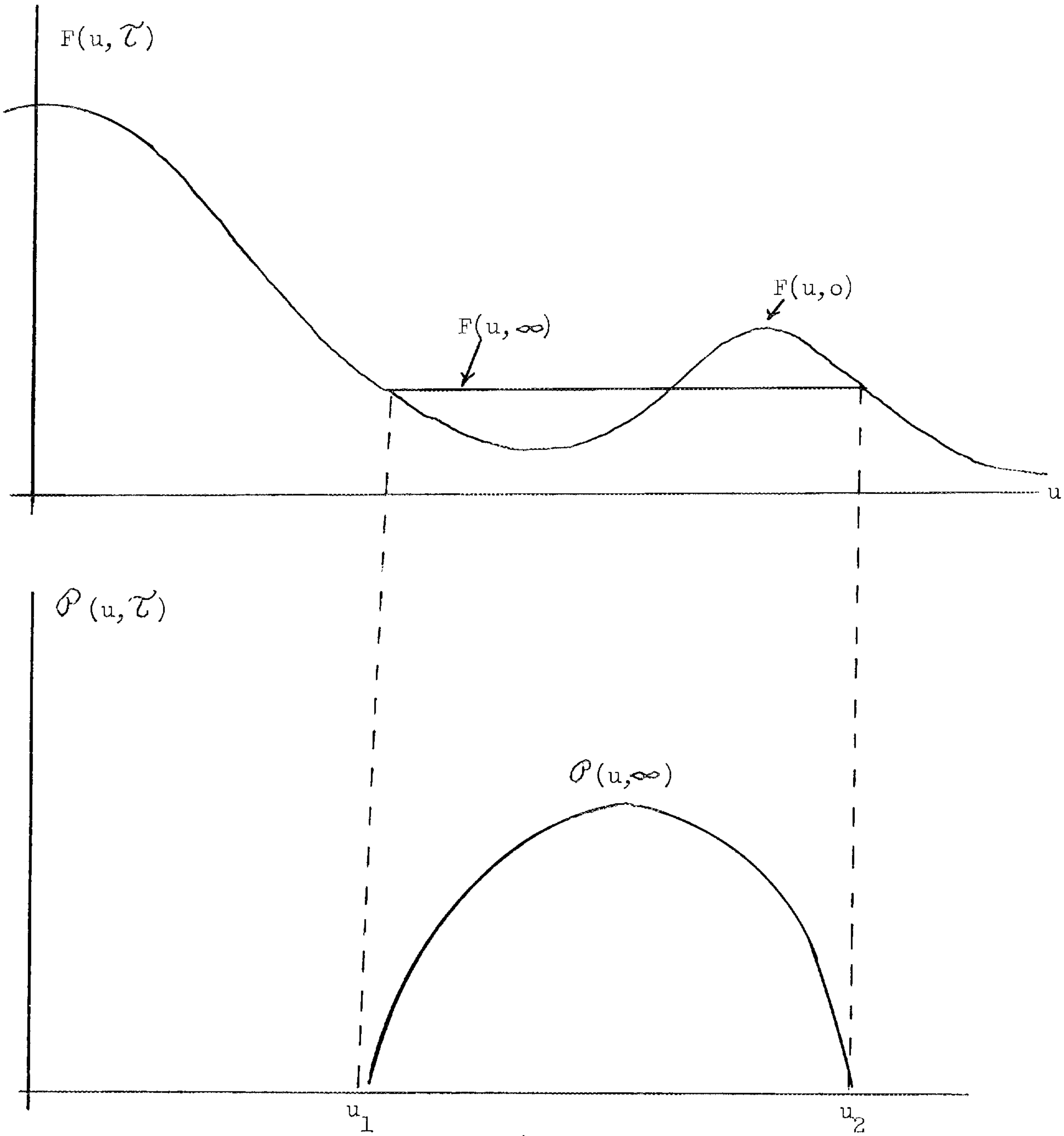


Fig. 4.1 Initial and final F(u, τ) and $\mathcal{P}$(u, τ) for an unstable plasma.

We shall assume that at τ = 0 $\mathcal{P}$ (u,o) is very small. Where F(u,o) has a negative slope, γ(u,o) will be negative and $\mathcal{P}$ (u, τ) will decay. Where F(u,o) has a positive slope, γ(u,o) will be positive and there will be initially an exponential growth of $\mathcal{P}$ (u, τ). This will cause $D(u,\tau)$ to grow rapidly for those values of u for which $\gamma(u,\tau)$ is positive. This will lead to a rapid change of $F(u,\tau)$ for those values of u.

From Eq. (4.51) we see that when the steady state is achieved, $\gamma(u,\infty) = 0$ except when $\mathcal{P}(u,\infty) = 0$. Also $\gamma = 0$ implies $\partial F/\partial u = 0$. It follows that at $\tau = \infty$, $F(u,\infty)$ will consist of constant sections and sections with negative slope for $u > 0$ and positive slope for $u < 0$. $\mathcal{P}(u,\infty)$ will be non-vanishing where $\partial F(u,\infty)/\partial u = 0$ and it will vanish elsewhere. The final form of $F(u,\infty)$ and $\mathcal{P}(u,\infty)$ must be as shown in Fig. 4.1.

The position of the horizontal line in Fig. 4.1 can be determined as follows. Since the number of particles is conserved, the area under the horizontal line must be the same as the area under the original curve; that is

$$\int_{u_1}^{u_2} F(u,\infty)\, du = \int_{u_1}^{u_2} F(u,o)\, du \tag{4.55}$$

If one imagines lowering a horizontal line until the two areas are equal, it is clear that u_1 and u_2 will be determined in this manner. One can then calculate $\mathcal{P}(u,\infty)$ from Eq. (4.54) obtaining

$$\mathcal{P}(u,\infty) = \frac{2\pi}{\mathcal{E}}\; u^3 \int_{u_1}^{u} du' \left[F(u',\infty) - F(u,o)\right] \tag{4.56}$$

Note that $\mathcal{P}(u,\infty)$ is of order $\mathcal{E}^{-1}$.

One cannot say as much about the asymptotic solution when the spontaneous emission terms are retained, but one can still say something. Eq. (4.54) is still valid but $F(u,\infty)$ cannot be determined so easily. If Eqs. (4.44) and (4.45) have stationary solutions at $\tau = \infty$, then

$$\mathcal{P}(u,\infty) = -\frac{uF(u,\infty)}{\frac{\partial F(u,\infty)}{\partial u}} \tag{4.57}$$

Now, $\mathcal{P}(u,\infty)$ must be positive (since it is an energy density) and finite (since energy is conserved), so it follows that $F(u,\infty)$ must be a monotonically decreasing function. The general picture one forms of the development of an instability due to a bump-on-the-tail type of distribution is that $F(u,\tau)$ and $\mathcal{P}(u,\tau)$ rapidly change approximately into the forms shown in Fig. 4.1 under the influence of the stimulated emission terms. When these forms are approached, the spontaneous emission terms become important and $F(u,\tau)$ evolves into a form which decreases monotonically with increasing $|u|$. This is accompanied by a spreading of $\mathcal{P}(u,\tau)$.

We shall next use the quasi-linear equations to discuss the absorption of the energy of a wave packet by a stable plasma. We refer to Fig. 4.2. We suppose that initially there is a wave packet with $\mathcal{P}(u,o)$ non-vanishing in the range $u_1 < u < u_2$. In this range $\partial F/\partial u < 0$, so $\gamma(u)$ is negative and $\mathcal{P}(u,\tau)$ will decay. The diffusion caused by $\mathcal{P}(u,\tau)$ will cause $F(u,\tau)$ to flatten out thus reducing $\gamma(u)$. If the amplitude of the packet is sufficiently large and if spontaneous emission terms are neglected, then $F(u,\tau)$ may develop a plateau and $\gamma(u)$ would become zero in the plateau. If the amplitude of the packet was insufficiently large, the wave packet would disappear before the plateau developed. The spontaneous emission terms should keep the plateau from ever completely developing.

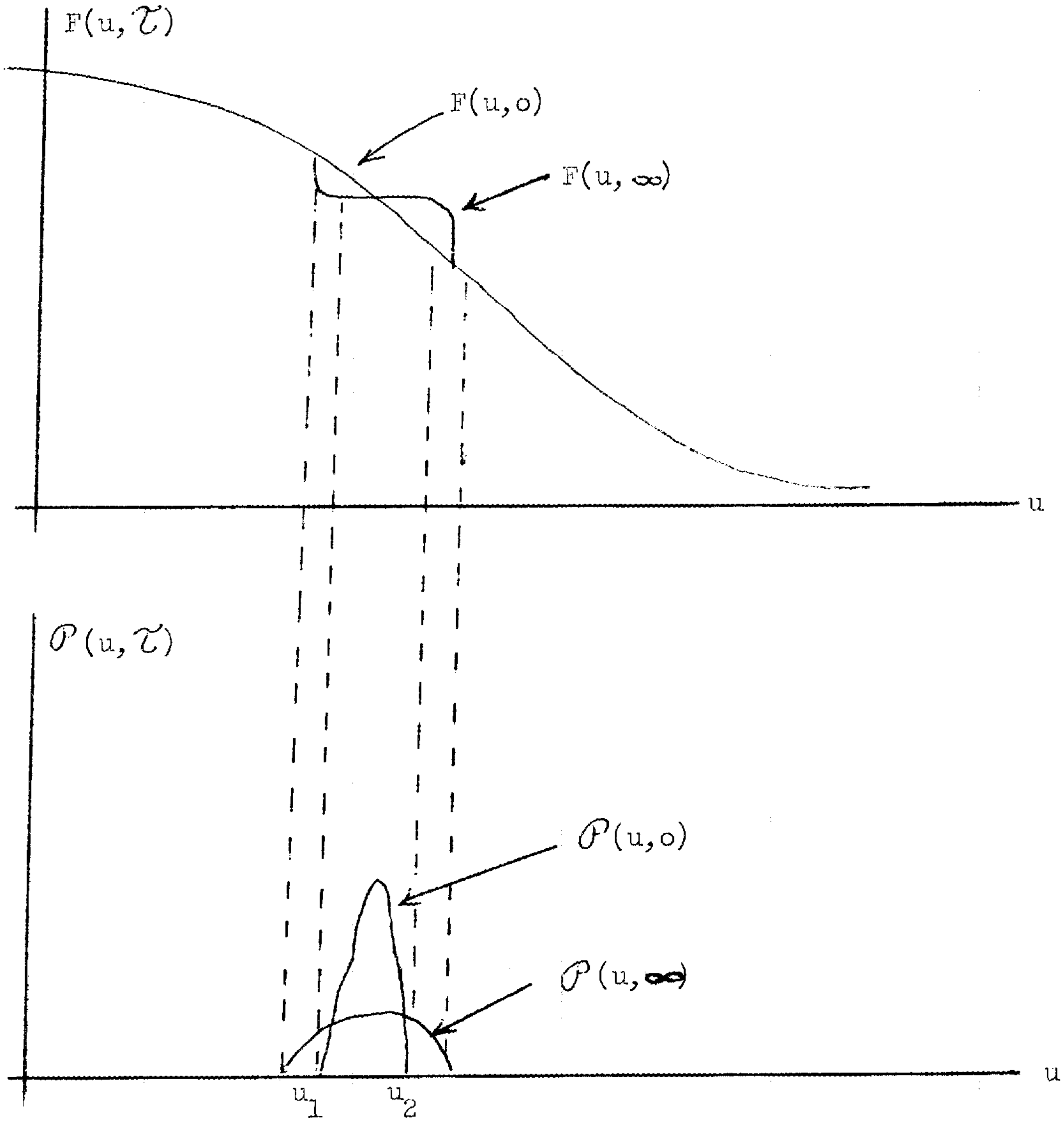


Fig. 4.2 Absorption of the energy of a wave packet by a stable plasma.

If a monochromatic wave is launched in the plasma, then the quasi-linear equations would not be applicable. They are based on the Fermi Golden Rule; the δ-function in Eq. (3.28) implies that one has a continuous spectrum to integrate over. What happens to a monochromatic wave is very interesting. Its main features can be understood by a simple physical argument. Suppose that by some magical process a wave with phase velocity v_p is imposed on the plasma at the time $t = 0$ without altering the electron distribution $f(v)$. (We are considering a one-dimensional plasma.) This is shown in Fig. 4.3.

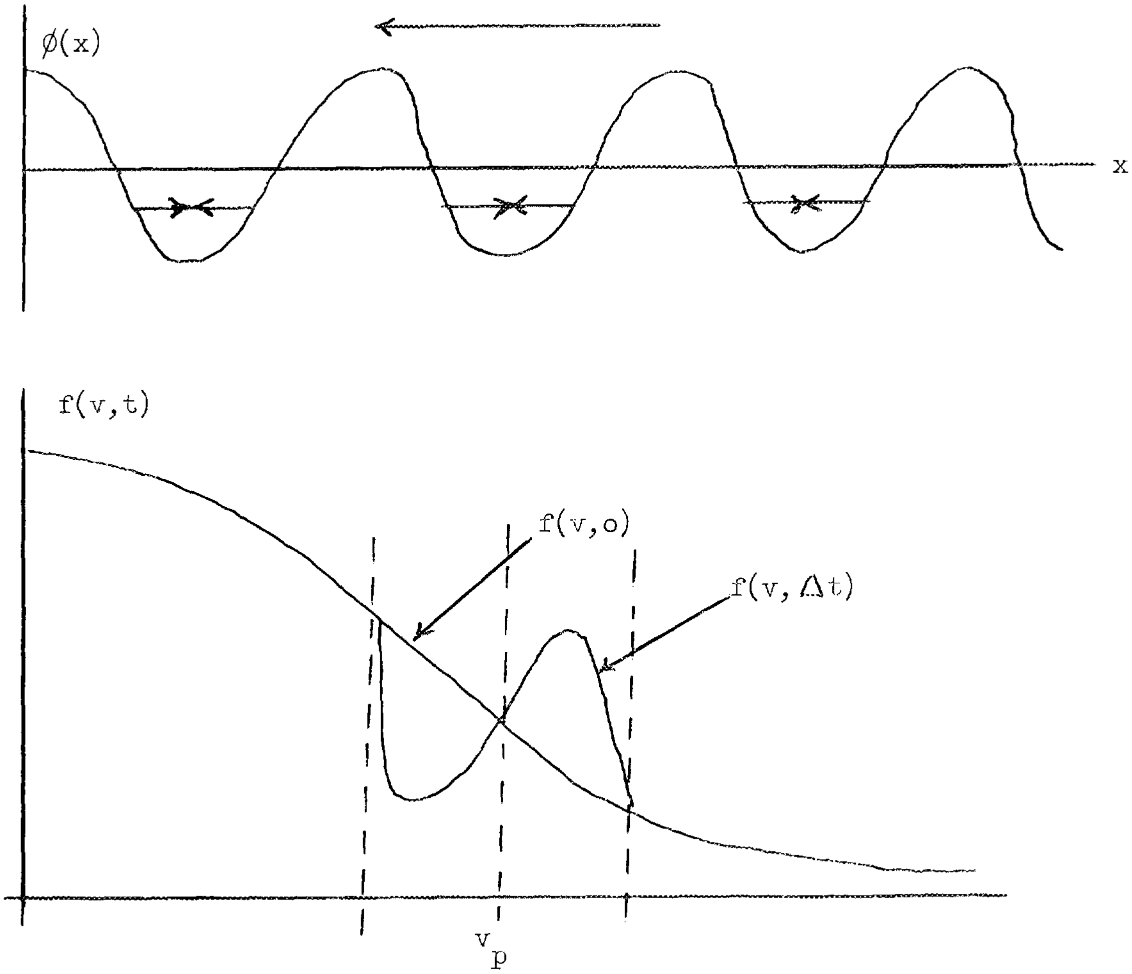


Fig. 4.3 Oscillatory damping of a monochromatic wave.

Consider the motion of the particles in a frame of reference moving with the phase velocity of the wave. Some of the particles will be trapped in the troughs of the waves; others with greater energy will not be trapped. After a time Δt equal to a half period of oscillation of the particles in the bottom of the troughs, the particles which were moving to the left will be moving to the right and those that were moving to the right will be moving to the left so the distribution function must look like the curve labeled $f(v, \Delta t)$. Since the kinetic energy of the particles whose distribution function is $f(v, \Delta t)$ is greater than it was when the distribution function was $f(v,o)$, the energy of the wave has decreased. The wave has been damped during this time interval. In the next interval Δt the velocities of trapped particles will again reverse, $f(v, 2\Delta t)$ is approximately the same as $f(v,o)$, the kinetic energy has decreased and so the amplitude of the wave has grown. Now, all of the trapped particles do not oscillate with the same period so as time goes on $f(v,t)$ develops more wiggles and eventually flattens out when the trapped particles have become randomized. While this is going on the amplitude of the waves alternately decays and grows, finally settling down to some smaller amplitude. This oscillatory Landau damping was predicted theoretically by O'Neil[27] and by Al'Tschul and Karpman[28] and has been observed experimentally by Malmberg and Wharton.[26] It has also been observed in numerical experiments by Armstrong.[37]

4.2 Quasi-Linear Theory of Longitudinal Waves in a Plasma in a Uniform Magnetic Field

Much of the arguments which led to Eqs. (4.17) and (4.19) can be taken over intact. The wave vector $\vec{p}$ for the particles must be replaced by the quantum numbers n, p_x, p_z (see Eq. (2.52)). The matrix element which is necessary for Eq. (3.23) has been given in Eq. (2.58).

Writing out the equation corresponding to the schematic Eq. (4.16) we find

$$\frac{\partial N_\sigma(\vec{k})}{\partial t} = \sum_s \sum_{n,p_x,p_z} \sum_{\ell=-\infty}^{+\infty} \frac{2\pi}{\hbar} \left[\frac{4\pi e_s^2\, h\Omega_{k\sigma}}{V\, k^2 \left|\frac{\partial}{\partial\omega}\, \omega\,\epsilon_1\right|_{\Omega_{k\sigma}}}\right]$$

$$J_\ell^2\left(\frac{k_\perp v_\perp}{\omega_{cs}}\right)\delta\left[\hbar\,\omega_{cs}(n+\ell+1/2) + \frac{\hbar^2}{2m_s}(p_z+k_z)^2\right.$$

$$\left.-\hbar\,\omega_{cs}(n+1/2) - \frac{\hbar^2}{2m_s}p_z^2 - \hbar\Omega_{k\sigma}\right]$$

$$\left\{N_s(n+\ell,\ p_x+k_x,\ p_z+k_z)\left[1-N_s(n,\ p_x,\ p_z)\right]\left[N_\sigma(\vec{k})+1\right]\right.$$

$$\left.-\left[1-N_s(n+\ell,\ p_x+k_x,\ p_z+k_z)\right]N_s(n,\ p_x,\ p_z)\,N_\sigma(\vec{k})\right\} \qquad (4.58)$$

We have written Eq. (4.58) without reduction in order to make clear the origin of all the terms. The transitions which are considered are those in which particles of species s and quantum numbers $n+\ell$, p_x+k_x, p_z+k_z emit a quasi-particle of type σ with wave vector $\vec{k}$ and the inverse of this transition. We have not restricted ourselves to a single species of particle or quasi-particle.

In taking the classical limit

$$N_s(n+\ell,\ p_x+k_x,\ p_z+k_z) - N_s(n,\ p_x,\ p_z)$$

$$\rightarrow f_s\left(v_\perp^2 + \frac{2\hbar\,\omega_{cs}}{m_s}\ell,\ v_z+\frac{\hbar}{m_s}k_z,\ y_{cs}-\frac{\hbar\,k_x}{m\,\omega_{cs}}\right)$$

$$- f_s(v_\perp^2,\ v_z,\ y_c)$$

$$\rightarrow \frac{\hbar}{m_s}\left[\frac{\ell\,\omega_{cs}}{v_\perp}\frac{\partial f_s}{\partial v_\perp} + k_z\frac{\partial f_s}{\partial v_z} - \frac{k_x}{\omega_{cs}}\frac{\partial f_s}{\partial y_c}\right] \qquad (4.59)$$

Here y_{cs} is the y-coordinate of the guiding center of the gyrating particle. It is related to the x-component of the particles momentum by Eq. (2.53). In the classical limit Eq. (4.58) becomes

$$\frac{\partial P_\sigma(\vec{k})}{\partial t} = 2\gamma_\sigma(\vec{k})\,P_\sigma(\vec{k}) + S_\sigma(\vec{k}) \qquad (4.60)$$

where

$$\gamma_\sigma(\vec{k}) = \sum_s \sum_{\ell=-\infty}^{+\infty} \frac{4\pi e_s^2 \Omega_{\sigma k}}{m_s k^2 \left| \frac{\partial}{\partial \omega} \omega \epsilon_1 \right|_{\Omega_{\sigma k}}}$$

$$\int d^3v \, J_\ell^2 \left(\frac{k_\perp v_\perp}{\omega_{cs}}\right) \delta(\Omega_{\sigma k} - \ell \omega_{cs} - k_z v_z)$$

$$\left[\frac{\ell \omega_{cs}}{v_\perp} \frac{\partial}{\partial v_\perp} + k_z \frac{\partial}{\partial v_z} - \frac{k_x}{\omega_{cs}} \frac{\partial}{\partial y_c}\right] f_s(v_\perp, v_z, y_c) \tag{4.61}$$

$$S_\sigma(\vec{k}) = \sum_s \frac{4\pi e_s^2 \Omega_{\sigma k}}{m_s k^2 \left| \frac{\partial}{\partial \omega} \omega \epsilon_1 \right|_{\Omega_{\sigma k}}} \sum_{\ell=-\infty}^{+\infty}$$

$$\int d^3v \, J_\ell^2 \left(\frac{k_\perp v_\perp}{\omega_{cs}}\right) \delta(\Omega_{\sigma k} - \ell \omega_{cs} - k_z v_z) \tag{4.62}$$

Equations similar to Eqs. (4.18) and (4.19) may be written for the rate of change of $N_s(n,p_x,p_z)$. We shall refer the reader to Walters et al.[15] for the details and give the results in the classical limit

$$\frac{\partial f_s}{\partial t}(v_\perp, v_z, y_c) = \sum_\sigma \sum_{\ell=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{8\pi^2 e_s^2}{k^2 \left[\frac{\partial}{\partial \omega} \omega \epsilon_{1L}\right]_{\Omega_{\sigma k}}}$$

$$\left[\frac{\ell \omega_{cs}}{v_\perp} \frac{\partial}{\partial v_\perp} + k_z \frac{\partial}{\partial v_z} - \frac{k_x}{\omega_{cs}} \frac{\partial}{\partial y_{cs}}\right] J_\ell^2 \left(\frac{k_\perp v_\perp}{\omega_{cs}}\right) \delta(\Omega_{\sigma k}$$

$$- \ell \omega_{cs} - k_z v_z) \left\{ \frac{1}{m_s} P_\sigma(\vec{k}) \left[\frac{\ell \omega_{cs}}{v_\perp} \frac{\partial}{\partial v_\perp} + k_z \frac{\partial}{\partial v_z}\right.\right.$$

$$\left.\left. - \frac{k_x}{\omega_{cs}} \frac{\partial}{\partial y_{cs}}\right] f_s(\vec{v}, y_c) + \Omega_{\sigma k} f_s(\vec{v}, y_c) \right\} \tag{4.63}$$

Eq. (4.61) agrees with the growth rate calculated from Eq. (2.110). It should be noted that the dependence of f_s on y_c gives the possibility of wave growth due to spatial gradients of guiding centers. This is the drift cyclotron instability of Mikhailovski and Timofeev.[38] It may be seen from Eq. (4.63) that this is accompanied by a diffusion of particles in the y-direction.

CHAPTER 5. SCATTERING OF PARTICLES AND PHOTONS BY PLASMAS

The scattering of particles and the scattering of light may be treated in much the same way. It is convenient to begin with the scattering of particles.

5.1 The Scattering of Particles[39]

We consider a test particle of charge e_r and mass m_r interacting with a plasma. The interaction Hamiltonian may be written as

$$H' = \sum_s \sum_{i=1}^{N_s} \frac{e_r e_s}{|\vec{x} - \vec{x}_{si}|} = e_r \phi(\vec{x}) \tag{5.1}$$

where $\vec{x}$ is the position of the test particle, $\vec{x}_{si}$ is the position of the i^{th} particle of species s in the plasma and $\phi(\vec{x})$ is the electrostatic potential at the position of the test particle. Only coulomb interactions are taken into account. As in Chapter 2 we expand the potential in a Fourier series in a box of volume V. Thus,

$$\phi(\vec{x}) = \sum_{\vec{q}} \phi(\vec{q})\; e^{i\vec{q}\cdot\vec{x}} \tag{5.2}$$

$$\begin{aligned}\phi(\vec{q}) &= \int \frac{d^3x}{V}\; e^{-i\vec{q}\cdot\vec{x}}\; \phi(\vec{x}) \\ &= \sum_s \sum_i \frac{4\pi e_s}{Vq^2}\; e^{-i\vec{q}\cdot\vec{x}_{si}}\end{aligned} \tag{5.3}$$

Eq. (5.1) may be written

$$H' = e_r \sum_{\vec{q}} \phi(\vec{q})\; e^{i\vec{q}\cdot\vec{x}} \tag{5.4}$$

We wish to calculate $W_r(a \to a')$, the transition probability per unit time that a test particle of species r initially in the state $|a>$ makes a transition to the state $|a'>$. We will denote initial and final states of the plasma by $|\alpha>$ and $|\alpha'>$. We calculate $W_r(a \to a')$ by summing

$$\frac{2\pi}{\hbar} \left| <\alpha'|< a'|H'|a>|\alpha> \right|^2 \delta[E_a - E_{a'} - \mathcal{E}_\alpha - \mathcal{E}_{\alpha'}]$$

over final states of the plasma and averaging over initial states. In the above E_a and $\mathcal{E}_\alpha$ denote energies of the test particle and the plasma respectively. We will let P_α be the probability that the initial state of the plasma is $|\alpha>$. We define ω by

$$\hbar\omega = E_a - E_{a'} \tag{5.5}$$

Then

$$W_r(a \to a') = \sum_\alpha P_\alpha \sum_{\alpha'} \frac{2\pi}{\hbar^2} \delta(\omega - \frac{\mathcal{E}_{\alpha'} - \mathcal{E}_\alpha}{\hbar})$$
$$<\alpha'|< a'|H'|a>|\alpha> <\alpha'|< a'|H'|a>|\alpha>^* \tag{5.6}$$

We now use

$$\delta(\omega - \frac{\mathcal{E}_{\alpha'} - \mathcal{E}_\alpha}{\hbar}) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i(\omega - \frac{\mathcal{E}_{\alpha'} - \mathcal{E}_\alpha}{\hbar})t} \tag{5.7}$$

and Eq. (5.3) to write

$$W_r(a \to a') = \frac{e_r^2}{\hbar^2} \sum_{\vec{q}'} \sum_{\vec{q}} \int_{-\infty}^{+\infty} dt\, e^{i\omega t}$$
$$< a'|e^{i\vec{q}\cdot\vec{x}}|a> < a'|e^{i\vec{q}'\cdot\vec{x}}|a>^*$$
$$\sum_\alpha \sum_{\alpha'} P_\alpha <\alpha|e^{i/\hbar\, \mathcal{E}_\alpha t}\, \phi^*(\vec{q}')\, e^{-i/\hbar\, \mathcal{E}_{\alpha'} t}|\alpha'>$$
$$<\alpha'|\phi(\vec{q})|\alpha>$$
$$= \frac{e_r^2}{\hbar^2} \sum_{\vec{q}'} \sum_{\vec{q}} \int dt\, e^{i\omega t}$$
$$< a'|e^{i\vec{q}\cdot\vec{x}}|a> < a'|e^{i\vec{q}'\cdot\vec{x}}|a>^* < \phi^*(\vec{q}',t)\ \phi(\vec{q},o) > \tag{5.8}$$

where $\phi^*(\vec{q}',t)$ is the time dependent operator with matrix elements

$$< \alpha | e^{i/\hbar \mathcal{E}_\alpha t} \, \phi^*(\vec{q}\,') \, e^{-i/\hbar \mathcal{E}_{\alpha'} t} | \alpha' >$$

and the angular brackets in the last factor are defined by

$$< A > = \sum_\alpha P_\alpha < \alpha | A | \alpha > \tag{5.9}$$

(The same average was used in Eq. (2.20).)

We now specialize to the case that $| a >$ and $| a' >$ are plane wave states of a free particle; that is

$$| a > = | \vec{k} >$$

$$< \vec{x} | \vec{k} > = \chi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \tag{5.10}$$

Then

$$< a' | e^{i\vec{q}\cdot\vec{x}} | a > = \delta_{\vec{k}', \vec{k} + \vec{q}} \tag{5.11}$$

and Eq. (5.8) becomes

$$W_r(\vec{k} \to \vec{k}') = \frac{e_r^2}{\hbar^2} \int_{-\infty}^{+\infty} dt \; e^{i\omega t} < \phi^*(\vec{k}' - \vec{k}, t) \; \phi(\vec{k}' - \vec{k}, 0) > \tag{5.12}$$

Now we will let

$$\hbar \vec{k} = m_r \vec{v} + \hbar \vec{q} \tag{5.13}$$

$$\hbar \vec{k}' = m_r \vec{v} \tag{5.14}$$

and use $\phi^*(-\vec{q}, t) = \phi(\vec{q}, t)$ (which follows from the reality of $\phi(\vec{x}, t)$) to obtain

$$W_r(\vec{v} + \frac{\hbar \vec{q}}{m_r} \to \vec{v}) = \frac{e_r^2}{\hbar^2} \int_{-\infty}^{+\infty} dt \; e^{i\omega t} < \phi(\vec{q}, t) \; \phi^*(\vec{q}, 0) > \tag{5.15}$$

for the transition probability for unit time that the indicated transition will occur with transfer of momentum $\hbar \vec{q}$ and transfer of energy $\hbar\omega$. We may use

$$\vec{E}(\vec{q}, t) = -i \, \vec{q} \, \phi(\vec{q}, t) \tag{5.16}$$

To write this in the form

$$W_r(\vec{v} + \frac{\hbar \vec{q}}{m_r} \rightarrow \vec{v}) = \frac{8\pi e_r^2}{\hbar^2 q^2} \int_{-\infty}^{+\infty} dt \, e^{i\omega t} < \frac{\vec{E}(\vec{q},t) \cdot \vec{E}^*(\vec{q},o)}{8\pi} > \tag{5.17}$$

$$= \frac{8\pi e_r^2}{\hbar^2 q^2} P_E(\vec{q},\omega) \tag{5.18}$$

where

$$P_E(\vec{q},\omega) = \int_{-\infty}^{+\infty} dt \, e^{i\omega t} < \frac{\vec{E}(\vec{q},t) \cdot \vec{E}(\vec{q},0)}{8\pi} > \tag{5.19}$$

is the spectral density of the electric field fluctuations. Note that

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} P_E(\vec{q},\omega) = < \frac{\vec{E}(\vec{q},0) \cdot \vec{F}^*(\vec{q},0)}{8\pi} > \tag{5.20}$$

so

$$V \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi} P_E(\vec{q},\omega)$$

may be interpreted as the energy per unit volume in electric field fluctuations.

In a plasma in which the field fluctuations are in equilibrium with the particles, $P_E(\vec{q},\omega)$ may be calculated by the dressed test particle model[29,30] which we shall now briefly describe.

In such an equilibrium plasma we expect

$$< \phi(\vec{q},t) \; \phi^*(q,t') > = < \phi(q,t - t') \; \phi(\vec{q},0) > \tag{5.21}$$

Taking the Fourier transform with respect to both t and t' gives

$$< \phi(\vec{q},\omega) \; \phi^*(\vec{q},\omega') > = \int_{-\infty}^{+\infty} dt \, e^{i\omega t} \int_{-\infty}^{+\infty} dt' \, e^{-i\omega t'}$$

$$< \phi(\vec{q},t) \; \phi^*(\vec{q},t')$$

$$= 2\pi \delta(\omega - \omega') \int_{-\infty}^{+\infty} d(t - t') \, e^{i\omega(t - t')}$$

$$< \phi(\vec{q}, t - t') \ \phi^*(q, 0) >$$

$$= \frac{16\pi^2}{q^2} \ \delta(\omega - \omega') \ P_E(\vec{q}, \omega) \tag{5.22}$$

By Poisson's equation

$$\phi(\vec{q}, \omega) = - \sum_s \frac{4\pi e_s}{q^2} n_s(\vec{q}, \omega) \tag{5.23}$$

where $n_s(\vec{q}, \omega)$ is the Fourier transform of the particle density. Then

$$< \phi(\vec{q}, \omega) \ \phi^*(\vec{q}, \omega') > = \sum_r \sum_s \frac{(4\pi)^2 e_s e_r}{q^4} \tag{5.24}$$

$$< n_r(\vec{q}, \omega) \ n_s^*(\vec{q}, \omega') >$$

Now, the essence of the dressed test particle model is that the particles may be regarded as uncorrelated, but that the contribution of each particle to the potential must be modified by the shielding factor $\epsilon^{-1}(q, \omega)$. That is

$$< \phi(q, \omega) \ \phi^*(q, \omega') > = \sum_r \sum_s \frac{(4\pi)^2 e_s e_r}{q^4 \ \epsilon(\vec{q}, \omega) \ \epsilon^*(\vec{q}, \omega')}$$

$$< n_r(\vec{q}, \omega) \ n_s^*(\vec{q}, \omega') >_o \tag{5.25}$$

where the subscript on $< \ldots >_o$ indicates that the average is to be calculated for uncorrelated (free) particles. This is easily calculated using the second quantization formalism introduced in Chapter 2. The density operator is

$$n_s(\vec{x}, t) = \Psi_s^+(\vec{x}, t) \ \Psi_s(x, t)$$

$$= \frac{1}{V} \sum_{\vec{q}} \sum_{\vec{p}} C^+_{s\vec{p}}(t) \ C_{s\vec{p} + \vec{q}}(t) \ e^{i\vec{q}\cdot\vec{x}} \tag{5.26}$$

For free particles the same time dependence of $C_{s\vec{p}}(t)$ is given. By

$$C_{sp}(t) = C_{sp}(o)\, e^{-\frac{i\hbar}{2m_s} p^2 t} \tag{5.27}$$

so

$$n_s(\vec{x},t) = \frac{1}{V} \sum_{\vec{q}} \sum_{\vec{p}} C^+_{sp}(o)\, C_{s\vec{p}+\vec{q}}(o)\, e^{i\vec{q}\cdot\vec{x} - i\nu_s t} \tag{5.28}$$

where

$$\nu_s(\vec{p}, \vec{q}) = \frac{\hbar}{2m} \left[|\vec{p} + \vec{q}|^2 - p^2 \right] \tag{5.29}$$

Taking the Fourier transform of Eq. (5.28) gives

$$n_s(\vec{q},\omega) = \frac{2\pi}{V} \sum_{\vec{p}} C^+_{s\vec{p}}(o)\, C_{s\vec{p}+\vec{q}}(o)\, \delta(\omega - \nu_s(\vec{p},\vec{q})) \tag{5.30}$$

Then

$$< n_r(\vec{q},\omega)\; n_s{*}(\vec{q},\omega') > = \frac{(2\pi)^2}{V^2} \sum_{\vec{p}} \sum_{\vec{p}\,'}$$

$$\delta(\omega - \nu_r(\vec{p},\vec{q}))\, \delta(\omega' - \nu_s(\vec{p}\,',\vec{q}))$$
$$< C^+_{r\vec{p}}(o)\, C_{r\vec{p}+\vec{q}}(o)\, C^+_{s\vec{p}\,'+\vec{q}}(o)\, C_{s\vec{p}\,'}(o) >_o \tag{5.31}$$

The last factor in Eq. (5.31) is just

$$\delta_{rs}\, \delta_{\vec{p}\,\vec{p}\,'} \sum_{\alpha} P_\alpha < \alpha | C^+_{s\vec{p}}\, C_{s\vec{p}}\, C_{s\vec{p}+\vec{q}}\, C^+_{s\vec{p}+\vec{q}} | \alpha >$$

$$= \delta_{rs}\, \delta_{pp'} \sum_{\alpha} P_\alpha\, N_s^{(\alpha)}(\vec{p}) \left[1 - N_s^{(\alpha)}(\vec{p} + \vec{q})\right] \tag{5.32}$$

where $N_s^{(\alpha)}(\vec{p})$ is the number of particles of species s with momentum $\hbar\vec{p}$ when the state of the plasma is $|\alpha>$. For a classical plasma we may set $\left[1 - N_s^{(\alpha)}(\vec{p} + \vec{q})\right] \simeq 1$ and write

$$< N_s(\vec{p}) > = \sum_{\alpha} P_\alpha\, N_s^{(\alpha)}(\vec{p}) \tag{5.33}$$

Eq. (5.31) now becomes

$$< n_r(q,\omega)\; n_s{*}(q,\omega') >_o = \delta_{rs} \frac{(2\pi)^2}{V^2} \sum_{\vec{p}}$$

$$\delta(\omega-\omega')\,\delta(\omega-\nu_s(\vec{p},\vec{q}))\,\langle N_s(\vec{p})\rangle$$

$$=\delta_{rs}\,\delta(\omega-\omega')\,\frac{(2\pi)^2}{V}\int d^3v\; f_s(\vec{v})\,\delta\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_s}\right) \tag{5.34}$$

In the last step we have applied the prescription of Eq. (4.29).

Eq. (5.25) becomes

$$\langle \phi(\vec{q},\omega)\;\phi^*(\vec{q},\omega')\rangle = \frac{(2\pi)^2}{V}\,\delta(\omega-\omega')\sum_s \frac{(4\pi)^2 e_s^2}{q^4\left|\epsilon(\vec{q},\omega)\right|^2}\int d^3v\; f_s(\vec{v})\,\delta\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_s}\right) \tag{5.35}$$

Comparison with Eq. (5.22) gives the result

$$P_E(\vec{q},\omega) = \sum_s \frac{4\pi^2 e_s^2}{q^2 V\left|\epsilon\;(\vec{q},\omega)\right|^2}\int d^3v\; f_s(\vec{v})\,\delta\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_s}\right) \tag{5.36}$$

Putting this result into Eq. (5.18) we find that the result can be put into the form

$$W_r\left(\vec{v}+\frac{\hbar\vec{q}}{m_r}\rightarrow\vec{v}\right) = \sum_s V\int d^3v'\; f_s(\vec{v}')$$

$$\frac{2\pi}{\hbar}\left|\frac{4\pi e_s e_r}{V\,q^2\,\epsilon(\vec{q},\omega)}\right|^2$$

$$\delta\left[\frac{m_r}{2}\left|\vec{v}+\frac{\hbar\vec{q}}{m_r}\right|^2+\frac{m_s}{2}v'^2-\frac{m_r}{2}v^2-\frac{m_s}{2}\left|v'+\frac{\hbar q}{m_s}\right|^2\right] \tag{5.37}$$

In this form the transition probability has an obvious interpretation. $W_r(\vec{v}+\hbar\,\vec{q}/m_r\rightarrow\vec{v})$ is obtained by adding the transition probabilities calculated by first order perturbation theory for the collisions of the test particle with the particles of the plasma. Such collisions may be represented by the diagram of Fig. 5.1.

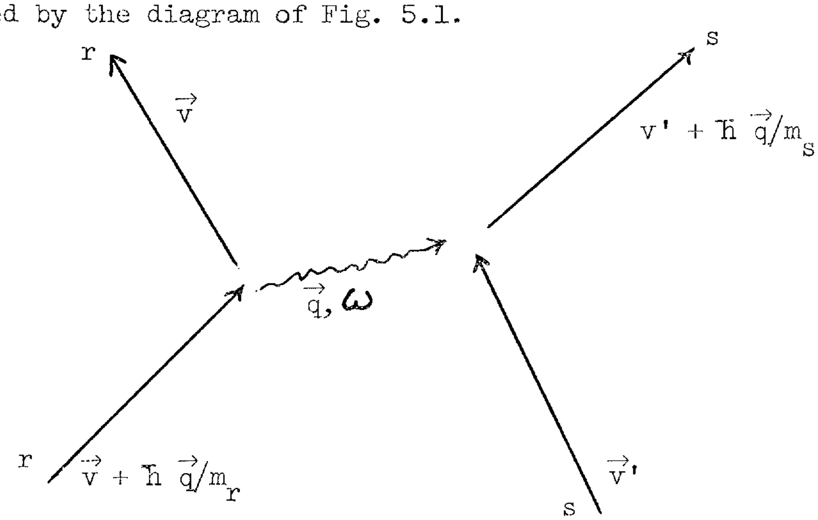


Fig. 5.1 Particle collision.

In each collision the matrix element for the transition is the shielded coulomb matrix element

$$\frac{4\pi e_r e_s}{V\, q^2\, \epsilon(\vec{q},\omega)} \tag{5.38}$$

where $\hbar\,\vec{q}$ is the momentum transfer and $\hbar\omega$ is the energy transfer.

We shall next examine $P_E(\vec{q},\omega)$ in somewhat greater detail. It is useful to multiply by the correction factor $\partial\omega\epsilon_1/\partial\omega$ and interpret

$$P(\vec{q},\omega) = \left(\frac{\partial}{\partial\omega}\ \omega\,\epsilon_1(\vec{q},\omega)\right) P_E(\vec{q},\omega) \tag{5.39}$$

as the spectral density of total energy including both electric field energy and kinetic energy. $P(\vec{q},\omega)$ will have peaks near the frequencies of electrostatic oscillations since $\epsilon(\vec{q},\omega) \simeq 0$. We shall use Eq. (4.4) for $\epsilon_2(\vec{q},\omega)$ to write

$$P(\vec{q},\omega) = \frac{\hbar}{V}\ \frac{|\epsilon_2(q,\omega)|}{|\epsilon(\vec{q},\omega)|^2}\ \left[\frac{\partial}{\partial\omega}\ \omega\,\epsilon_1(\vec{q},\omega)\right] R(\vec{q},\omega) \tag{5.40}$$

where

$$R(\vec{q},\omega) = \frac{\sum_s e_s^2 \int d^3v\ f_s(\vec{v})\ \delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}\right)}{\left|\sum_s e_s^2 \int d^3v \left[f_s\left(\vec{v} + \frac{\hbar\vec{q}}{m_s}\right) - f_s(\vec{v})\right] \delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}\right)\right|} \tag{5.41}$$

In the $\epsilon_2 \to 0$ limit (that is, the limit of small damping) we have

$$\frac{|\epsilon_2|}{\epsilon_1^2 + \epsilon_2^2} \xrightarrow[\epsilon_2 \to 0]{} \pi\,\delta\left(\epsilon_1(\vec{q},\omega)\right)$$

$$\to \pi \sum_{\Omega_{\vec{q}}} \frac{\delta(\omega - \Omega_{\vec{q}})}{\left|\frac{\partial}{\partial\omega}\epsilon_1(q,\omega)\right|} \tag{5.42}$$

where the $\Omega_{\vec{q}}$'s are the solutions of $\epsilon_1(\vec{q},\omega) = 0$. Now, from an examination of the symmetry of $\epsilon_1(q,\omega)$ as given by Eq. (4.3) it may be shown that if $\Omega_{\vec{q}}$ satisfies $\epsilon_1(\vec{q},\Omega_{\vec{q}}) = 0$ then $-\Omega_{-\vec{q}}$ is also a solution. For simplicity let us consider a single species plasma and suppose that there is only one weakly damped mode. Then using Eq. (5.42) in Eq. (5.40) gives

$$P(\vec{q},\omega) = \frac{\pi}{V}\Big\{\delta(\omega - \Omega_{\vec{q}})\ S_{\vec{q}}\hbar\Omega_{\vec{q}}\, R(\vec{q},\Omega_{\vec{q}})$$
$$+\ \delta(\omega + \Omega_{-\vec{q}})\ S_{-\vec{q}}\hbar\Omega_{-\vec{q}}\, R(\vec{q}, -\Omega_{-\vec{q}})\Big\} \tag{5.43}$$

where $S_{\vec{q}} = \pm 1$ is the sign of the energy of the wave defined in Eq. (3.8).

The significance of the factors $R(\vec{q},\Omega_{\vec{q}})$ and $R(\vec{q}, -\Omega_{-\vec{q}})$ becomes apparent if one takes $f_s(\vec{v})$ to be the Maxwell-Boltzmann distribution function. Then Eq. (5.41) gives

$$R(\vec{q},\Omega_{\vec{q}}) = \frac{1}{\left|e^{-\hbar\Omega_{\vec{q}}/T} - 1\right|} = \frac{1}{e^{+\hbar\Omega_{\vec{q}}/T} - 1} + 1 = N(\vec{q}) + 1 \tag{5.44}$$

where $N(\vec{q})$ is the Planck distribution, Eq. (4.28). Similarly

$$R(\vec{q}, -\Omega_{-\vec{q}}) = N(-\vec{q}) \tag{5.45}$$

and Eq. (5.43) becomes

$$P(\vec{q},\omega) = \frac{\pi}{V} \left\{ \delta\,(\omega - \Omega_{\vec{q}})\, S_{\vec{q}}\, \hbar\Omega_{\vec{q}} \left[N(\vec{q}) + 1\right] + \delta\,(\omega + \Omega_{-\vec{q}})\, S_{-\vec{q}}\, \hbar\Omega_{-\vec{q}}\, N(-\vec{q}) \right\} \tag{5.46}$$

When Eq. (5.46) is used in the expression for $W_r(\vec{v} + \hbar\vec{q}/m_s \to \vec{v})$ one finds one term proportional to $N(\vec{q}) + 1$ which may be interpreted as the stimulated plus spontaneous emission of a quasi-particle of momentum $\hbar\,\vec{q}$ and energy $\hbar\Omega_{\vec{q}}$. The test particle may also lose momentum $\hbar\,\vec{q}$ by absorbing a quasi-particle of momentum $-\hbar\,\vec{q}$ in which case the change in its energy is $\hbar\omega = -\hbar\Omega_{-\vec{q}}$. This absorption process contributes the term proportional to $N(-\vec{q})$.

When the state of the plasma departs from thermal equilibrium, the quasi-particle number $N(\vec{q})$ departs from its equilibrium value. In particular if the plasma approaches an unstable state $N(\vec{q})$ may become very large since the denominator in Eq. (5.41) approaches zero.

Returning to Eq. (5.8) we shall remark that a similar analysis can be made when $|\,a>$ and $|\,a'>$ are the states of a particle in a uniform magnetic field. We shall just quote the result. It is

$$W_r(n + \ell,\ k_x + q_x,\ k_z + q_z \to n,\ k_x,\ k_z) = \frac{8\pi e_r^2}{\hbar^2} \sum_{q_y} \frac{1}{q^2}\, J_n^2\left(\frac{q_\perp v_1}{\omega_{cr}}\right) P_E(\vec{q},\omega) \tag{5.47}$$

The summation over q_y in this equation is a consequence of the non-conservation of the y-component of momentum because of the magnetic field.

5.2 The Scattering of Photons[31,39]

The initial and final states of the system are taken to be

$$|i> = |\vec{k} + \vec{q}, \sigma> |\alpha> \tag{5.48}$$

$$|f> = |\vec{k}, \sigma'> |\alpha'> \tag{5.49}$$

Here, $|\vec{k}, \sigma>$ denotes the state of the radiation field when there is one photon of momentum $\hbar \vec{k}$ and polarization direction σ present, $|\alpha>$ is the state of the plasma before the scattering and $|\alpha'>$ is the state after the scattering. The term in the Hamiltonian responsible for the scattering is the A^2 term; namely

$$H' = \sum_s \sum_i \frac{e_s^2}{2m_s c^2} A^2 (\vec{x}_{si}) \tag{5.50}$$

where $\vec{A}(\vec{x})$ is given by Eq. (3.2).

As in the last section, we calculate $W(\vec{k} + \vec{q}, \sigma \to \vec{k}, \sigma')$ by summing

$$\frac{2\pi}{h} |< f | H' | i >|^2 \delta (E_i - E_f) \tag{5.51}$$

over final states and averaging over initial states. The term in A^2 containing $A^+_{\vec{k},\sigma'}$, $A_{\vec{k} + \vec{q}, \sigma}$ are responsible for the transition. In a derivation which parallels that of the last section we obtain

$$W(\vec{k} + \vec{q}, \sigma \to \vec{k}, \sigma') = (2\pi)^3 \frac{(\vec{u}_{\vec{k} + \vec{q}, \sigma} \cdot \vec{u}_{\vec{k} \sigma'})^2}{V^2 \Omega'_{\vec{k}\sigma'} \cdot \Omega'_{\vec{k} + \vec{q}, \sigma}}$$

$$\sum_r \sum_s \frac{e_r^2 e_s^2}{m_r m_s} \int_{-\infty}^{+\infty} dt\, e^{i\omega t} < n_r(\vec{q}, t)\, n_s^*(\vec{q}, o) > \tag{5.52}$$

where we have introduced the abbreviation

$$\Omega'_{\vec{k}\sigma} = \left[\frac{1}{2} \frac{\partial}{\partial \omega} \omega^2 \epsilon_{1T}(\vec{k}\omega)\right] \Omega_{\vec{k}\sigma} \tag{5.53}$$

In Eq. (5.52)

$$n_s(\vec{q},o) = \sum_i e^{-i\vec{q}\cdot\vec{x}_{si}} \tag{5.54}$$

is the Fourier transform of the density operator for species s

$$n_s(\vec{x},o) = \sum_i \delta(\vec{x} - \vec{x}_{si}) \tag{5.55}$$

The time dependent operators are introduced as they were in Eq. (5.8).

For a single species plasma one can use Poisson's equation to write

$$< n(\vec{q},t)\ n^*(\vec{q},o) > = \left(\frac{q^2}{4\pi e}\right)^2 < \phi(\vec{q},t)\ \phi^*(q,o) >$$

$$= \frac{q^2}{(4\pi e)^2} < \vec{E}(\vec{q},t) \cdot \vec{E}^*(\vec{q},o) > \tag{5.56}$$

so

$$\int_{-\infty}^{+\infty} dt\ e^{i\omega t} < n(\vec{q},t)\ n^*(q,o) > = \frac{q^2}{2\pi e^2} P_E(\vec{q},\omega) \tag{5.57}$$

Using Eq. (5.36) one obtains

$$\int_{-\infty}^{+\infty} dt\ e^{i\omega t} < n(\vec{q},t)\ n^*(\vec{q},o) > = \frac{2\pi}{V} \frac{1}{|\epsilon(\vec{q},\omega)|^2}$$

$$\int d^3v\ f(\vec{v})\,\delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m}\right) \tag{5.58}$$

This may be substituted into Eq. (5.52) to obtain

$$W(\vec{k}+\vec{q}, \sigma \rightarrow \vec{k}, \sigma') = \frac{(2\pi)^4(\ {}_{k+q,\sigma} \cdot \vec{u}_{k\sigma'})^2}{V^3\,\Omega'_{\vec{k}\sigma}\,\Omega'_{k+q,\sigma'}\,|\epsilon(\vec{q},\omega)|^2}$$

$$\frac{e^4}{m^2} \int d^3v \; f(\vec{v}) \, \delta \left(\omega - \vec{q} \cdot \vec{v} - \frac{\hbar q^2}{2m} \right) \tag{5.59}$$

Just as in the last section the peaks in W which occur at frequencies for which $\epsilon(\vec{q},\omega) \simeq 0$ may be interpreted in terms of emission and absorption of quasi-particles. As the plasma approaches an unstable state the number of quasi-particles increases and the peaks in W grow because of the increased stimulated emission and absorption.[31]

For an electron-ion plasma the expression for W is somewhat more complicated than Eq. (5.59). It has been discussed elsewhere and will not be given further consideration here.[31]

CHAPTER 6. NON-DIVERGENT KINETIC EQUATIONS

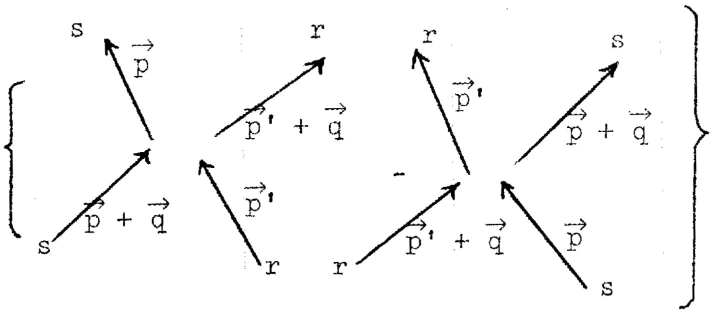
A variety of equations have been proposed for describing the evolution in time of $f(\vec{v})$ the particle velocity distribution function in a plasma. None of them are completely satisfactory. The earliest of these was the Boltzmann equation.[40,41] It is unsatisfactory because it neglects collective effects, and also it diverges for small momentum transfers in particle collisions. On the other hand, the Balescu-Lenard equation[13,14] does include collective effects but is divergent for large momentum transfers. Wyld and Pines[12] have derived an equation which is not divergent for either small or large momentum transfers and reduces to the Boltzmann equation in one limit and the Balescu-Lenard equation in another limit. We review this derivation in section 6.1. If the plasma is unstable the Wyld-Pines and Balescu-Lenard equations are divergent. The quasi-linear equations discussed in Chapter 3 describe the evolution of an unstable plasma but ignore particle collisions.

In recent years a number of equations have been proposed to deal with both stable and unstable plasmas.[42-47] We will add one more set of equations to the list. The equations we derive in section 6.2 have the attractive features that they conserve particles, energy and momentum, satisfy an H-theorem and reduce to Wyld-Pines, Balescu-Lenard, Boltzmann and quasi-linear equations in various limits.

6.1 The Wyld-Pines Equation

Consider the rate of change of $N_s(\vec{p})$, the number of particles of species s and momentum $\hbar\,\vec{p}$, due to collisions with the other particles of the plasma. Schematically we may write

$$\frac{\partial N_s}{\partial t}(\vec{p}) = \sum_{r,\vec{p}\,'}$$



(6.1)

As we did in Chapter 3, we replace each diagram by the corresponding transition probability per unit time which is calculated from first order perturbation theory. The matrix element is taken to be the shielded coulomb matrix element given in Eq. (5.38). We obtain

$$\frac{\partial N_s}{\partial t}(\vec{p}) = \sum_{r,\vec{p}\,',\vec{q}} \frac{2\pi}{\hbar} \left| \frac{4\pi e_r e_s}{V\, q^2\, \epsilon(\vec{q},\omega)} \right|^2$$

$$\delta\left[\frac{\hbar^2}{2m_s}|\vec{p}+\vec{q}|^2 + \frac{\hbar^2}{2m_r}p'^2 - \frac{\hbar^2}{2m_s}p^2 - \frac{\hbar^2}{2m_r}|\vec{p}\,'+\vec{q}|^2\right]$$

$$\Big\{N_s(\vec{p}+\vec{q})\, N_r(\vec{p}\,')\left[1 - N_s(\vec{p})\right]\left[1 - N_r(\vec{p}\,'+\vec{q})\right]$$

$$- N_r(\vec{p}\,'+\vec{q})\, N_s(\vec{p})\left[1 - N_r(\vec{p}\,')\right]\left[1 - N_s(\vec{p}+\vec{q})\right]\Big\} \qquad (6.2)$$

where ω in the argument of $\epsilon(\vec{q},\omega)$ is understood to be given by

$$\hbar\omega = \frac{\hbar^2}{2m_s}\left[|\vec{p}+\vec{q}|^2 - p^2\right] = \frac{\hbar^2}{2m_r}\left[|\vec{p}\,'+\vec{q}| - p'^2\right] \qquad (6.3)$$

Using Eqs. (4.29b), (4.29c), (4.29f) and (4.29g) we obtain

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \sum_r \frac{4e_r^2 e_s^2}{\hbar^2}\int d^3q \int d^3v'$$

$$\frac{\delta\left(\vec{q}\cdot\vec{v} - \vec{q}\cdot\vec{v}\,' + \frac{\hbar}{2m_s}q^2 - \frac{\hbar}{2m_r}q^2\right)}{q^4\left|\epsilon(\vec{q},\ \vec{q}\cdot\vec{v} + \hbar^2 q^2/2m_s)\right|^2}$$

$$\left\{ f_s(\vec{v} + \frac{\hbar \vec{q}}{m_s})\ f_r(\vec{v}') - f_s(\vec{v})\ f_r(\vec{v}' + \frac{\hbar \vec{q}}{m_r}) \right\} \tag{6.4}$$

This is the Wyld-Pines equation. The classical limit of this equation may be taken in two different ways. First, we let $\hbar \vec{q} = \vec{p}$ be the momentum transferred in the collision. This will be kept finite when the classical limit is being taken. In the classical limit

$$\epsilon(\vec{p}/\hbar,\ \frac{\vec{p} \cdot \vec{v}}{\hbar} + p^2/2m_s) \xrightarrow[\hbar \to 0]{} \epsilon(\infty, \infty) = 1 \tag{6.5}$$

and one obtains

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \sum_r 4e_r^2 e_s^2 \int d^3p \int d^3v'$$

$$\frac{1}{p^4} \delta \left[\frac{m_s}{2} \left| \vec{v} + \vec{p}/m_s \right|^2 + \frac{m_r}{2} v'^2 - \frac{m_s}{2} v^2 - \frac{m_r}{2} \left| \vec{v}' + \frac{\vec{p}}{m_r} \right| \right]$$

$$\left\{ f_s(\vec{v} + \vec{p}/m_s)\ f_r(\vec{v}') - f_s(\vec{v})\ f_r(\vec{v}' + p/m_s) \right\} \tag{6.6}$$

This is just the Boltzmann equation.

If one lets $\hbar$ approach zero in Eq. (6.4), expands the δ-function, $\epsilon(q, \omega)$ and the distribution functions, then Wyld and Pines have shown that one obtains

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \sum_r \frac{2e_r^2 e_s^2}{m_s} \frac{\partial}{\partial \vec{v}} \cdot \int d^3q \int d^3v'$$

$$\frac{\vec{q}\,\vec{q}}{q^4} \frac{\delta(\vec{q} \cdot \vec{v} - \vec{q} \cdot \vec{v}')}{\left| \epsilon(\vec{q},\ \vec{q} \cdot \vec{v}) \right|^2}$$

$$\cdot \left\{ \frac{1}{m_s} \frac{\partial f_s}{\partial \vec{v}}(\vec{v})\ f_r(\vec{v}') - \frac{1}{m_r} f_s(\vec{v}) \frac{\partial f_r}{\partial \vec{v}'}(\vec{v}') \right\} \tag{6.7}$$

which is the Balescu-Lenard equation.

Because of the p^{-4} in Eq. (6.6) the Boltzmann equation diverges for small momentum transfers. This is because by Eq. (6.5) the dielectric function has been set equal to unity, so the screening effect which is important at large impact parameters has been neglected.

Eq. (6.7) is divergent for large momentum transfers. This should not surprise us; when we let $\hbar$ approach zero, the momentum transfer $\hbar\,\vec{q}$ approached zero so collisions with large momentum transfers are not properly treated.

Both Eqs. (6.4) and (6.7) are divergent for an unstable plasma for then $|\epsilon(\vec{q},\omega)|^2$ will vanish for the ω corresponding to a marginally stable wave. It is this divergence which we now want to investigate further and find a cure for.

The Balescu-Lenard equation may be written in the form

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \frac{\partial}{\partial \vec{v}} \cdot \left(\overleftrightarrow{D}(\vec{v}) \cdot \frac{\partial f_s}{\partial \vec{v}}\right) + \frac{\partial}{\partial \vec{v}} \cdot \left(\vec{A}(\vec{v})\; f_s(\vec{v})\right) \tag{6.8}$$

where

$$\overleftrightarrow{D}(\vec{v}) = \frac{8\pi^2 e_s^2}{m_s^2}\, V \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \frac{\vec{q}\,\vec{q}}{q^2}\, P_E(\vec{q},\omega)\, \delta(\omega - \vec{q}\cdot\vec{v}) \tag{6.9}$$

$$\vec{A}(\vec{v}) = \frac{8\pi^2 e^2}{m_s} \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \frac{\vec{q}}{q^2}\, \delta(\omega - \vec{q}\cdot\vec{v})\, \frac{\epsilon_2(\vec{q},\omega)}{|\epsilon(\vec{q},\omega)|^2} \tag{6.10}$$

$$P_E(\vec{q},\omega) = \sum_r \frac{4\pi^2 e_r^2}{q^2\, V |\epsilon(\vec{q},\omega)|^2} \int d^3v\; f_r(\vec{v})\, \delta(\omega - \vec{q}\cdot\vec{v}) \tag{6.11}$$

as may easily be verified. In obtaining Eq. (6.10) we have used Eq. (4.6) for $\epsilon_2(\vec{q},\omega)$. Eq. (6.11) is just the classical limit of Eq. (5.36). We have written the Balescu-Lenard equation in this form in order to point out the similarities with the quasi-linear equations, Eqs. (4.31), (4.34) and (4.35).

In the quasi-linear expressions for $\overleftrightarrow{D}(\vec{v})$ and $\vec{A}(\vec{v})$ there is no integration over ω. This is essentially because quasi-linear theory treats the lifetime of the quasi-particles as infinite. In the Balescu-Lenard equation the quasi-particles are virtual particles which are interchanged in a collision as is shown in Fig. 5.1.

In the Balescu-Lenard equation the function $P_E(\vec{q},\omega)$ is given its equilibrium value. In the quasi-linear equations the corresponding function $P(\vec{q})$ must satisfy the differential equation, Eq. (4.30). This is the origin of the divergence of the Balescu-Lenard equation; $P_E(\vec{q},\omega)$ is assumed to be fixed at the value given by Eq. (6.11) which is infinite for an unstable plasma. Really, $P(\vec{q},\omega)$ should evolve in time as $P(\vec{q})$ does in the quasi-linear theory.

There is also a factor of two difference between the quasi-linear $\overleftrightarrow{D}$ and the Balescu-Lenard $\overleftrightarrow{D}$. This is because $P(\vec{q})$ includes the kinetic energy of the particles while $P_E(\vec{q},\omega)$ does not. The factor of two came from Eq. (4.11).

It seems clear that to find equations which preserve the best features of the Balescu-Lenard equation and the quasi-linear equations, one should find an equation which describes the evolution of $P_E(\vec{q},\omega)$. This we shall proceed to do.

6.2 Non-Divergent Kinetic Theory[48]

We shall write Eq. (5.18) as

$$W_r(\vec{v} + \frac{\hbar\,\vec{q}}{m_r} \to \vec{v}) = \frac{8\pi e_r^2}{\hbar^2\, q^2}\; P(\vec{q},\omega)\left[\frac{\partial}{\partial\omega}\;\omega\,\epsilon_1(\vec{q},\omega)\right]^{-1} \tag{6.12}$$

where $P(\vec{q},\omega)$ is given by Eq. (5.39). It will be more convenient to work with $P(q,\omega)$ rather than $P_E(\vec{q},\omega)$. The transition probability for the inverse transition is given by

$$W_r(\vec{v} \to \vec{v} + \frac{\hbar\,\vec{q}}{m_r}) = \frac{8\pi e_r^2}{\hbar^2\, q^2}\, P(-\vec{q},-\omega)\left[\frac{\partial}{\partial\omega}\;\omega\,\epsilon_1(\vec{q},\omega)\right]^{-1} \tag{6.13}$$

This is found by changing the sign of the momentum and energy transfers and then shifting the velocity by $\hbar\,\vec{q}/m_r$. The relation

$$\epsilon_1(\vec{q},\omega) = \epsilon_1(-\vec{q},-\omega) \tag{6.14}$$

may be seen from inspection of Eq. (4.3); it is used in restoring the last factor in Eq. (6.13) to the form it has in Eq. (6.12). It will be shown later that the difference between Eqs. (6.12) and (6.13) is related to spontaneous transition probabilities.

We can now write an equation for the rate of change of $f_s(\vec{v})$. By an obvious line of reasoning it is

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \sum_{\vec{q}} \left\{ W_s(\vec{v} + \frac{\hbar\,\vec{q}}{m_s} \to \vec{v})\; f_s(\vec{v} + \frac{\hbar\,\vec{q}}{m_s}) \right.$$

$$\left. - W_s(\vec{v} \to \vec{v} + \frac{\hbar\,\vec{q}}{m_s})\; f_s(\vec{v}) \right\}$$

$$= \frac{8\pi e_s^2}{\hbar^2}\, V \int \frac{d^3q}{(2\pi)^3} \int d\,\omega\; \delta(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar\, q^2}{2m_s})$$

$$\frac{1}{q^2}\left[\frac{\partial}{\partial\omega}\;\omega\;\epsilon_1(\vec{q},\omega)\right]^{-1}\left\{P(\vec{q},\omega)\; f_s(\vec{v} + \frac{\hbar\,\vec{q}}{m_s}) - P(-\vec{q},-\omega)\; f_s(\vec{v})\right\} \tag{6.15}$$

where Eqs. (6.12) and (6.13) have been used. Eq. (4.29f) has been used to convert a sum to an integral. It is convenient to introduce the integral over ω and the δ-function in order to give $\hbar\omega$ its value of the particles energy loss.

We shall assume that each time a particle makes a transition the energy it loses (or gains) goes into (or comes out of) the energy of fluctuations. It follows that the rate of change of $P(\vec{q},\omega)$ is given by

$$\frac{\partial}{\partial t} P(\vec{q},\omega) = \sum_s \sum_{\vec{v}} \frac{\pi\hbar\omega}{V} \delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar^2 q^2}{2m_s}\right)$$

$$\left\{ W_s\left(\vec{v} + \frac{\hbar\vec{q}}{m_s} \to \vec{v}\right) f_s\left(\vec{v} + \frac{\hbar\vec{q}}{m_s}\right) - W_s\left(\vec{v} \to \vec{v} + \frac{\hbar\vec{q}}{m_s}\right) f_s(\vec{v}) \right\}$$

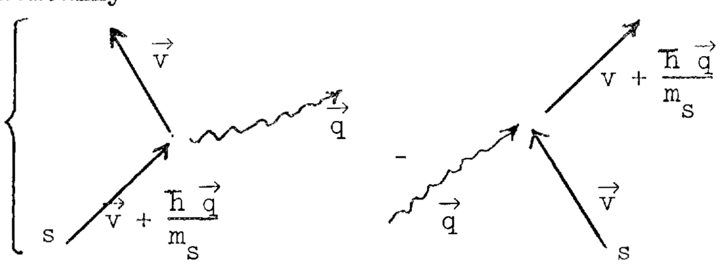
$$= \sum_s \frac{8\pi^2 e_s^2\,\omega}{\hbar\, q^2} \int d^3v\, \delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}\right)$$

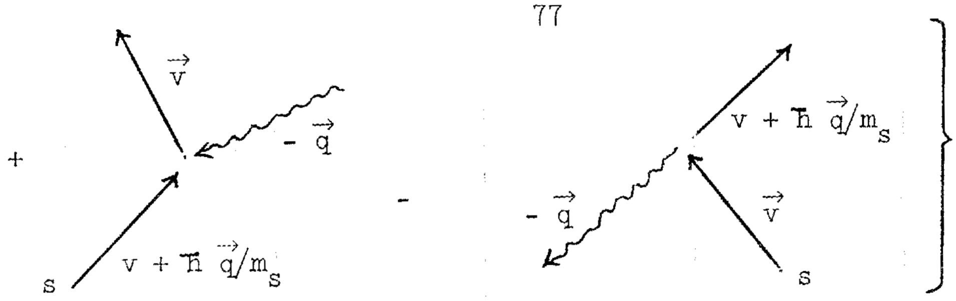
$$\left[\frac{\partial}{\partial\omega}\,\omega\epsilon_1(\vec{q},\omega)\right]^{-1} \left\{ P(\vec{q},\omega)\, f_s\left(\vec{v} + \frac{\hbar\vec{q}}{m_s}\right) - P(-\vec{q},-\omega)\, f_s(\vec{v}) \right\} \tag{6.16}$$

where Eq. (4.29g) has been used.

The physical content of Eqs. (6.15) may be made more apparent by writing them schematically

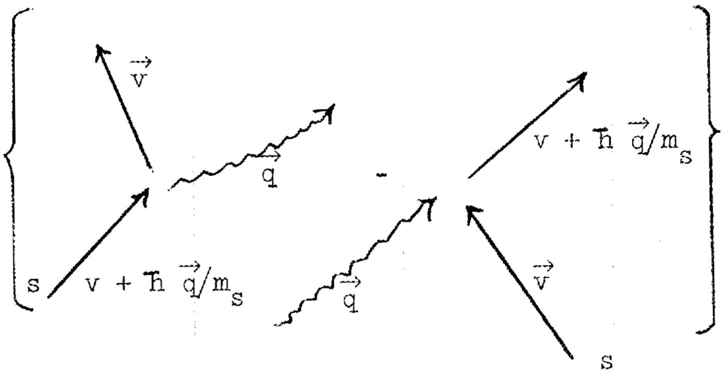
$$\frac{\partial f_s}{\partial t}(\vec{v}) = \sum_{\vec{q}}$$





(6.17)

$$\frac{\partial P}{\partial t}(\vec{q},\omega) \sum_s \sum_{\vec{v}}$$



(6.18

In this form they resemble the schematic equations of quasi-linear theory, Eqs. (4.18) and (4.16). We showed in Chapter 5 that $W_s(\vec{v} + \hbar\,\vec{q}/m_s \rightarrow \vec{v})$ contained terms which could be interpreted as due to emission of a quasi-particle of momentum $\hbar\,\vec{q}$ and absorption of a quasi-particle of momentum $-\hbar\,\vec{q}$, so both terms appear in Eq. (6.17).

We shall now examine some consequences of Eqs. (6.15) and (6.16). We define the particle density by

$$n_s = \int d^3v\; f_s(\vec{v}) \;, \tag{6.19}$$

the total momentum by

$$\vec{G} = \sum_s \int d^3v\; m_s\vec{v}\; f_s(\vec{v}) + V\int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi}\; \frac{\vec{q}}{\omega}\; P(\vec{q},\omega) \tag{6.20}$$

and total energy by

$$W = \sum_s \int d^3v \, \frac{m_s}{2} v^2 \, f_s(\vec{v}) + V \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi} P(\vec{q},\omega) \tag{6.21}$$

A straightforward calculation yields the gratifying result that

$$\frac{dn_s}{dt} = \frac{d\vec{G}}{dt} = \frac{dW}{dt} = 0 \tag{6.22}$$

Next, we shall prove an H-theorem. The entropy of the particles of species s is defined by

$$S_s = -K \int d^3v \, f_s(\vec{v}) \log f_s(\vec{v}) \tag{6.23}$$

(This is the classical definition. We have already omitted terms like $1 - f_s(\vec{v})$ in Eqs. (6.15) and (6.16) which should appear in a Fermion gas, so Eq. (6.23) is the appropriate definition rather than Eq. (4.24).) The entropy of the field fluctuations is defined to be

$$S_p = 2KV \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi} N(\vec{q},\omega) \log \left| N(\vec{q},\omega) \right| \tag{6.24}$$

where

$$N(\vec{q},\omega) = \frac{1}{\hbar\omega} P(\vec{q},\omega) \tag{6.25}$$

This is a definition which we have not found in the literature. It is motivated by the analogy with Eq. (6.23). Also, it permits us to prove an H-theorem. Also, in the limit of weak damping or growth it reduces to Eq. (4.25) for the entropy of a gas of Bosons. Now

$$\frac{d}{dt} S_s = -K \int d^3v \left[\log f_s(\vec{v}) + 1\right] \frac{\partial f_s}{\partial t}$$

$$= -K \frac{8\pi e_s^2}{\hbar^2} V \int d^3v \int \frac{d^3q}{(2\pi)^3} \int d\omega \frac{\delta(\omega - \vec{q} \cdot \vec{v} - \hbar q^2/2m_s)}{q^2}$$

$$\left[\frac{\partial}{\partial\omega}\,\omega\epsilon_1(\vec{q},\omega)\right]^{-1}\left\{P(\vec{q},\omega)\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\;-\;P(-\vec{q},-\omega)\;f_s(\vec{v})\right\}$$

$$\left[\log\,f_s(\vec{v})\;+\;1\right] \tag{6.26}$$

where Eq. (6.15) has been used. This equation may be written in a different form by making the change of variables $\vec{q}, \omega, \vec{v} \rightarrow -\vec{q}, -\omega, \vec{v} + \hbar\,\vec{q}/m_s$. Adding the two equations and dividing by two gives

$$\frac{d}{dt}\,S_s = -\,\frac{1}{2}\,K\,\frac{8\pi e_s^2 V}{\hbar^2}\int d^3v\int\frac{d^3q}{(2\pi)^3}\int d\omega\,\frac{\delta(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_s})}{q^2}$$

$$\left[\frac{\partial}{\partial\omega}\,\omega\epsilon_1(\vec{q},\omega)\right]^{-1}\left\{P(\vec{q},\omega)\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\;-\;P(-\vec{q},-\omega)\;f_s(\vec{v})\right\}$$

$$\left[\log\,f_s(\vec{v})\;-\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\right] \tag{6.27}$$

A similar calculation using Eq. (6.16) gives

$$\frac{dS_p}{dt} = \frac{1}{2}\,K\sum_s\frac{8\pi e_s^2 V}{\hbar^2}\int d^3v\int\frac{d^3q}{(2\pi)^3}\int d\omega\,\frac{\delta(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar\,q^2}{2m_s})}{q^2}$$

$$\left[\frac{\partial}{\partial\omega}\,\omega\epsilon_1(\vec{q},\omega)\right]^{-1}\left\{P(\vec{q},\omega)\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\;-\;P(-\vec{q},-\omega)\;f_s(\vec{v})\right\}$$

$$\left[\log\left|P(\vec{q},\omega)\right|\;-\;\log\left|P(-\vec{q},-\omega)\right|\right] \tag{6.28}$$

Summing Eq. (6.27) over s and adding to Eq. (6.28) gives the rate of change of the total entropy

$$\frac{dS}{dt} = \frac{1}{2}\,K\sum_s\frac{8\pi e_s^2 V}{\hbar^2}\int d^3v\int\frac{d^3q}{(2\pi)^3}\int d\omega\,\frac{\delta(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar\,q^2}{2m_s})}{q^2}$$

$$\left[\frac{\partial}{\partial\omega}\,\omega\epsilon_1(\vec{q},\omega)\right]^{-1}\left\{P(\vec{q},\omega)\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\;-\;P(-\vec{q},-\omega)\;f_s(\vec{v})\right\}$$

$$\left\{\log\left|P(\vec{q},\omega)\;f_s(\vec{v}+\frac{\hbar\,\vec{q}}{m_s})\right|\;-\;\log\left|P(-\vec{q},-\omega)\;f_s(\vec{v})\right|\right\} \tag{6.29}$$

The product of the last two factors is of the form

$$\left\{x - y\right\} \left\{\log |x| - \log |y|\right\}$$

which is always positive if x and y are positive. It can be negative if $P(\vec{q}, \omega)$ is negative, but this can only happen if

$$\left[\frac{\partial}{\partial \omega} \omega \epsilon_1(\vec{q}, \omega)\right]$$

is negative. Since the same factor occurs in the integrand of Eq. (6.29), the integrand will in any case be positive and we can conclude that

$$\frac{dS}{dt} \geq 0 \tag{6.30}$$

It is easily seen that equality holds when $f_s(\vec{v})$ is the Maxwell-Boltzmann distribution and $P(\vec{q}, \omega)$ is the equilibrium spectral density given by Eqs. (5.39) and (5.36). When these are substituted into Eqs. (6.15) and (6.16) the right-hand sides vanish showing that they are stationary distributions. It is tempting to conclude from this that the system evolves toward thermal equilibrium. What is missing from the proof is a proof that S has a single maximum.

Another consequence of Eqs. (6.15) and (6.16) is that

$$\frac{\partial}{\partial t} \left[P(\vec{q}, \omega) - P(-\vec{q}, -\omega)\right] = 0 \tag{6.31}$$

If the system approaches thermal equilibrium it follows that

$$P(\vec{q}, \omega) - P(-\vec{q}, -\omega) = P_o(\vec{q}, \omega) - P_o(-\vec{q}, -\omega)$$

$$= \frac{\hbar}{V} \left[\frac{\partial}{\partial \omega} \omega \epsilon_{10}(\vec{q}, \omega)\right] \frac{\epsilon_{20}(\vec{q}, \omega)}{|\epsilon_o(\vec{q}, \omega)|^2} \tag{6.32}$$

where the subscript o indicates that P_o and $\epsilon_o = \epsilon_{10} + i\epsilon_{20}$ are to be calculated using the Maxwell-Boltzmann distributions which the plasma approaches asymptotically. These distribution functions are characterized by number densities, n_s and a temperature T and mean velocity $\vec{V}$. The temperature and mean velocity are determined by W and $\vec{G}$ which we have already been shown to be constants of the motion. Therefore, it is possible in principle to determine the asymptotic state of the plasma from its state at any time.

Eq. (6.32) may be used to eliminate $P(-\vec{q},-\omega)$ from Eqs. (6.15) and (6.16) which then become

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \frac{8\pi e_s^2 V}{\hbar^2} \int \frac{d^3q}{(2\pi)^3} \int d\omega\, \delta(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}) \frac{1}{q^2}$$

$$\left[\frac{\partial}{\partial\omega}\, \omega\epsilon_1(\vec{q},\omega)\right]^{-1} \Bigg\{ P(\vec{q},\omega) \left[f_s(\vec{v} + \frac{\hbar\vec{q}}{m_s}) - f_s(\vec{v})\right]$$

$$+ \frac{\hbar}{V} \left[\frac{\partial}{\partial\omega}\, \omega\,\epsilon_{10}(\vec{q},\omega)\right] \frac{\epsilon_{20}(\vec{q},\omega)}{\epsilon_o(\vec{q},\omega)^2}\, f_s(\vec{v})\Bigg\} \qquad (6.33)$$

and

$$\frac{\partial P}{\partial t}(\vec{q},\omega) = 2\gamma(\vec{q},\omega)\, P(\vec{q},\omega) + S(\vec{q},\omega) \qquad (6.34)$$

where

$$2\gamma(\vec{q},\omega) = \sum_s \frac{8\pi^2 e_s^2\,\omega}{\hbar\, q^2} \left[\frac{\partial}{\partial\omega}\, \omega\epsilon_1(\vec{q},\omega)\right]^{-1}$$

$$\int d^3v\; \delta(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}) \left[f_s(\vec{v} + \frac{\hbar\vec{q}}{m_s}) - f_s(\vec{v})\right] \qquad (6.35)$$

$$S(\vec{q},\omega) = \sum_s \frac{8\pi^2 e_s^2\,\omega}{V q^2} \frac{\left[\frac{\partial}{\partial\omega}\, \omega\,\epsilon_{10}\right]}{\left[\frac{\partial}{\partial\omega}\, \omega\,\epsilon_1\right]} \frac{\epsilon_{20}(\vec{q},\omega)}{|\epsilon_o(\vec{q},\omega)|^2}$$

$$\int d^3v\; \delta(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s})\, f_s(\vec{v}) \qquad (6.36)$$

Eqs. (6.33) and (6.34) are the kinetic equations which we have been working toward. We will now inquire into their plausibility. First, note that when Eq. (4.4) is used for ϵ_2 we see that $\gamma(\vec{q},\omega)$ is given by

$$\gamma(\vec{q},\omega) = - \frac{\omega\,\epsilon_2(\vec{q},\omega)}{\frac{\partial}{\partial\omega}\,\omega\,\epsilon_1(\vec{q},\omega)} \tag{6.37}$$

which is just the linear growth rate expected from the arguments of section 2.3.

A stationary solution of Eq. (6.34) is

$$P(\vec{q},\omega) = - \frac{S(\vec{q},\omega)}{2\gamma(\vec{q},\omega)}$$
$$= \sum_s \frac{4\pi^2 e_s^2}{V\,q^2} \left[\frac{\partial}{\partial\omega}\,\omega\,\epsilon_{10}\right] \frac{\epsilon_{20}(\vec{q},\omega)}{\epsilon_2(\vec{q},\omega)}$$
$$\frac{1}{\left|\epsilon_0(\vec{q},\omega)\right|^2} \int d^3v\; \delta\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}\right) f_s(\vec{v}) \tag{6.38}$$

This agrees with the equilibrium $P(\vec{q},\omega)$ given by Eqs. (5.39) and (5.36) if the difference between $\epsilon(\vec{q},\omega)$ and $\epsilon_0(\vec{q},\omega)$ is negligible. It may be argued that it is only when this difference is negligible that one is justified in neglecting the time derivative of $P(\vec{q},\omega)$ in Eq. (6.34). If this equilibrium $P(\vec{q},\omega)$ is substituted into Eq. (6.33) one obtains the Wyld-Pines equation. As we have already remarked, the Wyld-Pines equation reduces to the Boltzmann and Balescu-Lenard equations in the appropriate limits.

To see how the quasi-linear equations emerge from Eqs. (6.33) and (6.34) we shall assume that the waves are so weakly damped or so weakly growing that the form of $P(\vec{q},\omega)$ is given by Eq. (5.43); namely

$$P(\vec{q},\omega) = \frac{\pi}{V}\Big\{\delta\left(\omega - \Omega_{\vec{q}}\right)\left[P(\vec{q}) + \hbar\,\Omega_{\vec{q}}\,S_{\vec{q}}\right]$$
$$+\,\delta\left(\omega + \Omega_{-\vec{q}}\right)\,P(-\vec{q}) \tag{6.39}$$

where we have used

$$P(\vec{q}) = S_q \hbar \Omega_{\vec{q}} N(\vec{q}) \tag{6.40}$$

For the moment we shall neglect the spontaneous emission terms. They will be discussed later. When Eq. (6.39) is substituted into Eq. (6.34) one obtains

$$\frac{\partial P}{\partial t}(\vec{q}) = 2\gamma(\vec{q})\, P(\vec{q}) \tag{6.41}$$

and a similar equation for $P(-\vec{q})$. Here

$$\gamma(\vec{q}) = -\left[\frac{\omega\epsilon_2(\vec{q},\omega)}{\frac{\partial}{\partial\omega}\,\omega\,\epsilon_1(\vec{q},\omega))}\right]_{\Omega_{\vec{q}}} \tag{6.42}$$

in agreement with the quasi-linear result. We have neglected the term $\hbar S_{\vec{q}} \Omega_{\vec{q}}$ in Eq. (6.39) since it vanishes in the classical limit. When Eq. (6.39) is substituted into Eq. (6.33), the integration over ω carried out and the change of variable $\vec{q} \to -\vec{q}$ made in the terms containing $P(-\vec{q})$ one obtains

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \frac{8\pi^2 e_s^2}{\hbar^2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \frac{P(\vec{q})}{\left[\frac{\partial}{\partial\omega}\,\omega\,\epsilon_1(\vec{q},\omega)\right]_{\Omega_q}}$$

$$\left\{\delta(\Omega_{\vec{q}} - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_s}) \left[f_s(\vec{v} + \frac{\hbar\vec{q}}{m_s}) - f_s(\vec{v})\right]\right.$$

$$\left. + \,\delta(\Omega_{\vec{q}} - \vec{q}\cdot\vec{v} + \frac{\hbar q^2}{2m_s}) \left[f_s(\vec{v} - \frac{\hbar\vec{q}}{m_s}) - f_s(\vec{v})\right]\right\} \tag{6.43}$$

Expanding $f_s(\vec{v} \pm \hbar\vec{q}/m_s)$ and the δ-functions and taking the $\hbar \to 0$ limit gives

$$\frac{\partial f_s}{\partial t}(\vec{v}) = \frac{\partial}{\partial\vec{v}} \cdot (\overleftrightarrow{D}(\vec{v}) \frac{\partial f_s}{\partial\vec{v}}) \tag{6.44}$$

where

$$\overleftrightarrow{D}(\vec{v}) = \frac{8\pi^2 e_s^2}{m_s^2} \int \frac{d^3q}{(2\pi)^3} \frac{\vec{q}\,\vec{q}}{q^2} \frac{P(\vec{q})}{\left[\frac{\partial}{\partial\omega} \omega \epsilon_1(\vec{q},\omega)\right]}$$

$$\delta(\Omega_q - \vec{q}\cdot\vec{v}) \tag{6.45}$$

in agreement with quasi-linear theory. (Note that in Eq. (4.35) we used $\partial\omega\epsilon_1/\partial\omega = 2$ which is appropriate for plasmons.)

We shall now discuss the spontaneous emission terms. A particle can change its velocity from $\vec{v} + \hbar\,\vec{q}/m_s$ to $\vec{v}$ by emitting a quasi-particle of momentum $\hbar\,\vec{q}$ or by absorbing a quasi-particle of momentum $-\hbar\,\vec{q}$. Similarly, a particle can change its momentum from $\vec{v}$ to $\vec{v} + \hbar\,\vec{q}/m_s$ by absorbing a quasi-particle of momentum $\hbar\,\vec{q}$ or by emitting a wave of momentum $-\hbar\,\vec{q}$. Since the transition probabilities for stimulated emission and absorption are equal, it follows that

$$W_s(\vec{v} + \frac{\hbar\,\vec{q}}{m_s} \to \vec{v}) - W_s(\vec{v} \to \vec{v} + \frac{\hbar\,\vec{q}}{m_s})$$

is equal to the difference between the transition probabilities per unit time for spontaneous emission of a wave of momentum $\hbar\,\vec{q}$ and spontaneous emission of a wave of momentum $-\hbar\,\vec{q}$. By Eqs. (6.12) and (6.32) this is

$$\frac{8\pi e_s^2}{\hbar^2\, q^2} \frac{1}{\left[\frac{\partial}{\partial\omega} \omega \epsilon_1(\vec{q},\omega)\right]} \left[P(\vec{q},\omega) - P(-\vec{q},-\omega)\right]$$

$$= \frac{8\pi e_s^2}{\hbar\, V\, q^2} \frac{\left[\frac{\partial}{\partial\omega} \omega\, \epsilon_{10}(\vec{q},\omega)\right]}{\left[\frac{\partial}{\partial\omega} \omega \epsilon_1(\vec{q},\omega)\right]} \frac{\epsilon_{20}(\vec{q},\omega)}{|\epsilon_o(\vec{q},\omega)|^2} \tag{6.46}$$

We may use Eq. (5.43) to write this as

$$\frac{2\pi}{\hbar^2}\left[\frac{4\pi e_s^2\, \hbar(\omega)}{V\, q^2 \left|\frac{\partial}{\partial\omega} \omega\epsilon_{10}(\vec{q},\omega)\right|}\right]\left[\frac{\frac{\partial}{\partial\omega} \omega\, \epsilon_{10}(\vec{q},\omega)}{\frac{\partial}{\partial\omega} \omega \epsilon_1(\vec{q},\omega)}\right]$$

$$\left[\delta(\omega - \Omega_{\vec{q}}) - \delta(\omega + \Omega_{-\vec{q}})\right] \tag{6.47}$$

Except for the factor

$$\frac{\frac{\partial}{\partial \omega}\, \omega \epsilon_{10}(\vec{q}, \omega)}{\frac{\partial}{\partial \omega}\, \omega \epsilon_{1}(\vec{q}, \omega)} \tag{6.48}$$

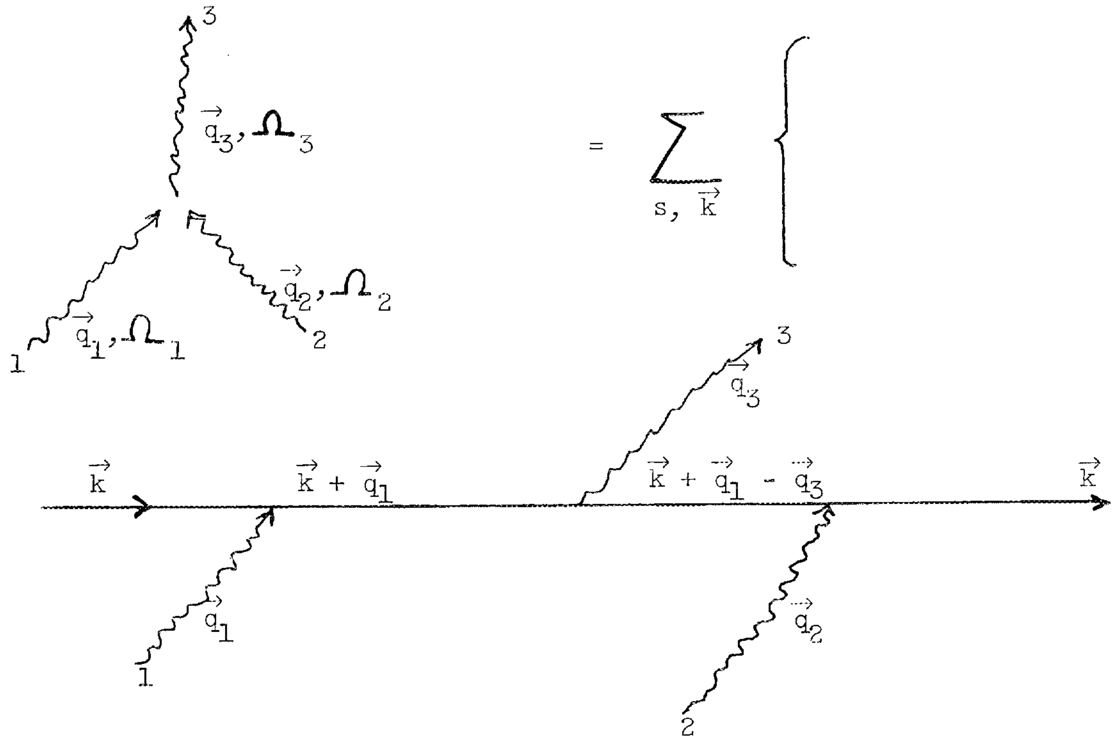
Eq. (6.47) is just what we would calculate for the difference in the spontaneous emission probabilities if we use the Fermi Golden Rule and Eq. (3.23) for the vertex part. It should be noted that the frequencies $\Omega_{\vec{q}}$ are to be calculated from $\epsilon_{10}(\vec{q}, \omega) = 0$ rather than $\epsilon_1(\vec{q}, \omega) = 0$. This difference in the characteristic frequencies is probably not significant in those problems for which quasi-linear theory is applicable. Also, in those problems the factor in Eq. (6.48) probably does not differ significantly from unity. Neglecting these discrepancies the spontaneous emission terms give the expected contributions to Eq. (6.41) and (6.44).

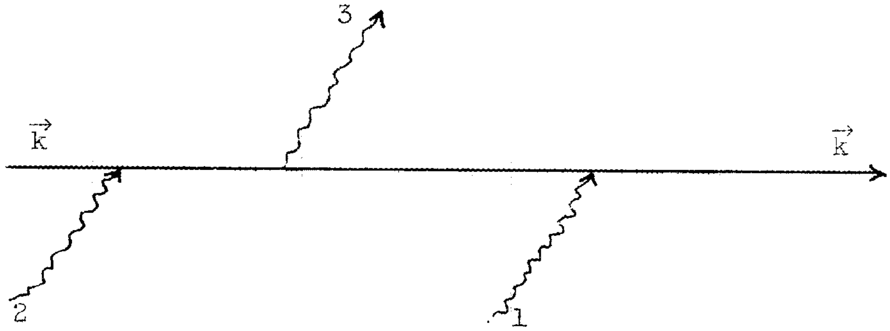
CHAPTER 7. HIGHER ORDER PROCESSES

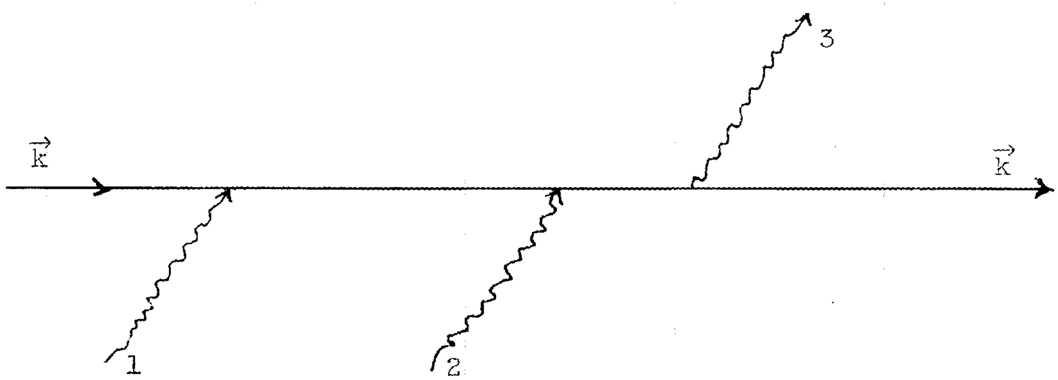
The wave-particle interaction vertex functions which we found in Eqs. (3.23), (3.26) and (3.27) can be used to construct the vertex functions for higher order processes by using the higher order terms in Eq. (3.29). As the first illustration of this, we shall calculate the vertex function for the interaction of three waves in an unmagnetized plasma.

7.1 Three-Wave Interaction

We shall consider the case in which a wave of type 1 with momentum $\hbar \vec{q}_1$ and energy $\hbar \Omega_1$ combines with a wave of type 2 and $\hbar \vec{q}_2$ and $\hbar \Omega_2$ to give a wave of type 3 with $\hbar \vec{q}_3$ and $\hbar \Omega_3$. Schematically we can write



+ 
3
$\vec{k}$
$\vec{k}$
2
1

+ 
3
$\vec{k}$
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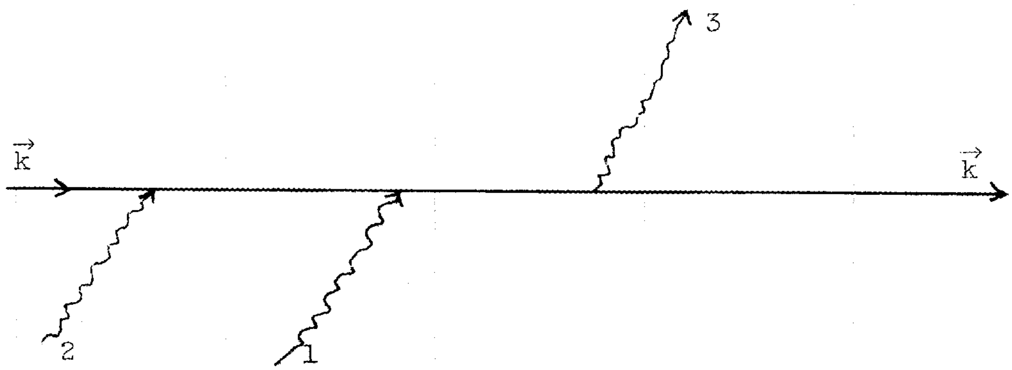
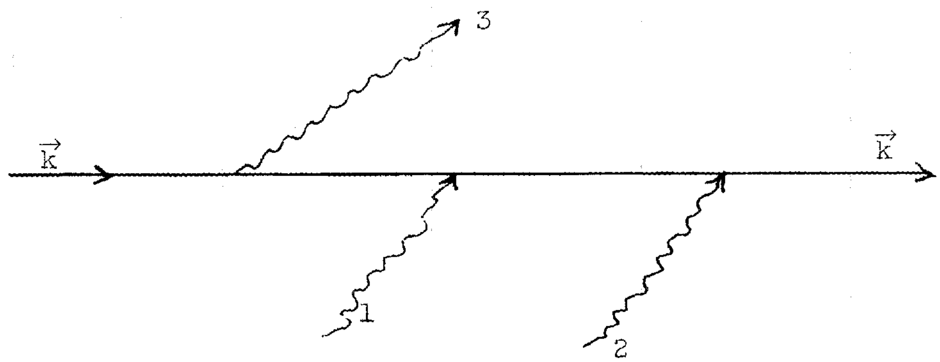
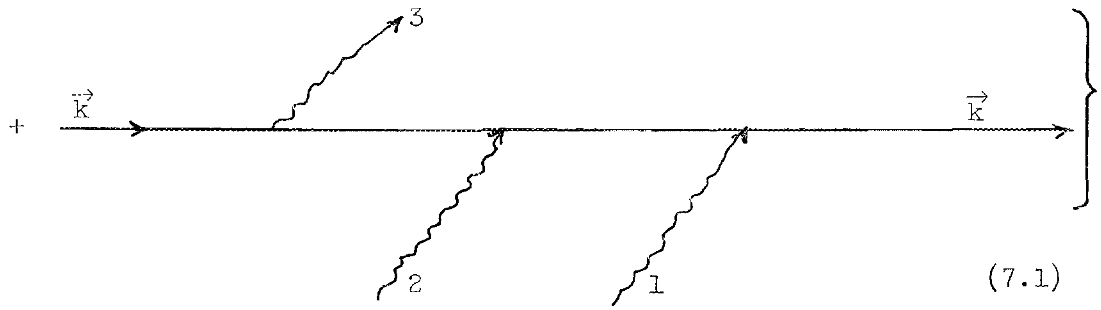
1
2

+ 
3
$\vec{k}$
$\vec{k}$
2
1

+ 
3
$\vec{k}$
$\vec{k}$
1
2



(7.1)

What we mean by this is that we shall use the third order term in Eq. (3.29)

$$M = \sum_{I} \sum_{II} \frac{< f|H'|I >< I|H'|II >< II|H'|i >}{(E_i - E_I + i\eta)\,(E_i - E_{II} + i\eta)} \tag{7.2}$$

to calculate M, the vertex function for the three wave interaction. The waves interact with one another through the wave-particle interactions as illustrated by the diagrams. The sum over intermediate states in Eq. (7.2) is a sum over all of the particles which can take part in the process; hence the sums over s and $\vec{k}$ in Eq. (7.1). The sum over intermediate states also includes a sum over the six time orders shown in Eq. (7.1). The particle which enters the process with momentum $\hbar\,\vec{k}$ leaves with the same momentum so there is no change in the particle distribution functions. The only change is that waves 1 and 2 disappear and wave 3 appears. At each vertex momentum (but not energy) is conserved. We have shown this explicitly in the first diagram of Eq. (7.1). There is overall conservation of energy and momentum so

$$\vec{q}_1 + \vec{q}_2 = \vec{q}_3 \tag{7.3}$$

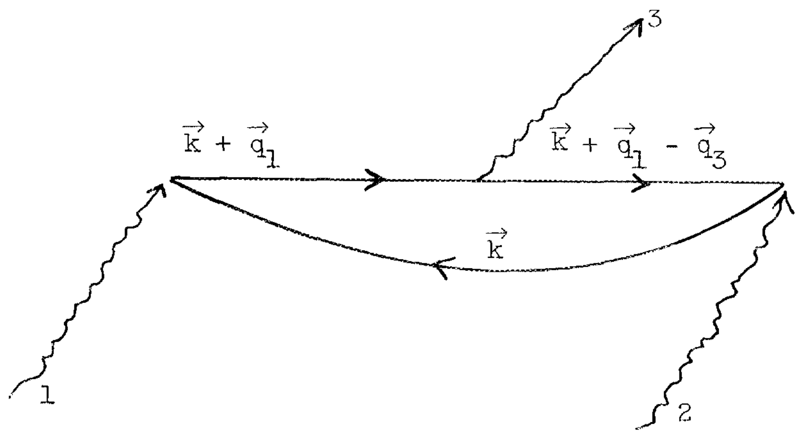
$$\Omega_1 + \Omega_2 = \Omega_3 \tag{7.4}$$

A straightforward calculation gives

$$
\begin{aligned}
M = \frac{V}{\hbar^2} \sum_s \int d^3v \; f_s(\vec{v}) \Bigg\{ & \frac{M_2(\vec{k},\vec{q}_2)\; M_3(\vec{k}+\vec{q}_1,\; \vec{q}_3)\; M_1(\vec{k},\vec{q}_1)}{\left[\Omega_1 - \vec{v}\cdot\vec{q}_1 - \frac{\hbar}{2m_s} q_1^2 + i\eta\right]\left[\Omega_1 - \Omega_3 + \vec{v}\cdot\vec{q}_2 - \frac{\hbar}{2m_s} q_2^2 + i\eta)\right]} \\
& + \frac{M_1(\vec{k},\vec{q}_1)\; M_3(\vec{k}+\vec{q}_2,\; \vec{q}_3)\; M_2(\vec{k},\vec{q}_2)}{\left[\Omega_2 - \vec{v}\cdot\vec{q}_2 - \frac{\hbar}{2m_s} q_2^2 + i\eta\right]\left[\Omega_2 - \Omega_3 + \vec{v}\cdot\vec{q}_1 - \frac{\hbar}{2m_s} q_1^2 + i\eta\right]} \\
& + \frac{M_3(\vec{k},\vec{q}_3)\; M_2(\vec{k}+\vec{q}_1,\; \vec{q}_2)\; M_1(\vec{k},\vec{q}_1)}{\left[\Omega_1 - \vec{v}\cdot\vec{q}_1 - \frac{\hbar}{2m_s} q_1^2 + i\eta\right]\left[\Omega_1 + \Omega_2 - \vec{v}\cdot\vec{q}_3 - \frac{\hbar}{2m_s} q_3^2 + i\eta\right]} \\
& + \frac{M_3(\vec{k},\vec{q}_3)\; M_1(\vec{k}+\vec{q}_2,\; \vec{q}_1)\; M_2(\vec{k},\vec{q}_2)}{\left[\Omega_2 + \vec{v}\cdot\vec{q}_2 - \frac{\hbar}{2m_s} q_2^2 + i\eta\right]\left[\Omega_1 + \Omega_2 - \vec{v}\cdot\vec{q}_3 - \frac{\hbar}{2m_s} q_3^2 + i\eta\right]} \\
& + \frac{M_2(\vec{k},\vec{q}_2)\; M_1(\vec{k}-\vec{q}_3,\; \vec{q}_1)\; M_3(\vec{k},\vec{q}_3)}{\left[-\Omega_3 + \vec{v}\cdot\vec{q}_3 - \frac{\hbar}{2m_s} q_3^2 + i\eta\right]\left[\Omega_1 - \Omega_3 + \vec{v}\cdot\vec{q}_2 - \frac{\hbar}{2m_s} q_2^2 + i\eta\right]} \\
& + \frac{M_1(\vec{k},\vec{q}_1)\; M_2(\vec{k}-\vec{q}_3,\; \vec{q}_2)\; M_3(\vec{k},\vec{q}_3)}{\left[-\Omega_3 + \vec{v}\cdot\vec{q}_3 - \frac{\hbar}{2m_s} q_3^2 + i\eta\right]\left[\Omega_2 - \Omega_3 + \vec{v}\cdot\vec{q}_1 - \frac{\hbar}{2m_s} q_1^2 + i\eta\right]} \Bigg\}
\end{aligned}
\tag{7.5}
$$

In this equation M_1, M_2 and M_3 are the appropriate wave-particle vertex functions taken from Eqs. (3.23) and (3.26). Their arguments are the wave vectors of the particles and quasi-particles involved in the process. We have used $\hbar\,\vec{k} = m_s\vec{v}$ and have simplified the energy denominators. Eq. (4.29g) has been used to replace the sum over $\vec{k}$ by an integral over $\vec{v}$.

Before considering some special cases of Eq. 7.5 we shall remark that diagrams such as the first one in Eq. (7.1) are sometimes drawn as



In this diagram the incoming particle lines are not drawn but a line is drawn from the last vertex to the first indicating that the particle of momentum $\hbar \vec{k}$ which was removed from this state at the first vertex has now been restored. This may be thought of as the creation at the first vertex of a particle and a "hole" (that is, a hole in the original particle distribution). At the final vertex a particle and a hole recombine.

a. Three-plasmon interaction. If all three of the waves are plasma oscillations with frequencies given by

$$\Omega_{\vec{q}} = \omega_{pe}(1 + \frac{3}{2} \mathrm{Le}^2 q^2), \tag{7.6}$$

then it is not possible to satisfy Eqs. (7.3) and (7.4). Therefore we need not concern ourselves with this case.

b. Plasmon-plasmon-phonon interaction. We shall suppose that quasi-particles 1 and 3 are plasmons and that 2 is a phonon with frequency given by Eq. (4.10). Then, Eqs. (7.3) and (7.4) can be satisfied. The plasmon-particle and phonon-particle vertex functions are given in Eqs. (4.13) and (4.14). They are functions of the quasi-particle wave vector only, so they can be taken outside of the velocity space integral in Eq. (7.5). We are interested in classical limits so we shall keep only the lowest order terms in h. After some tedious algebra we find

$$M = \sum_s \frac{M_1 M_2 M_3 V}{m_s^2} \int d^3v \, f_s(\vec{v})$$

$$\frac{1}{(\Omega_1 - \vec{v}\cdot\vec{q}_1)(\Omega_2 - \vec{v}\cdot\vec{q}_2)(\Omega_3 - \vec{v}\cdot\vec{q}_3)}$$

$$\left[\frac{(\vec{q}_1\cdot\vec{q}_2)\, q_3^2}{(\Omega_3 - \vec{v}\cdot\vec{q}_3)} + \frac{(\vec{q}_2\cdot\vec{q}_3)\, q_1^2}{(\Omega_1 - \vec{v}\cdot\vec{q}_1)} + \frac{(\vec{q}_1\cdot\vec{q}_3)\, q_2^2}{(\Omega_2 - \vec{v}\cdot\vec{q}_2)}\right] \tag{7.7}$$

With further approximations this can be put into an interesting and useful form. Since the plasmon frequencies are much greater than the phonon frequency, we neglect all but the last term in Eq. (7.7). Also, we make the approximation

$$(\Omega_1 - \vec{v}\cdot\vec{q}_1)(\Omega_3 - \vec{v}\cdot\vec{q}_3) \simeq \omega_{pe}^2 \tag{7.8}$$

In the integral

$$\int d^3v \, \frac{f_s(\vec{v})}{(\Omega_2 - \vec{v}\cdot\vec{q}_2)^2}$$

We assume that the phase velocity of the phonon is much greater than the thermal velocity of the ions and obtain

$$\int d^3v \, \frac{f_i(\vec{v})}{(\Omega_2 - \vec{v}\cdot\vec{q})^2} \simeq \frac{n_i}{\Omega_2^2} \simeq \frac{n_i \, m_i}{q_2^2 \, T_e} \tag{7.9}$$

for the ion term. We assume that the phase velocity of the phonon is much less than the thermal velocity of the electrons and obtain

$$\int d^3v \; \frac{f_e(\vec{v})}{(\Omega_2 - \vec{v}\cdot\vec{q})^2} \simeq \frac{n_e \, m_e}{q_2^2 \, T_e} \tag{7.10}$$

for the electron contribution. Finally, we assume

$$q_1, \; q_3 >> q_2 \tag{7.11}$$

so

$$\vec{q}_3 = \vec{q}_1 + \vec{q}_2 \simeq \vec{q}_1 \tag{7.12}$$

With these approximations Eq. (7.7) can be reduced to the form

$$M = \left(\frac{\hbar^3 \omega_{pe}^2 \, q_2^2}{8V \, m_i \, n \, \Omega_2} \right)^{1/2} \tag{7.13}$$

This is essentially the result obtained by Vedenov and Rudakov by another method.[49] It has also been obtained by a more elementary method by Harris.[50]

c. Plasmon-plasmon-photon interaction. We shall suppose that quasi-particles 1 and 2 are plasmons and that 3 is a photon. The plasmon-particle vertex function is again given by Eq. (4.13). The photon-particle vertex function is given by Eq. (3.26). In evaluating Eq. (3.26) we use

$$< \vec{k}_1 \left| \vec{u}_3 \cdot \vec{v} \, e^{i\vec{q}\cdot\vec{x}} \right| \vec{k}_2 >_s = \frac{\hbar}{m_s} \vec{u} \cdot \vec{k}_1 \, \delta_{\vec{k}_1 \vec{k}_2, \vec{q}} \tag{7.14}$$

and

$$\frac{1}{\omega} \frac{\partial}{\partial \omega} \omega^2 \epsilon_T = \frac{1}{\omega} \frac{\partial}{\partial \omega} \omega^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) = 2 \tag{7.15}$$

where we have used the cold plasma dielectric function. We obtain

$$M_3(\vec{k},\vec{q}_3) = \frac{\hbar}{m_s} \left[\frac{2\pi\,\hbar\,e_s^2}{V\,\Omega_3}\right]^{1/2} \vec{u}_3 \cdot \vec{k} \tag{7.16}$$

where $\hbar\,\vec{k}$ is the particle momentum and $\hbar\,\vec{q}_3$ is the photon momentum. Eq. (7.5) takes a simple form if the plasma is cold; that is

$$f_s(\vec{v}) = n_s\,\delta(\vec{v}) \tag{7.17}$$

Then, $\vec{k} = m_s\vec{v}/\hbar$ will be zero, and because of the factor $\vec{u}_3 \cdot \vec{k}$ in Eq. (7.16) the only terms in Eq. (7.5) that are non-zero are the terms containing $M_3(\vec{k}+\vec{q}_1,\ \vec{q}_3)$ and $M_3(\vec{k}+\vec{q}_2,\ \vec{q}_3)$. We obtain

$$M = \frac{V}{\hbar^2} \sum_s n_s \left(\frac{2\pi e_s^2\,\hbar\,\Omega_1}{V\,q_1^2}\right)^{1/2} \left(\frac{2\pi e_s^2\,\hbar\,\Omega_2}{V\,q_2^2}\right)^{1/2} \frac{\hbar}{m_s} \left(\frac{2\pi\,\hbar\,e_s^2}{V\,\Omega_3}\right)^{1/2}$$

$$\left\{ \frac{\vec{u}_3 \cdot \vec{q}_1}{(\Omega_1 - \frac{\hbar}{2m_s} q_1^2)(-\Omega_2 - \frac{\hbar}{2m_s} q_2^2)} \right.$$

$$\left. + \frac{\vec{u}_3 \cdot \vec{q}_2}{(-\Omega_1 - \frac{\hbar}{2m_s} q_1^2)(\Omega_2 - \frac{\hbar}{2m_s} q_2^2)} \right\} \tag{7.18}$$

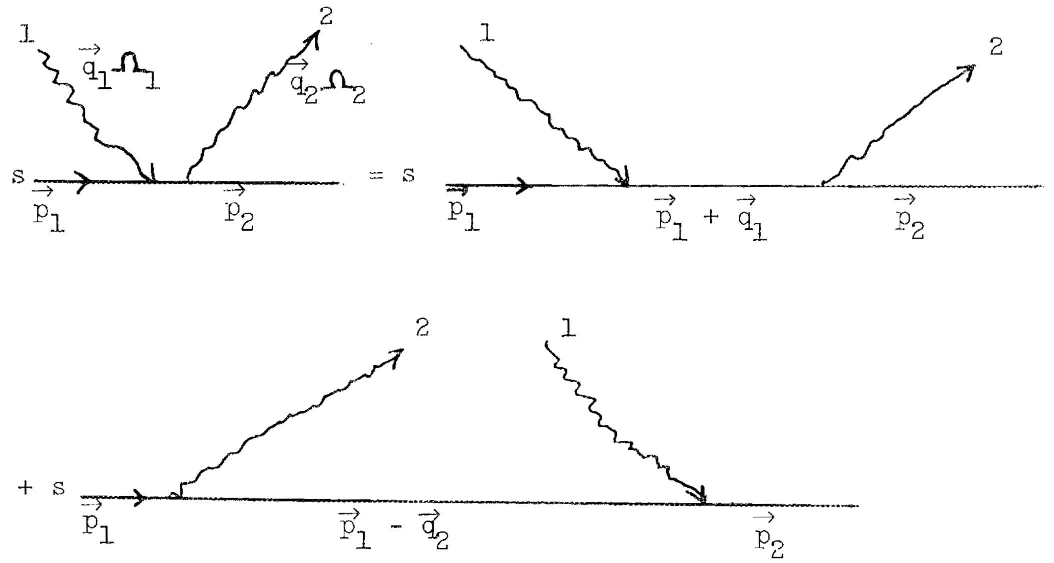
Using $\vec{u}_3 \cdot \vec{q}_3 = \vec{u}_3 \cdot (\vec{q}_1 + \vec{q}_2) = 0$ and keeping only the lowest order terms in $\hbar$ gives

$$M = -\frac{\hbar}{2m_e}\left(\frac{2\pi e^2}{V\,\Omega_3}\right)^{1/2} \frac{\vec{u}_3 \cdot \vec{q}_1}{q_1\,q_2}\,(q_1^2 - q_2^2) \tag{7.19}$$

We have approximated Ω_1 and Ω_2 by ω_{pe} and neglected the ion contribution.

7.2 Wave-Particle Scattering

We now consider the process in which a wave is scattered by a particle. This may be calculated as a second order process which may be written schematically as



(7.20)

It is not necessary that the quasi-particle labeled 2 be of the same type as the quasi-particle labeled 1. For instance, a plasmon may be converted to a phonon in the scattering process. Overall conservation of momentum and energy is assumed so

$$\vec{p}_1 + \vec{q}_1 = \vec{p}_2 + \vec{q}_2 \tag{7.21}$$

and

$$\frac{\hbar^2}{2m_s} p_1^2 + \hbar \Omega_1 = \frac{\hbar^2}{2m_s} p_2^2 + \hbar \Omega_2 \tag{7.22}$$

Applying second order perturbation theory gives the matrix element for the process of Eq. (7.20) as

$$M = \sum_I \frac{< f|H'|I > < I|H'|i >}{E_i - E_I + i\eta}$$

$$= \frac{M_2(\vec{p}_2,\vec{q}_2)\; M_1(\vec{p}_1,\vec{q}_1)}{\left(\frac{\hbar^2 p_1^2}{2m_s} + \hbar\,\Omega_1\right) - \frac{\hbar^2}{2m_s}\left|\vec{p}_1 + \vec{q}_1\right|^2 + i\eta}$$

$$+ \frac{M_1(\vec{p}_2,\vec{q}_1)\; M_2(\vec{p}_1,\vec{q}_2)}{\left(\frac{\hbar^2 p_1^2}{2m_s} + \hbar\,\Omega_1\right) - \left(\hbar\,\Omega_1 + \hbar\,\Omega_2 + \frac{\hbar^2}{2m_s}\left|\vec{p}_1 - \vec{q}_2\right|^2\right) + i\eta} \tag{7.23}$$

To be specific we shall restrict ourselves to plasmons and phonons; then M_1 and M_2 are given by Eq. (3.23)

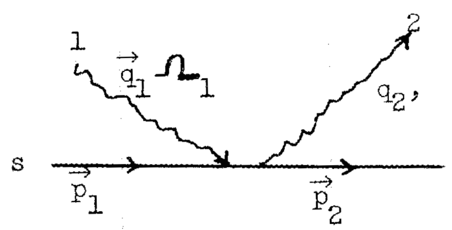
$$M_{1,2}(\vec{p},\vec{q}) = \left(\frac{4\pi e_s^2\, \hbar\;\Omega_{1,2}}{V\, q^2 \left|\frac{\partial}{\partial\omega}\;\omega\,\epsilon_1(\vec{q},\omega)\right|_{\Omega_{1,2}}}\right)^{1/2} \tag{7.24}$$

Letting $\vec{v} = h\; p_1/m_s$ we can write

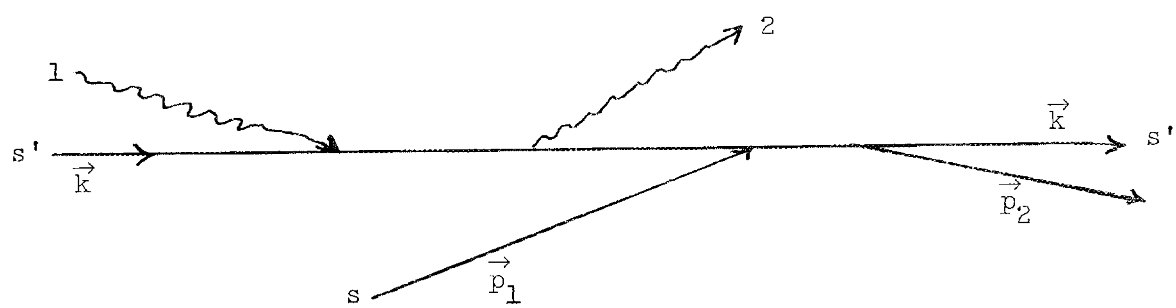
$$M = \frac{4\pi e_s^2\,\hbar}{V\, q_1\, q_2\, m_s}\left[\frac{\Omega_1\;\Omega_2}{\left|\frac{\partial}{\partial\omega}\;\omega\epsilon_1\right|_{\Omega_1}\left|\frac{\partial}{\partial\omega}\;\omega\epsilon_1\right|_{\Omega_2}}\right]^{1/2}$$

$$\frac{\vec{q}_1\cdot\vec{q}_2}{\left[\Omega_1 - \vec{q}_1\cdot\vec{v} - \frac{\hbar}{2m}\,q_1^2 + i\eta\right]\left[\Omega_2 - \vec{q}_2\cdot\vec{v} + \frac{\hbar}{2m_s}\,q_2^2 - i\eta\right]} \tag{7.25}$$

It is not sufficient to stop with Eq. (7.25). As we shall see there is a third order matrix element which makes a contribution of the same order. This is the process shown schematically in Eq. (7.26).



$$= \sum_{s',\vec{k}} \Bigg\{$$



+ five other time orderings (7.26)

In this process a particle of momentum $\hbar\,\vec{k}$ absorbs and emits the quasi-particles and collides with another particle. The other particle has its momentum changed from $\hbar\,\vec{p}_1$ to $\hbar\,\vec{p}_2$ while the original particle has its momentum restored to $\hbar\,\vec{k}$. There are six time orderings as there were for Eq. (7.1). The vertex functions for the quasi-particle emission and absorption are again given by Eq. (7.24). The vertex function for the particle-particle scattering may be taken to be the screened Coulomb vertex

$$M_3(\vec{p}_1,\vec{p}_2) = \frac{4\pi e_s e_{s'}}{V\left|\vec{p}_1 - \vec{p}_2\right|^2 \epsilon(\vec{p}_1 - \vec{p}_2,\ \Omega_1 - \Omega_2)} \tag{7.27}$$

The matrix element for this third order process may be written as

$$M = \sum_s M_1(\vec{q}_1, \Omega_1)\ M_2(\vec{q}_2, \Omega_2)\ M_3(\vec{p}_1,\vec{p}_2)$$

$$V \int d^3v\ f_s(\vec{v}) \sum_{I} \sum_{II} \frac{1}{(E_i - E_I + i\eta)(E_i - E_{II} + i\eta)} \tag{7.28}$$

where the last sum is over the six time orderings. We shall spare the reader the tedious details of collecting the terms and expressing the results in a convenient form. When only the lowest order terms in $\hbar$

are kept we find

$$M = \sum_s \frac{(4\pi)^2 e_s^2 e_{s'}^2}{m_s^2 V q_1 q_2} \left| \frac{\Omega_1 \Omega_2}{(\frac{\partial}{\partial\omega} \omega\epsilon_1)_{\Omega_1} (\frac{\partial}{\partial\omega} \omega\epsilon_1)_{\Omega_2}} \right|^{1/2}$$

$$\frac{1}{|\vec{q}_1 - \vec{q}_2|^2 \epsilon(\vec{q}_1 - \vec{q}_2, \Omega_1 - \Omega_2)} \int d^3v\, f(\vec{v})$$

$$\left\{ \frac{q_1^2 q_2^2}{\Lambda_1^2 \Lambda_2^2} + \frac{\vec{q}_1 - \vec{q}_2}{(\Lambda_1 - \Lambda_2)^2} \left[\frac{2|\vec{q}_1 - \vec{q}_2|^2}{\Lambda_1 \Lambda_2} - \left| \frac{\vec{q}_1}{\Lambda_1} - \frac{\vec{q}_2}{\Lambda_2} \right|^2 \right] \right. \tag{7.29}$$

where

$$\Lambda_1 = \Omega_1 - \vec{q}_1 \cdot \vec{v} \tag{7.30}$$

$$\Lambda_2 = \Omega_2 - \vec{q}_2 \cdot \vec{v} \tag{7.31}$$

If both quasi-particles are plasmons and we make the approximations

$$\left(\frac{\partial}{\partial\omega} \omega\epsilon_1\right)_{\Omega_1} = \left(\frac{\partial}{\partial\omega} \omega\epsilon_2\right)_{\Omega_2} \simeq 2 \tag{7.32}$$

$$\Lambda_1 \simeq \Lambda_2 \simeq \omega_{pe} \tag{7.33}$$

and neglect the motion of the ions, then Eq. (7.29) reduces to

$$M = \frac{\hbar \omega_{pe}}{2m V} \left\{ \frac{q_1 q_2}{|\vec{q}_1 - \vec{q}_2|^2 \epsilon(\vec{q}_1 - \vec{q}_2, \Omega_1 - \Omega_2)} \right.$$

$$+ \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left[\frac{1}{\epsilon(\vec{q}_1 - \vec{q}_2, \Omega_1 - \Omega_2)} - 1 \right] \tag{7.34}$$

When the same approximations are made in Eq. (7.25) and the results are added to Eq. (7.34), the last term cancels and one obtains

$$M = \frac{\hbar\,\omega_{pe}}{2m\,V}\left\{\frac{q_1\;q_2}{\left|\vec{q}_1 - \vec{q}_2\right|^2} + \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1\;q_2}\right\}\frac{1}{\epsilon(\vec{q}_1 - \vec{q}_2, \Omega_1 - \Omega_2)} \tag{7.35}$$

From Eqs. (7.21) and (7.22)

$$\Omega_1 - \Omega_2 = \frac{\hbar}{2m_s}\,(p_2^2 - p_1^2) \simeq \vec{v} \cdot (\vec{q}_1 - \vec{q}_2) \tag{7.36}$$

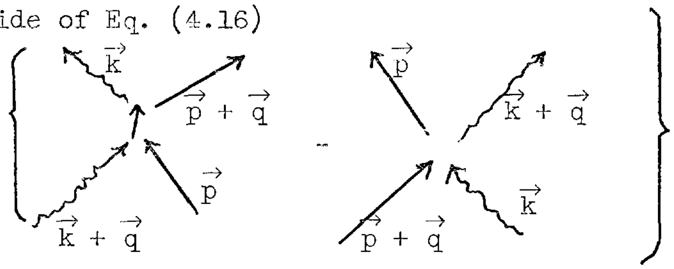
In the classical limit this is to be used in the argument of $\epsilon(\vec{q}_1 - \vec{q}_2, \Omega_1 - \Omega_2)$.

7.3 Extension of the Quasi-Linear Equations

We shall now discuss the corrections which must be made to the quasi-linear equations because of these higher order processes.

a. The unmagnetized plasma. We have already remarked that if $\vec{B}_o = 0$ there are no three-plasmon interactions which conserve momentum and energy. For the moment we restrict ourselves to a single species plasma so interactions involving phonons are ignored. We shall consider the corrections to the quasi-linear equations given in Eqs. (4.17) and (4.19). The non-linear processes of interest here are the wave-particle scattering processes of the last section. These are often referred to as non-linear Landau damping terms. We must add to the right hand side of Eq. (4.16)

$$\left(\frac{\partial N_\lambda}{\partial t}(\vec{k})\right)_{NLLD} = \sum_{\vec{p},\vec{q}}$$



$$= \sum_{\vec{p},\vec{q}} \frac{2\pi}{\hbar} |M|^2 \delta\left[\hbar\,\Omega_{k+q} + \frac{\hbar^2}{2m} p^2 - \hbar\,\Omega_k - \frac{\hbar^2}{2m} |\vec{p}+\vec{q}|^2\right]$$

$$\Big\{ N_\lambda(\vec{k}+\vec{q})\, N_e(\vec{p}) \left[1 - N_e(p+q)\right] \left[N_\lambda(\vec{k}) + 1\right]$$
$$- \left[N_\lambda(\vec{k}+\vec{q}) + 1\right] \left[1 - N_e(\vec{p})\right] N_e(\vec{p}+\vec{q}) \Big] N_\lambda(\vec{k}) \Big\} \tag{7.37}$$

where M is the matrix element discussed in section 7.2 and we have dropped some superfluous subscripts. A similar equation for $N_e(p)$ is

$$\left(\frac{\partial N_e}{\partial t}(\vec{p})\right)_{NLLD} = - \sum_{\vec{k},\vec{q}} \frac{2\pi}{\hbar} |M|^2 \delta \left[7.37\right] \left\{7.37\right\} \tag{7.38}$$

where we have emphasized that the arguments of the δ -functions and the quantity in curly brackets are the same in Eqs. (7.37) and (7.38).

It should be noted that in the classical unit the argument of the δ-function becomes

$$(\Omega_{\vec{k}+\vec{q}} - \Omega_{\vec{k}}) - \vec{v}\cdot\vec{q} = 0 \tag{7.39}$$

The particles which give the damping are those moving with the velocity of a wave whose frequency is the difference frequency and whose wave vector is the difference wave number. This may provide an effective damping mechanism for waves which are linearly weakly damped.

It is easily seen that the right hand sides of Eqs. (7.37) and (7.38) vanish when N_e and N_λ are the equilibrium distributions of Eqs. (4.27) and (4.28). Furthermore, it is not difficult to show that the entropy defined by Eq. (4.23) increases monotonically. The proof is quite similar to that which precedes Eq. (6.30).

Non-linear Landau has been discussed by Kadomtsev.[1]

Although three plasmon interactions cannot conserve energy and momentum, four plasmon processes can. For plasmon-plasmon scattering we have

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \tag{7.40}$$

$$\Omega_1 + \Omega_2 = \Omega_3 + \Omega_4 \tag{7.41}$$

where 1 and 2 refer to the plasmons before scattering and 3 and 4 refer to the plasmons after scattering. Using Eq. (7.6) for the frequencies gives

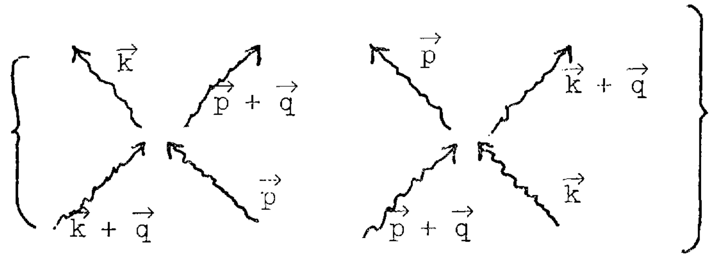
$$k_1^2 + k_2^2 = k_3^2 + k_4^2 \tag{7.42}$$

The scattering is just like that for particles of equal mass with Eq. (7.43) playing the role of the energy conservation equation

$$v_1^2 + v_2^2 = v_3^2 + v_4^2 \tag{7.43}$$

This four-wave interaction gives a correction to the right hand side of Eq. (4.17); namely

$$\left(\frac{\partial N_\lambda}{\partial t}(\vec{k})\right)_{4W} = \sum_{\vec{p},\vec{q}}$$



$$= \sum_{\vec{p},\vec{q}} \frac{2\pi}{\hbar^2} |M|^2 \delta\left[\Omega_{k+q} + \Omega_p - \Omega_k - \Omega_{p+q}\right]$$

$$\left\{ N_\lambda(\vec{k}+\vec{q})\, N_\lambda(\vec{p}) \left[N_\lambda(\vec{k}) + 1\right]\left[N_\lambda(\vec{p}+\vec{q}) + 1\right] - \left[N_\lambda(\vec{k}+\vec{q}) + 1\right]\left[N_\lambda(\vec{p}) + 1\right] N_\lambda(\vec{k})\, N_\lambda(\vec{p}+\vec{q}) \right\} \tag{7.44}$$

where M is the matrix element for the four-wave interaction. Once again it may be shown that the equilibrium distribution (Eq. 4.28), causes the right hand side to vanish and that the entropy defined by Eq. (4.25) increases monotonically.

Zakharov[20] has calculated the matrix element for the four plasmon interaction from the fluid equations for a cold plasma and derived the classical limit of Eq. (7.44). He found that the Rayleigh-Jeans distribution is an equilibrium solution. Rather curiously he found another equilibrium solution with

$$N_{\lambda}(\vec{k}) \sim \frac{1}{k^{13/3}} \tag{7.45}$$

This is analogous to the Kolmogorov spectrum of hydrodynamic turbulence.

b. The magnetized plasma. The interaction of three-plasmons in a cold plasma in a strong magnetic field has been treated by Walters[15,16] using the fluid equations, by Aamodt and Drummond[17] using the Vlasov equations and by Ross[21] using methods equivalent to those of section 7.1. In a strong magnetic field the motion of the electrons is essentially one dimensional. If the electrons are cold the dielectric function is

$$\epsilon(\vec{q},\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2} \frac{q_z^2}{q^2} \tag{7.46}$$

where the z-direction is the direction of the magnetic field. The frequency of a plasma oscillation is given by

$$\Omega_{\vec{q}} = \omega_{pe} \frac{|q_z|}{q} \tag{7.47}$$

Also

$$\left(\frac{\partial}{\partial\omega} \omega\epsilon_1\right)_{\Omega_{\vec{q}}} = 2 \tag{7.48}$$

Eq. (7.7) is easily adapted to this case. Since the motion of the electrons is one dimensional, the integration over velocity space is replaced by an integration over v_z. The vectors $\vec{q}_1$, $\vec{q}_2$ and $\vec{q}_3$ are to be replaced by their z-components except in M_1 M_2 and M_3 where inspection of Eq. (3.23) shows that q_1^2, q_2^2 and q_3^2 should appear. Eq. (7.7) becomes

$$M = \left(\frac{2\pi e^2 \hbar}{V}\right)^{3/2} \left[\frac{\Omega_1 \; \Omega_2 \; \Omega_3}{q_1^2 \; q_2^2 \; q_3^2}\right]^{1/2} \frac{V}{m_e^2} \int dv \; f_e(v)$$

$$\frac{1}{(\Omega_1 - q_{1z}v)\,(\Omega_2 - q_{2z}v)\,(\Omega_3 - q_{3z}v)}$$

$$\left[\frac{q_{1z}\; q_{2z}\; q_{3z}^2}{(\Omega_3 - q_{3z}v)} + \frac{q_{2z}\; q_{3z}\, q_{1z}^2}{(\Omega_1 - q_{1z}v)} + \frac{q_{1z}\; q_{3z}\; q_{2z}^2}{(\Omega_2 - q_{2z}v)}\right] \tag{7.49}$$

We have neglected the ions and have used Eq. (7.48) in Eq. (3.23). If the electrons are cold so that the electron distribution function is given by Eq. (7.17), then Eq. (7.49) gives

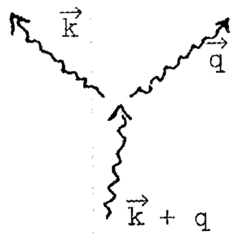
$$M = \left(\frac{2\pi e^2 \hbar}{V}\right)^{3/2} \frac{nV}{m_e^2} \quad \frac{1}{q_1\; q_2\; q_3 \left[\Omega_1 \; \Omega_2 \; \Omega_3\right]^{1/2}}$$

$$\left[\frac{q_{1z}\; q_{2z}\; q_{3z}^2}{\Omega_3} + \frac{q_{2z}\; q_{3z}\; q_{1z}^2}{\Omega_1} + \frac{q_{1z}\; q_{3z}\; q_{2z}^2}{\Omega_2}\right] \tag{7.50}$$

Eq. (7.49) agrees with the result of Ross,[21] and Eq. (7.50) agrees with the result of Walters.[15,16]

This three-wave interaction should add a term to the right hand side of Eq. (4.58). The term is

$$\left(\frac{\partial N_\lambda}{\partial t}(\vec{k})\right)_{3-W} = \sum_{\vec{q}} \Bigg\{$$



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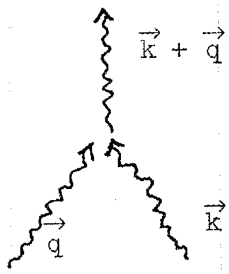

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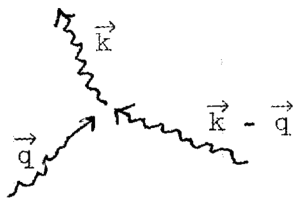

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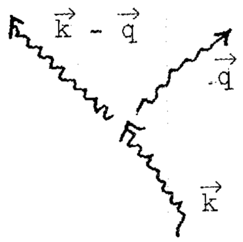


$$\Bigg\}$$

$$= \sum_{\vec{q}} \left[\frac{2\pi}{\hbar^2} \left|M(\vec{k} + \vec{q}, \vec{k}, \vec{q})\right|^2 \delta(\Omega_{\vec{k}+\vec{q}} - \Omega_{\vec{k}} - \Omega_{\vec{q}})\right.$$

$$\left\{N(\vec{k} + \vec{q}) \left[N(\vec{k}) + 1\right] \left[N(\vec{q}) + 1\right] - \left[N_{(\vec{k}+\vec{q})} + 1\right] N(\vec{k})\, N(\vec{q})\right\}$$

$$+ \frac{2\pi}{\hbar} \left|M(\vec{k}, \vec{q}, \vec{k} - \vec{q})\right|^2 \delta(\Omega_{\vec{k}} - \Omega_{\vec{q}} - \Omega_{\vec{k}-\vec{q}})$$

$$\left\{N(\vec{q})\, N(\vec{k} - \vec{q}) \left[N(\vec{k}) + 1\right] - \left[N(\vec{q}) + 1\right]\left[N(\vec{k} - \vec{q}) + 1\right] N(\vec{k}) \qquad (7.51)$$

The matrix elements are given by Eqs. (7.49) or (7.50).

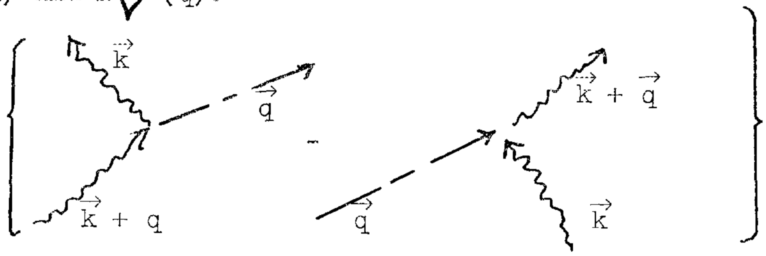
Once again it may be shown that the equilibrium distribution, Eq. (4.28), causes the three-wave interaction to vanish, and also that the entropy defined by Eq. (4.25) increases monotonically.

There should also be wave-particle scattering contributions similar to Eqs. (7.37) and (7.38). Ross[21] has calculated the vertex functions for these by methods equivalent to those of section 7.1. The classical calculation using the Vlasov equations was given by Aamodt and Drummond.[17]

7.4 A Wave-Vector Space Instability

We shall now discuss plasmon-plasmon-phonon interaction for which we derived the vertex function in section 7.1 a. We will denote plasmons by the symbol λ and a wavy line (λ ⟿) and phonons by the symbol ν and a broken line (ν- - - - ➝). We shall neglect all interactions but this three-wave interaction. We now write kinetic equations for $N_\lambda(\vec{k})$ and $N_\nu(\vec{q})$.

$$\frac{\partial N_\nu}{\partial t}(\vec{q}) = \sum_{\vec{k}}$$



$$= \sum_{\vec{k}} \frac{2\pi}{\hbar^2} |M|^2 \delta(\Omega_{\lambda\,\vec{k}+\vec{q}} - \Omega_{\lambda\vec{k}} - \Omega_{\nu\vec{q}})$$

$$\left\{ N_\lambda(\vec{k}+\vec{q}) \left[N_\lambda(\vec{k}) + 1\right] \left[N_\nu(\vec{q}) + 1\right] - \left[N_\lambda(\vec{k}+\vec{q}) + 1\right] N_\lambda(\vec{k})\, N_\nu(\vec{q}) \right\} \tag{7.52}$$

This is almost the same as Eqs. (4.16) and (4.17). Here the plasmons play the role of the electrons and the phonons play the role previously played by the plasmons. The matrix element, of course, is different; here M is given by Eq. (7.7) or Eq. (7.13). An equation quite similar to Eq. (4.19) can be written for $N_\lambda(\vec{k})$. Because it is so similar we shall not write it down. The similarity can be made more striking if we assume the approximations which led to Eq. (7.13). We write

$$\Omega_{\lambda\,\vec{k}+\vec{q}} - \Omega_{\lambda\vec{k}} \simeq \vec{q} \cdot \frac{\partial \Omega_{\vec{k}}}{\partial \vec{k}} = \vec{q} \cdot v_{\vec{k}} \tag{7.53}$$

where $\vec{v}_{\vec{k}}$ is the group velocity of a plasmon

$$N_\lambda(k+q) - N_\lambda(k) \simeq q \cdot \frac{\partial N_\lambda}{\partial \vec{k}}(\vec{k}) \tag{7.54}$$

We introduce spectral densities $P_\lambda(\vec{k}) = \hbar\Omega_{\lambda k} N_\lambda(\vec{k})$ and $P_\nu(\vec{q}) = \hbar\Omega_{\nu_q} N_\nu(\vec{q})$ and use Eq. (4.29f) to obtain

$$\frac{\partial P_\nu}{\partial t}(\vec{q}) = 2\gamma_\nu(\vec{q})\, P_\nu(\vec{q}) + S_\nu(\vec{q}) \tag{7.55}$$

where

$$\gamma_\nu(\vec{q}) = \frac{\pi q^2}{8m_i\, n\, \Omega_{\nu_q}} \int \frac{d^3k}{(2\pi)^3}\, \vec{q} \cdot \frac{\partial P_\lambda}{\partial \vec{k}}(\vec{k})$$

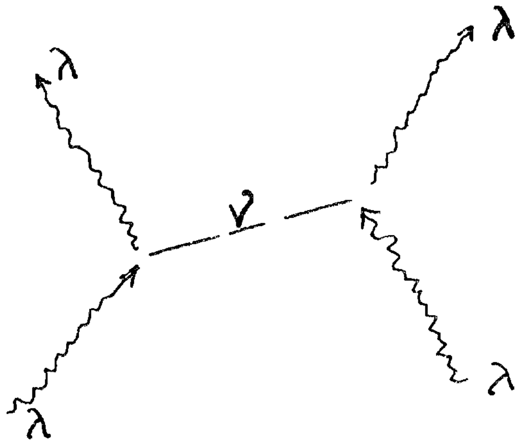
$$\delta(\vec{q} \cdot \vec{v}_k - \Omega_{\nu_{\vec{q}}}) \tag{7.56}$$

$$S_\nu(\vec{q}) = \frac{2\pi q^2}{8m_i\, n\, \omega_{pe}} \int \frac{d^3k}{(2\pi)^3}\, P_\lambda^2(\vec{k})\, \delta(\vec{q} - \vec{v}_k - \Omega_{\nu_{\vec{q}}}) \tag{7.57}$$

which are to be compared with Eqs. (4.30), (4.32) and (4.33).

If the plasmon distribution function is such that more plasmons emit than absorb then $\gamma_\nu(\vec{q})$ will be positive and the energy in the phonons will grow at the expense of the energy in the plasmons. We may call this a wave-vector-space instability in analogy with velocity space instabilities.

Eq. (7.56) without the spontaneous emission term was first derived by Vedenov and Rudakov.[49] Vedenov and Rudakov have also investigated the non-linear interaction between plasmons which occur through the exchange of a virtual phonon; that is, the process described by diagrams like

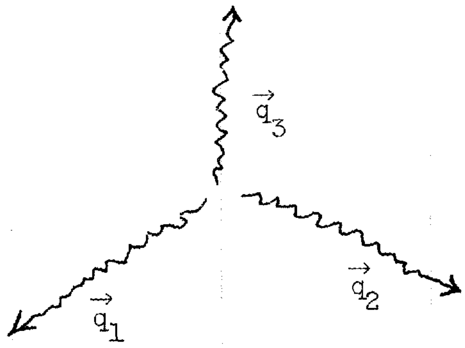


They find the interaction gives rise to an attractive force between plasmons causing plasma oscillations to tend to "bunch." This has also been discussed by Chang and Drummond[51] and by Harris.[50]

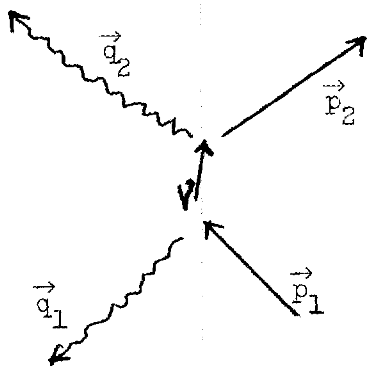
7.5 Negative Energy Waves and Explosive Instabilities

Until now the discussion in this chapter has been restricted to positive energy waves. We remarked in section 2.3 that some of the instabilities predicted by the linear equations could be viewed as the coupling of a negative energy wave to a positive energy wave as illustrated by the diagram in Fig. 7.1a. There are also non-linear interactions involving negative energy waves which give rise to instabilities. These are illustrated by the diagrams of Figs. 7.1b and 7.1c.

(a)



(b)



(c)

Fig. 7.1 Negative energy wave interactions which give rise to instabilities.

In each of the diagrams of Fig. 7.1 one of the waves is assumed to have negative energy. In Fig. 7.1a we shall suppose that it is wave 3 which has negative energy. Then conservation of energy and momentum require

$$\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0 \tag{7.58}$$

$$\Omega_1 + \Omega_2 + (-\Omega_3) = 0 \tag{7.59}$$

Similarly, for Fig. 7.1c

$$\vec{p}_1 = \vec{p}_2 + \vec{q}_1 + \vec{q}_2 \tag{7.60}$$

$$\frac{\hbar^2}{2m_s} p_1^2 = \frac{\hbar^2}{2m_s} p_2^2 + \hbar \Omega_1 + (-\hbar \Omega_2) \tag{7.61}$$

where we have assumed that wave 2 has negative energy.

Kinetic equations similar to Eqs. (7.51) and (7.37) can be written down for the processes of Fig. 7.1b and 7.1c. This has been done by Dikasov, Rudakov and Ryutov,[52] by Aamodt and Sloan[53,54] and by Coppi, Rosenbluth and Sudan.[55] These interactions give rise to explosive instabilities. They are called that because the wave amplitude grows to an infinite amplitude in a finite time. We shall discuss them further in the next chapter.

7.6 Radiation from Plasma Oscillations

In section 7.1 c we derived the vertex function for the process in which two plasmons are destroyed and a photon is created. The conservation of momentum and energy for this process require

$$\vec{q}_1 + \vec{q}_2 = \vec{q}_3 \tag{7.62}$$

$$\Omega_1 + \Omega_2 = \Omega_3 \tag{7.63}$$

Approximating Ω_1 and Ω_2 by ω_{pe} and Ω_3 by $(\omega_{pe}^2 + q_3^2 c^2)^{1/2}$ gives

$$\Omega_3 = (\omega_{pe}^2 + q_3^2 c^2)^{1/2} = 2\,\omega_{pe} \tag{7.64}$$

The frequency of the emitted photons is twice the plasma frequency. Solving Eq. (7.64) for q_3 gives

$$q_3 = q_{30} = \frac{\sqrt{3}\,\omega_{pe}}{c} \tag{7.65}$$

for the wave number of the emitted photon. The energy emitted per unit time due to this process is

$$\frac{\text{energy}}{\text{time}} = \sum_{\vec{u}_3} \int \frac{V\, d^3 q_3}{(2\pi)^3} \int \frac{V\, d^3 q_1}{(2\pi)^3}\, \hbar \Omega_3$$

$$\frac{2\pi}{\hbar^2} |M|^2 \delta(\Omega_3 - \Omega_1 - \Omega_2)\, N_\lambda(\vec{q}_1)\, N_\lambda(\vec{q}_3 - \vec{q}_1) \qquad (7.66)$$

where M is given by Eq. (7.19). To get this we have summed over the polarizations and integrated over the wave vectors of the emitted photons and have integrated over wave vectors of one of the plasmons. The wave vector of the other plasmon is fixed by Eq. (7.62). We can write

$$\delta(\Omega_3 - \Omega_1 - \Omega_2) = \delta\left(\sqrt{\omega_p^2 + c^2\, q_3^2} - 2\omega_{pe}\right)$$

$$= \frac{2\omega_p}{c^2\, q_3}\, \delta(q_3 - q_{30}) \qquad (7.67)$$

and

$$d^3 q_3 = q_3^2\, dq_3\, d\omega \qquad (7.68)$$

where $d\omega$ is the element of solid angle into which the photon is emitted. The sum over polarization can be carried out with the result

$$\sum_{\vec{U}_3} (\vec{q}_1 \cdot \vec{u}_3)^2 = q_1^2 \sin^2 \Theta = q_1^2 \left[1 - \left(\frac{\vec{q}_1 \cdot \vec{q}_{30}}{q_1\, q_{30}}\right)^2\right] \qquad (7.69)$$

where Θ is the angle between $\vec{q}_1$ and $\vec{q}_{30}$ the wave vector of the emitted photon. Carrying out the sum over $\vec{u}_3$ and the integration over dq_3 in Eq. (7.66) and then dividing by $d\omega$ gives the energy emitted per unit time per unit solid angle. The result is

$$\frac{dW}{dt\, d\omega} = \frac{\sqrt{3}}{(2\pi)^4} \frac{e^2 V}{m^2 c^3} \int d^3 q_1 \frac{P_\lambda(\vec{q}_1)\, P_\lambda(\vec{q}_3 - \vec{q}_1)}{|\vec{q}_3 - \vec{q}_1|^2}$$

$$\left[1 - \left(\frac{\vec{q}_1 \cdot \vec{q}_{30}}{q_1\, q_{30}}\right)^2\right]\left[q_1^2 - |\vec{q}_{30} - \vec{q}_1|^2\right]^2 \qquad (7.70)$$

We have expressed the result in terms of classical quantities. Classical calculations of this process have been published by Sturrock,[56] Aamodt and Drummond,[57] Boyd,[58] Birmingham, Dawson and Oberman,[59] and Tidman and Dupree.[60] Tidman[61] has discussed the relevance of this process to burst of radio emission from the sun which seem to occur at twice the plasma frequency.

CHAPTER 8. INTERACTION OF MONOCHROMATIC WAVES

In previous chapters we have been concerned with a continuous spectrum of waves. If there are only a few monochromatic waves in the plasma, then use of the Fermi Golden Rule to calculate transition probabilities is no longer justified and other techniques must be used.

If there are a large number of quasi-particles in the same state then one is justified as treating the wave as a classical wave. The creation and destruction operators, $B_{\vec{k}\sigma}$, $B^{+}_{\vec{k}\sigma}$, $A_{\vec{k}\sigma}$, and $A^{+}_{\vec{k}\sigma}$ introduced in Chapter 3 may be treated as classical wave amplitudes and their complex conjugates (which they were before we quantized the system). For simplicity we shall restrict our discussion to the interaction of three waves. We shall keep in the Hamiltonian only terms involving these three waves and shall discard all of the rest. Then the Hamiltonian can be written as

$$H = H_o + H' \tag{8.1}$$

$$H_o = S_1 \hbar \Omega_1 C_1^+ C_1 + S_2 \hbar \Omega_2 C_2^+ C_2 + S_3 \hbar \Omega_3 C_3^+ C_3 \tag{8.2}$$

$$\begin{aligned} H' &= \hbar M_1 C_1^+ C_2^+ C_3^+ + \hbar M_1^* C_1 C_2 C_3 \\ &+ \hbar M_2 C_1^+ C_2^+ C_3 + \hbar M_2^* C_1 C_2 C_3^+ \\ &+ \hbar M_3 C_1^+ C_2 C_3^+ + \hbar M_3^* C_1 C_2^+ C_3 \\ &+ \hbar M_4 C_1^+ C_2 C_3 + \hbar M_4^* C_1 C_2^+ C_3^+ \end{aligned} \tag{8.3}$$

Here C_1, C_2 and C_3 are the amplitudes of the waves and C_1^+, C_2 and C_3^+ are their complex conjugates. The frequencies of the three waves are Ω_1, Ω_2 and Ω_3 in the absence of interaction. The signs of the energies of the three waves are denoted by S_1, S_2 and S_3. In H' have been included all of the three-wave interactions. The vertex functions M_1, M_2, M_3 and M_4 and their complex conjugates M_1^* etc. have supposedly been calculated by the methods of Chapter 7. We have written H' so that it is real.

The equations of motion for the C_i's are obtained from the Heisenberg equations of motion

$$\frac{d}{dt} C_i = - \frac{i}{\hbar} \left[C_i, H\right]_- \tag{8.4}$$

using the Boson commutation relations, Eqs. (3.12) and (3.13). After the equations of motion are obtained they are interpreted as ordinary differential equations for the wave amplitudes rather than operator equations. We find

$$\dot{C}_1 = - i S_1 \Omega_1 C_1 - i M_1 C_2^+ C_3^+ - i M_2 C_2^+ C_3$$

$$- i M_3 C_2 C_3^+ - i M_4 C_2 C_3 \tag{8.5}$$

$$\dot{C}_2 = - i S_2 \Omega_2 C_2 - i M_1 C_1^+ C_3^+ - i M_2 C_1^+ C_3$$

$$- i M_3^* C_1 C_3 - i M_4^* C_1 C_3^+ \tag{8.6}$$

$$\dot{C}_3 = - i S_3 \Omega_3 C_3 - i M_1 C_1^+ C_2^+ - i M_2^* C_1 C_2$$

$$- i M_3 C_1^+ C_2 - i M_4^* C_1 C_2^+ \tag{8.7}$$

where the dot denotes a derivative with respect to time. Some special cases will now be considered.

8.1 Resonant Interaction of Positive Energy Waves

We shall assume that $S_1 = S_2 = S_3 = +1$, so all three are positive energy waves. It is convenient to let

$$C_1(t) = a_1(t)\, e^{-i\Omega_1 t} \tag{8.8}$$

$$C_2(t) = a_2(t)\ e^{-i\Omega_2 t} \tag{8.9}$$

$$C_3(t) = a_3(t)\ e^{-i\Omega_3 t} \tag{8.10}$$

Eq. (8.5) becomes

$$\begin{aligned}\dot{a}_1 = &-i\, M_1\, a_2^+\, a_3^+\, e^{i(\Omega_1 + \Omega_2 - \Omega_3)t} \\ &- i\, M_2\, a_2^+\, a_3\, e^{i(\Omega_1 + \Omega_2 - \Omega_3)t} \\ &- i\, M_3\, a_2^+\, a_3\, e^{i(\Omega_1 - \Omega_2 + \Omega_3)t} \\ &- i\, M_4\, a_2\, a_3\, e^{i(\Omega_1 - \Omega_2 - \Omega_3)t}\end{aligned} \tag{8.11}$$

There are similar equations for $\dot{a}_2$ and $\dot{a}_3$. Now we shall suppose that

$$\Omega_1 + \Omega_2 = \Omega_3 \tag{8.12}$$

so that the diagrams of Fig. 8.1 are the dominant processes.

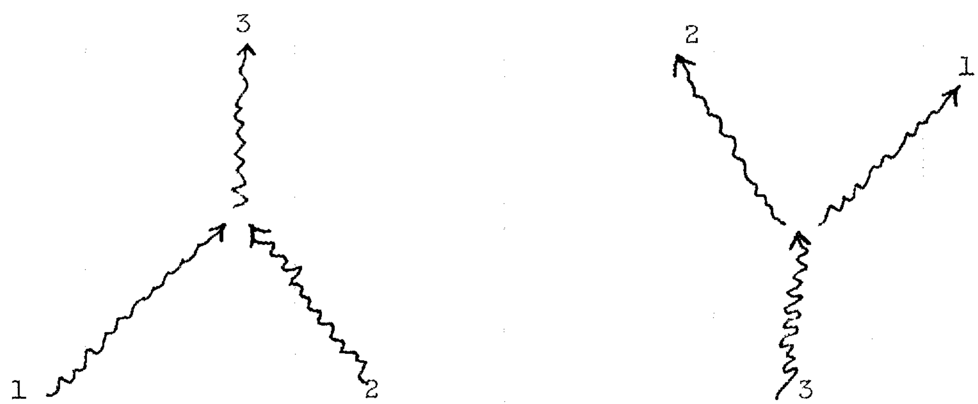


Fig. 8.1

All of the terms in Eq. (8.11) except one will contain as factors rapidly oscillating exponentials. If a_1, a_2 and a_3 vary slower then these terms will approximately average to zero and we will be justified in keeping only the term containing $M_2\, a_2^+\, a_3$. Assuming that this is the case, we write

$$\dot{a}_1 = - i M_2 a_2^+ a_3 \tag{8.13}$$

Similarly, when the same approximation is made in the equations for $\dot{a}_2$ and $\dot{a}_3$ we obtain

$$\dot{a}_2 = - i M_2 a_1^+ a_3 \tag{8.14}$$

$$\dot{a}_3 = - i M_2^* a_1 a_2 \tag{8.15}$$

Now, we see from Fig. 8.1 that whenever a quasi-particle of type 2 is destroyed, one of type 3 is created. This leads us to expect that

$$a_2^+ a_2 + a_3^+ a_3 = \text{constant} \tag{8.16}$$

Indeed, using Eqs. (8.14) and (8.15) we easily find that

$$\frac{d}{dt} (a_2^+ a_2 + a_3^+ a_3) = 0 \tag{8.17}$$

Similarly, conservation of energy would imply that

$$\Omega_1 a_1^+ a_1 + \Omega_2 a_2^+ a_2 + \Omega_3 a_3^+ a_3 = \text{constant} \tag{8.18}$$

Again, the equations of motion may be used to show that this is indeed true.

Next we write

$$a_i(t) = b_i(t) e^{-i\phi_i(t)} \quad i = 1, 2, 3 \tag{8.19}$$

where $b_i(t)$ and $\phi_i(t)$ are real functions. Eq. (8.13) gives

$$\dot{b}_i - i \dot{\phi}_1 b_1 = - i M_2 b_2 b_3 e^{i(\phi_1 + \phi_2 - \phi_3)}$$

$$= - i |M_2| b_2 b_3 e^{i(\phi_1 + \phi_2 - \phi_3 - \sigma)} \tag{8.20}$$

where we have let

$$M_2 = |M_2| e^{-i\sigma} \tag{8.21}$$

The real and imaginary parts of Eq. (8.20) yield

$$\dot{b}_i = |M_2| b_2 b_3 \sin \Phi \tag{8.22}$$

$$\dot{\phi}_1 = |M_2| \frac{b_2 b_3}{b_1} \cos \Phi \tag{8.23}$$

where

$$\Phi = \phi_1 + \phi_2 - \phi_3 - \sigma \tag{8.24}$$

In a similar manner we obtain

$$\dot{b}_2 = |M_2|\ b_1\ b_3 \sin \Phi \tag{8.25}$$

$$\dot{b}_3 = - |M_2| b_1\ b_2 \sin \Phi \tag{8.26}$$

The three equations like Eq. (8.23) can be combined to give

$$\dot{\Phi} = |M_2| \left(\frac{b_2\ b_3}{b_1} + \frac{b_1\ b_3}{b_2} - \frac{b_1\ b_2}{b_3} \right) \cos \Phi \tag{8.27}$$

The constants of the motion, Eqs. (8.16) and (8.17), may be used to show that

$$b_2^2 = b_1^2 + b_{20}^2 - b_{10}^2 \tag{8.28}$$

$$b_3^2 = b_{10}^2 + b_{30}^2 - b_1^2 \tag{8.29}$$

where b_{10}, b_{20} and b_{30} are the values of b_1, b_2 and b_3 at $t = 0$. These relations may be used to eliminate b_2 and b_3. Dividing Eq. (8.27) by Eq. (8.22) gives

$$\frac{d\ \Phi}{db_1} = \left(\frac{1}{b_1} + \frac{b_1}{b_2^2} - \frac{b_1}{b_3^2} \right) \frac{\cos \Phi}{\sin \Phi} \tag{8.30}$$

from which

$$\frac{\sin \Phi}{\cos \Phi}\ d\ \Phi = \left(\frac{1}{b_1} + \frac{b_1}{b_2^2} - \frac{b_1}{b_3^2} \right) db_1 \tag{8.31}$$

Using Eqs. (8.28) and (8.29), both sides of this equation can be integrated to obtain

$$b_1\ b_2\ b_3 \cos \Phi = \text{constant} = \Gamma \tag{8.32}$$

Instead of giving the general solution we shall consider a simple special case. Suppose that at $t = 0$, $b_1 = 0$. Then $\Gamma = 0$. At later times when b_1, b_2 and b_3 are all non-zero, Eq. (8.32) can be satisfied only by $\cos \Phi = 0$. Then $\sin \Phi = \pm 1$ and Eq. (8.22) becomes

$$\frac{db_1}{dt} = \pm |M_2| \, b_2 \, b_3$$

$$= \pm |M_2| \sqrt{b_1^2 + b_{20}^2} \sqrt{b_{30}^2 - b_1^2} \qquad (8.33)$$

Then

$$\pm \int_o^{b_1} \frac{db_1}{\sqrt{1 + \frac{b_1^2}{b_{20}^2}} \sqrt{b_{30}^2 - b_1^2}} = |M_2| \, b_{20} t \qquad (8.34)$$

The result can be expressed in terms of the Jacobi eliptic function[62]

$$b_1(t) = b_{30} \text{ sn } (|M_2| \, b_{20} t) \qquad (8.35)$$

The other quantities of interest b_2 and b_3 are obtained from Eq. (8.28) and (8.29) with $b_{10} = 0$. The intensity of wave number 1 is $b_1^2(t)$; it oscillates between its initial value of zero and b_{30}^2. The intensity of wave number 2 oscillates between b_{20}^2 and $b_{20}^2 + b_{30}^2$. The intensity of wave number 3 oscillates between b_{30}^2 and zero.

The general solution has been given by Danilkin[63] and by Sugihara.[64] The interaction of transverse electromagnetic waves with longitudinal plasma oscillations and ion sound waves has been discussed by Montgomery,[65] Danilkin,[63] Sugihara[64] and Dolinsky and Goldman.[66]

8.2 Parametric Excitation of Waves[67-69]

Some simplification of the equations of the last section is obtained if it is assumed that the amplitude of one of the waves is held fixed by some means. Let us suppose that

$$|C_1|^2 = |a_1|^2 = \text{constant} \qquad (8.36)$$

We shall assume that Eq. (8.12) is almost but not quite satisfied and write

$$\Omega_1 + \Omega_2 - \Omega_3 = \Delta \tag{8.37}$$

where Δ is a small quantity. Then, Eqs. (8.14) and (8.15) are replaced by

$$\dot{a}_2 = - i M_2 a_1^+ a_3 e^{i\Delta t} \tag{8.38}$$

$$\dot{a}_3 = - i M_2^* a_1 a_2 e^{-i\Delta t} \tag{8.39}$$

There is no equation for a_1 since it is assumed to be given. We let

$$a_1 = |a_1| e^{-i\alpha} \tag{8.40}$$

$$\dot{a}_2 = - i |M_2 a_1| a_3 e^{i(\Delta t + \alpha - \sigma)} \tag{8.41}$$

$$\dot{a}_3 = - i |M_2 a_1| a_2 e^{-i(\Delta t + \alpha - \sigma)} \tag{8.42}$$

These are coupled linear equations with variable coefficients. A solution is easily obtained. If we assume that

$$a_3(t) = b_3 e^{i\nu t} , \tag{8.43}$$

then inspection of Eq. (8.41) shows that a_2 must have the form

$$a_2(t) = b_2 e^{i[(\nu + \Delta)t + \alpha - \sigma]} \tag{8.44}$$

Substituting Eqs. (8.43) and (8.44) into Eqs. (8.41) and (8.42) gives the linear algebraic equations

$$(\nu + \Delta) b_2 + |M_2 a_1| b_3 = 0 \tag{8.45}$$

$$|M_2 a_1| b_1 + \nu b_3 = 0 \tag{8.46}$$

Setting the determinant of the coefficients equal to zero and solving the quadratic equation for ν gives the two roots

$$\nu_{\pm} = - \frac{1}{2}\Delta \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + |M_2 a_1|^2} \tag{8.47}$$

These frequencies are always real.

The general solution for $a_2(t)$ will have the form

$$a_2(t) = A_+ \; e^{i(\nu_+ + \Delta)t} + A_i \; e^{i(\nu_- + \Delta)t} \tag{8.48}$$

where A_+ and A_- are constants which must be determined by the initial conditions. As an example let us suppose that at $t = 0$

$$a_2(o) = 0 \tag{8.50}$$

$$a_3(o) = a_{30} \tag{8.51}$$

A simple calculation gives

$$\left|a_2(t)\right|^2$$

$$= \left|a_{30}\right|^2 \frac{\left|M_2 \; a_1\right|^2}{\left(\frac{\Delta}{2}\right)^2 + \left|M_2 \; a_1\right|^2} \sin^2 \sqrt{\left(\frac{\Delta}{2}\right)^2 + \left|M_2 \; a_1\right|^2} \; t \tag{8.52}$$

From Eq. (8.16) we find

$$\left|a_3(t)\right|^2 = \left|a_{30}\right|^2 - \left|a_2(t)\right|^2 \tag{8.53}$$

Note that in the case of exact resonance, $\Delta = 0$, these equations reduce to

$$\left|a_2(t)\right|^2 = \left|a_{30}\right|^2 \sin^2 \left|M_2 \; a_1\right| t \tag{8.54}$$

$$\left|a_3(t)\right|^2 = \left|a_{30}\right|^2 \cos^2 \left|M_2 \; a_1\right| t \tag{8.55}$$

A quite different result is obtained if it is supposed that wave number 3 has a fixed intensity and let

$$a_3 = \left|a_3\right| e^{-i\alpha} \tag{8.56}$$

Then the equations become

$$\dot{a}_1 = - i \left|M_2 \; a_3\right| a_2^+ \; e^{i(\Delta t - \alpha - \sigma)} \tag{8.57}$$

$$\dot{a}_2 = - i \left|M_2 \; a_3\right| a_1^+ \; e^{i(\Delta t - \alpha - \sigma)} \tag{8.58}$$

If we assume that

$$a_2 = b_2 \, e^{i\nu t} \,, \tag{8.59}$$

then inspection of Eq. (8.57) shows that a_1 must have the form

$$a_1 = b_1 \, e^{i[(\Delta - \nu)\, t - \alpha - \sigma]} \tag{8.60}$$

Substituting Eqs. (8.59) and (8.60) into Eqs. (8.57) and (8.58) gives

$$(\Delta - \nu)\, b_1 + |M_2 \, a_3|\, b_2 = 0 \tag{8.61}$$

$$|M_2 \, a_3|\, b_1 + \nu \, b_2 = 0 \tag{8.62}$$

Setting the determinant of the coefficients equal to zero and solving the quadratic equation for gives the two roots

$$\nu_{\pm} = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 - |M_2 \, a_3|^2} \tag{8.63}$$

If we assume the initial conditions

$$a_1(o) = 0 \tag{8.64}$$

$$a_2(o) = a_{20} \,, \tag{8.65}$$

then we find

$$|a_1(t)|^2 = |a_{20}|^2 \left| \frac{(M_2 \, a_3)^2}{\left(\frac{\Delta}{2}\right)^2 - |M_2 \, a_3|^2} \right| \sin^2 \sqrt{\left(\frac{\Delta}{2}\right)^2 - |M_2 \, a_3|^2} \; t \tag{8.66}$$

$$|a_2(t)|^2 = |a_{20}|^2 + |a_1(t)|^2 \tag{8.67}$$

If $|M_2 \, a_3|^2 > (\Delta/2)^2$ then the frequencies given by Eq. (8.63) are complex. The sine function in Eq. (8.66) is replaced by a hyperbolic sine and both $|a_1|^2$ and $|a_2|^2$ grow exponentially. Energy is continuously fed from wave number 3 to waves 1 and 2.

8.3 Explosive Instabilities[52-55]

Finally, we consider the case that one of the waves, say wave number 3, has a negative energy. The terms in Eq. (8.3) which we wish to consider are those containing $C_1^+ \ C_2^+ \ C_3^+$ and $C_1 \ C_2 \ C_3$. The corresponding diagrams are shown in Fig. 8.2.

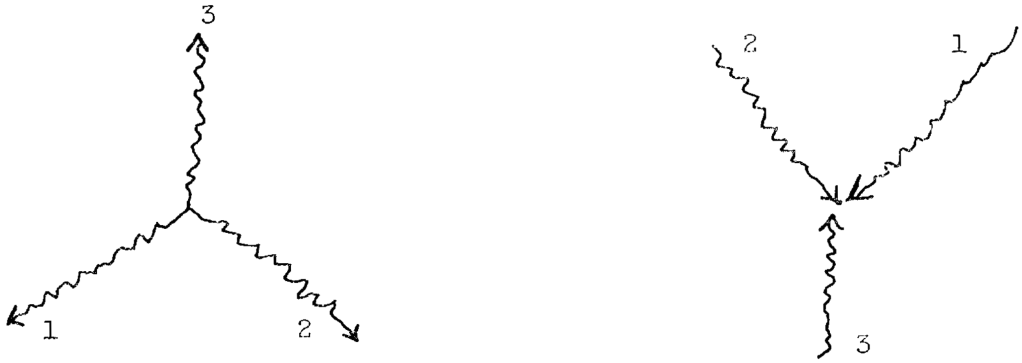


Fig. 8.2

Conservation of energy requires

$$S_1 \Omega_1 + S_2 \Omega_2 + S_3 \Omega_3 = 0 \tag{8.68}$$

We assume $S_1 = S_2 = +1$ and $S_3 = -1$ so this is just

$$\Omega_1 + \Omega_2 - \Omega_3 = 0 \tag{8.69}$$

Let $C_1(t)$ and $C_2(t)$ be given by Eqs. (8.8) and (8.9) and let

$$C_3(t) = a_3(t) \ e^{+i\Omega_3 t} \tag{8.70}$$

As before we neglect terms in the equations of motion which have rapidly oscillating factors. Now the equations of motion become

$$\dot{a}_1 = - i M_1 a_2^+ a_3^+ \tag{8.71}$$

$$\dot{a}_2 = - i M_1 a_1^+ a_3^+ \tag{8.72}$$

$$\bar{a}_3 = - i M_1 a_1^+ a_2^+ \tag{8.73}$$

Inspection of Fig. 8.2 suggests that

$$\frac{d}{dt} a_1^+ a_1 = \frac{d}{dt} a_2^+ a_2 = \frac{d}{dt} a_3^+ a_3 \tag{8.74}$$

and this is easily shown to be the case.

The analysis of the equation of motion is very similar to that of section 8.1. We use Eq. (8.19) and find

$$\dot{b}_1 = \left| M_1 \right| b_2\, b_3 \sin \Phi \tag{8.75}$$

$$\dot{\Phi} = \left|M_1\right| \left(\frac{b_2\, b_3}{b_1} + \frac{b_1\, b_3}{b_2} + \frac{b_1\, b_2}{b_3} \right) \cos \Phi \tag{8.76}$$

where

$$\Phi = \phi_1 + \phi_2 + \phi_3 - \sigma \tag{8.77}$$

Note that Eqs. (8.76) and (8.77) differ from Eqs. (8.27) and (8.24) by the sign of one term. From Eq. (8.74)

$$b_1^2 - b_2^2 = \text{constant} = b_{10}^2 - b_{20}^2 \tag{8.78}$$

$$b_1^2 - b_3^2 = \text{constant} = b_{10}^2 - b_{30}^2 \tag{8.79}$$

When Eq. (8.76) is divided by Eq. (8.75) one obtains

$$\frac{\sin \Phi}{\cos \Phi}\, d\,\Phi = db_1 \left(\frac{1}{b_1} + \frac{b_1}{b_2^2} + \frac{b_1}{b_3^2} \right)$$

$$= \left(\frac{db_1}{b_1} + \frac{db_2}{b_2} + \frac{db_3}{b_3} \right) \tag{8.80}$$

so

$$b_1\, b_2\, b_3 \cos \Phi = \Gamma = \text{constant} \tag{8.81}$$

as before. Let us suppose that $b_1 = 0$ at $t = 0$; then $\Gamma = 0$ and $\cos \Phi = 0$ at future times. Eq. (8.75) gives

$$\dot{b}_1 = \left| M_1 \right| \sqrt{b_{20}^2 + b_1^2}\ \sqrt{b_{30}^2 + b_1^2} \tag{8.82}$$

from which

$$\int_0^{b_1} \frac{db_1}{\sqrt{b_{20}^2 + b_1^2}\ \sqrt{b_{30}^2 + b_1^2}} = \left|M_1\right| t \tag{8.83}$$

Note that b_1 grows to an infinite amplitude in a finite time since

$$\int_0^{\infty} \frac{db_1}{\sqrt{b_{20}^2 + b_1^2}\ \sqrt{b_{30}^2 + b_1^2}} = \text{finite} \tag{8.84}$$

The integral in Eq. (8.83) becomes elementary if $b_{20} = b_{30}$. Then

$$\int_0^{b_1} \frac{db_1}{b_{20}^2 + b_1^2} = \frac{1}{b_{20}} \tan^{-1} \frac{b_1}{b_{20}} \tag{8.85}$$

so

$$b_1(t) = b_{20} \tan b_{20} \left| M_1 \right| t \tag{8.86}$$

The amplitude becomes infinite when

$$t = \frac{\pi}{2b_{20} \left| M_1 \right|} \tag{8.87}$$

Such instabilities which grow to an infinite amplitude in finite times are called explosive. They are probably stabilized by the instability changing the distribution function so that the frequencies change so that Eq. (8.69) is not satisfied or by changing the sign of the energy. Very little is known quantitatively about the stabilization of these instabilities.

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REFERENCES

1. B. B. Kadomtsev, Plasma Turbulence (Academic Press, New York, 1965).
2. A. A. Vedenov, Theory of Turbulent Plasma, Achievements of Science: Plasma Physics, 1965 (English translation by Z. Lerman available from U. S. Department of Commerce).
3. V. N. Tsytovich, Soviet Physics Uspekhi 9, 805 (1967).
4. T. H. Dupree, Phys. Fluids 9, 1773 (1966).
5. Leo P. Kadanoff and Gordon Baym, Quantum Statistical Mechanics (W. A. Benjamin, New York, 1962).
6. D. Pines, The Many Body Problem (W. A. Benjamin, New York, 1961).
7. A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (Prentice-Hall, New Jersey, 1963).
8. D. F. DuBois, Lectures in Theoretical Physics, IXC, Edited by Wesley E. Britten (Gordon and Breach, Inc., New York, 1967).
9. D. Pines and J. R. Schrieffer, Phys. Rev. 125, 804 (1962).
10. W. E. Drummond and D. Pines, Nucl. Fusion Suppl. Pt. 3, 1049 (1962).
11. A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Nucl. Fusion Suppl. Pt. 2, 465 (1962).
12. H. H. Wyld, Jr. and D. Pines, Phys. Rev. 127, 1851 (1962).
13. R. Balescu, Phys. Fluids 3, 52 (1960).
14. A. Lenard, Ann. Phys. (New York), 10, 390 (1960).

15. G. M. Walters and E. G. Harris, Phys. Fluids 11, 112 (1968).

16. G. M. Walters, Ph. D. Thesis, The University of Tennessee, 1967.

17. R. E. Aamodt and W. E. Drummond, Phys. Fluids 7, 1816 (1964).

18. S. Krishan, Plasma Physics 10, 201 (1968).

19. S. Krishan and A. A. Selim, Plasma Physics 10, 931 (1968).

20. V. E. Zakharov, Soviet Phys. JETP 24, 455 (1967).

21. David W. Ross, University of Texas Center for Plasma Physics and Thermonuclear Research Report ORO-3458-12, June 1968 (to be published).

22. A. Gailitis, L. M. Gorbunov, L. Kovrishnich, V. V. Pustovalov, V. P. Silin and V. N. Tsytovich, Proceedings of Seventh International Congress on Phenomena in Ionized Gases (Gradevinska Knjiga Publishing House, Beograd, 1966), II, 471.

23. A. S. Davidov, Quantum Mechanics (Addison-Wesley, Reading, Mass., 1965), Chapters 14 and 15.

24. A. L. Alekseev and Yu. P. Nikitin, Soviet Physics JETP 23, 608 (1966).

25. Taro Kihara, Asamu Aono and Taro Dodo, Nuclear Fusion 2 (1962).

26. J. H. Malmberg and C. B. Wharton, Phys. Rev. Letters, 19, 775 (1967).

27. Thomas O'Neil, Phys. Fluids 8, 2255 (1965).

28. L. M. Altshul and V. I. Karpman, Soviet Phys. JETP 22, 361 (1966).

29. N. Rostoker, Nuclear Fusion 1, 101 (1961).

30. S. Ichamaru, Ann. Phys. (New York) 20, 78 (1962).

31. M. N. Rosenbluth and N. Rostoker, Phys. Fluids 5, 776 (1962).

32. J. A. Armstrong, N. Bloembergen, J. Dusuing and P. S. Pershan, Phys. Rev. 127, 1918 (1962).

33. G. Bekefi, Radiation Processes in Plasmas (John Wiley, New York, 1966).

34. Oldwig Von Roos, Phys. Rev. 119, 1174 (1960).

35. E. Wigner, Phys. Rev. 40, 749 (1932).

36. L. D. Landau and E. M. Lifschitz, Statistical Physics (Pergamon Press, London, 1958), Chapter 5.

37. Thomas P. Armstrong, Phys. Fluids 10, 1269 (1967).

38. A. B. Mikhailovskii and A. V. Timofeev, Zh. Eksp. Teor. Fiz. 44, 919 (1963) (Soviet Physics JETP 17, 626 (1963)).

39. This topic has been treated in many papers. See A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko and K. N. Stepanov, Collective Oscillations in a Plasma (The M. I. T. Press, Cambridge, Mass., 1967) and references given in sections 20, 21 and 22.

40. L. Boltzmann, Wien. Sitz. 66, 275 (1872).

41. Ionized gases are treated by the Boltzmann equation in S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge University Press, Second Edition, 1952), Chapter 18.

42. E. A. Frieman and P. Rutherford, Ann. Phys. (N. Y.) 28, 134 (1964).

43. K. Nishikawa and Y. Osaka, Progr. Theoret. Phys. (Kyoto) 33, 402 (1965).

44. J. Price, Phys. Fluids 10, 1923 (1967).

45. A. L. Rogister and C. Oberman, J. Plasma Phys. 2, 33 (1968).

46. C. S. Wu, Phys. Fluids 11, 1733 (1968).

47. Tadatsugu Hatori, Phys. Fluids 11, 1584 (1968).

48. E. G. Harris, International Symposium on Fluctuations and Diffusion in Plasmas, Princeton University, June 1967. Paper C5.

49. A. A. Vedenov and L. I. Rudakov, Dokl. Akad. Nauk. SSSR 159, 767 (1964) (Soviet Physics Doklady 9, 1073 (1965)).

50. E. G. Harris, Oak Ridge National Laboratory Report 3871 (1965).

51. David B. Chang and William E. Drummond, Phys. Fluids 9, 2254 (1966).

52. V. M. Dikasov, L. I. Rudakov and D. D. Ryutov, Soviet Phys. JETP 21, 608 (1965).

53. R. E. Aamodt and M. L. Sloan, Phys. Rev. Letters 19, 1227 (1967).

54. M. L. Sloan, University of Texas Center for Plasma Physics and Thermonuclear Research Report CPP12, ORO-3458-14, 1968.

55. M. N. Rosenbluth, B. Coppi and R. N. Sudan, Paper CN-24/E-13 presented at the Conference on Plasma Physics and Controlled Nuclear Fusion Research, Novosibirsk USSR, August 1968 (to be published).

56. P. A. Sturrock, J. Nucl. Energy C (Plasma Phys.) 2, 158 (1961).

57. R. E. Aamodt and W. E. Drummond, J. Nucl. Energy C (Plasma Phys.) 6, 147 (1964).

58. T. J. M. Boyd, Phys. Fluids 7, 59 (1964).

59. T. Birmingham, J. Dawson and C. Oberman, Phys. Fluids 8, 297 (1965).

60. D. A. Tidman and T. H. Dupree, Phys. Fluids 8, 1860 (1965).

61. D. A. Tidman, Planetary Space Sci. 13, 781 (1965).

62. Handbook of Mathematical Functions (National Bureau of Standards, 1964), Formula 17.2.7.

63. I. S. Danilkin, Soviet Phys.-Tech. Phys. 11, 605 (1966).

64. Ryo Sugihara, Phys. Fluids 11, 178 (1968).

65. D. Montgomery, Physica 31, 693 (1965).

66. A. Dolinsky and R. Goldman, Phys. Fluids 10, 1251 (1967).

67. V. P. Silin, Soviet Phys. JETP 21, 1127 (1965).

68. Yu. M. Aliev and V. P. Silin, Soviet Phys. JETP 21, 601 (1965).

69. David Montgomery and Igor Alexeff, Phys. Fluids 9, 1362 (1966).

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1. I. Alexeff
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59-60. Thermonuclear Division Library

61-62. CR Library

63. Document Reference Section

64-87. Laboratory Records

88. ORNL-RC
89. ORNL Patent Office
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