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ANALYTICAL SOLUTIONS TO EDDY-CURRENT PROBE COIL PROBLEMS

C. V. Dodd
W. E. Deeds

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C. V. Dodd and W. E. Deeds[1]

ABSTRACT

Solutions have been obtained for axially symmetric eddy-current problems in two configurations of wide applicability. In both cases the eddy currents are assumed to be produced by a circular coil of rectangular cross section, driven by a constant amplitude alternating current. One solution is for a coil above a semi-infinite conducting slab with a plane surface, covered with a uniform layer of another conductor. This solution includes the special cases of a coil above a single infinite plane conductor or above a sheet of finite thickness, as well as the case of one metal clad on another. The other solution is for a coil surrounding an infinitely long circular conducting rod with a uniformly thick coating of another conductor. This includes the special cases of a coil around a conducting tube or rod, as well as one metal clad on a rod of another metal. The solutions are in the form of integrals of first-order Bessel functions giving the vector potential, from which the other electromagnetic quantities of interest can be obtained. The coil impedance has been calculated for the case of a coil above a two-conductor plane. The agreement between the calculated and experimental values is excellent.

INTRODUCTION

Electromagnetic problems are usually divided into three categories: low frequency, intermediate frequency, and high frequency. At low frequencies, static conditions are assumed; at high frequencies, wave equations are used. Both of these regions have been studied extensively. However, in the intermediate frequency range, where diffusion equations are used, very few problems have actually been solved. Eddy-current coil problems fall into this intermediate frequency region. This report presents an accurate technique for analyzing the problems of eddy-current testing.

[1]Consultant from the University of Tennessee.

Eddy-current testing has been used in industry for many years. As early as 1879, D. E. Hughes[2] used an induction coil to sort metals. There have been numerous articles on the testing of materials with eddy currents. Some of the first papers dealing with both the theory and the practical aspects of eddy-current testing are by Förster,[3] Förster and Stambke,[4] and Förster.[5] In this series of papers, analyses are made of a coil above a conducting surface, assuming the coil to be a magnetic dipole, and of an infinite coil encircling an infinite rod. Hochschild[6] also gives an analysis of an infinite coil including some eddy-current distributions in the metal. Waidelich and Renken[7] made an analysis of the coil impedance using an image approach. Their theoretical results agreed well with theory for relatively high frequencies. Libby[8] presented a theory in which he assumed the coil was a transformer with a network tied to the secondary. This network representation gave good results when compared to experiment. The diffusion of eddy-current pulses (Atwood and Libby[9]) can be represented in this manner. Russell, Schuster and Waidelich[10] gave an analysis of a cup core coil where they assumed the flux was entirely coupled into the conductor. The semi-empirical results agreed fairly well with the experimental measurements.

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[2]D. E. Hughes, Phil. Mag. 8(5), 50 (1879).

[3]Friedrich Förster, Z. Metallk. 43, 163–171 (1952).

[4]Friedrich Förster and Kurt Stambke, Z. Metallk. 45(4), 166–179 (1954).

[5]Friedrich Förster, Z. Metallk. 45(4), 197–199 (1954).

[6]R. Hochschild, "Electromagnetic Methods of Testing Metals," Progress in Nondestructive Testing, Vol. 1, Macmillan Company, New York, 1959.

[7]D. L. Waidelich and C. J. Renken, Proc. Natl. Electron Conf. 12, 188–196 (1956).

[8]H. L. Libby, Broadband Electromagnetic Testing Methods, HW-59614 (1959).

[9]K. W. Atwood and H. L. Libby, Diffusion of Eddy Currents, HW-79844 (1963).

[10]T. J. Russell, V. E. Schuster, and D. L. Waidelich, J. Electron. Control 13, 232–237 (1962).

Vein,[11] Cheng,[12] and Burrows[13] gave treatments based on delta function coils, and Burrows continued with the development of an eddy-current flaw theory. Dodd and Deeds,[14] Dodd,[15] and Dodd[16] gave a relaxation theory to calculate the vector potential of a coil with a finite cross section. Here we extend a "closed form" solution to such coils.

The vector potential is used as opposed to the electric and magnetic fields. The differential equations for the vector potential will be derived from Maxwell's equations, with the assumption of cylindrical symmetry. This differential equation will then be solved to obtain a "closed form" solution.

For the "closed form" solution, sinusoidal driving currents and linear, isotropic, and homogeneous media will be assumed. Solutions will be obtained for two different conductor geometries: a rectangular cross-section coil above a plane with one conductor clad on another and a rectangular cross-section coil encircling a two-conductor rod. The solutions for both geometries will be given in terms of integrals of Bessel functions. Once the vector potential has been determined, it can be used to calculate any physically observable electromagnetic quantity.

Equations to calculate eddy-current density, induced voltage, coil impedance, and effect of defects will be given. Measured values of coil impedance as compared with calculated values show excellent agreement.

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[11] P. R. Vein, J. Electron. Control **13**, 471–494 (1962).

[12] David H.S. Cheng, "The Reflected Impedance of a Circular Coil in the Proximity of a Semi-Infinite Medium," Ph.D. Dissertation, University of Missouri, 1964.

[13] Michael Leonard Burrows, A Theory of Eddy Current Flaw Detection, University Microfilms, Inc., Ann Arbor, Michigan, 1964.

[14] C. V. Dodd and W. E. Deeds, "Eddy Current Impedance Calculated by a Relaxation Method," pp. 300–314 in Proceedings of the Symposium on Physics and Nondestructive Testing, Southwest Research Institute, San Antonio, Texas, 1963.

[15] C. V. Dodd, A Solution to Electromagnetic Induction Problems, ORNL-TM-1185 (1965) and M.S. Thesis, the University of Tennessee, 1965.

[16] C. V. Dodd, Solutions to Electromagnetic Induction Problems, ORNL-TM-1842 (1967) and Ph.D. Dissertation, the University of Tennessee, 1967.

DERIVATION OF VECTOR POTENTIAL

The differential equations[17] for the vector potential will be derived from Maxwell's equations which are:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \cdot \vec{D} = \rho \tag{4}$$

The medium is taken to be linear and isotropic, but not homogeneous. In a linear and isotropic medium, the following relations between $\vec{D}$ and $\vec{E}$ and $\vec{B}$ and $\vec{H}$ hold:

$$\vec{B} = \mu \vec{H} \tag{5}$$

$$\vec{D} = \epsilon \vec{E} \tag{6}$$

The current density $\vec{J}$ can be expressed in terms of Ohm's law:

$$\vec{J} = \sigma \vec{E} \tag{7}$$

Equations (6) and (7) may be substituted into Equation (1) to obtain the curl of $\vec{H}$ in terms of $\vec{E}$:

$$\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \epsilon \vec{E}}{\partial t} \tag{8}$$

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[17]A list of symbols is given in Appendix A.

The term $\sigma\vec{E}$ is much greater than $\frac{\partial\epsilon\vec{E}}{\partial t}$, so the latter may be neglected for frequencies below about 10 Mc/sec (ref. 18). The magnetic induction field $\vec{B}$ may be expressed as the curl of a vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A} \tag{9}$$

Substituting this into Equation (2) gives

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A} = -\nabla \times \frac{\partial\vec{A}}{\partial t} \tag{10}$$

or

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\psi = \vec{E}_{induced} + \vec{E}_{applied} \tag{11}$$

$$\sigma\vec{E} = -\sigma\frac{\partial\vec{A}}{\partial t} - \sigma\nabla\psi \tag{12}$$

The term ψ is interpreted as an applied scalar potential. The coil may be driven by a voltage generator with an applied voltage ψ and an internal resistivity, $\frac{1}{\sigma}$. However, for the purpose of this problem, the driving function is expressed as an alternating current density of constant amplitude, $\vec{i}_o$ rather than an applied potential, where

$$\underset{\substack{\sigma \to 0. \\ \nabla\psi \to \infty}}{\text{Limit}} (-\sigma\nabla\psi) = \vec{i}_o \tag{13}$$

This provides a current which is not affected by induced voltages or the presence of other coils. Making this substitution gives:

$$\sigma\vec{E} = -\sigma\frac{\partial\vec{A}}{\partial t} + \vec{i}_o \tag{14}$$

Substituting Equations (5) and (9) into the left side of Equation (8) and Equation (14) into the right side gives:

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[18]For sinusoidal waves, $\frac{\partial\epsilon\vec{E}}{\partial t} = \frac{\epsilon\partial\vec{E}}{\partial t} = j\epsilon\omega\vec{E}$. The term σE is much greater than $\epsilon\omega E$ or $\sigma \gg \epsilon\omega$. $\sigma \approx 10^7$ mhos/meter for metals, $\epsilon \approx 10^{-11}$. For frequencies on the order of 10^7 cps, $\omega \approx 10^8$, $10^7 \gg 10^8 \times 10^{-11}$, or $\sigma \approx 10^{10}\ \epsilon\omega$.

$$\nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu} = \nabla \times [(1/\mu)\ \nabla \times \vec{A}] = -\sigma \frac{\partial \vec{A}}{\partial t} + \vec{i}_o \tag{15}$$

The vector identities (Morse and Feshbach[19])

$\nabla \times (\psi \vec{F}) = (\nabla \psi) \times \vec{F} + \psi \nabla \times \vec{F}$ and $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$,

can be used to expand the left side of Equation (15):

$$\begin{aligned}\nabla \times (1/\mu)(\nabla \times \vec{A}) &= \nabla(1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) \\ &= \nabla(1/\mu) \times (\nabla \times \vec{A}) + \frac{1}{\mu} \nabla(\nabla \cdot \vec{A}) - \frac{1}{\mu} \nabla^2 \vec{A}\ .\end{aligned} \tag{16}$$

In the definition of the vector potential the divergence of the vector potential was not defined, so it can be defined to be anything convenient. For induction problems $\nabla \cdot \vec{A}$ is set to zero. (This corresponds to the Coulomb gage.) Equation (16) will then yield the following results when substituted into Equation (15).

$$\nabla^2 \vec{A} = -\mu \vec{i}_o + \mu\sigma \frac{\partial \vec{A}}{\partial t} + \mu \nabla(1/\mu) \times (\nabla \times \vec{A}) \tag{17}$$

This is the equation for the vector potential in an isotropic, linear, inhomogeneous medium. For most coil problems it is possible to assume axial symmetry as shown in Fig. 1. The vector potential will be symmetric about the axis of the coil. Since this assumption is valid for most problems and the alternative to this assumption is a much more complicated and impractical problem, axial symmetry is assumed. With axial symmetry, there is only a θ component of $\vec{I}$ and therefore of $\vec{A}$. Expanding the θ component of Equation (17) gives:

$$\begin{aligned}\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} &= -\mu i_o + \mu\sigma \frac{\partial A}{\partial t} \\ &\quad - \mu \left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A}{\partial z}\right]\end{aligned} \tag{18}$$

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[19] Philip M. Morse and Herman Feshbach, Methods of Theoretical Physics, McGraw-Hill Book Company, New York, 1953.

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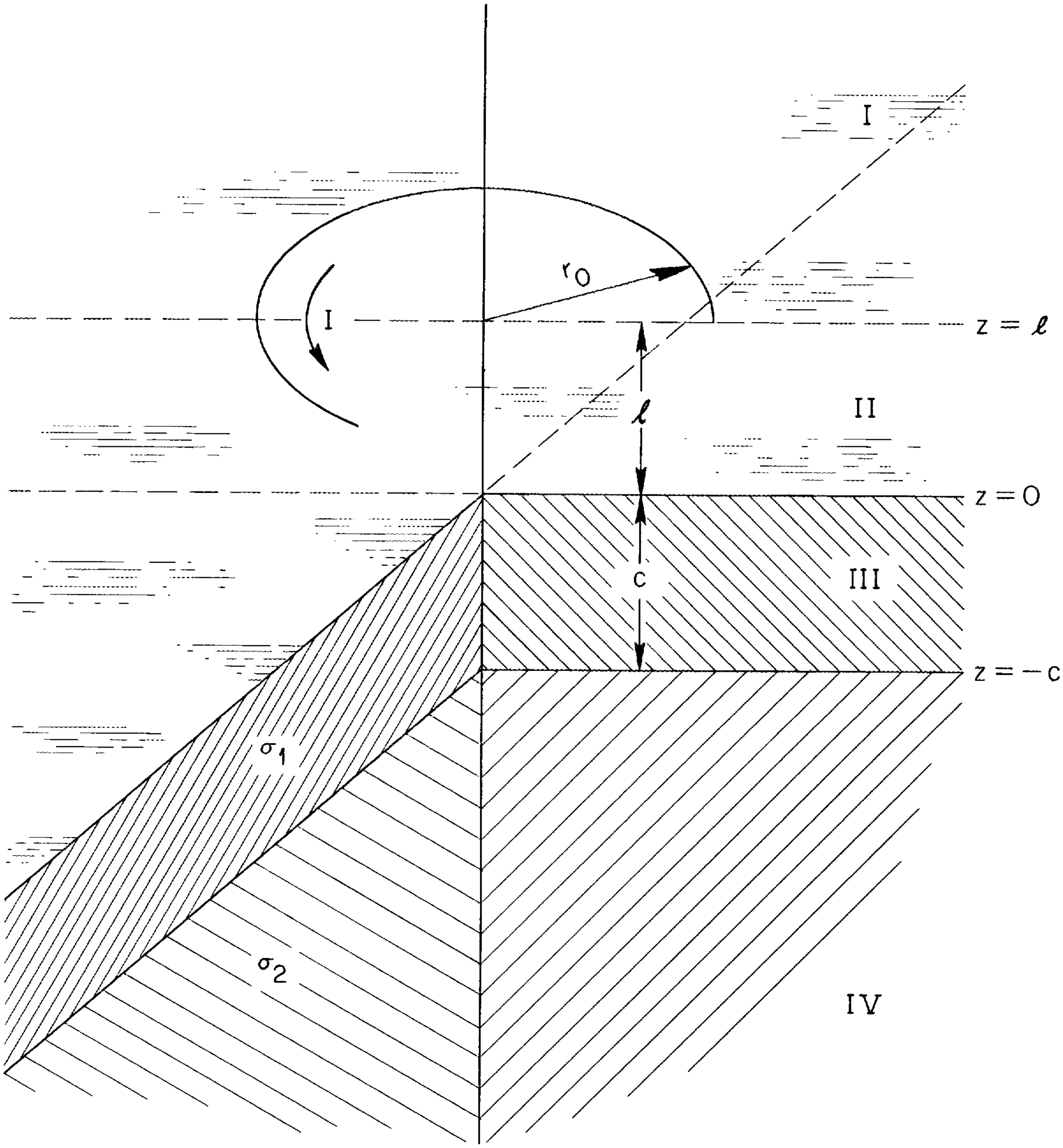

Fig. 1. Delta Function Coil above a Two-Conductor Plane.

Assume that i_o is a sinusoidal function of time, $i_o = i'_o\, e^{j\omega t}$. Then the vector potential is likewise a sinusoidal function of time,

$$A = A' e^{j(\omega t + \phi)} = A'' e^{j\omega t}.$$

Substituting into Equation (18) gives:

$$\frac{\partial^2 A''}{\partial r^2} e^{j\omega t} + \frac{1}{r}\frac{\partial A''}{\partial r} e^{j\omega t} + \frac{\partial^2 A''}{\partial z^2} e^{j\omega t} - \frac{A''}{r^2} e^{j\omega t} = -\mu\, i'_o\, e^{j\omega t} + j\omega\mu\sigma A''\, e^{j\omega t}$$
$$- \mu\left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA''}{\partial r} e^{j\omega t}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A''}{\partial z} e^{j\omega t}\right]$$

Canceling out the term $e^{j\omega t}$ and dropping the prime gives:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = -\mu i_o + j\omega\mu\sigma A - \mu\left[\frac{\partial(1/\mu)}{\partial r}\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right) + \left(\frac{\partial(1/\mu)}{\partial z}\right)\frac{\partial A}{\partial z}\right] \tag{19}$$

This is the general differential equation for the vector potential in a linear, inhomogeneous medium with a sinusoidal driving current. We shall now obtain a "closed form" solution of Equation (19).

CLOSED FORM SOLUTIONS OF THE VECTOR POTENTIAL

We shall assume the medium to be linear, isotropic, and homogeneous. When I is the total driving current in a delta function coil at (r_o, z_o), the general Equation (19) then becomes:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} - j\omega\mu\sigma A + \mu I\, \delta(r - r_o)\, \delta(z - z_o) = 0 \tag{20}$$

Once we have solved this linear differential equation for a particular conductor configuration, we can then superimpose any number of delta function coils to build any desired shape of coil (provided that the current in each coil is known).

We shall solve the problem for two different conductor configurations: a coil above a two-conductor plane and a coil encircling a two-conductor rod. These two configurations apply to a large number of practical problems.

Coil above a Two-Conductor Plane

The coil above a two-conductor plane is shown in Fig. 1. We have divided the problem into four regions. The differential equation in air (regions I and II) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} = 0 \tag{21}$$

The differential equation in a conductor (regions III and IV) is:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{\partial^2 A}{\partial z^2} - \frac{A}{r^2} - j\omega\mu\sigma_i A = 0 \tag{22}$$

Setting $A(r,z) = R(r)\,Z(z)$ and dividing by $R(r)\,Z(z)$ gives:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega\mu\sigma_i = 0 \tag{23}$$

We write for the z dependence:

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = \text{"constant"} = \alpha^2 + j\omega\mu\sigma_i \tag{24}$$

or

$$Z(z) = A\ e^{+\sqrt{\alpha^2 + j\omega\mu\sigma_i}\ z} + B\ e^{-\sqrt{\alpha^2 + j\omega\mu\sigma_i}\ z} \tag{25}$$

We define:

$$\alpha_i \equiv \sqrt{\alpha^2 + j\omega\mu\sigma_i} \tag{26}$$

Equation (23) then becomes:

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} + \alpha^2 - \frac{1}{r^2} = 0 \tag{27}$$

This is a first-order Bessel equation and has the solutions:

$$R(r) = C\ J_1(\alpha r) + D\ Y_1(\alpha r) \tag{28}$$

Combining the solutions we have:

$$A(r,z) = [A\ e^{+\alpha_i z} + B\ e^{-\alpha_i z}][C\ J_1(\alpha r) + D\ Y_1(\alpha r)] \tag{29}$$

We now need to determine the constants A, B, C, and D. They are functions of the separation "constant" α and are usually different for each value of α. Our complete solution would be a sum of all the individual solutions, if α were a discrete variable; but, since α is a continuous variable, the complete solution is an integral over the entire range of α. Thus, the general solution is:

$$A(r,z) = \int_0^\infty [A(\alpha)\ e^{\alpha_1 z} + B(\alpha)\ e^{-\alpha_1 z}][C(\alpha)\ J_1(\alpha r) + D(\alpha)\ Y_1(\alpha r)]\ d\alpha\ . \qquad (30)$$

We must take $A(\alpha) = 0$ in region I, where z goes to plus infinity. Due to the divergence of Y_1 at the origin, $D(\alpha) = 0$ in all regions. In region IV, where z goes to minus infinity, $B(\alpha)$ must vanish. The solutions in each region then become:

$$A^{(1)}(r,z) = \int_0^\infty B_1(\alpha) e^{-\alpha z}\ J_1(\alpha r)\ d\alpha \qquad (31)$$

$$A^{(2)}(r,z) = \int_0^\infty [C_2(\alpha)\ e^{+\alpha z} + B_2(\alpha)\ e^{-\alpha z}]\ J_1(\alpha r)\ d\alpha \qquad (32)$$

$$A^{(3)}(r,z) = \int_0^\infty [C_3(\alpha)\ e^{\alpha_1 z} + B_3(\alpha)\ e^{-\alpha_1 z}]\ J_1(\alpha r)\ d\alpha \qquad (33)$$

$$A^{(4)}(r,z) = \int_0^\infty [C_4(\alpha)\ e^{\alpha_2 z}\ J_1(\alpha r)\ d\alpha \qquad (34)$$

The boundary conditions between the different regions are:

$$A^{(1)}(r,\ell) = A^{(2)}(r,\ell) \qquad (35)$$

$$\frac{\partial A^{(1)}}{\partial z}(r,z)\Big]_{z=\ell} = \frac{\partial A^{(2)}}{\partial z}(r,z)\Big]_{z=\ell} - \mu I\ \delta(r - r_o) \qquad (36)$$

$$A^{(2)}(r,o) = A^{(3)}(r,o) \qquad (37)$$

$$\frac{\partial A^{(2)}}{\partial z}(r,z)\Big]_{z=o} = \frac{\partial A^{(3)}}{\partial z}(r,z)\Big]_{z=o} \qquad (38)$$

$$A^{(3)}(r,-c) = A^{(4)}(r,-c) \qquad (39)$$

$$\frac{\partial A^{(3)}}{\partial z}(r,z)\Big]_{z=-c} = \frac{\partial A^{(4)}}{\partial z}(r,z)\Big]_{z=-c} \qquad (40)$$

Equation (35) gives:

$$\int_0^\infty B_1(\alpha)\ e^{-\alpha\ell}\ J_1(\alpha r)\ d\alpha = \int_0^\infty [C_2(\alpha)\ e^{\alpha\ell} + B_2(\alpha)\ e^{-\alpha\ell}]\ J_1(\alpha r)\ d\alpha \tag{41}$$

If we multiply both sides of Equation (41) by $\int_0^\infty J_1(\alpha' r) r\ dr$ and then reverse the order of integration, we obtain:

$$\int_0^\infty \frac{B_1(\alpha)\ e^{-\alpha\ell}}{\alpha} [\int_0^\infty J_1(\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr]\ d\alpha = \int_0^\infty \frac{1}{\alpha} [C_2(\alpha)\ e^{\alpha\ell}$$

$$+ B_2(\alpha)\ e^{-\alpha\ell}][\int_0^\infty J_1(\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr]\ d\alpha \tag{42}$$

We can simplify Equation (42) by use of the Fourier-Bessel equation, which is:

$$F(\alpha') = \int_0^\infty F(\alpha) \int_0^\infty J_1(\alpha r)\ J_1(\alpha' r)\ \alpha r\ dr\ d\alpha \tag{43}$$

Equation (42) then becomes:

$$\frac{B_1}{\alpha'}\ e^{-\alpha'\ell} = \frac{C_2}{\alpha'}\ e^{\alpha'\ell} + \frac{B_2}{\alpha'}\ e^{-\alpha'\ell} \tag{44}$$

We can evaluate the other integral equations in a similar manner. We get (after dropping the primes on the α):

$$-B_1 e^{-\alpha\ell} = C_2\ e^{\alpha\ell} - B_2\ e^{-\alpha\ell} - \mu I r_o\ J_1(\alpha r_o) \tag{45}$$

$$\frac{C_2}{\alpha} + \frac{B_2}{\alpha} = \frac{C_3}{\alpha} + \frac{B_3}{\alpha} \tag{46}$$

$$C_2 - B_2 = \frac{\alpha_1}{\alpha}\ C_3 - \frac{\alpha_1}{\alpha}\ B_3 \tag{47}$$

$$\frac{C_3}{\alpha}\ e^{-\alpha_1 c} + \frac{B_3}{\alpha}\ e^{+\alpha_1 c} = \frac{C_4}{\alpha}\ e^{-\alpha_2 c} \tag{48}$$

$$\frac{\alpha_1}{\alpha}\ C_3\ e^{-\alpha_1 c} - \frac{\alpha_1}{\alpha}\ B_3\ e^{\alpha_1 c} = \frac{\alpha_2}{\alpha}\ C_4\ e^{-\alpha_2 c} \tag{49}$$

We now have six equations with six unknowns. Their solution is:

$$B_1 = \frac{1}{2}\mu I r_o\, J_1(\alpha r_o)\left\{e^{\alpha\ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right] e^{-\alpha\ell}\right\} \tag{50}$$

$$C_2 = \frac{1}{2}\mu I r_o\, J_1(\alpha r_o)\; e^{-\alpha\ell} \tag{51}$$

$$B_2 = \frac{1}{2}\mu I r_o\, J_1(\alpha r_o)\left\{\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right\} e^{-\alpha\ell} \tag{52}$$

$$C_3 = \mu I r_o\, J_1(\alpha r_o)\left\{\frac{\alpha(\alpha_2+\alpha_1)\; e^{-\alpha\ell + 2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right\} \tag{53}$$

$$B_3 = \mu I r_o\, J_1(\alpha r_o)\left\{\frac{\alpha(\alpha_1-\alpha_2)\; e^{-\alpha\ell}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right\} \tag{54}$$

$$C_4 = \mu I r_o\, J_1(\alpha r_o)\left\{\frac{2\alpha_1\alpha\; e^{(\alpha_2+\alpha_1)c}\; e^{-\alpha\ell}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right\} \tag{55}$$

We can now write the expressions for the vector potential in each region:

$$A^{(1)}(r,z) = \frac{\mu I r_o}{2}\int_0^\infty J_1(\alpha r_o)\, J_1(\alpha r)\; e^{-\alpha\ell-\alpha z} \times \left\{e^{2\alpha\ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right]\right\} d\alpha \tag{56}$$

$$A^{(2)}(r,z) = \frac{\mu I r_o}{2}\int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\; e^{-\alpha\ell} \times \left\{e^{\alpha z} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right] e^{-\alpha z}\right\} d\alpha\,. \tag{57}$$

$$A^{(3)}(r,z) = \mu I r_o \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\, e^{-\alpha \ell} \alpha$$

$$\times \left\{ \frac{(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}\, e^{\alpha_1 z} + (\alpha_1-\alpha_2)\, e^{-\alpha_1 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}} \right\} d\alpha \qquad (58)$$

$$A^{(4)}(r,z) = \mu I r_o \int_0^\infty J_1(\alpha r_o) J_1(\alpha r)\, e^{-\alpha \ell} \alpha$$

$$\times \left\{ \frac{2\alpha_1\, e^{(\alpha_2+\alpha_1)c}\, e^{+\alpha_2 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}} \right\} d\alpha \qquad (59)$$

These are the equations for the vector potential of a delta function coil above a two-conductor plane. Next we shall consider the derivation of the vector potential of a delta function coil encircling a two-conductor rod.

Coil Encircling a Two-Conductor Rod

We shall assume a delta function coil encircling an infinitely long, two-conductor rod, as shown in Fig. 2.

The general differential equation is the same as Equation (23) for a coil above a conducting plane.

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} - \frac{1}{r^2} - j\omega\mu\sigma = 0 \qquad (60)$$

Now, however, we shall assume the separation constant to be negative:

$$\frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = \text{"constant"} = -\alpha^2 \qquad (61)$$

Then

$$Z(z) = F \sin\alpha(z-z_o) + G \cos\alpha(z-z_o) \qquad (62)$$

and Equation (60) becomes:

$$r^2 \frac{\partial^2 R(r)}{\partial r^2} + \frac{r\partial R(r)}{\partial r} - [(\alpha^2 + j\omega\mu\sigma)r^2 + 1]\, R(r) = 0 \qquad (63)$$

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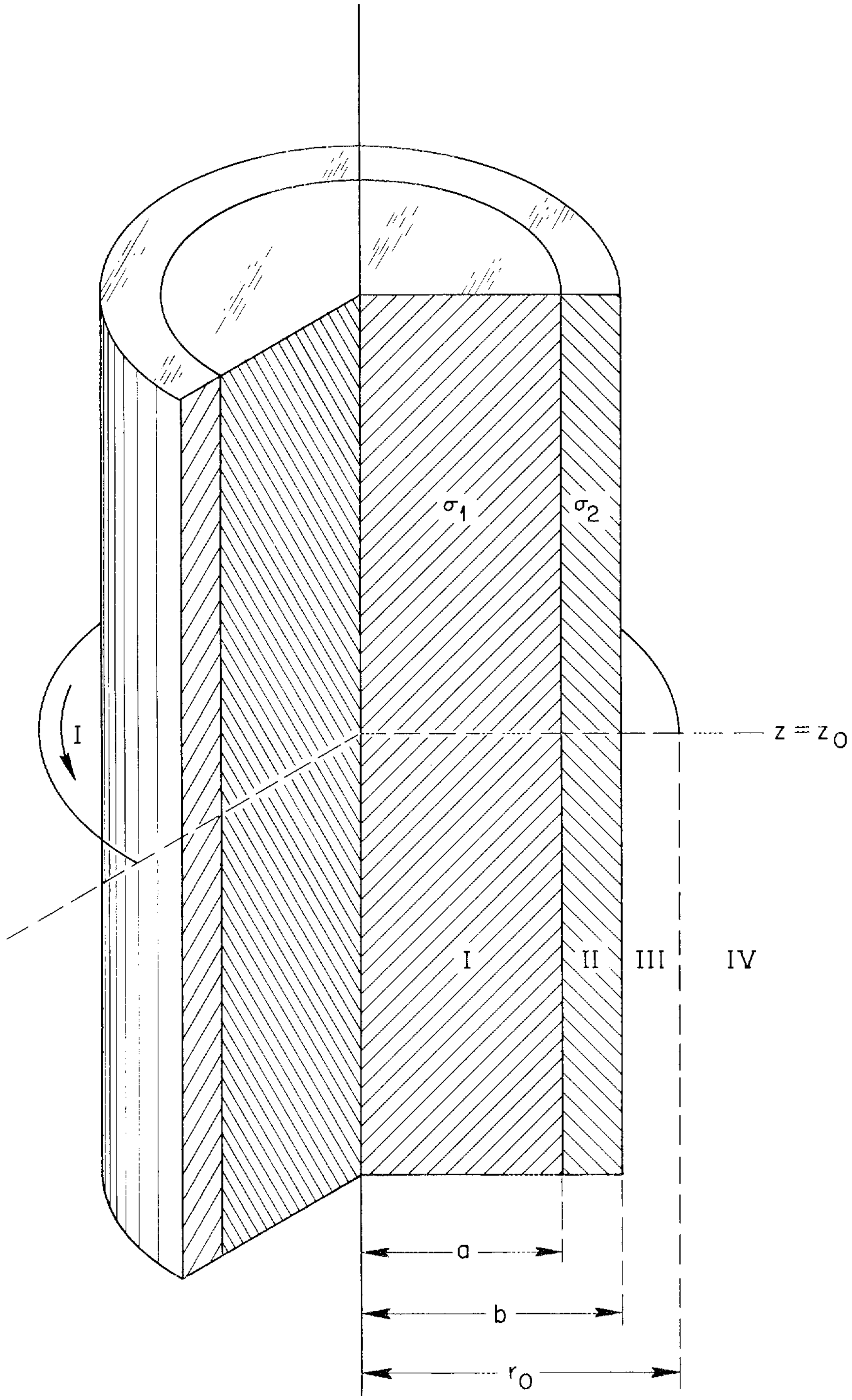

Fig. 2. Delta Function Coil Encircling a Two-Conductor Rod.

The solution to Equation (63) in terms of modified Bessel functions is:

$$R(r) = CI_1[(\alpha^2 + j\omega\mu\sigma)^{\frac{1}{2}}r] + DK_1[(\alpha^2 + j\omega\mu\sigma)^{\frac{1}{2}}r] \tag{64}$$

We can now write the vector potential in each region. We shall use the fact that it is symmetric (with respect to $z-z_o$) to eliminate the sine terms, and the fact that $K_1(o)$ and $I_1(\infty)$ both diverge to eliminate their coefficients in regions I and IV, respectively. Thus we have:

$$A^{(1)}(r,z-z_o) = \int_o^\infty C_1(\alpha)I_1[(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}r] \cos\alpha(z-z_o)\, d\alpha \tag{65}$$

$$A^{(2)}(r,z-z_o) = \int_o^\infty \{C_2(\alpha)I_1[(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r]$$

$$+ D_2(\alpha)K_1[(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r]\} \cos\alpha(z-z_o)\, d\alpha \tag{66}$$

$$A^{(3)}(r,z-z_o) = \int_o^\infty [C_3(\alpha)I_1(\alpha r) + D_3(\alpha)K_1(\alpha r)] \cos\alpha(z-z_o)\, d\alpha \tag{67}$$

$$A^{(4)}(r,z-z_o) = \int_o^\infty D_4(\alpha)K_1(\alpha r) \cos\alpha(z-z_o)\, d\alpha \tag{68}$$

The boundary conditions between the different regions are:

$$A^{(1)}(a,z-z_o) = A^{(2)}(a,z-z_o) \tag{69}$$

$$\frac{\partial}{\partial r} A^{(1)}(r,z-z_o)\Big]_{r=a} = \frac{\partial}{\partial r} A^{(2)}(r,z-z_o)\Big]_{r=a} \tag{70}$$

$$A^{(2)}(b,z-z_o) = A^{(3)}(b,z-z_o) \tag{71}$$

$$\frac{\partial}{\partial r} A^{(2)}(r,z-z_o)\Big]_{r=b} = \frac{\partial}{\partial r} A^{(3)}(r,z-z_o)\Big]_{r=b} \tag{72}$$

$$A^{(3)}(r_o,z-z_o) = A^{(4)}(r_o,z-z_o) \tag{73}$$

$$\frac{\partial}{\partial r} A^{(3)}(r,z-z_o)\Big]_{r=r_o} = \frac{\partial}{\partial r} A^{(4)}(r,z-z_o)\Big]_{r=r_o} + \mu I\delta(z-z_o) \tag{74}$$

If we multiply both sides of Equation (69) by $\cos\alpha'(z-z_o)$ and integrate from zero to infinity, we obtain:

$$\int_0^\infty\int_0^\infty C_1(\alpha)I_1[(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}r] \cos\alpha(z-z_o) \cos\alpha'(z-z_o)\ d\alpha$$

$$= \int_0^\infty\int_0^\infty[C_2(\alpha)I_1[\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r] + D_2(\alpha)K_1[(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}r)]$$

$$\times [\cos\alpha(z-z_o) \cos\alpha'(z-z_o)]\ d\alpha\ d(z-z_o) \tag{75}$$

We can reverse the order of integration and use the orthogonality properties of the cosine integral or use the Fourier integral theorem:

$$\frac{1}{\pi}\int_0^\infty f(\alpha)[\int_0^\infty \cos\alpha(z-z_o) \cos\alpha'(z-z_o)\ d(z-z_o)]\ d\alpha = f(\alpha') \tag{76}$$

Thus, we can solve the integral equations (69) through (74). We shall use α_1 and α_2 to designate $(\alpha^2 + j\omega\mu\sigma_1)^{\frac{1}{2}}$ and $(\alpha^2 + j\omega\mu\sigma_2)^{\frac{1}{2}}$. We shall use primes to designate derivatives with respect to the argument. We get from the integral equations (69) through (74):

$$C_1I_1(\alpha_1 a) = C_2I_1(\alpha_2 a) + D_2K_1(\alpha_2 a) \tag{77}$$

$$C_1\alpha_1 I_1'(\alpha_1 a) = C_2\alpha_2 I_1'(\alpha_2 a) + D_2\alpha_2 K_1'(\alpha_2 a) \tag{78}$$

$$C_2I_1(\alpha_2 b) + D_2K_1(\alpha_2 b) = C_3I_1(\alpha b) + D_3K_1(\alpha b) \tag{79}$$

$$C_2\alpha_2 I_1'(\alpha_2 b) + D_2\alpha_2 K_1'(\alpha_2 b) = C_3\alpha I_1'(\alpha b) + D_3\alpha K_1'(\alpha b) \tag{80}$$

$$C_3I_1(\alpha r_o) + D_3K_1(\alpha r_o) = D_4K_1(\alpha r_o) \tag{81}$$

$$C_3\alpha I_1'(\alpha r_o) + D_3\alpha K_1'(\alpha r_o) = D_4\alpha K_1'(\alpha r_o) + \frac{\mu I}{\pi} \tag{82}$$

Now we have six equations with six unknown constants. The equations may be solved to give the constants. We define:

$$D \equiv [\alpha_2 K_0(\alpha_2 b)K_1(\alpha b) - \alpha K_0(\alpha b)K_1(\alpha_2 b)][\alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a)] + [\alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)][\alpha I_1(\alpha_2 b)K_0(\alpha b) + \alpha_2 I_0(\alpha_2 b)K_1(\alpha b)] \tag{83}$$

The constants are

$$C_1 = \frac{\mu r_0 I K_1(\alpha r_0)}{ab\pi D} \tag{84}$$

$$D_2 = \frac{\mu r_0 I K_1(\alpha r_0)}{b\pi D}[(\alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a)] \tag{85}$$

$$C_2 = \frac{\mu I r_0 K_1(\alpha r_0)}{b\pi D}[\alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)] \tag{86}$$

$$C_3 = \frac{\mu I r_0 K_1(\alpha r_0)}{\pi} \tag{87}$$

$$D_3 = -\frac{\mu r_0 I}{\pi}\frac{K_1(\alpha r_0)}{K_1(\alpha b)}\left\{\frac{K_1(\alpha_2 b)}{bD}[\alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a)] - \frac{I_1(\alpha_2 b)}{bD}[\alpha_2 K_0(\alpha_2 a)I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)] + I_1(\alpha b)\right\} \tag{88}$$

$$D_4 = \frac{\mu I r_0 K_1(\alpha r_0)}{\pi}\left\{\frac{K_1(\alpha_2 b)[\alpha_2 I_1(\alpha_1 a)I_0(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a)I_0(\alpha_1 a)]}{K_1(\alpha b)bD} + \frac{I_1(\alpha_2 b)}{K_1(\alpha b)bD}[\alpha_2 I_1(\alpha_1 a)K_0(\alpha_2 a) + \alpha_1 K_1(\alpha_2 a)I_0(\alpha_1 a)] - \frac{I_1(\alpha b)}{K_1(\alpha b)} + \frac{I_1(\alpha r_0)}{K_1(\alpha r_0)}\right\} \tag{89}$$

We can now write for the vector potential in each region:

$$A^{(1)}(r,z-z_o) = \frac{\mu I}{\pi} \int_0^\infty \frac{r_o}{ab} \frac{K_1(\alpha r_o)}{D} I_1(\alpha_1 r) \cos\alpha(z-z_o)\, d\alpha \tag{90}$$

$$A^{(2)}(r,z-z_o) = \frac{\mu I}{\pi} \int_0^\infty \frac{r_o K_1(\alpha r_o)}{bD} \Big\{[(\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)]$$

$$\times K_1(\alpha_2 r) + [\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)] I_1(\alpha_2 r)\Big\} \cos\alpha(z-z_o)\, d\alpha \tag{91}$$

$$A^{(3)}(r,z-z_o) = \frac{\mu I}{\pi} \int_0^\infty r_o K_1(\alpha r_o) \Bigg\{I_1(\alpha r) - \Bigg[\frac{K_1(\alpha_2 b)}{bD\ K_1(\alpha b)} \Big(\alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)$$

$$- \alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a)\Big) - \frac{I_1(\alpha_2 b)}{bD\ K_1(\alpha b)} \Big(\alpha_2 K_o(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)\Big)$$

$$+ \frac{I_1(\alpha b)}{K_1(\alpha b)}\Bigg] K_1(\alpha r)\Bigg\} \cos\alpha(z-z_o)\, d\alpha \tag{92}$$

$$A^{(4)} = \frac{\mu I}{\pi} \int_0^\infty r_o K_1(\alpha r_o) K_1(\alpha r) \Bigg\{\frac{K_1(\alpha_2 b)[\alpha_2 I_1(\alpha_1 a) I_o(\alpha_2 a) - \alpha_1 I_1(\alpha_2 a) I_o(\alpha_1 a)]}{K_1(\alpha b)\ bD}$$

$$+ \frac{I_1(\alpha_2 b)}{K_1(\alpha b)\ bD} [\alpha_2 I_1(\alpha_1 a) K_o(\alpha_2 a) + \alpha_1 K_1(\alpha_2 a) I_o(\alpha_1 a)]$$

$$- \frac{I_1(\alpha b)}{K_1(\alpha b)} + \frac{I_1(\alpha r_o)}{K_1(\alpha r_o)}\Bigg\} \cos\alpha(z-z_o)\, d\alpha \tag{93}$$

Equations (90) through (93) are the equations for the vector potential of a delta function coil encircling a two-conductor rod. We will now consider the superposition of the delta function coils to form "real" coils.

Coils of Finite Cross Section

We have the equations for the vector potential produced by a single delta function coil. We can now approximate any coil such as the ones shown in Figs. 3 and 4 by the superposition of a number of delta function coils.

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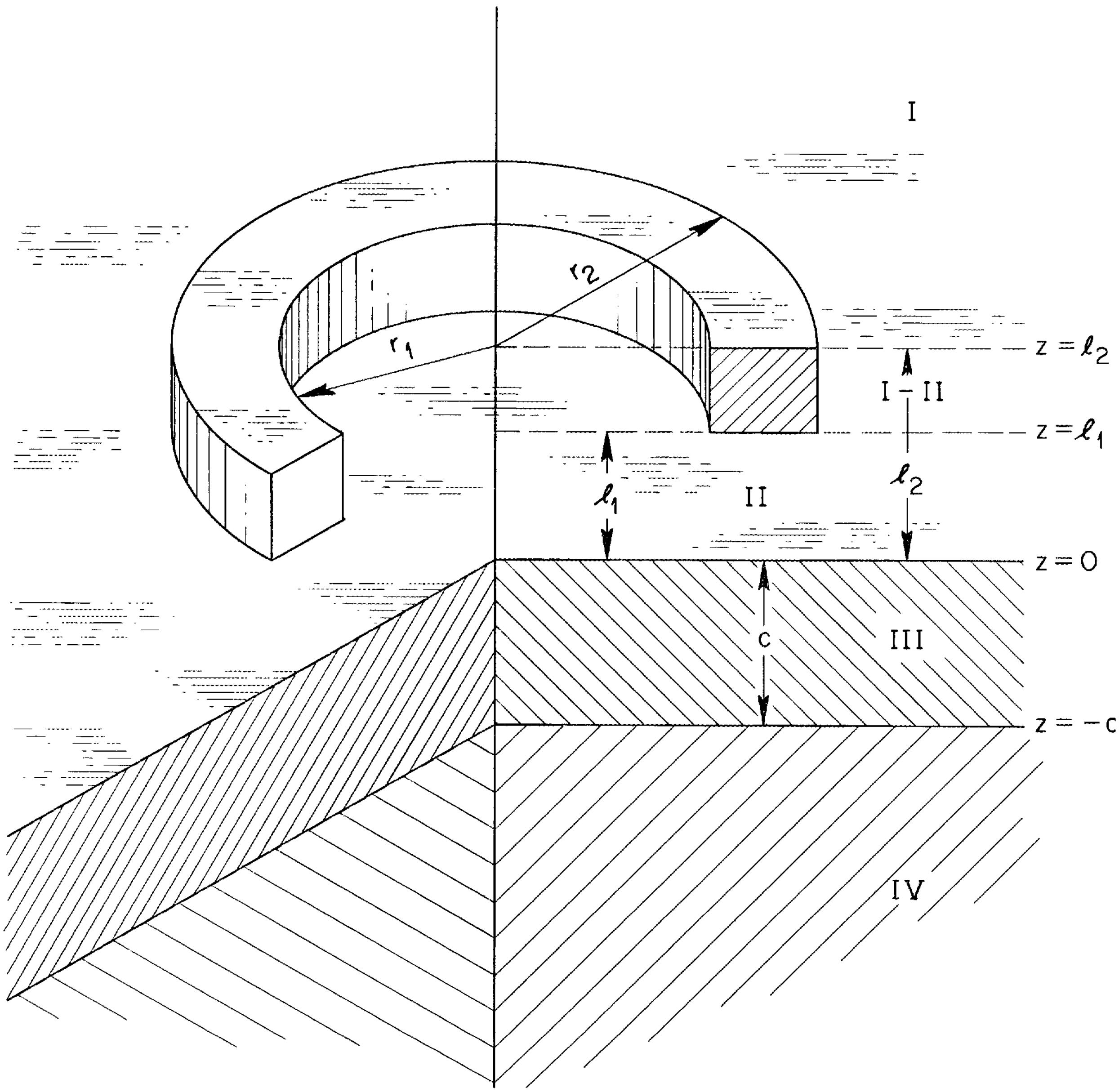

Fig. 3. Rectangular Cross-Section Coil above a Two-Conductor Plane.

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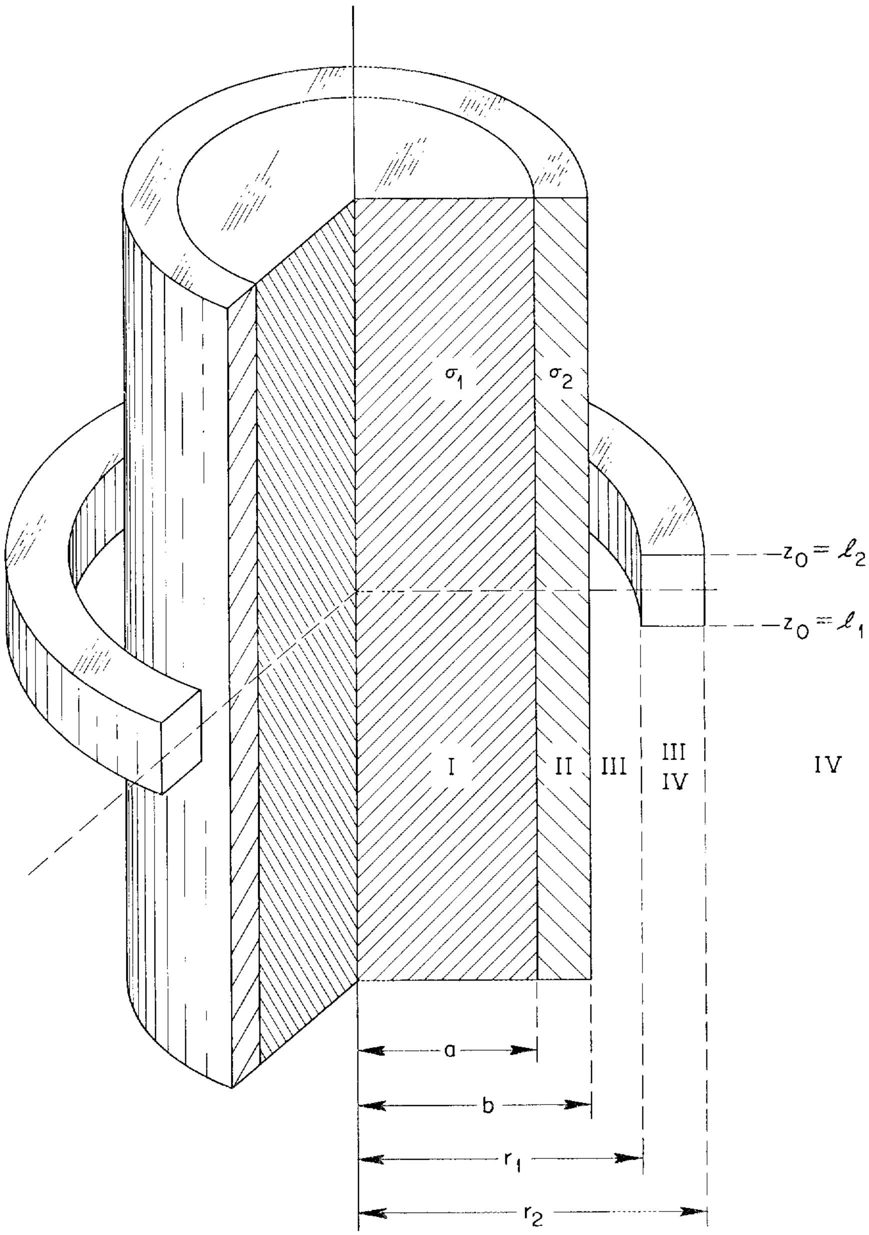

Fig. 4. Rectangular Cross-Section Coil Encircling a Two-Conductor Rod.

In general, we have:

$$A(r,z)(\text{total}) = \sum_{i=1}^{n} A_i(r,z) = \sum_{i=1}^{n} A(r,z,\ell_i,r_i) \tag{94}$$

This equation is good for coils of any cross section. If we let the current distribution in the delta function coils approach a continuous current distribution, we obtain:

$$A(r,z)(\text{total}) = \int_{\substack{\text{coil} \\ \text{cross section}}} A(r,z,r_o,\ell)\ d(\text{area}) \tag{95}$$

where $A(r,z,\ell,r_o)$ is the vector potential produced by an applied current density $i_o(\ell,r_o)$. If the coil has a rectangular cross section, as in Figs. 3 and 4, we have:

$$A(r,z)(\text{total}) = \int_{r_1}^{r_2} \int_{\ell_1}^{\ell_2} A(r,z,r_o,\ell)\ dr_o d\ell \tag{96}$$

We will now assume that the applied current density $i_o(\ell,r_o)$ is a constant over the dimensions of the coil; that is, the current in each loop has the same phase and amplitude. We shall apply these results to Equation (56), the case of a probe coil above a two-conductor plane.

After reversing the order of integration, we write:

$$A^{(1)}(r,z) = \int_0^\infty \int_{r_1}^{r_2} \int_{\ell_1}^{\ell_2} \frac{\mu i_o r_o}{2} J_1(\alpha r_o) J_1(\alpha r)\ e^{-\alpha(\ell+z)} \Bigg\{ e^{2\alpha\ell} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}\right] \Bigg\}\ d\alpha\ dr_o\ d\ell \tag{97}$$

We shall express the integral over r_o as:

$$\int_{r_o=r_1}^{r_2} r_o J_1(\alpha r_o) dr_o = \frac{1}{\alpha^2} \int_{\alpha r_o=\alpha r_1}^{\alpha r_2} \alpha r_o J_1(\alpha r_o) d\alpha r_o = \frac{1}{\alpha^2} \int_{x=\alpha r_1}^{\alpha r_2} x J_1(x)\ dx \equiv \frac{1}{\alpha^2} I(r_2,r_1) \tag{98}$$

The integral over ℓ is:

$$\int_{\ell=\ell_1}^{\ell_2} e^{-\alpha(\ell+z)} \left\{e^{2\alpha\ell} + 1\right\} d\ell = e^{-\alpha z} \int_{\ell=\ell_1}^{\ell_2} \left\{e^{\alpha\ell} + e^{-\alpha\ell}\right\} d\ell$$

$$= \frac{e^{-\alpha z}}{\alpha} \left[\left(e^{\alpha\ell_2} - e^{\alpha\ell_1}\right) - \left(e^{-\alpha\ell_2} - e^{-\alpha\ell_1}\right)\right] \qquad (99)$$

Upon applying Equations (98) and (99), the equations for the vector potential in the various regions for a rectangular cross-section coil become:

$$A^{(1)}(r,z) = \frac{\mu i_o}{2}\int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1)\, J_1(\alpha r)\, e^{-\alpha z} \left\{e^{\alpha\ell_2} - e^{\alpha\ell_1} - \left(e^{-\alpha\ell_2} - e^{-\alpha\ell_1}\right)\right.$$

$$\left. \times \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)\,(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right]\right\} d\alpha \qquad (100)$$

$$A^{(2)}(r,z) = \frac{\mu i_o}{2} \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left(e^{-\alpha\ell_1} - e^{-\alpha\ell_2}\right) \times \left\{e^{\alpha z}\right.$$

$$\left. + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right] e^{-\alpha z}\right\} d\alpha \qquad (101)$$

$$A^{(3)}(r,z) = \mu i_o \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left(e^{-\alpha\ell_1} - e^{-\alpha\ell_2}\right)$$

$$\times \left\{\frac{\alpha(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}\, e^{\alpha_1 z} + \alpha(\alpha_1-\alpha_2)\, e^{-\alpha_1 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right\} d\alpha \qquad (102)$$

$$A^{(4)}(r,z) = \mu i_o \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1) J_1(\alpha r) \left(e^{-\alpha\ell_1} - e^{-\alpha\ell_2}\right)$$

$$\times \left\{\frac{2\alpha\alpha_1\, e^{(\alpha_2+\alpha_1)c}\, e^{\alpha_2 z}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\, e^{2\alpha_1 c}}\right\} d\alpha \qquad (103)$$

Equation (100) for $A^{(1)}$ is valid in the region above the coil and Equation (101) for $A^{(2)}$ is valid for the region below the coil. We have to give special treatment to region I-II, between the top and bottom of the coil. For a point (r,z) in region I-II, we can use the equation $A^{(1)}(r,z)$ for the portion of the coil from z down to ℓ_1 and the equation $A^{(2)}(r,z)$ for the portion of the coil from z up to ℓ_2. If we substitute $\ell_2 = z$ in Equation (100) and $\ell_1 = z$ in Equation (101) and add the two equations, we get:

$$A^{(1,2)}(r,z) = \frac{\mu i_o}{2} \int_0^\infty \frac{1}{\alpha^3} I(r_2,r_1)J_1(\alpha r) \left\{2 - e^{\alpha(z-\ell_2)} - e^{-\alpha(z-\ell_1)} + e^{-\alpha z} \right.$$

$$\left. \left(e^{-\alpha\ell_1} - e^{-\alpha\ell_2}\right)\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}\right]\right\} d\alpha \tag{104}$$

We now have the equations for the vector potential in all the regions.

CALCULATION OF PHYSICAL PHENOMENA

Once we have determined the vector potential, we can calculate any physically observable electromagnetic induction phenomenon. We shall now give the equations and perform the calculations for some of the phenomena of interest in eddy-current testing.

Induced Eddy Currents

We have, from Ohm's law:

$$\vec{J} = \sigma\vec{E} = -\sigma\frac{\partial\vec{A}}{\partial t} = -j\omega\sigma\vec{A} \tag{105}$$

From the axial symmetry, Equation (105) becomes:

$$J = -j\omega\sigma A(r,z) \tag{106}$$

where A(r,z) is given by either Equation (102) or (103), depending on the region of interest.

Induced Voltage

We have, for the voltage induced in a length of wire,:

$$V = j\omega \int \vec{A} \cdot d\vec{s} \tag{107}$$

For an axially symmetric coil with a single loop of radius r, Equation (107) becomes:

$$V = j\omega \; 2\pi r \; A(r,z) \tag{108}$$

The total voltage induced in a coil of n turns is then:

$$V = j \; 2\pi\omega \sum_{i=o}^{n} r_i A(r_i, z_i) \tag{109}$$

We can approximate the above summation by an integral over a turn density of N turns per unit cross-sectional area:

$$V \approx j \; 2\pi\omega \iint\limits_{\substack{\text{coil} \\ \text{cross section}}} rA(r,z) \; Ndrdz \tag{110}$$

For coils with a constant number of turns per unit cross-sectional area:

$$V = \frac{j \; 2\pi\omega \; n}{\text{coil cross section}} \iint\limits_{\substack{\text{coil} \\ \text{cross section}}} rA(r,z) \; drdz \tag{111}$$

This is the equation for the voltage induced in a coil by any coaxial coil.

When the two coils are one and the same, with cross-sectional area equal to $(\ell_2 - \ell_1)(r_2 - r_1)$, the self-induced voltage is:

$$V = \frac{j \; 2\pi\omega \; n}{(\ell_2 - \ell_1)(r_2 - r_1)} \int_{\ell_1}^{\ell_2} \int_{r_1}^{r_2} rA^{(1,2)}(r,z) \; drdz \tag{112}$$

Coil Impedance

From the self-induced voltage, we can calculate the coil impedance

$$V = ZI, \text{ or } Z = \frac{V}{I} \tag{113}$$

The current in a single loop is related to the applied current density, i_o, by:

$$i_o = \frac{n\ I}{(\ell_2-\ell_1)(r_2-r_1)} \tag{114}$$

The coil impedance becomes:

$$Z = \frac{j\omega\pi\ \mu\ n^2}{(\ell_2-\ell_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}\ I^2(r_2,r_1)\ \Bigg\{2(\ell_2-\ell_1) + \frac{1}{\alpha}$$

$$\Bigg[2e^{-\alpha(\ell_2-\ell_1)} - 2 + \left(e^{-2\alpha\ell_2} + e^{-2\alpha\ell_1} - 2e^{-\alpha(\ell_2+\ell_1)}\right)$$

$$\times\left(\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}\right)\Bigg]\Bigg\}\ d\alpha \tag{115}$$

This equation can be made more general by normalizing all the dimensions in terms of a mean coil radius, $\bar{r}$.

$$\bar{r} = \frac{r_1+r_2}{2} \tag{116}$$

All lengths are divided by $\bar{r}$ and all α's are multiplied by $\bar{r}$.

Upon normalization, Equation (115) becomes:

$$Z = \frac{j\omega\pi\ \mu\ n^2\ \bar{r}}{(\ell_2-\ell_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}\ I^2(r_2,r_1)\ \Biggl\{2(\ell_2-\ell_1) + \frac{1}{\alpha}$$

$$\Biggl[2e^{-\alpha(\ell_2-\ell_1)} - 2 + \biggl(e^{-2\alpha\ell_2} + e^{-2\alpha\ell_1} - 2e^{-\alpha(\ell_2+\ell_1)}\biggr)$$

$$\times\left(\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\ e^{2\alpha_1 c}}\right)\Biggr]\Biggr\}\ d\alpha \qquad (117)$$

The impedance may be normalized by dividing it by the magnitude of the air impedance. For the air impedance $\alpha_1 = \alpha_2 = \alpha$ and:

$$Z_{air} = \frac{2\pi\omega\ \mu\ n^2\bar{r}}{(\ell_2-\ell_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}\ I^2(r_2,r_1)\ \Biggl\{(\ell_2-\ell_1)$$

$$+ \frac{1}{\alpha}\left[e^{-\alpha(\ell_2-\ell_1)} - 1\right]\Biggr\}\ d\alpha \qquad (118)$$

Flaw Impedance

Once the eddy-current density is known, we can simulate a flaw by superimposing a small current flowing in the opposite direction. The normalized impedance change due to a small, spherical defect not too close to the surface (Burrows[20]) is:

$$Z' = \frac{3}{2}\ \sigma\ \text{vol}\left(\frac{A_{defect}}{I}\right)^2 \qquad (119)$$

where A_{defect} is the vector potential at the defect, given by the equations for either $A^{(3)}$ and $A^{(4)}$ and "vol" is the volume of the defect.

Coil Inductance

The coil inductance is related to the magnitude of the air impedance by:

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[20] Michael Leonard Burrows, A Theory of Eddy Current Flaw Detection, University Microfilms, Inc., Ann Arbor, Michigan, 1964.

$$\omega L = |Z_{air}| \tag{120}$$

or

$$L = \frac{2\pi\mu\ n^2\ \bar{r}}{(\ell_2-\ell_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}\ I^2(r_2,r_1)\left\{(\ell_2-\ell_1) + \frac{1}{\alpha}\left[e^{-\alpha(\ell_2-\ell_1)} - 1\right]\right\} d\alpha \tag{121}$$

Mutual Inductance

The voltage generated in a "pickup" coil with dimensions r_2', r_1', ℓ_2', ℓ_1' by a current I flowing in a "driver" coil with dimensions r_2, r_1, ℓ_2, ℓ_1 is:

$$V = M\frac{dI}{dt} = j\omega MI \tag{122}$$

or

$$M = \frac{V}{j\omega I} \tag{123}$$

Using Equation (111) to calculate the voltage we have:

$$M = \frac{2\pi\ n'}{(\text{coil cross section})'}\iint\limits_{\substack{(\text{coil}\\ \text{cross section})'}} rA(r,z)\ drdz \tag{124}$$

The equation for A will vary, depending on the region where the pickup coil is located. If the pickup coil is located in region I-II, the mutual inductance is:

$$M = \frac{\mu\pi\, n\, n'\, \bar{r}}{(\ell_2'-\ell_1')(r_2'-r_1')(\ell_2-\ell_1)(r_2-r_1)} \int_0^\infty \frac{1}{\alpha^5}\, I(r_2,r_1)I(r_2',\, r_1') \times \Bigg\{2(\ell_2' - \ell_1')$$

$$+ \frac{1}{\alpha}\Bigg[e^{-\alpha(\ell_2-\ell_1')} - e^{-\alpha(\ell_2-\ell_2')} + e^{-\alpha(\ell_2'-\ell_1)} - e^{-\alpha(\ell_1'-\ell_1)} + \Big(e^{-\alpha(\ell_2'+\ell_2)}$$

$$- e^{-\alpha(\ell_2'+\ell_1)} - e^{-\alpha(\ell_1'+\ell_2)} + e^{-\alpha(\ell_1'+\ell_1)}\Big)$$

$$\times \left(\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\; e^{2\alpha_1 c}}\right)\Bigg]\Bigg\}\, d\alpha \quad (125)$$

This is the mutual inductance between the driver coil and the pickup coil in the presence of a clad conductor. By the reciprocity theorem, this is equal to the mutual inductance between the pickup coil and the driver coil.

Evaluation of Integrals

The normalized impedance has been calculated using a C-E-I-R time-sharing computer to evaluate integral equations (117) and (118). The solutions have been programmed for any rectangular coil dimensions and lift-off as well as for a metal of any conductivity clad (in varying thickness) onto a base metal of any conductivity. The programs, in "BASIC" language, and their descriptions are given in Appendix B.

Figure 5 shows how the normalized impedance varies as a function of clad thickness.

EXPERIMENTAL VERIFICATION

A family of four coils was constructed with different mean radii but all with the same normalized dimensions. The coil impedance was measured at various values of normalized lift-off and at various values of $\bar{r}^2\omega\mu\sigma$. The values of the experimental normalized coil impedance and

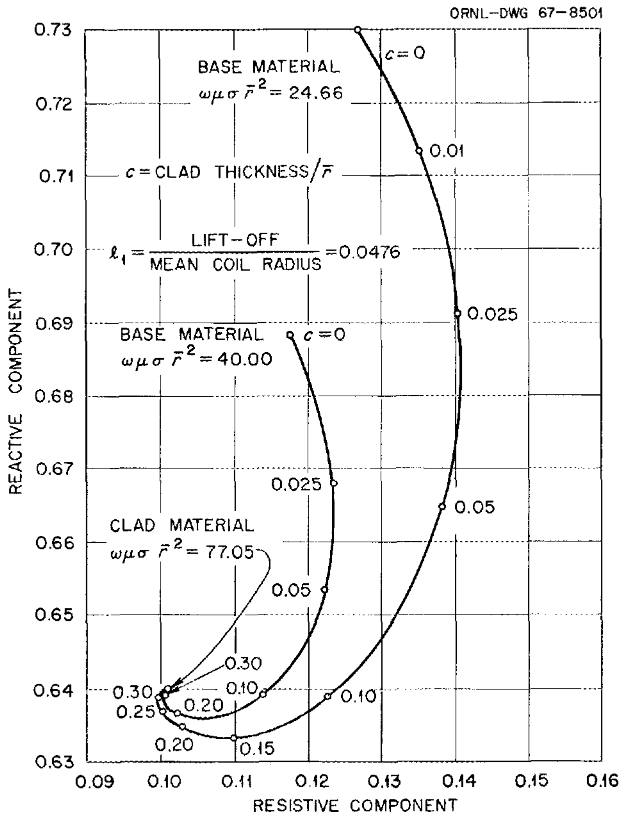

Fig. 5. Variation of Normalized Impedance with Clad Thickness.

the calculated normalized coil impedance are plotted in Fig. 6. The agreement between the calculated and measured values is excellent at the higher frequencies. At the lower frequencies the measurements are very difficult to make, and the accuracy of the measured values becomes very poor. (Because of this, few eddy-current tests are made at these frequencies.) Thus the theory is in excellent agreement with experimental values at the frequencies of interest in eddy-current testing.

ACCURACY OF CALCULATIONS

This technique, like most others used in engineering, is "exact, except for a few assumptions we have to make in order to work the problem." We will now discuss the probable errors in some of these assumptions.

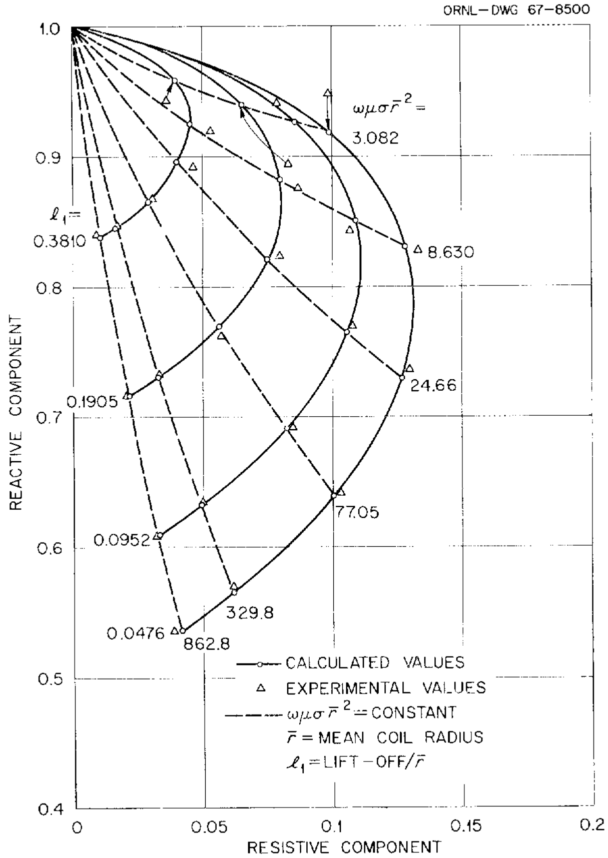

Fig. 6. Variation of Experimental and Calculated Values of Normalized Impedance with Frequency and Lift-Off.

Axial Symmetry

This is a very good assumption, but we cannot easily wind coils that have perfect axial symmetry. This error will vary with the winding technique and will decrease as the number of turns on the coil and the coil-to-conductor spacing increases. This error will be effectively reduced when normalized impedance is calculated. For a typical coil it should be less than 0.01%.

Current Sheet Approximation

This error arises because we have assumed a current sheet, while we actually have a coil wound with round, insulated wire. Some correction formulas are given by Rosa and Grover[21] for the inductance of a coil in air. From Equations (87) and (93) by Rosa and Grover, we have calculated the following correction formula:

$$\frac{\Delta L}{L} = \frac{[0.5058r_2 - 0.2742r_1 + 0.44\ (\ell_2 - \ell_1)]}{n}\left(\ell n\ \frac{D}{d} + 0.155\right) \tag{126}$$

where all dimensions are normalized by the mean coil radius. The symbols D and d are the wire diameters with and without insulation, respectively. For a typical coil with 100 turns the change in inductance is 0.19%. The change in normalized impedance will be a small fraction of the change in inductance.

High Frequency Effects

These are probably the most serious sources of error in this calculation technique. As the frequency increases, the current density ceases to be uniformly distributed over the cross section of the wire but becomes concentrated near the surface. The resistance of the coil increases, and the inductance decreases. The current is capacitively coupled between the turns in the coil, tending to flow across the loops of wire rather than through them. Both the interwinding capacitance and the coil-to-metal sample capacitance increase. The coil-to-sample capacitance can be reduced by winding the coil such that the turns nearest the sample are electrically near alternating-current ground. The coil-to-sample capacitance will be much less than the interwinding capacitance. If the coil is used at frequencies where the interwinding capacitance has a small effect, the error in calculated normalized impedance will be a much smaller effect.

---

[21]Edward B. Rosa and Frederick W. Grover, "Formulas and Tables for the Calculation of Mutual and Self-Inductance," Nat. Bur. Std. (U.S.), Tech. News Bull. 8(1), 1–237 (1912).

CONCLUSIONS

This technique presents a quick and easy way to calculate the observed effects of actual eddy-current tests to a high degree of accuracy.

ACKNOWLEDGMENTS

The authors wish to express their appreciation to W. A. Simpson for performing the experimental measurements, to J. W. Luquire and W. G. Spoeri for editing and checking the equations and their assistance in programming the stepwise solution of the integral equations.

APPENDIX A

List of Symbols

In the first column the symbol used is given, and in the second column the name. In the third column the meter-kilogram-second (MKS) units are given. In the last column the dimensions are given in terms of mass (M), length (L), time (T), and electric charge (Q).

Symbol	Name	MKS Units	Dimensions
A	vector potential	$\frac{\text{webers}}{\text{meter}}$	$\frac{ML}{TQ}$
B	magnetic induction	$\frac{\text{webers}}{\text{meter}^2}$	$\frac{M}{TQ}$
c	clad thickness	meter	L
D	electric displacement	$\frac{\text{coulomb}}{\text{meter}^2}$	$\frac{Q}{L^2}$
E	electric intensity	$\frac{\text{volt}}{\text{meter}}$	$\frac{ML}{T^2Q}$
H	magnetic intensity	$\frac{\text{ampere}}{\text{meter}}$	$\frac{Q}{TL}$
I	applied current	ampere	$\frac{Q}{T}$
i_o	applied current density	$\frac{\text{ampere}}{\text{meter}^2}$	$\frac{Q}{TL^2}$
J	current density	$\frac{\text{ampere}}{\text{meter}^2}$	$\frac{Q}{TL^2}$
j	square root of minus one		
L	inductance	henries	$\frac{ML^2}{Q^2}$
ℓ	distance from metal to delta function coil	meter	L
ℓ_2	distance from metal to top of the coil	meter	L
ℓ_1	distance from metal to bottom of the coil	meter	L

Symbol	Name	MKS Units	Dimensions
N	turns per unit area	$\frac{\text{turns}}{\text{meter}^2}$	$\frac{1}{L^2}$
n	number of turns	turns	
r_1	coil inner radius	meters	L
r_2	coil outer radius	meters	L
$\bar{r}$	mean coil radius	meters	L
t	time	seconds	T
V	voltage	volt	$\frac{ML^2}{T^2Q}$
Z	impedance	ohms	$\frac{ML^2}{TQ^2}$
α	separation constant	meter^{-1}	$\frac{1}{L}$
α_i	$(\alpha^2+j\omega\mu\sigma_i)^{\frac{1}{2}}$	meter^{-1}	$\frac{1}{L}$
ϵ	dielectric constant	$\frac{\text{farad}}{\text{meter}}$	$\frac{T^2Q^2}{ML^3}$
μ	permeability	$\frac{\text{henry}}{\text{meter}}$	$\frac{ML}{Q^2}$
σ	conductivity	$\frac{\text{mho}}{\text{meter}}$	$\frac{TQ^2}{ML^3}$
ω	angular frequency	$\frac{\text{radians}}{\text{second}}$	$\frac{1}{T}$

APPENDIX B

This appendix contains two "BASIC" programs which were used to calculate eddy-current coil impedance using the C-E-I-R time-sharing computer system. The first program, CLADT5, is the more general program and will calculate the impedance of a coil of any rectangular cross section positioned any distance above one conductor of any conductivity clad on another conductor of any conductivity. The second program is a special case of the first program where the two metals have the same conductivity. While the integral over α is from o to ∞, the integrals converge to within about 0.03% of their final value for the integral of o to α=35.

```
CLADT5     9:27    CEIR    08/25/67

1 REM THIS IS A PROGRAM TO CALCULATE EDDY CURRENT COIL IMPEDANCE
2 REM FOR A COIL ABOVE A CONDUCTING PLANE.  THE COIL INNER AND
3 REM OUTER RADII,R1 AND R2, AND THE SPACING OF THE BOTTOM AND
4 REM TOP OF THE COIL ABOVE THE PLANE, L1 AND L2, MUST BE GIVEN.
5 REM THE VALUE OF R↑2*MU*FREQ*COND MUST BE GIVEN FOR BOTH THE BASE
6 REM MATERIAL, M1, AND THE CLAD MATERIAL, M2.  THE THICKNESS, C,
7 REM OF THE CLAD MATERIAL MUST ALSO BE GIVEN.
10 LET R1=.8333
20 LET R2=1.1667
30 LET L1=.0476
40 LET L2=.3809
50 LET M1=77.05
60 LET M2=40
70 LET C=.05
80 PRINT "R1=";R1,"R2=";R2,"L1=";L1,"L2=";L2
90 PRINT "CLAD THICKNESS IS ";C,"M1=";M1,"M2=";M2
100 PRINT "X", "AIR VALUE", "REAL PART", "IMAG PART"
110 LET S1=1E-2
120 LET S2=5
130 LET I6=0
140 LET I7=0
150 LET I8=0
160 LET I9 =0
170 LET B1=0
180 LET B2 =S2
190 FOR X = B1 +S1/2 TO B2 STEP S1
200 LET Z=R2*X
210 LET Q1=R2
220 GOSUB 790
230 LET I2=F2
240 LET Z=R1*X
250 LET Q1=R1
260 GOSUB 790
270 LET I1=F2
280 LET I3=I2-I1
290 LET S3=S1*I3*I3/X
300 IF 2*X*L1>10 THEN 630
310 LET Y2=.707107*SQR(SQR(X*X*X*X+M2*M2)-X*X)
320 LET X1 = .707107*SQR(SQR(X*X*X*X+M1*M1)+X*X)
330 LET Y1 = .707107*SQR(SQR(X*X*X*X+M1*M1)-X*X)
340 IF 2*X1*C>30 THEN 510
350 LET X2 = .707107*SQR(SQR(X*X*X*X+M2*M2)+X*X)
360 LET Y2=.707107*SQR(SQR(X*X*X*X+M2*M2)-X*X)
370 LET X3=EXP(2*X1*C)
380 LET Y3=COS(2*Y1*C)
390 LET Y4=SIN(2*Y1*C)
400 LET A6=(X-X1)*(X1+X2)+Y1*(Y1+Y2)
410 LET A7=(X-X1)*(Y1+Y2)-Y1*(X1+X2)
420 LETA5=(X+X1)*(X1-X2)-Y1*(Y1-Y2)+(A6*Y3-A7*Y4)*X3
430 LET B5=Y1*(X1-X2)+(X+X1)*(Y1-Y2)+(A7*Y3+A6*Y4)*X3
```

```
440 LET C6=(X+X1)*(X1+X2)-Y1*(Y1+Y2)
450 LET C7=Y1*(X1+X2)+(X+X1)*(Y1+Y2)
460 LET C5=(X-X1)*(X1-X2)+Y1*(Y1-Y2)+(C6*Y3-C7*Y4)*X3
470 LET D5=(X-X1)*(Y1-Y2)-Y1*(X1-X2)+(C7*Y3+C6*Y4)*X3
480 LET K1=(A5*C5+B5*D5)/(C5*C5+D5*D5)
490 LET K2=(C5*B5-A5*D5)/(C5*C5+D5*D5)
500 GOTO 560
510 LET A5 =X-X1
520 LET B5=-Y1
530 LET C5=X+X1
540 LET D5=Y1
550 GOTO 480
560 LET G=EXP(-2*X*L1)+EXP(-2*X*L2)-2*EXP(-X*(L1+L2))
570 LET G1=G*K1
580 LET G2 =G*K2
590 LET A1=S3*G1/(2*X)
600 LET A2=S3*G2/(2*X)
610 LET I6=I6+A1
620 LET I8=I8+A2
630 LET G3=EXP(-X*(L2-L1))-1
640 LET A3=S3*(G3/X+L2-L1)
650 LET I9=I9+A3
660 NEXT X
670 LET B1=B1+S2
680 LET B2=B2+S2
690 LET Q3=X+S1/2
700 LET I7=I9+I6
710 PRINT Q3,I9,I7,I8
720 IF X < 3 THEN 190
730 LET S1=5E-2
740 IF X < 30 THEN 190
750 LET Q1=-I8/I9
760 LET Q2=I7/I9
770PRINT"NORMALIZED IMAG PART";Q2,"NORMALIZED REAL PART";Q1
780 GOTO 940
790 IFZ>3THEN 880
800 LETL5=INT(2*Z)+3
810 LETF1=.5*Q1*Q1*Z
820 LET F2=F1/3
830 FORN=1TOL5
840 LETF1=-F1*.250*Z*Z/(N*N+N)
850 LETF2=F2+F1/(2*N+3)
860 NEXTN
870 GOTO 930
880 LET P1=.8069*Z↑.497+.1738*EXP(-.2675*Z)
890 LET P2=SIN(Z-2.340+.106*EXP(-.068*Z)+.32*EXP(-.3*Z))
900 LET P3=.156*EXP(-.9*Z)*SIN(2.2*Z-.31)
910 LET P4=1
920 LET F2=(P1*P2+P3+P4)/(X*X)
930 RETURN
940 END
```

```
CVDØØ5     13:48    CEIR    Ø8/18/67

1 REM THIS IS A PROGRAM TO CALCULATE EDDY CURRENT COIL IMPEDANCE
2 REM FOR A COIL ABOVE A CONDUCTING PLANE.  THE COIL INNER AND
3 REM OUTER RADII,R1 AND R2, AND THE SPACING OF THE BOTTOM AND
4 REM TOP OF THE COIL ABOVE THE PLANE, L1 AND L2, MUST BE GIVEN.
5 REM THE VALUE OF R↑2*FREQ*MU*COND MUST ALSO BE GIVEN.
1Ø LET R1=.8333
2Ø LET R2=1.1667
3Ø LET L1=.Ø952
4Ø LET L2=.4285
5Ø LET M=77.Ø5
6Ø LET S1=1E-2
7Ø LET S2=1
8Ø PRINT"R1=";R1,"R2=";R2,"L1=";L1,"L2=";L2,"M=";M
9Ø  PRINT "X","AIR VALUE","REAL PART","IMAG PART"
1ØØ LET I3=Ø
11Ø LET I4=Ø
12Ø LET I7=Ø
13Ø LET I8=Ø
14Ø LET I9 =Ø
15Ø LET B1=Ø
16Ø LET B2 =S2
17Ø FOR X = B1 +S1/2 TO B2 STEP S1
18Ø LET Z=R2*X
19Ø LET Q1=R2
2ØØGOSUB68Ø
21Ø LET I2=F2
22Ø LET Z=R1*X
23Ø LET Q1=R1
24Ø GO SUB 68Ø
25Ø LET I1=F2
26Ø LET I3=I2-I1
27Ø LET S3=S1*I3*I3/X
43Ø LET X1=SQR(X*X*X*X+M*M)
44Ø LET K1=1.41421*X*SQR(X1-X*X)/M-1
45Ø LET K2=(2*X-1.41421*SQR(X1+X*X))*X/M
46Ø LET G=EXP(-2*X*L1)+EXP(-2*X*L2)-2*EXP(-X*(L1+L2))
47Ø LET G1=G*K1
48Ø LET G3=EXP(-X*(L2-L1))-1
49Ø LET G2 =G*K2
5ØØ LET A1=S3*G1/(2*X)
51Ø LET A3=S3*(G3/X+L2-L1)
52Ø LET A2=S3*G2/(2*X)
53Ø LFT I7=I7+A1+A3
54Ø LET I9=I9+A3
55Ø LET I8=I8+A2
56Ø NEXT X
57Ø LET B1=B1+S2
58Ø LET B2=B2+S2
59Ø LET Q3=X+S1/2
```

```
600 PRINT Q3,I9,I7,I8
610 IF X < 3 THEN 170
620 LET S1=5E-2
630 IF X < 30 THEN 170
640 LET Q1=-I8/I9
650 LET Q2=I7/I9
660PRINT"NORMALIZED IMAG PART";Q2,"NORMALIZED REAL PART";Q1
670 GOTO830
680 IF Z>3THEN770
690 LET L5=INT(2*Z)+3
700 LET F1=.5*Q1*Q1*Z
710 LET F2=F1/3
720 FOR N=1TOL5
730 LET F1=-F1*.250*Z*Z/(N*N+N)
740 LET F2=F2+F1/(2*N+3)
750 NEXT N
760 GOTO820
770 LET P1=.8069*Z↑.497+.1738*EXP(-.2675*Z)
780 LET P2=SIN(Z-2.340+.106*EXP(-.068*Z)+.32*EXP(-.3*Z))
790 LET P3=.156*EXP(-.9*Z)*SIN(2.2*Z-.31)
800 LET P4=1
810 LET F2=(P1*P2+P3+P4)/(X*X)
820 RETURN
830 END
```

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