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EXACT EXPRESSIONS FOR THE COVARIANCES BETWEEN PRODUCTS OF RANDOM VARIABLES

John W. Wilkinson

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EXACT EXPRESSIONS FOR THE COVARIANCES BETWEEN PRODUCTS OF RANDOM VARIABLES

John W. Wilkinson

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ABSTRACT

In evaluating the variances of sums of products of random variables one needs the variances of products and the covariances between products of random variables. In this report, exact formulae are developed for these variances and covariances for products of k random variables. These expressions are compared with the usual approximate formulae.

1. Introduction

One often encounters sums of products of random variables in which the evaluation of the variances of such sums is of interest. In the expressions for the variances there are two types of terms; (i) variances of products, and (ii) covariances between products.

Discussion of (i) is handled in detail by Goodman in [1] and [2]. Using procedures and notation similar to his, we will devote attention to (ii). In a sense, (i) is a special case of (ii).

In Section 2 is discussed the case where the random variables that form the products are independent. For this case the exact formulae for the covariances are immediately usable since they involve only variances and covariances of the original random variables. However, for the case where the random variables forming the products need not be independent (discussed in Section 3) the exact covariance expressions are, in our opinion, (because of the involvement of higher order moments) of little practical use, except for gaining some appreciation of the character of the errors involved in using the approximate formulae.

2. The Case Where the Variables That Form The Products Are Independent.

Let (x_{11}, x_{21}) and $x_{12}, x_{22})$ be two sets of two independent random variables. That is, x_{1i} is independent of x_{2i}, $i = 1,2$. Nothing is required concerning any independences among other combinations. In keeping with the notation introduced by Goodman, [1], [2], let us make the following definitions:

$$
\begin{aligned}
E(x_{ji}) &= X_{ji}, \\
G(x_{ji}) &= V(x_{ji})/X_{ji}^2, \\
\Delta x_{ji} &= x_{ji} - X_{ji}, \\
\delta x_{ji} &= \Delta x_{ji}/X_{ji}, \\
D(x_{1i}, x_{2i}) &= E(\delta x_{1i})(\delta x_{2i}) = \frac{\operatorname{cov}(x_{1i}, x_{2i})}{X_{1i}\, X_{2i}},
\end{aligned}
\tag{2.1}
$$

where E denotes expectation and V denotes variance.

(For simplicity we shall assume that all expectations differ from zero.)

Since

$$
\begin{aligned}
(x_{1i}x_{2i} - X_{1i}X_{2i}) &= X_{1i}X_{2i}[(\delta x_{1i} + 1)(\delta x_{2i} + 1) - 1] \\
&= X_{1i}X_{2i}[\delta x_{1i} + \delta x_{2i} + \delta x_{1i}\delta x_{2i}], \quad i = 1,2,
\end{aligned}
\tag{2.2}
$$

it follows that

$$
\begin{aligned}
\operatorname{cov}(x_{11}x_{21}, x_{12}x_{22}) &= E(x_{11}x_{21} - X_{11}X_{21})(x_{12}x_{22} - X_{12}X_{22}) \\
&= X_{11}X_{21}X_{12}X_{22}\; E(\delta x_{11} + \delta x_{21} + \delta x_{11}\delta x_{21})(\delta x_{12} \\
&\qquad + \delta x_{22} + \delta x_{12}\delta x_{22}).
\end{aligned}
\tag{2.3}
$$

Because of the independence of x_{1i} and x_{2i}, this simplifies to

$$\begin{aligned}\operatorname{cov}(x_{11}x_{21}, x_{12}x_{22}) &= X_{11}X_{21}X_{12}X_{22}\{D(x_{21},x_{22}) \\ &\quad + D(x_{11},x_{12}) + D(x_{11},x_{12})\, D(x_{21},x_{22})\} \\ &= X_{11}X_{12}\operatorname{cov}(x_{21},x_{22}) + X_{21}X_{22}\operatorname{cov}(x_{11},x_{12}) \\ &\quad + \operatorname{cov}(x_{11},x_{12})\operatorname{cov}(x_{21},x_{22}).\end{aligned} \tag{2.4}$$

The usual approximate formula for this situation is

$$\begin{aligned}\operatorname{cov}(x_{11}x_{21}, x_{12}x_{22}) &\cong X_{11}X_{12}\operatorname{cov}(x_{21},x_{22}) \\ &\quad + X_{21}X_{22}\operatorname{cov}(x_{11},x_{12}).\end{aligned} \tag{2.5}$$

The expressions for the variance of the product of two independent random variables can be obtained directly from (2.4) and (2.5) by dropping the second subscripts. Doing this yields

$$V(x_1x_2) = X_1^2\, V(x_2) + X_2^2\, V(x_1) + V(x_1)V(x_2), \text{ and} \tag{2.6}$$

$$V(x_1x_2) \cong X_1^2\, V(x_2) + X_2^2\, V(x_1). \tag{2.7}$$

For the case concerning the product of k random variables we have

$$\left(\prod_{j=1}^{k} x_{ji} - \prod_{j=1}^{k} X_{ji}\right) = \prod_{j=1}^{k} X_{ji}\left[\prod_{j=1}^{k}(\delta x_{ji} + 1) - 1\right],\ i = 1,2, \tag{2.8}$$

from which it follows that

$$\operatorname{cov}\left(\prod_{j=1}^{k} x_{j1},\ \prod_{j=1}^{k} x_{j2}\right) = \left(\prod_{j=1}^{k} X_{j1}X_{j2}\right)\Big[\sum_{j=1}^{k} D(x_{j1},x_{j2}) \tag{2.9}$$

$$+ \sum_{j_1<j_2} D(x_{j_1 1},x_{j_1 2})D(x_{j_2 1},x_{j_2 2})$$

$$+ \sum_{j_1<j_2<j_3} D(x_{j_1 1},x_{j_1 2})D(x_{j_2 1},x_{j_2 2})D(x_{j_3 1},x_{j_3 2})$$

$$+ \cdot\ \cdot\ \cdot\ \cdot$$

$$+ D(x_{11},x_{12})D(x_{21},x_{22})\ldots D(x_{k1},x_{k2})\Big],$$

where $\sum_{j_1<j_2<\cdots<j_s}$ denotes summation over all values of $j_1 < j_2 \cdots < j_s$ ranging over $1,2,\ldots,k$.

Specifically for $k = 3$, (2.9) reduces to

$$\operatorname{cov}(x_{11}x_{21}x_{31},x_{12}x_{22}x_{32}) = X_{11}X_{21}X_{31}X_{12}X_{22}X_{32}[D(x_{11},x_{12}) \tag{2.10}$$

$$+ D(x_{21},x_{22}) + D(x_{31},x_{32}) + D(x_{11},x_{12})\ D(x_{21},x_{22})$$

$$+ D(x_{11},x_{12})D(x_{31},x_{32}) + D(x_{21},x_{22})D(x_{31},x_{32})$$

$$+ D(x_{11},x_{12})D(x_{21},x_{22})D(x_{31},x_{32})].$$

The usual approximate formulae for (2.9) and (2.10) are, respectively,

$$\operatorname{cov}\left(\prod_{j=1}^{k} x_{j1},\ \prod_{j=1}^{k} x_{j2}\right) \cong \left[\prod_{j=1}^{k} X_{j1}X_{j2}\right]\left[\sum_{j=1}^{k} D(x_{j1},x_{j2})\right], \text{ and} \tag{2.11}$$

$$\operatorname{cov}(x_{11}x_{21}x_{31},x_{12}x_{22}x_{32}) \cong X_{11}X_{21}X_{31}X_{12}X_{22}X_{32}[D(x_{11},x_{12}) \tag{2.12}$$

$$+ D(x_{21},x_{32}) + D(x_{31},x_{32})].$$

By dropping the second subscripts in (2.9) - (2.12) we obtain the expressions (exact and approximate, respectively) for the variance of the product of k independent random variables. These are, in corresponding order,

$$V(\prod_{j=1}^{k} x_j) = (\prod_{j=1}^{k} X_j^2)[\sum_{j=1}^{k} G(x_j) + \sum_{j_1<j_2} G(x_{j1})G(x_{j2}) \tag{2.13}$$

$$+ \sum_{j_1<j_2<j_3} G(x_{j1})G(x_{j2})G(x_{j3})$$

$$+ \ldots.$$

$$+ G(x_1)G(x_2)\ldots G(x_k)],$$

$$V(x_1x_2x_3) = X_1^2\, X_2^2\, X_3^2\, [G(x_1) + G(x_2) + G(x_3) \tag{2.14}$$

$$+ G(x_1)G(x_2) + G(x_1)G(x_3) + G(x_2)G(x_3)$$

$$+ G(x_1)G(x_2)G(x_3)],$$

$$V(\prod_{j=1}^{k} x_j) \cong (\prod_{j=1}^{k} x_j^2)(\sum_{j=1}^{k} G(x_j)), \text{ and} \tag{2.15}$$

$$V(x_1x_2x_3) \cong X_1^2\, X_2^2\, X_3^2[G(x_1) + G(x_2) + G(x_3)]. \tag{2.16}$$

3. The Case Where the Variables That Form The Products Need Not Be Independent

Since

$$x_{1i}x_{2i} - E(x_{1i}x_{2i}) = X_{1i}X_{2i}\left[(\delta x_{1i} + 1)(\delta x_{2i} + 1) - \frac{E(x_{1i}x_{2i})}{X_{1i}X_{2i}}\right]$$

$$= X_{1i}X_{2i}[\delta x_{1i} + \delta x_{2i} + \delta x_{1i}\delta x_{2i} - D(x_{1i},x_{2i})], \tag{3.1}$$

then

$$\begin{aligned}
\operatorname{cov}(x_{11}x_{21}, x_{12}x_{22}) &= E[x_{11}x_{21} - E(x_{11}x_{21})][x_{12}x_{22} - E(x_{12}x_{22})] \qquad (3.2)\\
&= X_{11}X_{21}X_{12}X_{22}\, E[\delta x_{11} + \delta x_{21} + \delta x_{11}\delta x_{21}\\
&\quad - D(x_{11},x_{21})][\delta x_{12} + \delta x_{22} + \delta x_{12}\delta x_{22} - D(x_{12},x_{22})]\\
&= X_{11}X_{21}X_{12}X_{22}[D(x_{11},x_{12}) + D(x_{11},x_{22})\\
&\quad + D(x_{21},x_{12}) + D(x_{21},x_{22}) + D(x_{11},x_{12},x_{22})\\
&\quad + D(x_{11},x_{21},x_{12}) + D(x_{21},x_{12},x_{22})\\
&\quad + D(x_{11},x_{21},x_{22}) + D(x_{11},x_{21},x_{12},x_{22})\\
&\quad - D(x_{11},x_{21})D(x_{12},x_{22})],
\end{aligned}$$

where

$$D(z_1,z_2,\ldots,z_k) = E[(\delta z_1)(\delta z_2)\ldots(\delta z_k)]. \qquad (3.3)$$

The usual approximate formula is

$$\begin{aligned}
\operatorname{cov}(x_{11}x_{21}, x_{12}x_{22}) \cong X_{11}X_{21}X_{12}X_{22}[&D(x_{11},x_{12}) + D(x_{11},x_{22}) \qquad (3.4)\\
&+ D(x_{21},x_{12}) + D(x_{21},x_{22})].
\end{aligned}$$

The exact and approximate expressions for the variance of a product of two random variables can be obtained directly from (3.2) and (3.4), respectively, by dropping the second subscripts. Hence

$$V(x_1x_2) = X_1^2\, X_2^2\, [G(x_1) + G(x_2) + 2D(x_1,x_2) + 2D(x_1,x_1,x_2) \tag{3.5}$$

$$+ 2D(x_1,x_2,x_2) + D(x_1,x_1,x_2,x_2)$$

$$- \{D(x_1,x_2)\}^2], \text{ and}$$

$$V(x_1x_2) \cong X_1^2\, X_2^2\, [G(x_1) + G(x_2) + 2D(x_1,x_2)]. \tag{3.6}$$

For the case concerning the product of k random variables we have

$$[\prod_{j=1}^{k} x_{ji} - E(\prod_{j=1}^{k} x_{ji})] = \prod_{j=1}^{k} X_{ji}\left[\prod_{j=1}^{k}(\delta x_{ji} + 1) - \frac{E(\prod_{j=1}^{k} x_{ji})}{\prod_{j=1}^{k} X_{ji}}\right], \tag{3.7}$$

$$i = 1,2,$$

$$= \prod_{j=1}^{k} X_{ji}[\sum_{j=1}^{k} \delta x_{ji} + \sum_{j_1<j_2} \delta x_{j_1 i}\delta x_{j_2 i}$$

$$+ \sum_{j_1<j_2<j_3} \delta x_{j_1 i}\delta x_{j_2 i}\delta x_{j_3 i} + \ldots$$

$$+ \delta x_{1i}\delta x_{2i}\ldots\delta x_{ki} - D(x_{1i},x_{2i},\ldots,x_{ki})].$$

Although Goodman in [2] has introduced a notation which permits a more compact presentation we have chosen not to use it because of the complexity of its introduction. Hence, we will proceed to obtain the covariance expression in terms of notation already established.

From the definition of covariance and (3.7) it follows that

$$\mathrm{cov}(\prod_{j=1}^{k} x_{j1}, \prod_{j=1}^{k} x_{j2}) = \prod_{j=1}^{k} X_{j1}X_{j2}\, E[\sum_{j=1}^{k} \delta x_{j1} + \sum_{j_1<j_2} \delta x_{j_1 1}\, \delta x_{j_2 1}$$

$$+ \sum_{j_1<j_2<j_3} \delta x_{j_1 1}\, \delta x_{j_2 1}\, \delta x_{j_3 1} + \ldots + \delta x_{11}\delta x_{21}\ldots\delta x_{k1} -$$

$$- D(x_{11},x_{21},\ldots,x_{k1})][\sum_{j=1}^{k} \delta x_{j2} + \sum_{j_1<j_2} \delta x_{j_1 2}\, \delta x_{j_2 2}$$

$$+ \sum_{j_1<j_2<j_3} \delta x_{j_1 2}\, \delta x_{j_2 2}\, \delta x_{j_3 2} + \ldots + \delta x_{12}\, \delta x_{22} \ldots \delta x_{k2}$$

$$- D(x_{12},x_{22},\ldots,x_{k2})]$$

$$= (\prod_{j=1}^{k} X_{j1}X_{j2})[\sum_{j=1}^{k}\{\sum_{\ell=1}^{k} D(x_{j1},x_{\ell 2})$$

$$+ \sum_{j_1<j_2} \Big(D(x_{j1},x_{j_1 2},x_{j_2 2}) + D(x_{j_1 1},x_{j_2 1},x_{j2})\Big)$$

$$+ \sum_{j_1<j_2<j_3} \Big(D(x_{j1},x_{j_1 2},x_{j_2 2},x_{j_3 2}) + D(x_{j_1 1},x_{j_2 1},x_{j_3 1},x_{j2})\Big)$$

$$+ \ldots + D(x_{j1},x_{12},x_{22},\ldots,x_{k2}) + D(x_{11},x_{21},\ldots,x_{k1},x_{j2})\}$$

$$+ \sum_{i_1<i_2} \{\sum_{j_1<j_2} D(x_{i_1 1},x_{i_2 1},x_{j_1 2},x_{j_2 2}) \tag{3.8}$$

$$+ \sum_{j_1<j_2<j_3} \Big(D(x_{i_1 1},x_{i_2 1},x_{j_1 2},x_{j_2 2},x_{j_3 2})$$

$$+ D(x_{j_1 1},x_{j_2 1},x_{j_3 1},x_{i_1 2},x_{i_2 2})\Big) + \ldots . D(x_{i_1 1},x_{i_2 1},x_{12},x_{22},\ldots x_{k2})$$

$$+ D(x_{11},x_{21},\ldots,x_{k1},x_{i_1 2},x_{i_2 2})\} + \ldots .$$

$$+ D(x_{11},x_{21},\ldots,x_{k1},x_{12},x_{22},\ldots .,x_{k2})$$

$$- D(x_{12},x_{22},\ldots,x_{k2})\{\sum_{j_1<j_2} D(x_{j_1 1},x_{j_2 1})$$

$$+ \sum_{j_1<j_2<j_3} D(x_{j_1 1},x_{j_2 1},x_{j_3 1}) + \ldots + D(x_{11},x_{21},\ldots,x_{k1})\}$$

$$-D(x_{11},x_{21},\ldots,x_{k1})\{\sum_{j_1<j_2} D(x_{j_1 2},x_{j_2 2},) + \sum_{j_1<j_2<j_3} D(x_{j_1 2},x_{j_2 2},x_{j_3 2})$$

$$+\ldots+ D(x_{12},x_{22},\ldots,x_{k2})\}+ D(x_{11},x_{21},\ldots,x_{k1})D(x_{12},x_{22},\ldots,x_{k2})]$$

The approximate formula is a delight to look at after the above exact formula. It is

$$\mathrm{cov}(\prod_{j=1}^{k} x_{j1}, \prod_{j=1}^{k} x_{j2}) \cong (\prod_{j=1}^{k} X_{j1}X_{j2})(\sum_{j,\ell=1}^{k} D(x_{j1},x_{\ell 2})) \tag{3.9}$$

The corresponding expressions for the variances, obtained by dropping the second subscripts are, respectively,

$$V(\prod_{j=1}^{k} x_j) = (\prod_{j=1}^{k} X_j^2)[\sum_{j=1}^{k} G(x_j) + 2\sum_{j_1<j_2} D(x_{j_1},x_{j_2}) \tag{3.10}$$

$$+ 2\sum_{j=1}^{k}\{\sum_{j_1<j_2} D(x_j,x_{j_1},x_{j_2}) + \sum_{j_1<j_2<j_3} D(x_j,x_{j_1},x_{j_2},x_{j_3}) +\ldots$$

$$+ D(x_j,x_1,x_2,\ldots,x_k)\}$$

$$+ 2\sum_{i_1<i_2}\{\frac{1}{2}\sum_{j_1<j_2} D(x_{i_1},x_{i_2},x_{j_1},x_{j_2}) + \sum_{j_1<j_2<j_3} D(x_{i_1},x_{i_2},x_{j_1},x_{j_2},x_{j_3})$$

$$+\ldots+ D(x_{i_1},x_{i_2},x_1,x_2,\ldots,x_k)\}$$

$$+\ldots+ D(x_1,x_2,\ldots,x_k,x_1,x_2,\ldots,x_k)$$

$$- 2D(x_1,x_2,\ldots,x_k)\{\sum_{j_1<j_2} D(x_{j_1},x_{j_2}) + \sum_{j_1<j_2<j_3} D(x_{j_1},x_{j_2},x_{j_3})$$

$$+\ldots+ D(x_1,x_2,\ldots,x_k)\}$$

$$+ \{D(x_1,x_2,\ldots,x_k)\}^2], \text{ and}$$

$$V(\prod_{j=1}^{k} x_j) \cong (\prod_{j=1}^{k} X_j^2)[\sum_{j=1}^{k} G(x_j) + 2 \sum_{j_1<j_2} D(x_{j_1}, x_{j_2})]. \tag{3.11}$$

One will note that in (3.10), all the terms with more than two entries for the arguments of the D's involve higher order moments of both sets of variables.

For example

$$D(x_1,x_2,x_1,x_2,x_3) = E(\delta x_1)^2(\delta x_2)^2\delta(x_3)$$
$$= \frac{E(x_1 - X_1)^2(x_2 - X_2)^2(x_3 - X_3)}{X_1^2\, X_2^2\, X_3}.$$

It is for this reason that we made the earlier comment that the use of the exact formulae for variance and covariance of products is impractical when the variables forming the products are dependent.

To help to appreciate the form of (3.8), (3.9), (3.10) and (3.11) the rewriting of these expressions for k = 3 may help. The corresponding formulae become, respectively,

$$\mathrm{cov}(x_{11}x_{21}x_{31}, x_{12}x_{22}x_{32}) = (X_{11}X_{21}X_{31}X_{12}X_{22}X_{32})[\sum_{j=1}^{3}\{\sum_{\ell=1}^{3} D(x_{j1}, x_{\ell 2}) \tag{3.12}$$

$$+ \sum_{j_1<j_2} (D(x_{j1}, x_{j_1 2}, x_{j_2 2},) + D(x_{j_1 1}, x_{j_2 1}, x_{j2})$$

$$+ D(x_{j1}, x_{12}, x_{22}, x_{32}) + D(x_{11}, x_{21}, x_{31}, x_{j2})\}$$

$$+ \sum_{i_1<i_2} \{ \sum_{j_1<j_2} D(x_{i_1 1}, x_{i_2 1}, x_{j_1 2}, x_{j_2 2})$$

$$+ D(x_{i_1 1}, x_{i_2 1}, x_{12}, x_{22}, x_{32}) + D(x_{11}, x_{21}, x_{31}, x_{i_1 2}, x_{i_2 2})\} +$$

$$+ D(x_{11},x_{21},x_{31},x_{12},x_{22},x_{32})$$

$$- D(x_{11},x_{21},x_{31})\{D(x_{12},x_{22}) + D(x_{12},x_{32}) + D(x_{22},x_{32})$$

$$+ D(x_{12},x_{22},x_{32})\}$$

$$- D(x_{12},x_{22},x_{32})\{D(x_{11},x_{21}) + D(x_{11},x_{31}) + D(x_{21},x_{31})$$

$$+ D(x_{11},x_{21},x_{31})\}$$

$$+ D(x_{11},x_{21},x_{31})\, D(x_{12},x_{22},x_{32})],$$

$$\mathrm{cov}(x_{11}x_{21}x_{31},x_{12}x_{22}x_{32}) \cong (X_{11}X_{21}X_{31}X_{12}X_{22}X_{32})[\sum_{j,\ell=1}^{3} D(x_{j1},x_{\ell 2})], \quad (3.13)$$

$$V(x_1x_2x_3) = X_1^2\, X_2^2\, X_3^2[G(x_1) + G(x_2) + G(x_3) + 2D(x_1,x_2) \quad (3.14)$$

$$+ 2D(x_1,x_3) + 2D(x_2,x_3) + 2\sum_{j=1}^{3} \sum_{j_1<j_2} D(x_j,x_{j_1},x_{j_2})$$

$$+ 4 \sum_{j=1}^{3} D(x_j,x_1,x_2,x_3) + \sum_{j_1<j_2} D(x_{j_1},x_{j_1},x_{j_2},x_{j_2})$$

$$+ 2 \sum_{j_1<j_2} D(x_{j_1},x_{j_2},x_1,x_2,x_3)$$

$$+ D(x_1,x_1,x_2,x_2,x_3,x_3)$$

$$- 2D(x_1,x_2,x_3)\{D(x_1,x_2) + D(x_1,x_3) + D(x_2,x_3)$$

$$+ D(x_1,x_2,x_3)\} + \{D(x_1,x_2,x_3)\}^2],$$

$$V(x_1x_2x_3) \cong (X_1X_2X_3)^2[G(x_1) + G(x_2) + G(x_3) \quad (3.15)$$

$$+ 2D(x_1,x_2) + 2D(x_1,x_3) + 2D(x_2,x_3)].$$

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