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STABILITY OF THE GUIDING CENTER PLASMA
(Thesis)

David B. Nelson

A dissertation in the department of mathematics submitted to the faculty of the Graduate School of Arts and Science in partial fulfillment of the requirements for the degree of Doctor of Philosophy at New York University.

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OAK RIDGE NATIONAL LABORATORY
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Abstract

Using a variational principal for the guiding center plasma, the implications for absolute stability of local stability plus interchange stability are examined. The geometry chosen for this analysis is open ended, with magnetic lines free to interchange at the ends.

The method of analysis uses the fact that any admissable variation away from an equilibrium can be decomposed into an interchange plus a variation which vanishes at one end of the plasma domain. Then the second variation, as a quadratic form, is accordingly separated into three terms which are examined separately.

We find that if the spatial magnetic field and plasma gradients are sufficiently small, then local stability plus a strong form of interchange stability do suffice for absolute variational stability. This strong form is actually weaker than most of the sufficient conditions for interchange stability which have previously been derived.

It is shown that for a plasma which satisfies these criteria stability is independent of the plasma length; i.e. lengthening such a plasma does not destroy its stability.

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1. Introduction

This thesis is addressed to the general problem of stability of a plasma adequately described by the zero order guiding center theory. This theory is rigorously valid only in the ideal limit where the Larmor radius vanishes, since the plasma particles are identified with their guiding centers. However, it has been shown [1,2] that the true particle orbits are approximated asymptotically (in the sense of Poincaré) by the guiding center equation of motion. Guiding center motion is Hamiltonian, akin to the motion of a bead on a wire; hence we can write fluid equations (Liouville's equation) for a collection of such particles [3,4,5]. The salient features of such a fluid are that it is perfectly conducting, flux preserving (which implies the particles are "stuck" to a field line and move with it), and is macroscopically described by two pressure components, one parallel and another perpendicular to the magnetic field.

We are concerned with the stability of static equilibria of such a plasma when confined by a magnetic field of "mirror machine" (i.e., open ended as opposed to toroidal) geometry. In particular, we investigate the relationship between stability against interchanges (these are motions which leave the magnetic field unchanged and merely alter the assignment of plasma) and general stability of low pressure plasma. There is considerable confusion in the literature on this score. The usual argument which relates the two types of stability runs somewhat as follows: the plasma is stable if its energy can only be increased by any motion. Now the energy change is composed of a magnetic variation plus a plasma variation. The equilibrium field of a low pressure plasma is essentially a vacuum field, hence a minimum energy configuration. For any motion which is not an interchange the plasma variation will be dominated by the positive magnetic variation below some pressure level. Thus, the only possible unstable motions for a low pressure plasma are interchanges, and in this case interchange stability implies absolute stability.

The fallacy in this argument has been pointed out in [6]. In particular, we must always require local stability (defined in this thesis on page 10). But if we do adjoin this requirement then under suitable conditions we will show that it and interchange stability do imply absolute stability at low pressure (or sometimes even at moderate pressure). Thus the physical intuition can be salvaged and made precise.

Caution should be given that our analysis, while precise analytically, encompasses only a small corner of plasma physical reality. For one thing, the guiding center model, while sophisticated magnetohydrodynamics, distinguishes only some of the microinstabilities which plague plasma physics. For another, we confine ourselves to variational stability against infinitesimal perturbation. Our definition of stability is really positivity of the second variation, which is at most equivalent to boundedness of solutions to the linearized equations of motion. But in general our definition is even more restrictive than this, for the equivalence requires self adjointness of the associated operator and a discreet spectrum in the neighborhood of the origin. Explicit examples can be given which violate these criteria.

Although many of our results, most notably those in section 5 on the plasma with lines tied at one end, will generalize easily to the two fluid case, we consider only a neutral single fluid plasma in which charge separation is neglected and it is assumed that $E \cdot B = 0$. We assume that the magnetic moment μ is constant for a particle but we do not require constancy of the second adiabatic invariant J.

The general plan of the thesis is as follows: in section 2 the variational principle and the geometry of the problem are discussed along with the relevant boundary conditions and some of their implications. Some of the more important concepts are introduced and clarified.

Section 3 is devoted to a derivation of the second variation in a form suited to the problem. This is by far the most tedious of all the sections since many of the formulas are of necessity long and involved. The most important equation in this section is (3.6) which displays the form of the second variation which we use. It should be mentioned that there is exact equivalency between (3.6) and the original form; no approximations have been made.

Section 4 outlines the method of attack on the problem. This involves splitting a general variation into two parts, one an interchange and the other vanishing on one end. Then the second variation, as a quadratic form, is likewise split into a second variation for each of the parts plus a cross term.

In the following sections each of the three terms is analyzed separately. Section 5 shows when a plasma with one end tied is stable. Two theorems are proven. Section 6 considers the cross term and an estimate is found for its magnitude. In section 7 the second variation for an interchange is examined in light of the above estimate. Conditions are derived for positiveness of the whole second variation. These conditions form the main result of the thesis, showing under what restrictions interchange stability plus local stability imply absolute stability.

Section 8 discusses the results of lengthening the plasma and indicates the difficulties to be overcome before this analysis can be extended to toroidal geometries. We then examine our results and compare them with others in the literature. We find them to be quite general, encompassing and extending even results obtained by purely formal perturbation methods.

Acknowledgement: I would like to thank my advisor Professor Harold Grad who suggested this problem and contributed many valuable ideas to its solution. Thanks are also due to Professor Harold Weitzner who saved me from at least one serious mistake, and to Dr. Gareth Guest who broadened my perspective on the physics of plasma physics. This research was sponsored by Oak Ridge National Laboratory, operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

2. Formulation of the Problem

The terminology and variational principal which we employ have been developed by H. Grad [6,7,8], and whenever possible we follow the same notation. Since we will constantly refer to a variation as a motion, it is wise to review the origin of this motion. From the field equations

$$\frac{\partial B}{\partial t} + \text{curl } E = 0$$

and from Ohm's law for a perfectly conducting fluid

$$E + U \times B = 0.$$

Combining these two equations

$$\frac{\partial B}{\partial t} = \text{curl } (U \times B) \tag{2.1}$$

This equation indicates that the magnetic flux is frozen into the plasma (as can be seen by calculating the change in flux through a moving surface); it also relates the change in B to the motion of the plasma. For the variational principal we invert the logic of this relation. As is well known we can imbed an equilibrium field in a family of fields admissable to the variational problem and parameterized by a variable t such that the equilibrium field is $B(0,x)$, all others are identified by $B(t,x)$, and the variational notation δB is replaced by

$$\delta B = \partial B/\partial t \Big|_{t=0}$$

Alternatively we can use (2.1) to characterize the variation by a velocity field U; for given any variation δB we can find a U which yields that variation. In fact it is only the component of U perpendicular to B which enters, and we shall take U to be perpendicular to B in order to make use of the relation

$$U \cdot B = 0.$$

That U does represent a motion is seen by introducing the representation (sometimes referred to as a Clebsch transformation) of the solenoidal B field:

$$B = \nabla\alpha \times \nabla\beta$$

with which (α,β) constant identifies a field line. Then

$$\partial B/\partial t = \nabla(\partial\alpha/\partial t) \times \nabla\beta + \nabla\alpha \times \nabla(\partial\beta/\partial t) \tag{2.2}$$

and (2.2) together with (2.1) imply

$$\frac{D\alpha}{Dt} \equiv \frac{\partial\alpha}{\partial t} + U \cdot \nabla\alpha = 0$$

$$\frac{D\beta}{Dt} \equiv \frac{\partial\beta}{\partial t} + U \cdot \nabla\beta = 0$$

so that the field lines move with velocity U. (We shall reserve D/Dt for the particle derivative moving with U.)

The quantities α and β can be used to define a Lagrangian coordinate system moving with the field (α, β, σ) where σ can be arclength at some initial time and then is carried by the motion. To relate σ to arclength s we introduce the factor ζ:

$$\zeta = \partial\sigma/\partial s$$

As stated before the plasma is constrained to remain on its initial field line as this is carried by U. The motion along the given line is governed by a Hamiltonian $H(\sigma,p,t)$ where p is canonically conjugate to σ. Thus the fluid is described by a single distribution f, a function of (σ,p,t) directly and (α,β,μ) as parameters. This distribution $f(\sigma,p,\alpha,\beta,\mu,t)$ is related to the usual particle distribution $F(x,\xi,t)$ by

$$f = 2\pi F m^{-2}$$

and the Jacobian relations

$$dx = \frac{d\alpha d\beta d\sigma}{B}, \qquad d\xi = 2\pi B\zeta \frac{dp d\mu}{m^2}$$

$$dx d\xi = \frac{2\pi}{m^2} d\alpha d\beta d\sigma dp d\mu = \frac{2\pi}{m^2} d\Omega$$

which imply

$$F d\xi = B\zeta f dp d\mu, \quad F dx d\xi = f d\Omega$$

(see [7]).

The variation of the distribution function is well described in [7]. Essentially we admit any function f which is obtainable from a given reference function by an incompressible mapping w in the (σ,p) plane. Thus the variational class is broader than the class accessible by a dynamic motion for we admit any Hamiltonian motion, and not just that derivable from the particular known Hamiltonian. A complete variation is then given by two velocity fields (U,w) which describe $(\delta B, \delta f)$, but by using the pessimistic variation we can find w as a function of U, leaving U as the only variational field.

The success of the variational principle depends upon the conservation of total energy which is the sum of kinetic, internal, and magnetic components defined by

$$\mathcal{K} = \int \frac{1}{2} \rho \, U^2 \, dx$$

$$\mathcal{U} = \int \epsilon f \, d\Omega$$

$$\mathfrak{M} = \int \frac{1}{2\mu_0} B^2 \, dx$$

where

$$\epsilon = \frac{1}{2m} \zeta^2 p^2 + \mu B$$

If we are careful to refer to the equilibrium when $t = 0$, then $\zeta = 1$, and letting v represent speed along the line we can also write

$$\epsilon = \frac{1}{2} mv^2 + \mu B$$

Conservation of energy is verified in [7]; this is not trivial because of the singular zero order guiding center limit.

The variational principle is classical in form; we take as variational function Φ, defined by

$$\Phi = \mathcal{U} + \mathfrak{M}$$

A static equilibrium is defined as a stationary value of Φ; it is also a stationary solution of the equations of motion for which

$\mathcal{K} = 0$. To study stability we form $\delta^2\Phi$ and ask whether it is positive for any variation U.

Of course a variational analysis depends as much on the admissable class of functions and the geometry of the problem as it does on the variational function; our problem is BC II as defined in [7].

We assume that we are given a fixed tubular domain D shown in figure 1 with lateral side S_0 and ends S_1 and S_2. For B we admit any field which is topologically simple (i.e., the field lines traverse D from S_1 to S_2) and satisfies div B = 0 as well as certain boundary conditions. On the lateral side S_0 we have $B_n = 0$ (S_0 is a flux surface). On the ends S_1 and S_2 B_n is specified arbitrarily subject to $B_n > 0$ on S_1 and $B_n < 0$ on S_2. The flux requirement is imposed that $\int_{S_1} B\cdot dS + \int_{S_2} B\cdot dS = 0$.

Physically these boundary conditions correspond to a perfectly conducting plasma vessel with ends insulated from the plasma.

In addition we impose the boundary condition on the ends S_1 and S_2 that $J_n = 0$. This is a natural boundary condition for an equilibrium which results from the variational analysis; we impose it as well for any variation away from the equilibrium. This is physically reasonable, for no current can flow through an insulator.

These boundary conditions imply certain relations which will be used later. B_n fixed means $n\cdot\partial B/\partial t = 0$ or by (2.1)

$$n \cdot \text{curl } (U \times B) = 0 \tag{2.3}$$

The natural boundary condition $J_n = 0$ implies

$$n \cdot \text{curl } B = 0 \tag{2.4}$$

on S_1 and S_2, since $\mu_0 J$ = curl B. Differentiating (2.3) and (2.4) yields

$$n \cdot \text{curl}\ \partial(U \times B)/\partial t = 0 \tag{2.5}$$

$$n \cdot \text{curl}\ \partial B/\partial t = 0 \tag{2.6}$$

Relations (2.3) to (2.6) indicate that B and U xB as well as their time derivatives are all surface gradients (see e.g. [9]) wherever the corresponding boundary conditions hold. This fact means that such integrals as $\int B \times \partial(U\times B)/\partial t \cdot dS$, which appears in (3.4), vanish, since we have the calculus formula $\oint \nabla\varphi \times \nabla\psi \cdot dS = 0$.

The concept of line tying enters into this analysis, and it is best defined by comparing (see [7]) the two boundary conditions BC I and BC II. In BC I α and β are specified on the ends, which requires $U = 0$ there. In this case the lines are tied on the ends; they are not free to move since α and β are fixed. The physical mechanism to support this line tying must be good electrical contact with the vessel ends through a perfectly conducting medium such as cold plasma. On the other hand in BC II (our case) we merely require B_n fixed on the ends. This allows a motion of the lines which is equivalent to an incompressible mapping of the (α,β) plane through the relation

$$\int_S B\cdot dS = \int_S d\alpha d\beta$$

We also introduce the concept of local stability. In [8] it is shown that the necessary and sufficient conditions for stability against variations which are sufficiently localized in extent along a line are the following inequalities

$$B^2/\mu_0 + p_2 - p_1 > 0$$

$$B^2/\mu_0 + 2p_2 - C_2 B^3 > 0 \tag{2.7}$$

$$\partial f^0/\partial\epsilon > 0$$

where f^0 is the equilibrium distribution, of necessity a function of σ and p only through ϵ, and C_2 is a certain moment of f^0. For sufficiently localized variations certain terms dominate in $\delta^2 \Phi$; these inequalities arise naturally to make those terms positive. Local stability is then equivalent with (2.7).

Finally we say a word about interchanges. We have defined them as variations which leave B fixed, but often we speak of them as incompressible mappings of (α,β) to (α^1,β^1). It is easy to show the equivalence of these. If $\partial(\alpha^1,\beta^1)/\partial(\alpha,\beta) = 1$, then there is a flux function $\psi(\alpha,\beta)$ such that

$$\frac{\partial\alpha}{\partial t} = \frac{\partial\psi}{\partial\beta} \qquad \frac{\partial\beta}{\partial t} = -\frac{\partial\psi}{\partial\alpha} \tag{2.8}$$

Now

$$\frac{\partial B}{\partial t} = \nabla\frac{\partial\alpha}{\partial t} \times \nabla\beta + \nabla\alpha \times \nabla\frac{\partial\beta}{\partial t} \tag{2.9}$$

and $\partial B/\partial t = 0$ is equivalent to (2.8). We exhibit the proof in one direction: inserting (2.8) into (2.9) yields

$$\frac{\partial B}{\partial t} = \nabla\frac{\partial\psi}{\partial\beta} \times \nabla\beta - \nabla\alpha \times \nabla\frac{\partial\psi}{\partial\alpha} = \frac{\partial^2\psi}{\partial\alpha\partial\beta} \nabla\alpha \times \nabla\beta - \nabla\alpha \times \nabla\beta \frac{\partial^2\psi}{\partial\alpha\partial\beta}$$

and the conclusion follows.

3. Derivation of the Second Variation

From among the various (equivalent) forms for the second variation we select for our analysis one which manifests the role of local stability. For the fluid variation we can use the results of [7] and [8], but we shall derive the magnetic variation in a form appropriate to our boundary conditions. Reference [7] gives in equation (10.24) the following form for the fluid variation, and the brackets <> denote an average defined by (7.7) of [7].

$$\delta^2 \mathcal{U} = \int \left[p_1 G_1 + p_2 G_2 + \frac{DU}{Dt} \cdot \operatorname{div} P \right] dx$$

$$+ \int (g^2 - \langle g \rangle^2) \frac{\partial f^0}{\partial \epsilon} d\Omega$$

where

$$G_1 = 4 \quad (b \cdot \frac{\partial U}{\partial s})^2$$

$$G_2 = (\frac{\partial U}{\partial s})^2 - 2 \; (b \cdot \frac{\partial U}{\partial s})^2 + (b \cdot \frac{\partial U}{\partial s} - \operatorname{div} U)^2 + \frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i}$$

$$g = \partial \epsilon / \partial t$$

If we transform this as suggested in [8] by evaluating the term involving g^2 we find

$$\delta^2 \mathcal{U} = \int \{ (p_2 - p_1) \; (b \times \frac{\partial U}{\partial s})^2 + (2p_2 - C_2 B^3) \; (b \cdot \frac{\partial U}{\partial s} - \operatorname{div} U)^2 \} \, dx$$

$$+ \int \{ p_2 \left[\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} - (\operatorname{div} U)^2 \right] + \frac{DU}{Dt} \cdot \operatorname{div} P \} \, dx - \int \langle g \rangle^2 \frac{\partial f^0}{\partial \epsilon} d\Omega$$

The most direct way of deriving the second magnetic variation is to use Lagrangian coordinates. We parallel [7], but there are subtleties in handling the boundary terms.

The domain D is fixed in Eulerian coordinates, but not in Lagrangian. The velocity of the boundary in Lagrangian coordinates is $-(U \cdot n)$. We have

$$\mathfrak{M} = \frac{1}{2\mu_0} \int \frac{B}{\zeta}\, d\sigma d\alpha d\beta$$

$$\frac{d\mathfrak{M}}{dt} = \frac{1}{2\mu_0} \int \frac{D}{Dt} \left(\frac{B}{\zeta}\right) d\sigma d\alpha d\beta - \frac{1}{2\mu_0} \oint B^2\ U \cdot dS$$

$$\frac{d^2\mathfrak{M}}{dt^2} = \frac{1}{2\mu_0} \int \frac{D^2}{Dt^2} \left(\frac{B}{\zeta}\right) d\sigma d\alpha d\beta - \frac{1}{2\mu_0} \oint \zeta B \frac{D}{Dt} \left(\frac{B}{\zeta}\right) U \cdot dS \quad (3.1)$$

$$+ \frac{1}{2\mu_0} \oint \frac{\partial}{\partial t} (B^2 U) \cdot dS$$

The final boundary term vanishes since

$$\oint \frac{\partial}{\partial t} (B^2 U) \cdot dS = \oint \frac{\partial}{\partial t} (B \times U \times B) \cdot dS$$

and as shown in section 2 the boundary conditions imply that both B and U x B as well as their time derivatives are surface gradients. For the other boundary term in (3.1) a direct calculation yields

$$-\frac{1}{2\mu_0} \oint \zeta B \frac{D}{Dt} \left(\frac{B}{\zeta}\right) U \cdot dS = \frac{1}{\mu_0} \oint \left[\frac{1}{2} B^2 \operatorname{div} U - B \cdot B \cdot \nabla U\right] U \cdot dS$$

If we work out the volume term as in equation (10.32) of [7] then the second magnetic variation is

$$\frac{d^2\mathfrak{M}}{dt^2} = \int \tilde{G}_m \, dx + \frac{1}{\mu_o} \int \frac{DU}{Dt} \cdot B \times \text{curl } B \, dx$$

$$+ \frac{1}{\mu_o} \oint \left[\frac{1}{2} B^2 \frac{DU}{Dt} - B \times \frac{DU}{Dt} \times B \right] \cdot dS \tag{3.2}$$

$$+ \frac{1}{\mu_o} \oint \left[\frac{1}{2} B^2 \text{ div } U - B \cdot B \cdot \nabla U \right] U \cdot dS$$

where

$$\tilde{G}_m = \frac{1}{2\mu_o} B^2 \left[\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} - (\text{div } U)^2 + 2\left(\frac{\partial U}{\partial S} - b \text{ div } U\right)^2 \right]$$

To achieve the desired form for the second variation requires further manipulation of (3.2). The first boundary term is expanded as follows:

$$\int \left[\frac{1}{2} B^2 \frac{DU}{Dt} - B \times \frac{DU}{Dt} \times B \right] \cdot dS$$

$$= \int \left[- \frac{1}{2} B^2 \frac{DU}{Dt} + B\left(B \cdot \frac{DU}{Dt}\right) \right] \cdot dS \tag{3.3}$$

$$= \int \left[- \frac{1}{2} B^2 \frac{\partial U}{\partial t} - \frac{1}{2} B^2 \, (U \cdot \nabla U) + B\left(B \cdot \frac{\partial U}{\partial t}\right) + B(B \cdot U \cdot \nabla U) \right] \cdot dS \, .$$

This can be simplified using the identity

$$B^2 \frac{\partial U}{\partial t} = B \times \frac{\partial}{\partial t} (U \times B) - \frac{\partial B}{\partial t} \times U \times B - 2B\left(U \cdot \frac{\partial B}{\partial t}\right)$$

which follows immediately from the relations

$$B \times \frac{\partial}{\partial t} (U \times B) = B^2 \frac{\partial U}{\partial t} - B\left(B \cdot \frac{\partial U}{\partial t}\right) + U\left(B \cdot \frac{\partial B}{\partial t}\right)$$

$$- \frac{\partial B}{\partial t} \times U \times B = - U\left(B \cdot \frac{\partial B}{\partial t}\right) + B\left(U \cdot \frac{\partial B}{\partial t}\right) \, .$$

This identity and (3.3) then yield the form desired for the first boundary term of (3.2):

$$\frac{1}{\mu_0}\int\left[\frac{1}{2}B^2\frac{DU}{Dt} - B\times\frac{DU}{Dt}\times B\right]\cdot dS$$
$$= -\frac{1}{2\mu_0}\int\left[B\times\frac{\partial}{\partial t}(U\times B) - \frac{\partial B}{\partial t}\times U\times B\right]\cdot dS \tag{3.4}$$
$$+\frac{1}{\mu_0}\int\left[-\frac{1}{2}B^2(U\cdot\nabla U) + B(B\cdot U\cdot\nabla U)\right]\cdot dS\ .$$

Now the first of these two term vanishes because of the boundary conditions; thus the second magnetic variation is

$$\frac{d^2\mathfrak{M}}{dt^2} = \int\tilde{G}_m\ dx + \frac{1}{\mu_0}\int\frac{DU}{Dt}\cdot B\times\mathrm{curl}\ B\ dx$$
$$+\frac{1}{\mu_0}\oint[B\ (B\cdot U\cdot\nabla U) - \frac{1}{2}B^2\ (U\cdot\nabla U)]\cdot dS$$
$$+\frac{1}{\mu_0}\oint[\frac{1}{2}B^2\ \mathrm{div}\ U - B\cdot B\cdot\nabla U]\ U\cdot dS\ .$$

Combining this form of the magnetic variation with the fluid variation, and using the equilibrium condition

$$\mathrm{div}\ P = J\times B, \tag{3.5}$$

yields for the full second variation

$$\delta^2\Phi = \int(\frac{B^2}{\mu_0} + p_2 - p_1)\ (b\times\frac{\partial U}{\partial s})^2\ dx$$
$$+\int(\frac{B^2}{\mu_0} + 2p_2 - B^3C_2)\ (b\cdot\frac{\partial U}{\partial s} - \mathrm{div}\ U)^2\ dx$$
$$+\int p_*\left[\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} - (\mathrm{div}\ U)^2\right]dx$$

$$- \int \langle g \rangle^2 \frac{\partial f^o}{\partial \varepsilon} d\Omega + \frac{1}{\mu_o} \oint \left[B(B \cdot U \cdot \nabla U) - \frac{1}{2} B^2 (U \cdot \nabla) U \right] \cdot dS$$

$$+ \frac{1}{\mu_o} \oint [\frac{1}{2} B^2 \text{ div } U - B \cdot B \cdot \nabla U] \, U \cdot dS$$

We shall have the desired form after performing two integrations by parts on the term involving p_*,

$$\int p_* \left[\frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} - (\text{div } U)^2 \right] dx = \int \nabla p_* \cdot (U \text{ div } U - U \cdot \nabla U) \cdot dx$$

$$- \frac{1}{2\mu_o} \oint B^2 (U \text{ div } U - U \cdot \nabla U) \cdot dS$$

$$= 2 \int p_* \cdot U \text{ div } U \, dx + \int U_i \frac{\partial^2 p_*}{\partial x_i \partial x_j} U_j \, dx$$

$$- \frac{1}{2\mu_o} \oint B^2 (U \text{ div } U - U \cdot \nabla U) \cdot dS - \frac{1}{\mu_o} \oint (B \cdot U \cdot \nabla B) U \cdot dS \, .$$

Making this substitution gives

$$\delta^2 \Phi = \int (\frac{B^2}{\mu_o} + p_2 - p_1)(b \times \frac{\partial U}{\partial s})^2 dx + \int (\frac{B^2}{\mu_o} + 2p_2 - B^3 C_2)(b \cdot \frac{\partial U}{\partial s} - \text{div } U)^2 dx$$

$$+ 2 \int \nabla p_* \cdot U \text{ div } U \, dx + \int U_i \frac{\partial^2 p_*}{\partial x_i \partial x_j} U_j \, dx - \int \langle g \rangle^2 \frac{\partial f^o}{\partial \varepsilon} d\Omega \qquad (3.6)$$

$$+ \int (U \cdot J \times B) \, U \cdot dS - \frac{1}{\mu_o} \oint (U \cdot U \cdot \nabla B) \, B \cdot dS$$

where we have used the identity

$$U \cdot B \cdot \nabla B - B \cdot U \cdot \nabla B = -U \cdot B \times \text{curl } B \, .$$

We shall give for reference a form for the second variation where we do not perform the second integration by parts on the p_* term. This is

$$\delta^2\Phi = \int\left(\frac{B^2}{\mu_0} + p_2 - p_1\right)\left(b \times \frac{\partial U}{\partial s}\right)^2 dx + \int\left(\frac{B^2}{\mu_0} + 2p - B^3C_2\right)\left(b\cdot\frac{\partial U}{\partial s} - \operatorname{div} U\right)^2 dx$$

$$+ \int \nabla p_* \cdot (U \operatorname{div} U - U \cdot \nabla U)\, dx - \int <g>^2 \frac{\partial f^0}{\partial \epsilon}\, d\Omega \tag{3.7}$$

$$+ \frac{1}{\mu_0}\oint (U \cdot B \cdot \nabla B)U \cdot dS - \frac{1}{\mu_0}\oint (U \cdot U \cdot \nabla B)B \cdot dS$$

To allow later estimates and to make explicit the effects of scaling, it is helpful to introduce a characteristic length and field strength and to define new variables in terms of them. Let R_0 be the radius of the domain D and B_0 the maximum field strength in D. Then define new primed variables by

$$R_0 x_i{}' = x_i$$

$$\frac{B_0{}^2}{\mu_0}\beta_i = p_i$$

$$B_0 B_i{}' = B_i$$

In terms of these variables (3.6) becomes

$$\delta^2\Phi = R_0\frac{B_0{}^2}{\mu_0}\Big\{\int(B'^2 + \beta_2 - \beta_1)\left(b \times \frac{\partial U}{\partial s'}\right)^2 dx' \tag{3.6a}$$

$$+ \int(B'^2 + 2\beta_2 - B'^3C_2')\left(b \cdot \frac{\partial U}{\partial s'} - \operatorname{div}' U\right)^2 dx'$$

$$+ 2\int \nabla'\beta_* \cdot U \operatorname{div}' U\, dx' + \int U_i \frac{\partial^2 \beta_*}{\partial x_i' \partial x_j'} U_j\, dx' - \int <g>^2 \frac{\partial f^0}{\partial \epsilon}\, d\Omega'$$

$$+ \oint (U \cdot J' \times B')U \cdot dS' - \oint (U \cdot U \cdot \nabla B')B' \cdot dS'\Big\}$$

However it is only in section 6 where we require this form to make estimates such as $2ab < a^2 + b^2$ where in the original form a and b have different dimensions. Sacrificing rigor for clarity, we use the unprimed variables and (3.6) instead of (3.6a); for the change merely introduces dimensional constants which are unimportant for the success of the estimates.

4. Method of Analysis

The boundary condition B_n fixed on the ends means that on the end any variation U is limited to an incompressible mapping of the (α,β) plane, i.e., an interchange. We exploit this fact by selecting (to be definite) the end S_1 and extend the interchange produced on S_1 by U throughout D. Thus, given any variation U, we can define an interchange U^i, such that $U-U^i$ vanishes on S_1. Naming this difference U^o we have

$$U = U^i + U^o$$

where U^i is an interchange and U^o vanishes on S_1. Then $\delta^2\Phi$, as a quadratic form Q(U), is written as

$$\delta^2\Phi = Q(U^i) + Q(U^i,U^o) + Q(U^o) \tag{4.1}$$

To find when $\delta^2\Phi$ is positive we examine each term separately. $Q(U^i)$ is the full second variation for an interchange. If we postulate interchange stability (and we do) it is positive by hypothesis. $Q(U^o)$ is the full second variation for a general variation which vanishes one end. This corresponds to BC I on end S_1 and BC II on S_2, i.e., a plasma with one end tied. In section 5 we consider this problem and show that $Q(U^o)$ is positive definite for (speaking imprecisely) either a short plasma or a smooth plasma which is locally stable. Accepting these conditions we estimate $Q(U^i,U^o)$ in section 6. $\delta^2\Phi$ will be positive if

$$Q(U^i,U^o) < Q(U^i) + Q(U^o) \tag{4.2}$$

The estimate which is found is

$$Q(U^i,U^o) < \delta_1 Q(U^o) + \delta_2 \int (U^i)^2\,dx \tag{4.3}$$

where δ_1 and δ_2, functions of the tensors ∇B and $\partial^2 p_* /\partial x_i \partial x_j$, can be made arbitrarily small by choosing their arguments small enough. To derive (4.2) from (4.3) requires the estimate

$$Q(U^i) > C \int (U^i)^2\ dx \tag{4.4}$$

This will not be true in every equilibrium. For example if there are neutral surfaces (4.4) will obviously fail.

In section 7 interchanges are studied to find when (4.4) is valid. We are able to weaken (4.4) and strengthen (4.3) to include equilibria with neutral surfaces. We find a simple and general requirement on the equilibrium to permit these estimates. This includes as a special case Grad's sufficient condition for interchange stability [10].

5. $Q(U^O)$: Plasma With One End Tied

Following [8] we assume local stability and seek conditions under which the obviously positive terms in $\delta^2\Phi$ dominate all others. The crucial fact in this analysis is that since U^O vanishes on S_1 we can estimate both $\int (U^O)^2\,dx$ and $\int_{S_2}(U^O)^2\,dS$ in terms of $\int (b \times \partial U^O/\partial s)^2\,dx$.* To establish these estimates we let L be a quantity larger than the length of any magnetic line in D, i.e.,

$$\max \int ds < L \quad .$$

Then from $U = \int (\partial U/\partial s)\,ds$ we have

$$\begin{aligned} U^2 &< \int ds \int \left(\frac{\partial U}{\partial s}\right)^2 ds \\ &\leq L \int (U \cdot \kappa)^2\,ds + L \int \left(b \times \frac{\partial U}{\partial s}\right)^2 ds \\ &< L^2 \bar{\kappa}^2 \bar{U}^2 + L \int \left(b \times \frac{\partial U}{\partial s}\right)^2 ds \end{aligned}$$

or

$$\bar{U}^2 < \frac{L}{1 - \bar{\kappa}^2 L^2} \int \left(b \times \frac{\partial U}{\partial s}\right)^2 ds \ , \tag{5.1}$$

where $\bar{U}$ is the maximum of U on a given line and $\bar{\kappa}$ the maximum of κ in D. Integrating (5.1) over any smooth surface S cutting all lines (but not multiply and nowhere tangent to a line)

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*In this section we shall drop the superscript on U^O.

$$\int U^2 dS < \frac{B_{max}}{(B\cdot n)_{min}} \frac{L}{1-\bar{\kappa}^2 L^2} \int (b \times \frac{\partial U}{\partial s})^2 \, dx$$

the factor involving B arises since $dS = d\alpha d\beta/(B\cdot n)$ where n is the local normal to the surface, and $dx = d\alpha d\beta ds/B(\alpha,\beta,s)$. The maximum is taken over D and the minimum over S. Calling this factor γ, we have in particular on the end S_2

$$\int_{S_2} U^2 \, dS < \frac{L\gamma}{1-\bar{\kappa}^2 L^2} \int \left(b \times \frac{\partial U}{\partial s}\right)^2 dx \,. \tag{5.2}$$

Integrating (5.1) over $d\alpha d\beta$ and then ds (and recalling $dx = d\alpha d\beta ds/B$)

$$\int U^2 \, dx < \frac{L^2 \gamma'}{1-\bar{\kappa}^2 L^2} \int (b \times \frac{\partial U}{\partial s})^2 \, dx \tag{5.3}$$

where $\gamma' = B_{max}/B_{min}$, both taken over D.

With these estimates we can prove the following theorems:

Theorem 1: If the plasma is short enough so L can be made suitably small, then local stability plus boundedness of the tensors ∇B and $\partial^2 p_*/\partial x_i \partial x_j$ ensure stability of the plasma.

Theorem 2: If the tensors ∇B and $\partial^2 p_*/\partial x_i \partial x_j$ are sufficiently small then local stability ensures stability of the plasma.

Theorems 1 and 2 are parallel, in both cases the small quantities allow the terms in $\delta^2 \Phi$ made positive by local stability to dominate all others. The analog of theorem 1 for both ends tied (no boundary terms) is proven in [8]; accordingly we prove only theorem 2 and then indicate the estimate of boundary terms required for theorem 1.

Referring to the expression (3.6) for $\delta^2\Phi$, local stability (see (2.7)) implies that the first two terms plus the term involving $\partial f^0/\partial\epsilon$ are all positive. We ignore the latter term and show the first two dominate all others under the conditions of the theorem. Now local stability indicates there is a positive number M such that

$$\frac{1}{M} < \frac{B^2}{\mu_0} + p_2 - p_1 < M \tag{5.4}$$

$$\frac{1}{M} < \frac{B^2}{\mu_0} + 2p_2 - B^3 C_2 < M$$

From the equilibrium equation (see e.g. (9.7) of [6])

$$U \cdot \nabla p_* = U \cdot \varkappa (B^2/\mu_0 + p_2 - p_1) \tag{5.5}$$

Thus for the third term of (3.6)

$$\int \nabla p_* \cdot U \operatorname{div} U \, dx \le M\bar{\varkappa} \left[\int |U| \, (U\cdot\varkappa + \operatorname{div} U) dx + \int |U| (U\cdot\varkappa) dx\right] \tag{5.6}$$

$$\le M\bar{\varkappa} \left[\frac{1}{2}\int U^2 dx + \frac{1}{2}\int (U\cdot\varkappa + \operatorname{div} U)^2 dx + \bar{\varkappa}\int U^2 dx\right]$$

which (5.3) and (5.4) show to be smaller than the dominant terms if $\bar{\varkappa}$ is small enough.

The other volume term is easily estimated by

$$\int U_i \frac{\partial^2 p_*}{\partial x_i \partial x_j} U_j \, dx \le \left|\overline{\frac{\partial^2 p_*}{\partial x_i \partial x_j}}\right| \int U^2 \, dx \tag{5.7}$$

which again is made arbitrarily smaller than the dominant terms if $\partial^2 p_*/\partial x_i \partial x_j$ is small enough.

The boundary terms are estimated as

$$\int_{S_2} (U \cdot U \cdot \nabla B) B \cdot dS \leq \bar{B} |\overline{\nabla_{\perp} B}| \int_{S_2} U^2 dS \tag{5.8}$$

$$\int_{S_2} (U \cdot J \times B) U \cdot dS \leq |\overline{J \times B}| \int_{S_2} U^2 dS$$

which (5.2) shows to be dominated if ∇B is small enough, proving theorem 2.

In proving theorem 1 the boundary terms are estimated exactly as in (5.8); in this case (5.2) indicates the integral $\int U^2 dS$ becomes arbitrarily small with L compared with the dominant terms.

The conditions of theorems 1 and 2 guarantee that $\delta^2\Phi$ is actually positive definite, since (5.2) and (5.3) show that $U \equiv 0$ if $\delta^2\Phi = 0$.

6. Estimating the Cross Terms

The inequality to be derived in this section was stated in section 4:

$$Q\,(U^i, U^o) \;<\; \delta_1\; Q(U^o) + \delta_2 \int (U^i)^2\, dx \tag{4.3}$$

where δ_1 and δ_2 become small with their arguments ∇B and $\partial^2 p_*/\partial x_i \partial x_j$, and $Q(U^i, U^o)$ are the cross terms corresponding to (3.6) which are defined by (4.1).

For an interchange curl $(U^i \times B) = 0$, and the following identities hold

$$\begin{aligned} b \times \frac{\partial U^i}{\partial s} &= U^i \cdot \nabla b \\ b \cdot \frac{\partial U^i}{\partial s} - \operatorname{div} U^i &= U^i \cdot \frac{\nabla B}{B} \\ \operatorname{div} U^i &= -\,U^i \cdot \left(\frac{\nabla B}{B} + \varkappa\right) \end{aligned} \tag{6.1}$$

Hence the cross terms have the form

$$\begin{aligned} Q(U^i, U^o) = {} & 2 \int \left(\frac{B^2}{\mu_o} + p_2 - p_1\right) (U^i \cdot \nabla b) \cdot \left(b \times \frac{\partial U^o}{\partial s}\right) dx \\ & + 2 \int \left(\frac{B^2}{\mu_o} + 2p_2 + B^3 C_2\right) \left(U^i \cdot \frac{\nabla B}{B}\right) \left(b \cdot \frac{\partial U^o}{\partial s} - \operatorname{div} U^o\right) dx \\ & + 2 \int [(U^i \cdot \nabla p_*) \operatorname{div} U^o - (U^o \cdot \nabla p_*)\{U^i \cdot (\frac{\nabla B}{B} + \varkappa)\}]\, dx \\ & + 2 \int U^i_i \frac{\partial^2 p_*}{\partial x_i \partial x_j} U^o_j\, dx - 2 \int \langle g^i g^o \rangle \frac{\partial f^o}{\partial \epsilon}\, d\Omega \\ & + \oint (U^i \cdot J \times B)\, U^o \cdot dS + \oint (U^o \cdot J \times B)\, U^i \cdot dS \\ & - \frac{1}{\mu_o} \oint (U^i \cdot U^o \cdot \nabla B)\, B \cdot dS - \frac{1}{\mu_o} \oint (U^o \cdot U^i \cdot \nabla B) B \cdot dS\,. \end{aligned} \tag{6.2}$$

To establish the estimate (4.3) for the volume terms of (6.1) we note that all these terms have the form

$$\int U^i \cdot \eta\, f(U^o)\, dx$$

where η is a vector (or tensor) which becomes small with its argument, ∇B or $\partial^2 p_*/\partial x_i \partial x_j$. For example, in the first term η is ∇b, while $f(U^o)$ is $b \times \partial U^o/\partial s$. Thus for each of the volume terms the following estimate holds:

$$\int U^i \cdot \eta\, f\,(U^o)\, dx \le |\bar{\eta}| \left[\int (U^i)^2\, dx \int f(U^o)^2\, dx \right]^{\frac{1}{2}}$$

$$\le \frac{1}{2} |\bar{\eta}| \left[\int (U^i)^2\, dx + \int f(U^o)^2\, dx \right] \qquad (6.3)$$

(Here we have used the inequality $2ab < a^2 + b^2$ which requires that U^i and $f(U^o)$ have the same dimension. It is for this reason that a dimensionless length, field strength, and pressure were introduced in section 3.) But $\int f(U^o)^2\, dx$ is in every case bounded in terms of the dominant contribution to $Q(U^o)$, hence for the volume terms (4.3) does hold.

The term in (6.2) containing $\partial f^o/\partial \epsilon$ can be estimated using the identity (where g^i and g^o refer to their respective arguments U^i and U^o)

$$\int < g^i g^o > \frac{\partial f^o}{\partial \epsilon}\, d\Omega = \int g^i\ < g^o > \frac{\partial f^o}{\partial \epsilon}\, d\Omega$$

which is easily proven and is implicit in (10.10) of [7]. Now

$$\int g^i < g^o > \frac{\partial f^o}{\partial \epsilon}\, d\Omega \le \left[\int (g^i)^2\, \frac{\partial f^o}{\partial \epsilon}\, d\Omega \int < g^o >^2\, \frac{\partial f^o}{\partial \epsilon}\, d\Omega \right]^{\frac{1}{2}}$$

and using (I.25) of [8]

$$\int (g^i)^2 \frac{\partial f^o}{\partial \epsilon} d\Omega = \int \{- 3p_1 (U^i \cdot \kappa)^2 - 2p_2 (U^i \cdot \kappa)(U^i \cdot \frac{\nabla B}{B}) - C_2 B^3 (U^i \cdot \frac{\nabla B}{B})^2\} dx$$

$$\leq \text{const. max } (|\bar{\kappa}|, |\overline{\nabla B}|) \int (U^i)^2 dx$$

whence follows immediately an estimate of the form

$$\int < g^i g^o > \frac{\partial f^o}{\partial \epsilon} d\Omega \leq \delta \int (U^i)^2 dx + \delta \int < g^o >^2 \frac{\partial f^o}{\partial \epsilon} d\Omega \tag{6.4}$$

which again has the form of (4.3) since δ is small with ∇B.

To achieve the same type of estimate for the boundary terms we must show $\int_{S_2} (U^i)^2 dS$ is bounded in terms of $\int (U^i)^2 dx$. Since U^i is an interchange, the existence of such a bound is obvious; it is derived in the Appendix and we find

$$\int_{S_2} (U^i)^2 dS < \text{const.} \int (U^i)^2 dx \tag{6.5}$$

which is similar to the estimate in (5.2) for U^o. Then the boundary terms are estimated (we do only the first term as a typical case) as:

$$\int (U^i \cdot J \times B) U^o \cdot dS \leq |\overline{J \times B}| \int |U^i| |U^o| dS$$

$$\leq |\overline{J \times B}| \frac{1}{2} \left[\int (U^i)^2 dS + \int (U^o)^2 dS\right]$$

Then using (5.2) and (6.5) it follows

$$(\text{boundary terms}) < \delta_1 \int (U^i)^2 dx + \delta_2 Q(U^o) \tag{6.6}$$

Combining (6.3), (6.4) and (6.6) yields the desired estimate (4.3).

7. Stability Theorems

Thus far we have shown

$$\delta^2 \Phi > Q(U^i) + Q(U^o) - \delta_1 \int (U^i)^2 \, dx - \delta_2 \, Q(U^o)$$

If $Q(U^i)$ satisfies the inequality

$$Q(U^i) > \text{const.} \int (U^i)^2 \, dx \tag{7.1}$$

which can be interpreted as a strong form of interchange stability, then under the conditions we have found in sections 5 and 6 $\delta^2 \Phi$ can be made positive definite. We summarize these conditions in the following theorem:

Theorem: an equilibrium, i.e., a stationary solution of the variational problem, is a stable equilibrium if the following conditions are satisfied:

1. It is locally stable.
2. The tensors ∇B and $\partial^2 p_* / \partial x_i \partial x_j$ are sufficiently small throughout D, where x_i and x_j are directions locally perpendicular to B.
3. It is strongly interchange stable in the sense that the second variation for an interchange U^i has the property:

$$\frac{\delta^2 \Phi}{\int (U^i)^2 \, dx} \geq C > 0$$

In cases such as axial symmetry where there are neutral surfaces, requirement (7.1) will not be met (merely let U^i be a neutral interchange so $\delta^2 \Phi = 0$). However, the result of a neutral interchange is again an equilibrium. Thus if U_n^i is a neutral interchange and U any other variation

$$Q(U + U_n^i) = Q(U) \tag{7.2}$$

Expanding (7.2), we find the cross terms must vanish, or

$$Q(U, U_n^i) = 0$$

We can use this fact to weaken the requirement (7.1) on $Q(U^i)$. Splitting an interchange into its components

$$U^i = U_n^i + U_\perp^i$$

where now $U_\perp^i$ is perpendicular to the neutral surface, the original form for the second variation becomes

$$\begin{aligned} \delta^2 \Phi &= Q(U^o + U_n^i + U_\perp^i) \\ &= Q(U^o) + Q(U^o, U_\perp^i) + Q(U_\perp^i) \end{aligned}$$

Thus requirement (7.1) can be weakened to

$$Q(U_\perp^i) > \text{const.} \int (U_\perp^i)^2 \, dx \tag{7.3}$$

Now (7.3) is a fairly weak requirement and is included in a number of sufficient conditions for interchange stability. To show this we analyze the second variation for interchanges using the methods of [10]. Since an interchange is an incompressible rearrangement of magnetic lines in the $\lambda = (\alpha, \beta)$ plane, the analysis is simplified by using these coordinates and representing the variation of lines by an incompressible velocity field $u = \partial\lambda/\partial t$. This flux velocity u is related to U by

$$U_i = (\partial x_i/\partial\lambda_j)u_j, \qquad U \cdot \nabla = u \cdot (\partial/\partial\lambda)$$

The pessimistic variation requires f to be a function of (σ,p) only through the action integral J if we impose the necessary condition $\partial f/\partial\epsilon \leq 0$. Since the magnetic field is unchanged we consider only the fluid variation and set

$$\Phi = \int \epsilon(J,\mu,\lambda)\ f\ (J,\mu,\lambda)\ d\Omega$$

where

$$d\Omega = dJ d\mu d\lambda\ .$$

Performing the first variation

$$\delta\Phi = -\int \epsilon\ u \cdot \frac{\partial f}{\partial\lambda}\, d\Omega = \int u \cdot \frac{\partial\epsilon}{\partial\lambda}\ f\ d\Omega$$

No boundary terms appear since $f = 0$ at the boundary of D. A necessary and sufficient condition that $\int u \cdot \alpha\ d\lambda$ vanish for arbitrary incompressible u is that α be a gradient, $\alpha = \partial\varphi/\partial\lambda$. Thus $\delta\Phi = 0$ implies $\epsilon\partial f/\partial\lambda = \partial\varphi/\partial\lambda$ so that φ and then ϵ are constant on constant f contours. Taking the second variation

$$\delta^2\Phi = -\int \left(u \cdot \frac{\partial\epsilon}{\partial\lambda}\right)\left(u \cdot \frac{\partial f}{\partial\lambda}\right) d\Omega$$

Specializing now to equilibria with axial symmetry or other neutral surfaces so that f is a function of (α,β) only through ψ $[f=f(J,\mu,\psi(\alpha,\beta))]$, the vanishing first variation implies also

$$u \cdot \partial\varepsilon/\partial\lambda = (u \cdot \partial\psi/\partial\lambda)\, \partial\varepsilon/\partial\psi$$

so that

$$\delta^2 \Phi = -\int (u \cdot \frac{\partial\psi}{\partial\lambda})^2 \, \frac{\partial\varepsilon}{\partial\psi} \frac{\partial f}{\partial\psi}\, d\Omega$$

It is obvious that a neutral interchange is one which satisfies $u \cdot \partial\psi/\partial\lambda = 0$. Thus U^i is equivalent to the component of u (denoted by $u_\perp$) parallel to $\partial\psi/\partial\lambda$, and

$$\delta^2 \Phi = -\int u_\perp^2 \, \frac{\partial\varepsilon}{\partial\psi} \frac{\partial f}{\partial\psi}\, d\Omega \tag{7.4}$$

since $(\partial\psi/\partial\lambda)^2 = 1$. From (7.4) we see that the requirement (7.3) is equivalent to

$$\int \frac{\partial\varepsilon}{\partial\psi} \frac{\partial f}{\partial\psi}\, dJ d\mu < 0 \tag{7.5}$$

on each neutral surface ψ. (Interchange stability alone requires non-positivity of 7.5.) Inequality (7.5) is independent of the variation and is instead a property of the equilibrium alone.

This condition is satisfied if Grad's sufficient condition for interchange stability [10] holds. When interpreted properly such conditions as that of Rosenbluth and Longmire [11], namely

$$\int (p_2 + p_1) \frac{\partial B}{\partial\psi} \frac{ds}{B^2} > 0$$

or that quoted by Furth [12]

$$\int \frac{\partial B}{\partial\psi} \frac{\partial}{\partial\psi} (p_2 + p_1) \frac{ds}{B^2} > 0$$

are also stronger than (7.5) and are all less general.

We can summarize this analysis in the following theorem:

Theorem: an equilibrium with neutral surfaces (e.g. axial symmetry) is a stable equilibrium if the following conditions are satisfied:

1. It is locally stable.
2. The tensors ∇B and $\partial^2 p_*/\partial x_i \partial x_j$ are sufficiently small.
3. On each neutral surface the inequality holds that

$$\int \frac{\partial \varepsilon}{\partial \psi} \frac{\partial f}{\partial \psi} \, dJ d\mu < 0$$

8. Scaling Analysis and Conclusions

There is a psychological belief that a short plasma will be easier to stabilize than a long one. It is therefore of value to examine our estimates more closely to determine the effect of stretching the domain D lengthwise while keeping the same β and plasma radius.

First we note that as L becomes long the behavior of $\varkappa$ is $\varkappa \sim L^{-2}$. For if b is almost parallel to the plasma axis, the component perpendicular to the plasma axis scales as L^{-1} and the derivative $\partial/\partial s$ also scales as L^{-1}.

To analyze the effects of lengthening the plasma we must write down the specific quantities involved in the estimates of $Q(U^0)$ and $Q(U^1, U^0)$. Looking first at $Q(U^0)$, we focus on (5.6) and (5.7) which estimate the volume terms. Now $\int U^2 \, dx \sim L^2$ and $\varkappa \sim L^{-2}$, so the estimate of the term (5.6) becomes no worse. To expand $\partial^2 p^* / \partial x_i \partial x_j$ we use (5.5) to form

$$\partial p_* / \partial x_i = \varkappa_i (B^2/\mu_0 + p_2 - p_1)$$

$$\frac{\partial^2 p_*}{\partial x_i \partial x_j} = \frac{\partial \varkappa_i}{\partial x_j}\left(\frac{B^2}{\mu_0} + p_2 - p_1\right) + \varkappa_i \varkappa_j \left(\frac{B^2}{\mu_0} + p_2 - p_1\right) - \varkappa_i \frac{\partial}{\partial x_j}(p_2 + p_1) \tag{8.1}$$

which implies $\partial^2 p_* / \partial x_i \partial x_j \sim \varkappa$. Thus (5.7) is also unchanged as L becomes larger.

The estimate (5.8) of the first boundary term is similarly done; for

$$U \cdot (U \cdot \nabla) B \le |B| \, U^2 \, |\partial b / \partial \psi|$$

which goes down like L^{-1}, while $\int U^2 \, dS \sim L$. In the second boundary term

$$U \cdot J \times B = U \cdot B \cdot \nabla B - U \cdot \nabla \frac{B^2}{2}$$

Now $U \cdot B \cdot \nabla B = B^2 \, U \cdot \varkappa \sim \varkappa$, but $U \cdot \nabla B^2$ is unchanged as the plasma is lengthened. However, if we initially orient the surface S_2 to be perpendicular to the axis of the plasma, then $U \cdot dS \sim L^{-1}$ and the estimate is unchanged as L increases.

We summarize this analysis in the following theorem:

> Theorem: If in a plasma with one end tied the end plate on the free end is oriented perpendicular to the plasma axis, then an equilibrium stable by the estimates of sections 5, 6 and 7 remains stable if the plasma is lengthened keeping the same β; $\delta^2\Phi$ remains positive.

We can prove a similar theorem for the plasma with both ends free (BC II) by analyzing the estimate of $Q(U^i,U^o)$. But since there are so many terms in $Q(U^i,U^o)$, it is easier to write down all the terms at once instead of considering each term separately. In detail the estimate (4.3) is

$$\begin{aligned} Q(U^i,U^o) < \Bigg\{ & |\overline{\nabla_\perp b}|\, M + \left|\overline{\frac{\nabla_\perp B}{B}}\right| M + M\, |\overline{\varkappa}| + \left|\overline{\frac{\nabla_\perp B}{B} + \varkappa}\right| + \left|\overline{\frac{\partial^2 p_*}{\partial x_i \partial x_j}}\right| \\ & + (3\overline{p_1} + 2\overline{p_2} + \overline{C_2 B^3})\left(|\overline{\varkappa}| + \left|\overline{\frac{\nabla_\perp B}{B}}\right|\right) \Bigg\} \int (U^i)^2 \, dx \qquad (8.2) \\ & + \Big\{ |\overline{J \times B}| + |\overline{B}_n| \; |\overline{\nabla_\perp B}| \Big\} \int (U^i)^2 \, dS \end{aligned}$$

$$+ \left\{ |\overline{\nabla_\perp b}| + |\overline{\frac{\nabla_\perp B}{B}}| + |\overline{\varkappa}| + \left[|\overline{\varkappa}| \, M + |\overline{\frac{\partial^2 p_*}{\partial x_i \partial x_j}}| \right] \frac{L^2 \gamma'}{1-\overline{\varkappa}^2 L^2} \right.$$

$$\left. + \left(|\overline{\varkappa}| + |\overline{\frac{\nabla_\perp B}{B}}| \right) \right\} Q(U^o)$$

$$+ \left\{ |\overline{J \times B}| + |\overline{B}_n| \, |\nabla_\perp B| \right\} \frac{L\gamma}{1-\overline{\varkappa}^2 L^2} Q(U^o)$$

All the terms in the factor multiplying $\int (U^i)^2 \, dx$ scale as L^{-1} with the exception of $\nabla_\perp B$ which is unchanged. Thus this estimate becomes no worse. Similarly (A.3) shows that the factor multiplying $\int (U^i)^2 dS$ is unchanged while that integral scales as $L^{-1} \int (U^i)^2 dx$. So lengthening the plasma leaves the estimate involving U^i unchanged.

The terms involving $Q(U^o)$ are also unchanged. In the first of these there is a factor L^2, but $\varkappa$ and $\partial^2 p_* / \partial x_i \partial x_j$ scale as L^{-2} so these cancel. The second term, resulting from the boundary integrals, scales just as in the plasma with one end tied. The estimate is unchanged as L increases if the end is oriented perpendicular to the plasma axis. In summary we state the following theorem:

> Theorem: If the estimates of sections 5, 6, and 7 yield a positive $\delta^2 \Phi$, so the plasma equilibrium is stable, then lengthening the plasma while keeping the same β will not produce instability if the end plates are oriented perpendicular to the plasma axis; $\delta^2 \Phi$ remains positive.

Since recent results [13] indicate that in mirror machines the classical scattering into the loss cone is perhaps much worse than originally predicted [14], attention is turning to toroidal machines as the prime hope for controlled thermonuclear fusion. It would be of value therefore to extend our results to the toroidal case. However, at least two difficulties preclude doing this with the present analysis. The first of these is the fact that the perfectly uniform field in the straight machine becomes an azimuthal $1/R$ field in the toroidal machine. Thus in the latter case $\varkappa \sim 1/R$ at best. And since L is now the circumference, $L \sim R$. So it is not possible to reduce $\varkappa$ independent of the plasma length, and further the estimates become worse as the length increases; for there are terms in (8.2) involving L^2. This difficulty could be circumvented if the estimate $\int (U^0)^2 \, dx \sim L^2 Q(U^0)$ could be improved; efforts to do this have been unsuccessful.

The second difficulty is that we have used end plates to restrict the variation on the ends to an interchange. This artifice is obviously impractical in a toroidal machine. However, we might still make use of the technique of splitting up $\delta^2 \Phi$ by introducing a mathematical cut through the torous and writing U on that surface as an interchange plus a compression. As pointed out in [15] there are difficulties in producing guiding center equilibria in a torus, so this analysis might be vacuous except for simple geometries such as axial symmetry.

In a recent paper Taylor and Hastie [16] considered a problem similar to ours: a perturbation about a uniform magnetic field B_0 with pressure ordering $p_1 \ll p_2 \ll B_0^2$ and the gradient ordering $\partial p_1/\partial x \ll \partial p_2/\partial x_\perp \sim \partial B/\partial x_\perp$. They find, in addition to the local stability conditions, the following sufficient condition (in (4.2) of [16])

$$\int \left(\frac{B_0^2}{\mu_0} - C_2 B_0^3\right)^{-1} \frac{\partial p_*}{\partial x_\perp} \cdot \left[\frac{\partial p_*}{\partial x_\perp} - \left(\frac{B_0}{\mu_0} - C_2 B_0^2\right) \frac{\partial B}{\partial x_\perp}\right] ds < 0 \qquad (8.3)$$

Our sufficient condition (7.5) includes as a special case Grad's sufficient condition for interchange stability (in (8.4) of [10]), which in the above limit can be written as *

$$B_0^{-4} \int \frac{\partial B}{\partial x_1} \cdot \left[\frac{\partial p_*}{\partial x_1} - \left(\frac{B_0}{\mu_0} - C_2 B_0^2\right) \frac{\partial B}{\partial x_1}\right] ds < 0 \qquad (8.4)$$

Inequality (8.3) is much more restrictive than (8.4) because it includes a term lacking in (8.4):

$$\int \frac{\partial p_2}{\partial x_1} \cdot \left[C_2 B_0^2 \frac{\partial B}{\partial x_1} + \frac{\partial p_2}{\partial x_1}\right] ds$$

which is almost always positive except in a magnetic well. In fact (8.3) is very hard to satisfy except in a magnetic well, whereas (8.4) can be satisfied in any long thin plasma, almost independent of field geometry [6].

Our exact analysis confirms that the Taylor and Hastie condition is approximately correct (modulo the details of the field and pressure gradients) but is rather crude for it is unduly restrictive of field geometry. Also their analysis is only formal, being an expansion around a uniform field and zero β which has not been proven to converge (or be asymptotic) for any finite β or field non-uniformity.

*see [6] for a similar comparison.

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Appendix. Estimating $\int (U^i)^2 dS$ in Terms of $\int (U^i)^2 dx$

This estimate results from the familiar one-dimensional inequality relating the max norm of a function to its integral norms:

$$\max_{0 \le x \le L} |f(x)| \le \frac{1}{\sqrt{L}} \left[\int_0^L |f|^2 \, dx \right]^{1/2} + \sqrt{L} \left[\int_0^L \left|\frac{df}{dx}\right|^2 \, dx \right]^{1/2}$$

To apply this inequality we choose ℓ and L such that

$$\ell < \min \int ds \qquad\qquad L > \max \int ds$$

with both min and max taken over all lines. Then along any line

$$\max |U| \le \frac{1}{\sqrt{\ell}} \left[\int U^2 \, ds \right]^{1/2} + \sqrt{L} \left[\int \left(\frac{\partial U}{\partial s}\right)^2 \, ds \right]^{1/2} .$$

Squaring,

$$\bar{U}^2 \le \frac{1}{\ell} \int U^2 ds + L \int \left(\frac{\partial U}{\partial s}\right)^2 \, ds + 2\sqrt{L/\ell} \left[\int U^2 \, ds \int \left(\frac{\partial U}{\partial s}\right)^2 \, ds \right]^{1/2}$$

Integrating as in (5.2) over any transverse surface and as there letting $\gamma = B_{max}/B_{n_{min}}$

$$\int U^2 dS \le \frac{\gamma}{\ell} \int U^2 dx + \gamma L \int \left(\frac{\partial U}{\partial s}\right)^2 dx + 2\gamma\sqrt{L/\ell} \left[\int U^2 \, dx \int \left(\frac{\partial U}{\partial s}\right)^2 dx \right]^{1/2} \qquad \text{(A.1)}$$

Here use is made of the identity $\int f \, g \, dS \le \left[\int f^2 \, dS \int g^2 \, dS \right]^{1/2}$.

For an interchange, curl $(U^i \times B) = 0$, and from this we find

$$\frac{\partial U^i}{\partial s} = U_i \cdot b - (U^i \cdot \varkappa) b \qquad \text{(A.2)}$$

Using (A.2)with (A.1) it is easy to see that

$$\int_{S_2} (U^i)^2 dS < \text{const.} \int (U^i)^2 \, dx$$

The precise estimate, which is required for section 8, is

$$\int (U^{i})^{2}\,dS \le \left\{\frac{\gamma}{\ell} + \gamma L\,\left(|\overline{\nabla_{\perp} b}| + |\overline{\varkappa}|\right)^{2} + 2\gamma\,\sqrt{L/\ell}\,\left(|\overline{\nabla_{\perp} b}| + |\overline{\varkappa}|\right)\right\} \int (U^{i})^{2}\,dx \qquad (A.3)$$

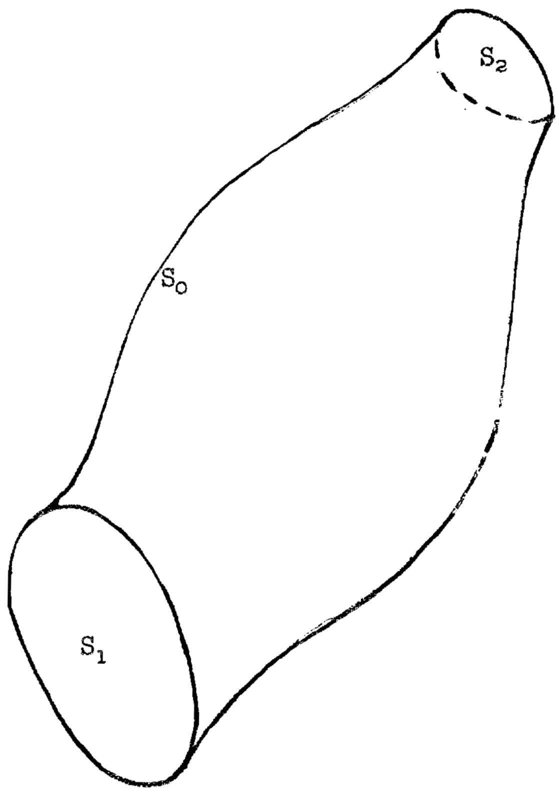

Figure 1. Plasma Domain D

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