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CONCERNING THE MOMENT PROBLEM

(THESIS)

G. G. Johnson

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G. G. Johnson

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CONTENTS

ACKNOWLEDGEMENT iv

1. Introduction 1
2. Some Properties of Bernstein Polynomials 4
3. Algebraic Properties of a Sequence of Step Functions 5
4. Properties of a Bounded Sequence of Associated Step Functions 8
5. Local Properties of a Bounded Sequence of Associated Step Functions 16
6. Global Properties of a Bounded Sequence of Associated Step Functions 27
7. Converse Theorems of Section 5 28
8. Two Additional Theorems 34
9. Conclusion 38

BIBLIOGRAPHY 39

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1. INTRODUCTION

By the moment problem for the real number sequence $c_0, c_1, \ldots$ and the interval [0,1], one means the problem of determining a real-valued function ϕ on [0,1] such that $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$. The integrals involved are ordinary Stieltjes integrals, and j is the function such that $j(x) = x$ for each number x. If each of f and g is a function, then fg denotes the function h so that $h(x) = f(x)g(x)$ for all numbers x for which $f(x)$ and $g(x)$ are defined. If n is a positive integer and f is a function, then f^n is the function h so that $h(x) = [f(x)]^n$ for all x for which $f(x)$ is defined.

A motivation for studying the moment problem lies in functional analysis. Frédéric Riesz has shown [1]* that every bounded linear functional T on the set $C_{[0,1]}$ of all continuous functions on [0,1] has a representation $Tf = \int_0^1 f \, d\alpha$, where α is of bounded variation on [0,1]. One way of establishing this result was found by T. H. Hildebrant and I. J. Schoenberg in 1933 [2]. Given a bounded linear functional T on $C_{[0,1]}$, they considered the moments $c_0, c_1, \ldots$, where $c_n = Tj^n$, $n = 0, 1, 2 \ldots$. Then, a function ϕ of bounded variation was found so that $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$. In effect, the moment problem for $c_0, c_1, \ldots$ was solved. It was then shown that this function ϕ gave a representation for T; that is, $Tf = \int_0^1 f \, d\phi$ for all f in $C_{[0,1]}$.

It is felt that the results of this paper make a contribution to extension theory for unbounded linear functionals on $C_{[0,1]}$.

In this paper, the theory of Bernstein polynomials is used extensively. In fact, the motivation for the present work is based on some considerations of Bernstein polynomials. If f is a function on [0,1] and n is a positive integer, the nth Bernstein polynomial associated with f is $B_n^f(x) = \sum_{t=0}^{n} \binom{n}{t} f\left(\frac{t}{n}\right) x^t (1 - x)^{n-t}$, $0 \leqq x \leqq 1$. Suppose ϕ is a function on [0,1], $\phi(0) = 0$, and $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$. Suppose $0 < x < 1$. Then,

$$\phi(x) = -\int_0^1 \phi \, dh \text{ , where}$$

$$h(t) = \begin{cases} 1 & \text{if } 0 \leqq t \leqq x \, , \\ 0 & \text{if } x < t \leqq 1 \, . \end{cases}$$

An integration by parts gives

$$\phi(x) = \int_0^1 h \, d\phi \, .$$

The nth Bernstein polynomial associated with h is $B_n^h(y) = \sum_{t=0}^{k} \binom{n}{t} y^t (1 - y)^{n-t}$ where $x \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$. (See Fig. 1.)

One might expect that $\phi(x)$ is "approximated" by

$$\int_0^1 B_n^h \, d\phi \text{ , which is}$$

*The numbers in square brackets refer to the Bibliography at the end of this paper.

$$\int_0^1 \sum_{t=0}^{k} \binom{n}{t} j^t (1-j)^{n-t}\, d\phi$$

$$= \int_0^1 \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i j^t\, d\phi$$

$$= \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t}\,,$$

since $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$. Hence, for each $0 < x < 1$, one might expect that $\phi(x)$ is "approximated" by

$$\sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i\, c_{i+t}, \text{ where } x \in \left[\frac{k}{n}, \frac{k+1}{n}\right) .$$

Define step functions $\phi_1, \phi_2, \ldots$ on $[0,1]$ by $\phi_n(0) = 0$ and $\phi_n(x) = \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t}$ if $x \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap (0,1]$ for each positive integer n (see Fig. 2). Note that the step function sequence $\phi_1, \phi_2 \ldots$ is defined in terms of $c_0, c_1, \ldots$. Hence, given a real number sequence $c_0, c_1, \ldots$, heuristic considerations lead one to expect that the step function sequence $\phi_1, \phi_2, \ldots$ so formed might approximate some function ϕ on $[0,1]$ such that $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$. Felix Hausdorff [3] in 1921 used these step functions to give necessary and sufficient conditions on a sequence

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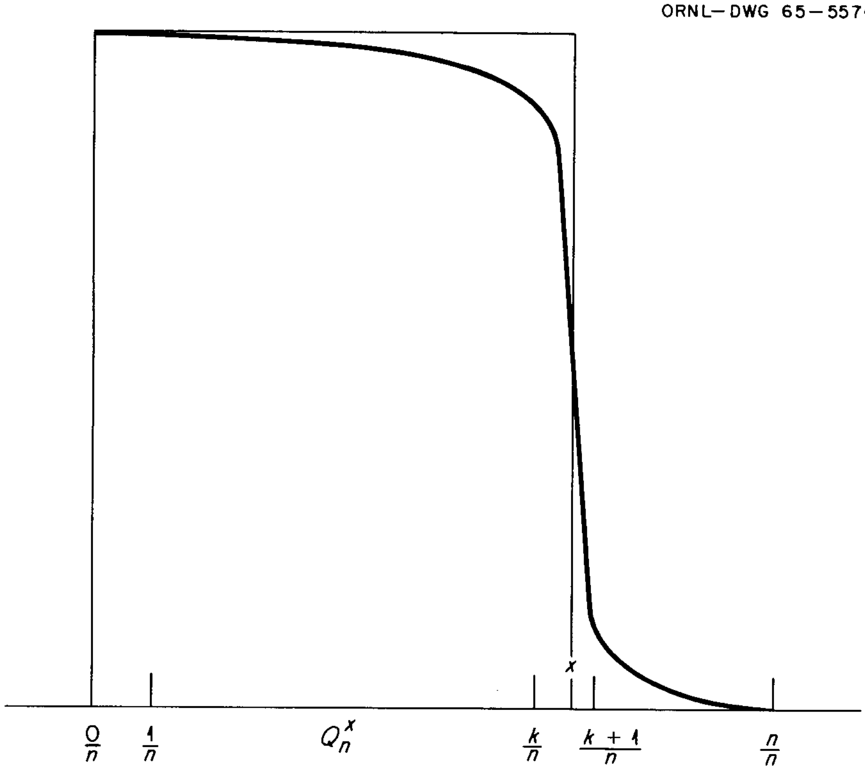

Fig. 1. Illustrating Bernstein Polynomial of a Simple Step Function.

ORNL—DWG 65—5572

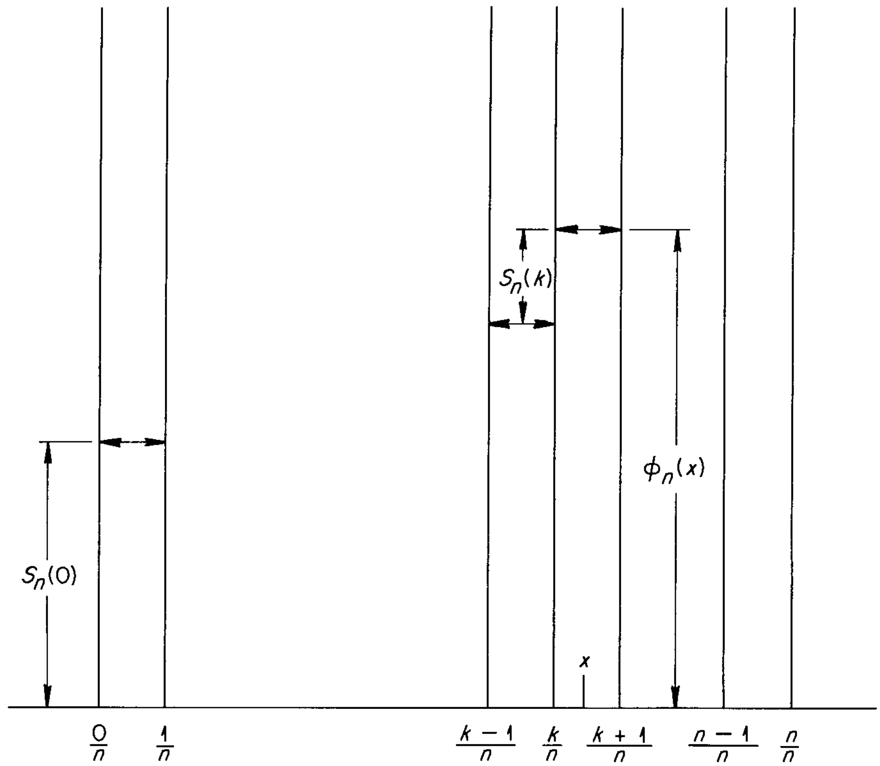

Fig. 2. Illustrating Definition ϕ_n of an Associated Step Function.

$c_0, c_1, \ldots$ in order that there be a function ϕ of bounded variation on [0,1] which generates the moments $c_0, c_1, \ldots$. Hubert S. Wall [4] stated and settled the moment problem for nondecreasing functions ϕ on [0,1]; I. J. Schoenberg [11] studied the completely monotonic case.

Higher dimensional moment problems are treated in Shohat and Tamarkin [12] as well as moment problems for the half line and the entire line. Also, necessary and sufficient conditions are given so that the moment problem on [0,1] has a solution $\phi(t) = \int_0^t \Psi(u)\,du$, where $\Psi \in L_p[0,1]$, $p > 1$ as well as a solution $\phi(t) = \int_0^t \Psi(u)\,du$, where Ψ is bounded on [0,1]. Ahiezer and Krein [13] treat various problems on [0,1] in greater detail than Shohat and Tamarkin. It should be noted that the results given, above all, deal with moment sequences which are generated by functions of bounded variation.

James H. Wells in [6] gives the following necessary and sufficient condition in order that there be a quasi-continuous function ϕ on [0,1] which is a solution to the moment problem: (1) There is a number $M > 0$ so that

$$\left| \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t} \right| < M, \quad 0 \leqq i \leqq n, \quad n = 1, 2, 3, \ldots,$$

and (2) if $x \in [0,1]$ and $\epsilon > 0$, there is a number $\delta > 0$ such that, if $x - \delta < z < w < x$ or $x < z < w < x + \delta$, then there is a positive integer N such that, if $n > N$,

$$\left| \sum_{nz<p\leqq nw} \binom{n}{p} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i c_{i+p} + \sum_{nz\leqq p<nw} \binom{n}{p} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i c_{i+p} \right| < \epsilon .$$

Since a quasi-continuous function is not necessarily of bounded variation (but all functions of bounded variation are quasi-continuous), Wells's result represents a departure from the study of moment sequences generated by functions of bounded variation. This paper gives a second contribution to the moment problem in the nonbounded variation case. Local properties of the sequence of step functions noted above are studied. These results are then applied to extend the study of moment problems beyond the quasi-continuous case.

2. SOME PROPERTIES OF BERNSTEIN POLYNOMIALS

Definition 2.1. Suppose f is a function on $[0,1]$ and n is a positive integer. The statement that B_n^f is the nth Bernstein polynomial associated with f means

$$B_n^f(x) = \sum_{t=0}^{n} \binom{n}{t} f\left(\frac{t}{n}\right) x^t(1-x)^{n-t} \text{ if } x \in [0,1] .$$

The proofs of the following three theorems can be found in [5] Chapter 1 and are omitted here.

Theorem 2.1. *Suppose* f *is a bounded function on* $[0,1]$ *and* x *is in* $[0,1]$. *If* f *is continuous at* x, *then* $\lim_{n\to\infty} B_n^f(x) = f(x)$. *If* f *is continuous on* $[0,1]$, *then this relation holds uniformly on* $[0,1]$.

Theorem 2.2. *Suppose* f *is a bounded function on* $[0,1]$ *and* x *is in* $(0,1)$. *If* f *has a discontinuity of the first kind at* x, *then* $\lim_{n\to\infty} B_n^f(x) = \frac{1}{2}[f(x-) + f(x+)]$.

Theorem 2.3. *If* f *is a nondecreasing (nonincreasing) function on* $[0,1]$ *and* n *is a positive integer, then* B_n^f *is nondecreasing (nonincreasing) on* $[0,1]$.

Definition 2.2. In the case

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \leqq x < z , \\ 0 & z \leqq x \leqq 1 , \end{cases}$$

define $B_n^f = Q_n^z$.

Theorem 2.4. (Wells [6].) *If* $\epsilon > 0$ *and* $0 < d \leqq 1/2$, *there is a positive number* N *such that, if* n *is an integer greater than* N *and* z *is in* $[d, 1-d]$,

$$1 - Q_n^z(x) < \epsilon \quad if \quad 0 \leqq x \leqq z - d ,$$

and

$$Q_n^z(x) < \epsilon \quad if \quad z + d \leqq x \leqq 1 .$$

Theorem 2.5. *If* $0 \leqq y \leqq z \leqq 1$ *and* n *is a positive integer, then* $Q_n^y(x) \leqq Q_n^z(x)$, $x \in [0,1]$.

Proof.

$$Q_n^y(x) = \sum_{t=0}^{n_y} \binom{n}{t} x^t(1-x)^{n-t} \quad \text{if} \quad y \in \left[\frac{n_y}{n}, \frac{n_{y+1}}{n}\right) ,$$

and

$$Q_n^z(x) = \sum_{t=0}^{n_z} \binom{n}{t} x^t(1-x)^{n-t} \quad \text{if} \quad z \in \left[\frac{n_z}{n}, \frac{n_{z+1}}{n}\right) .$$

Since $y \leqq z$, $\frac{n_y}{n} \leqq \frac{n_z}{n}$, and hence $n_y \leqq n_z$. If $n_y = n_z$, then $Q_n^z(x) = Q_n^y(x)$. If $n_y < n_z$, then $Q_n^z(x) = Q_n^y(x) + \sum_{t=n_y+1}^{n_z} \binom{n}{t} x^t(1-x)^{n-t}$. Since $j^t(1-j)^{n-t}$ is nonnegative on $[0,1]$ if $0 \leqq t \leqq n$, it follows that $Q_n^z(x) \geqq Q_n^y(x)$ if $x \in [0,1]$.

Theorem 2.6. *If each of* a, b, *and* c *is a number in* $[0,1]$ *such that* $\mathrm{a} \leqq \mathrm{b}$, *and if* n *is a positive integer, then*

$$V_{\mathrm{a}}^{\mathrm{b}} Q_{\mathrm{n}}^{\mathrm{c}} = Q_{\mathrm{n}}^{\mathrm{c}}(\mathrm{a}) - Q_{\mathrm{n}}^{\mathrm{c}}(\mathrm{b})\ .$$

Proof. If c is in $[0,1]$ and n is a positive integer, Q_n^c is nonnegative and nonincreasing on $[0,1]$.

3. ALGEBRAIC PROPERTIES OF A SEQUENCE OF STEP FUNCTIONS

Definition 3.1. Suppose $c_0, c_1, c_2, \ldots$ is a real number sequence and n is a positive integer. The statement that ϕ_n is the nth step function associated with $c_0, c_1, c_2, \ldots$ means that

$$\phi_n(0) = 0$$

and

$$\phi_n(x) = \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t} \quad \text{if} \quad x \in (0,1]\ ,$$

where k is the nonnegative integer such that x is in $\left[\frac{k}{n}, \frac{k+1}{n}\right)$. If t is an integer and $0 \leqq t \leqq n$, then, by the tth step of ϕ_n, one means $\binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t}$; this number is denoted by $S_n(t)$. The sequence $\phi_1, \phi_2, \ldots$ is called the associated step function sequence for $c_0, c_1, c_2, \ldots$ or when it is clear, simply the associated step function sequence. Theorems 3.1 and 3.3 can be found in [3].

Theorem 3.1. *If* n *is a positive integer, if* t *is an integer, and if* $0 \leqq \mathrm{t} \leqq \mathrm{n}$, *then*

$$S_n(t) = \binom{n}{t} \left[\frac{S_{n+1}(t)}{\binom{n+1}{t}} + \frac{S_{n+1}(t+1)}{\binom{n+1}{t+1}} \right].$$

Proof.

$$\begin{aligned}
\frac{S_{n+1}(t)}{\binom{n+1}{t}} + \frac{S_{n+1}(t+1)}{\binom{n+1}{t+1}} &= \sum_{i=0}^{n+1-t} \binom{n+1-t}{i} (-1)^i c_{i+t} + \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t+1} \\
&= c_t + \sum_{i=1}^{n+1-t} \binom{n+1-t}{i} (-1)^i c_{i+t} + \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t+1} \\
&= c_t + \sum_{i=0}^{n-t} \binom{n+1-t}{i+1} (-1)^{i+1} c_{i+t+1} + \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t+1} \\
&= c_t + \sum_{i=0}^{n-t} \left[\binom{n+1-t}{i} - \binom{n-t}{i} \right] (-1)^{i+1} c_{i+t+1} \\
&= c_t + \sum_{i=0}^{n-t-1} \left[\binom{n+1-t}{i+1} - \binom{n-t}{i} \right] (-1)^{i+1} c_{i+t+1}
\end{aligned}$$

$$= c_t + \sum_{i=0}^{n-t-1} \binom{n-t}{i+1} (-1)^{i+1} c_{i+t+1}$$

$$= c_t + \sum_{i=1}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t}$$

$$= \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i c_{i+t}$$

$$= \frac{S_n(t)}{\binom{n}{t}} .$$

Theorem 3.2. *If each of* m, n, r, *and* s *is a nonnegative integer such that* $m > r \geqq s$ *and* $m - r + s \geqq n \geqq 0$, *then*

$$\phi_m\left(\frac{n}{m}\right) - \phi_r\left(\frac{s}{r}\right) = \frac{1}{\binom{m}{r}} \sum_{k=s+1}^{n} \sum_{t=0}^{r-s-1} \binom{m-r+s-k+t}{t} \binom{r-s-1+k-t}{k-s-1} S_m(k)$$

$$- \frac{1}{\binom{m}{r}} \sum_{k=n+1}^{m-r+s} \sum_{t=r-s}^{r} \binom{m-r+s-k+t}{t} \binom{r-s-1+k-t}{k-s-1} S_m(k) .$$

If $s + 1 \leqq k \leqq m - r + s$, then

$$\binom{m}{r} = \sum_{t=0}^{r} \binom{m-r+s-k+t}{t} \binom{r-s-1+k-t}{k-s-1}.$$

A special case, when $m = r + 1$, was proved by Hausdorff in [3].

A proof of Theorem 3.2 is omitted since only the special case is used in this paper and the only available proof of Theorem 3.2 is very long.

It is remarked, however, that a study of the formula in Theorem 3.2 led to many of the results of this paper. In every such case, however, a shorter argument independent of Theorem 3.2 was found.

Theorem 3.3 *If* n *is a positive integer,*

$$(1) \quad V_0^1 \phi_n = \sum_{t=0}^{n} |S_n(t)| ,$$

and

$$(2) \quad V_0^1 \phi_n \leqq V_0^1 \phi_{n+1} .$$

Proof. Statement one is true since the right side of (1) is the sum of the absolute value of "jumps" of ϕ_n. To show that (2) is true note that

$$V_0^1 \phi_n = \sum_{t=0}^{n} |S_n(t)|$$

$$= \sum_{t=0}^{n} \binom{n}{t} \left| \frac{S_{n+1}(t)}{\binom{n+1}{t}} + \frac{S_{n+1}(t+1)}{\binom{n+1}{t+1}} \right|$$

$$\leqq \sum_{t=0}^{n} \binom{n}{t} \left[\frac{|S_{n+1}(t)|}{\binom{n+1}{t}} + \frac{|S_{n+1}(t+1)|}{\binom{n+1}{t+1}} \right]$$

$$= |S_{n+1}(0)| + \sum_{t=1}^{n} |S_{n+1}(t)| \left[\frac{\binom{n}{t-1}}{\binom{n+1}{t}} + \frac{\binom{n}{t}}{\binom{n+1}{t}} \right] + |S_{n+1}(n+1)|$$

$$= |S_{n+1}(0)| + \sum_{t=1}^{n} |S_{n+1}(t)| + |S_{n+1}(n+1)|$$

$$= V_0^1 \phi_{n+1} \, .$$

Theorem 3.4. *For each nonnegative integer* n,

$$\lim_{k\to\infty} \int_0^1 j^n \, d\phi_k = c_n \, .$$

This theorem is attributed to MacNerney by Wells, and a proof can be found in [7].

Theorem 3.5. *For each positive integer* n,

$$(1) \quad \lim_{k\to\infty} \int_0^1 \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i j^{i+t} \, d\phi_k = S_n(t) \, ,$$

$$(2) \quad \lim_{k\to\infty} \int_0^1 Q_n^x \, d\phi_k = \phi_n(x) \quad \textit{if} \quad x \in [0,1] \, ,$$

and

$$(3) \quad \lim_{k\to\infty} -\int_0^1 \phi_k \, dQ_n^x = \phi_n(x) \quad \textit{if} \quad x \in [0,1) \, .$$

Proof. The proof of statement one follows directly from Theorem 3.4 and the definition of $S_n(t)$. To show that statement two is true, note that

$$\phi_n(x) = \sum_{t=0}^{n_x} S_n(t) \quad \text{if} \quad x \in \left[\frac{n_x}{n}, \frac{n_x+1}{n}\right) \cap (0,1] \, .$$

$$\phi_n(x) = \sum_{t=0}^{n_x} \lim_{k\to\infty} \int_0^1 \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i j^{i+t} \, d\phi_k$$

$$= \lim_{k\to\infty} \int_0^1 \sum_{t=0}^{n_x} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i j^{i+t} \, d\phi_k$$

$$= \lim_{k\to\infty} \int_0^1 Q_n^x \, d\phi_k \, ,$$

since $Q_n^x = \sum_{t=0}^{n_x} \binom{n}{t} j^t(1-j)^{n-t} = \sum_{t=0}^{n_x} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i}(-1)^i j^{i+t}$. To establish that (3) is true, note that, on integrating (2) by parts, one has that $\lim_{k\to\infty} [\phi_k(1)Q_n^x(1) - \phi_k(0)Q_n^x(0) - \int_0^1 \phi_k \, dQ_n^x] = \phi_n(x)$. Since $\phi_k(0) = 0$, $k = 1, 2, \ldots$, and $Q_n^x(1) = 0$ if $x \neq 1$, $n = 1, 2, \ldots$, it follows that $\lim_{k\to\infty} \int_0^1 \phi_k \, dQ_n^x = -\phi_n(x)$ if $0 \leqq x < 1$.

4. PROPERTIES OF A BOUNDED SEQUENCE OF ASSOCIATED STEP FUNCTIONS

In this and the following two sections, it is assumed that $c_0, c_1, \ldots$ is a real number sequence such that the associated step function sequence $\phi_1, \phi_2, \ldots$ is uniformly bounded on [0,1].

Theorem 4.1. *If each of* n *and* t *is a nonnegative integer such that* $1 + t \leqq \frac{n}{2}$, *then* $\binom{n}{t}\left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t} > \binom{n}{t+1}\left(\frac{t+1}{n}\right)^{t+1}\left(1 - \frac{t+1}{n}\right)^{n-t-1}$.

Proof. If a is a positive integer, $\left(1 + \frac{1}{a}\right)^a < \left(1 + \frac{1}{1+a}\right)^{1+a}$, and $\left(\frac{a+1}{a}\right)^a < \left(\frac{a+2}{a+1}\right)^{a+1}$. Inverting, $\left(\frac{a+1}{a+2}\right)^{a+1} < \left(\frac{a}{a+1}\right)^a$. Hence, if d is a positive integer,

$$\left(\frac{a+d}{a+d+1}\right)^{a+d} < \left(\frac{a}{a+1}\right)^a .$$

Suppose n is an integer so that $n - a - 1 \geqq a + 1$, i.e., $\frac{n}{2} \geqq a + 1$. Then,

$$\left(\frac{n-a-1}{n-a}\right)^{n-a-1} < \left(\frac{a}{a+1}\right)^a .$$

Hence, $(a+1)^a(n-a-1)^{n-a-1} < a^a(n-a)^{n-a-1}$ and

$$\binom{n}{a}(n-a)\frac{(a+1)^a}{n^{a+1}}\frac{(n-a-1)^{n-a-1}}{n^{n-a-1}} < \binom{n}{a}\frac{a^a}{n^a}\frac{(n-a)^{n-a}}{n^{n-a}} .$$

Therefore,

$$\binom{n}{a}\left(\frac{n-a}{a+1}\right)\left(\frac{a+1}{n}\right)^{a+1}\left(1 - \frac{a+1}{n}\right)^{n-a-1} < \binom{n}{a}\left(\frac{a}{n}\right)^a\left(1 - \frac{a}{n}\right)^{n-a} ,$$

and hence,

$$\binom{n}{a+1}\left(\frac{a+1}{n}\right)^{a+1}\left(1 - \frac{a+1}{n}\right)^{n-a-1} < \binom{n}{a}\left(\frac{a}{n}\right)^a\left(1 - \frac{a}{n}\right)^{n-a}$$

if $\frac{n}{2} \geqq a + 1$, since $\binom{n}{a}\left(\frac{n-a}{a+1}\right) = \binom{n}{a+1}$.

Theorem 4.2. *If each of* n *and* t *is a positive integer,* $0 < t < n$, *then*

$$\frac{1}{4\sqrt{\pi n}} \frac{1}{\sqrt{\left(\frac{t}{n}\right)\left(1 - \frac{t}{n}\right)}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n+t}} < \binom{n}{t} < \sqrt{\frac{4}{m}} \frac{1}{\sqrt{\left(\frac{t}{n}\right)\left(1 - \frac{t}{n}\right)}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t}} .$$

Proof. By Stirling's formula,

$$\frac{1}{\sqrt{2}}\sqrt{2n\pi}\,n^n e^{-n} < n! < \sqrt{2}\,\sqrt{2n\pi}\,n^n e^{-n},$$

and hence,

$$\binom{n}{t} = \frac{n!}{t!(n-t)!} < \frac{2\sqrt{n\pi}\,n^n e^{-n}}{\sqrt{t\pi}\,t^t e^{-t}\,\sqrt{(n-t)\pi}(n-t)^{n-t}e^{-n+t}}$$

$$= \sqrt{\frac{4}{\pi n}}\;\frac{1}{\sqrt{\frac{t}{n}\left(1-\frac{t}{n}\right)}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t}}, \text{ and}$$

similarly for the remaining portion.

Theorem 4.3. *There is a positive number* A *such that*

$$\sum_{t=0}^{n}\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t} > A\sqrt{n} \text{ for } n = 1, 2, 3, \ldots.$$

Proof. Let $A = \frac{3}{\sqrt{32\pi}}$ and $k = \left[\frac{n}{2}\right]$ for $n = 2, 3, \ldots$. By Theorems 4.1 and 4.2,

$$\sum_{t=0}^{n}\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t} > (n+1)\binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}$$

$$> \left(\frac{n+1}{n}\right)\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}\frac{n}{4\sqrt{\pi n}}\;\frac{1}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}\;\frac{1}{\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}}$$

$$= \frac{\sqrt{n}}{4\sqrt{\pi}}\;\frac{n+1}{n\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}} \geqq \frac{\sqrt{n}}{4\sqrt{\pi}}\;\frac{1}{\sqrt{\frac{1}{3}\cdot\frac{2}{3}}} = \sqrt{n}\frac{3}{\sqrt{32\pi}}.$$

Theorem 4.4. *There is a positive number* B *such that*

$$\sum_{t=0}^{n}\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t} < B\sqrt{n} \text{ for } n = 1, 2, \ldots.$$

Proof. Let $B = \frac{22+20\sqrt{2}}{\sqrt{\pi}}$, and for each integer $n \geqq 4$ let the sequence $k_{n,i} = \frac{n}{2^i}$ for $i = 1, 2, \ldots, t_n$, where $\frac{n}{2^{t_n}} < \sqrt{\frac{n}{\pi}} \leqq \frac{n}{2^{t_{n-1}}}$. By Theorems 4.1 and 4.2,

$$\sum_{t=0}^{n}\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t} \leqq 2\sum_{t=0}^{k_{n,1}}\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1-\frac{t}{n}\right)^{n-t}$$

$$= 2 \left\{ \sum_{t=k_{n,2}}^{k_{n,1}} \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right) + \sum_{t=k_{n,3}}^{k_{n;2}-1} \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t} + \ldots \right.$$

$$+ \sum_{t=k_{n,i+1}}^{k_{n,i}-1} \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t} + \ldots + \sum_{t=k_{n,t_n}}^{k_{n,t_n-1}-1} \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t}$$

$$\left. + \sum_{t=0}^{k_{n,t_n}-1} \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t} \right\}$$

$$\leqq 2 \left\{ [k_{n,1} - k_{n,2} + 1] \binom{n}{k_{n,2}} \left(\frac{k_{n,2}}{n}\right)^{k_{n,2}} \left(1 - \frac{k_{n,2}}{n}\right)^{n-k_{n,2}} \right.$$

$$+ [k_{n,2} - 1 + k_{n,3} + 1] \binom{n}{k_{n,3}} \left(\frac{k_{n,3}}{n}\right)^{k_{n,3}} \left(1 - \frac{k_{n,3}}{n}\right)^{n-k_{n,3}} + \ldots$$

$$+ [k_{n,i} - 1 - k_{n,i+1} + 1] \binom{n}{k_{n,i+1}} \left(\frac{k_{n,i+1}}{n}\right)^{k_{n,i+1}} \left(1 - \frac{k_{n,i+1}}{n}\right)^{n-k_{n,i+1}} + \ldots$$

$$+ [k_{n,t_n-1} - 1 - k_{n,t_n} + 1] \binom{n}{k_{n,t_n}} \left(\frac{k_{n,t_n}}{n}\right)^{k_{n,t_n}} \left(1 - \frac{k_{n,t_n}}{n}\right)^{n-k_{n,t_n}}$$

$$\left. + [k_{n,t_n} - 1 + 1] \right\} \leqq 2 \left\{ [k_{n,1} - k_{n,2} + 1] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\frac{k_{n,2}}{n}\left(1 - \frac{k_{n,2}}{n}\right)}} \right.$$

$$+ [k_{n,2} - k_{n,3}] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\frac{k_{n,3}}{n}\left(1 - \frac{k_{n,3}}{n}\right)}} + \ldots$$

$$+ [k_{n,i} - k_{n,i+1}] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\frac{k_{n,i+1}}{n}\left(1 - \frac{k_{n,i+1}}{n}\right)}} + \ldots$$

$$\left. + [k_{n,t_n-1} - k_{n,t_n}] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\frac{k_{n,t_n}}{n}\left(1 - \frac{k_{n,t_n}}{n}\right)}} + k_{n,t_n} \right\}.$$

Note that

$$[k_{n,i} - k_{n,i+1} + 1] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\frac{k_{n,i+1}}{n}\left(1 - \frac{k_{n,i+1}}{n}\right)}}$$

$$\leqq \left[\frac{n}{2^i} - \frac{n}{2^{i+1}} + 2\right] \sqrt{\frac{4}{\pi n}} \frac{1}{\sqrt{\left(\frac{1}{2^{i+1}}\right)\left(1 - \frac{1}{2^i}\right)}} \leqq \frac{10}{\sqrt{2^i}} \sqrt{\frac{n}{\pi}}$$

for $i = 1, 2, \ldots, t_n - 1$.

Hence,

$$\sum_{t=0}^{n} \binom{n}{t}\left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t} \leqq 2\left[k_{n,t_n} + \sum_{i=1}^{t_n - 1} \sqrt{\frac{n}{\pi}} \frac{10}{\sqrt{2^i}}\right]$$

$$\leqq 2\left[\sqrt{\frac{n}{\pi}} + 10\sqrt{\frac{n}{\pi}} \sum_{i=1}^{t_n-1} \frac{1}{\sqrt{2^i}}\right] \leqq 2\sqrt{\frac{n}{\pi}}\left[1 + \frac{10}{\sqrt{2}} + \frac{10}{\sqrt{4}} + \ldots\right]$$

$$\leqq 2\sqrt{\frac{n}{\pi}}\,[1 + 10(1 + \sqrt{2})] = B\sqrt{n}\,.$$

Theorem 4.5. *There is a positive number* C *such that* $|S_n(t)| \leqq C\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t}$ *for each positive integer* n *and* $0 \leqq t \leqq n$.

Proof. By Theorem 3.5, $S_n(t) = \lim_{k\to\infty} \int_0^1 \binom{n}{t} j^t(1 - j)^{n-t}\, d\phi_k$. If $0 \leqq t < n$, $|S_n(t)| = \lim_{k\to\infty} \left|\int_0^1 \phi_k\, d\binom{n}{t} j^t(1 - j)^{n-t}\right|$, since $[(1 - j)^{n-t}](1) = 0$ if $0 \leqq t < n$ and $\phi_a(0) = 0$ if $a = 1, 2, \ldots$. For each positive integer a, $\left|\int_0^1 \phi_a\, d\binom{n}{t} j^t(1 - j)^{n-t}\right| \leqq B_0 V_0^1 \binom{n}{t} j^t(1 - j)^{n-t}$, where $B_0 \geqq |\phi_i(x)|$ if $x \in [0,1]$ and $i = 1, 2, \ldots$. Now $\binom{n}{t} j^t(1 - j)^{n-t}$ has a maximum point at $\frac{t}{n}$, is increasing on $\left[0, \frac{t}{n}\right]$, and is decreasing on $\left[\frac{t}{n}, 1\right]$, and $0 = \left[\binom{n}{t} j^t(1 - j)^{n-t}\right](0) = \left[\binom{n}{t} j^t(1 - j)^{n-t}\right](1)$ if $0 < t < n$. Hence, $V_0^1 \binom{n}{t} j^t\,(1 - j)^{n-t} \leqq 2\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t}$. Hence, $|S_n(t)| \leqq 2\,B_0 \binom{n}{t}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t}$ if $0 \leqq t < n$. Also, $|S_n(n)| = |C_n| \leqq 2B_0$. Let $C = 2B_0$. It follows that $|S_n(t)| \leqq C\binom{n}{t}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t}$ if $0 \leqq t \leqq n$ and $n = 1, 2, \ldots$.

Theorem 4.6. *There is a positive number* D *so that* $V_0^1\, \phi_n \leqq D\sqrt{n}$ *for* $n = 1, 2, \ldots$.

Proof. By Theorems 3.3, 4.4, and 4.5,

$$V_0^1\, \phi_n = \sum_{t=0}^{n} |S_n(t)| \leqq C \sum_{t=0}^{n} \binom{n}{t}\left(\frac{t}{n}\right)^t\left(1 - \frac{t}{n}\right)^{n-t} \leqq B \cdot C\sqrt{n} = D\sqrt{n}\,.$$

Theorem. 4.7. *Suppose* a *and* b *are numbers,* $0 \leqq a < b \leqq 1$, *and* $\epsilon > 0$. *There is a number* $N > 0$ *so that if* n *is an integer greater than* N, *there is a number* $M > 0$ *so that, if* m *is an integer greater than* M, then $V_a^b \phi_n < V_{a-\epsilon}^{b+\epsilon} \phi_m + \epsilon$.

Proof. If $a = 0$ and $b = 1$, then the result follows by Theorem 3.3. Suppose $a = 0$ and $b < 1$. Let B be a number such that $B \geqq |\phi_i(x)|$ if $x \in [0,1]$ and $i = 1, 2, \ldots$. Let d be a number such that $0 < d < \epsilon$, $b + d < 1$, $d < b$, and $d < 1/2$. By Theorem 2.4, there is a number $N' > 0$ such that, if $n > N'$ and $z \in [d, 1 - d]$, then

$$1 - Q_n^z(x) < \frac{\epsilon}{4B} \text{ if } 0 \leqq x \leqq z - d,$$

and

$$Q_n^z(x) < \frac{\epsilon}{4B} \text{ if } z + d \leqq x \leqq 1.$$

Choose $N > N'$ such that $\dfrac{N-1}{N} > b$. Let n be an integer greater than N. By Theorems 3.2 and 3.5, there is a number $M > 0$ such that, if m is an integer greater than M, then

$$\left| V_0^b \phi_n - \sum_{t=0}^{n_b} \left| \int_0^1 \binom{n}{t} j^t (1-j)^{n-t} \, d\phi_m \right| \right| < \frac{\epsilon}{2} \text{ where } b \in \left[\frac{n_b}{n}, \frac{n_b + 1}{n} \right).$$

Hence, $V_0^b \phi_n < \dfrac{\epsilon}{2} + \sum_{t=0}^{n_b} \left| \int_0^1 \binom{n}{t} j^t(1-j)^{n-t} \, d\phi_m \right|$. Note that

$$\sum_{t=0}^{n_b} \left| \int_0^1 \binom{n}{t} j^t(1-j)^{n-t} \, d\phi_m \right| \leqq \sum_{t=0}^{n_b} \left| \int_0^{b+d} \binom{n}{t} j^t(1-j)^{n-t} \, d\phi_m \right|$$

$$+ \sum_{t=0}^{n_b} \left| \int_{b+d}^1 \binom{n}{t} j^t(1-j)^{n-t} \, d\phi_m \right| \leqq \sum_{t=0}^{n_b} \int_0^{b+d} \binom{n}{t} j^t(1-j)^{n-t} \, |d\phi_m|$$

$$+ \sum_{t=0}^{n_b} \left| \int_{b+d}^1 \binom{n}{t} j^t(1-j)^{n-t} \, d\phi_m \right|$$

since $\binom{n}{t} j^t(1-j)^{n-t}$ is nonnegative on $[0,1]$ $t = 0, 1, 2, \ldots, n$. Integrating each term in the sum by parts and recalling that $t \leqq n_b < n$, one has that this last expression is less than or equal to

$$\int_0^{b+d} \sum_{t=0}^{n_b} \binom{n}{t} j^t(1-j)^{n-t} \, |d\phi_m| + \sum_{t=0}^{n_b} \left\{ |\phi_m(b+d)| \binom{n}{t} (b+d)^t [1-(b+d)]^{n-t} \right.$$

$$\left. + \left| \int_{b+d}^1 \phi_m \, d \binom{n}{t} j^t(1-j)^{n-t} \right| \right\}.$$

By definition, $Q_n^b = \sum_{t=0}^{n_b} \binom{n}{t} j^t(1-j)^{n-t}$. Since $0 \leqq Q_n^b(x) \leqq 1$ and $|\phi_m(x)| < B$ for $x \in [0,1]$, the above becomes

$$\leqq V_0^{b+d} \phi_m + BQ_n^b(b+d) + B \sum_{t=0}^{n_b} V_{b+d}^1 \binom{n}{t} j^t(1-j)^{n-t} \leqq V_0^{b+d} \phi_m + 2BQ_n^b(b+d).$$

Since $\binom{n}{t} j^t(1-j)^{n-t}$ is decreasing on $\left[\frac{t}{n}, 1\right]$ and $b + d > \frac{n_b}{n}$ and hence $V^1_{b+d}\binom{n}{t} j^t(1-j)^{n-t} \leqq \binom{n}{t}(b+d)^t[1-(b+d)]^{n-t}$ for $0 \leqq t \leqq n$, one has that $\sum_{t=0}^{n_b} V^1_{b+d}\binom{n}{t} j^t(1-j)^{n-t} \leqq \sum_{t=0}^{n_b}\binom{n}{t}(b+d)^t[1-(b+d)]^{n-t} = Q^b_n(b+d)$. Since $b \in [d, 1-d]$ and $n > N$, $Q^b_n(b+d) < \frac{\epsilon}{4B}$, and hence $\sum_{t=0}^{n_b}\left|\int_0^1 \binom{n}{t} j^t(1-j)^{n-t} d\phi_m\right| < V_0^{b+d}\,\phi_m + \frac{\epsilon}{2}$. Therefore, $V_0^b\,\phi_n < V_0^{b+d}\,\phi_m + \epsilon$, and, since $0 < d < \epsilon$,

$$V_0^b\,\phi_n < V_0^{b+\epsilon}\,\phi_m + \epsilon\,,$$

and similarly for the case $0 < a$ and $b = 1$.

Now consider the case $0 < a < b < 1$. Let d be a number such that $0 < d < \epsilon$, $0 < a - d$, and $b + d < 1$. By Theorem 2.9, there is a number $N' > 0$ such that, if $n > N'$ and $z \in [d, 1-d]$,

$$1 - Q^z_n(x) < \frac{\epsilon}{16B} \quad \text{if} \quad 0 \leqq x \leqq z - d\,,$$

and

$$Q^z_n(x) < \frac{\epsilon}{16B} \quad \text{if} \quad z + d \leqq x \leqq 1\,.$$

Choose $N > N'$ such that $\frac{1}{N} < a$ and $b < \frac{N-1}{N}$. Let n be an integer greater than N. By Theorems 3.3 and 3.5, there is a number $M > 0$ such that, if m is an integer greater than M,

$$\left| V_a^b\,\phi_n - \sum_{t=n_a+1}^{n_b}\left|\int_0^1 \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right|\right| < \frac{\epsilon}{2}\,,$$

where $a \in \left[\frac{n_a}{n}, \frac{n_a+1}{n}\right)$ and $b \in \left[\frac{n_b}{n}, \frac{n_b+1}{n}\right)$.

Hence,

$$V_a^b\phi_n < \frac{\epsilon}{2} + \sum_{t=n_a+1}^{n_b}\left|\int_0^1 \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right|\,.$$

Note that

$$\sum_{t=n_a+1}^{n_b}\left|\int_0^1 \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right| \leqq \sum_{t=n_a+1}^{n_b}\left|\int_0^{a-d} \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right|$$

$$+ \sum_{t=n_a+1}^{n_b}\left|\int_{a-d}^{b+d} \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right| + \sum_{t=n_a+1}^{n_b}\left|\int_{b+d}^{1} \binom{n}{t} j^t(1-j)^{n-t}\, d\phi_m\right|.$$

Integrating by parts and recalling that $0 < n_a \leqq t \leqq n_b < n$, one has

$$\sum_{t=n_a+1}^{n_b} \left|\int_0^{a-d} \binom{n}{t} j^t (1-j)^{n-t}\, d\phi_m\right|$$

$$\leqq \sum_{t=n_a+1}^{n_b} \left\{ \left|\phi_m(a-d)\right| \binom{n}{t} (a-d)^t [1-(a-d)]^{n-t} + \left|\int_0^{a+d} \phi_m\, d\binom{n}{t} j^t (1-j)^{n-t}\right| \right\}$$

$$\leqq B \sum_{t=n_a+1}^{n_b} \binom{n}{t} (a-d)^t [1-(a-d)]^{n-t} + B \sum_{t=n_a+1}^{n_b} V_0^{a-d} \binom{n}{t} j^t (1-j)^{n-t}$$

$$\leqq 2B \sum_{t=n_a+1}^{n_b} \binom{n}{t} (a-d)^t [1-(a-d)]^{n-t} \leqq 2B[Q_n^b(a-d) - Q_n^a(a-d)]$$

since (1) $\binom{n}{t} j^t(1-j)^{n-t}$ is increasing on $\left[0, \frac{t}{n}\right]$, $0 < t \leqq n$, and $a - d < \frac{n_a+1}{n}$, and so $V_0^{a-d}\binom{n}{t} j^t(1-j)^{n-t} \leqq \binom{n}{t}(a-d)^t [1-(a-d)]^{n-t}$ for $n_a < t \leqq n$, and (2) $Q_n^b(a-d) - Q_n^a(a-d) = \sum_{t=n_a+1}^{n_b} \binom{n}{t}(a-d)^t [1-(a-d)]^{n-t}$. Now $a \in [d, 1-d]$, $b \in [d, 1-d]$, and $n > N$, and hence, $Q_n^b(a-d) < \frac{\epsilon}{16B}$ and $Q_n^a(a-d) < \frac{\epsilon}{16B}$.

Therefore,

$$\sum_{t=n_a+1}^{n_b} \left|\int_0^{a-d} \binom{n}{t} j^t (1-j)^{n-t}\, d\phi_m\right| < 2B \left[\frac{2\epsilon}{16B}\right] = \frac{\epsilon}{4}.$$

Similarly,

$$\sum_{t=n_a+1}^{n_b} \left|\int_{b+d}^{1} \binom{n}{t} j^t (1-j)^{n-t}\, d\phi_m\right| < \frac{\epsilon}{4}.$$

Since $\binom{n}{t} j^t (1-j)^{n-t}$ is nonnegative on $[0,1]$, $t = 0, 1, \ldots, n$,

$$\sum_{t=n_a+1}^{n_b} \left|\int_{a-d}^{b+d} \binom{n}{t} j^t (1-j)^{n-t}\, d\phi_m\right| \leqq \int_{a-d}^{b+d} \sum_{t=n_a+1}^{n_b} \binom{n}{t} j^t (1-j)^{n-t}\, |d\phi_m|$$

$$= \int_{a-d}^{b+d} [Q_n^b - Q_n^a]\, |d\phi_m| \leqq V_{a-d}^{b+d}\, \phi_m ,$$

since $Q_n^b - Q_n^a$ is by definition $\sum_{t=n_a+1}^{n_b} \binom{n}{t} j^t (1-j)^{n-t}$, and since $0 \leqq Q_n^b(x) - Q_n^a(x) \leqq 1$ for $x \in [0,1]$.

The above results combined give

$$V_a^b\, \phi_n < \frac{\epsilon}{2} + \frac{\epsilon}{4} + V_{a-d}^{b+d}\, \phi_m + \frac{\epsilon}{4} = \epsilon + V_{a-d}^{b+d}\, \phi_m .$$

Since $0 < d < \epsilon$,

$$V_a^b \phi_n < \epsilon + V_{a-\epsilon}^{b+\epsilon} \phi_m .$$

Theorem 4.8. *Suppose* $a_0, a_1, \ldots$ *is a bounded real number sequence such that, if* $a_{n_1}, a_{n_2}, \ldots$ *is a subsequence converging to a number* p, $a_{n_1+1}, a_{n_2+1}, \ldots$ *converges to* p. *Then* $\{x|x$ *is a sequential limit point of a subsequence of* $a_0, a_1, a_2, \ldots\} = C$ *is a continuum.*

Proof. Since $a_0, a_1, \ldots$ is bounded, there is a set C. It is clear that C is closed and bounded. Suppose C is not connected, i.e., $C = A \cup B$, where A and B are disjoint and closed. Consider two numbers a and b in A and B respectively. Suppose $b > a$. If $[a,b] \subseteq C$, A and B are not disjoint. Hence, there is a number p in $[a,b] \setminus C$. Since $p \notin C$, there is a segment (u,v) containing p, no point of C, and, at most, a finite number of terms of $a_0, a_1, \ldots$. Let N be a positive number such that, if n is an integer greater than N, $a_n \notin (u,v)$. Let n_0 be the least positive integer greater than N such that $a_{n_0} \geqq v$ and $a_{n_0+1} \leqq u$. That there is such an integer n_0 follows from the facts that $b > v > u > a$ and that each of a and b is in C. Consider the sequence $a_{n_0}, a_{n_1}, \ldots$, where, for each positive i, $n_i + 1$ is the least positive integer greater than n_i such that $a_{n_{i+1}} \geqq v$ and $a_{n_{i+1}+1} \leqq u$. Since $a_0, a_1, \ldots$ is a bounded sequence, so is $a_{n_0}, a_{n_1}, \ldots$. Hence, some subsequence $a_{n_{m_0}}, a_{n_{m_1}}, \ldots$ converges to a number $q \geqq v$ since $a_{n_{m_i}} \geqq v$, $i = 0, 1, 2, \ldots$. By hypothesis, $a_{n_{m_0}+1}, a_{n_{m_1}+1}, \ldots$ also converges to q. This is a contradiction, since $a_{n_{m_i}+1} \leqq u < v \leqq q$ for $i = 0, 1, 2, \ldots$. Hence, C is a continuum.

Theorem 4.9. *If* x *is in* $[0,1]$, *then* $C = \{y|y$ *is a sequential limit point of a subsequence of* $\phi_1(x), \phi_2(x), \ldots\}$ *is a continuum.*

Proof. By Theorem 4.5, there is a number B such that $|S_n(t)| \leqq B \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t}$. By Theorem 4.2, if $0 < t < n$, then

$$\binom{n}{t} \leqq \frac{2}{\sqrt{\pi n}} \frac{1}{\left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t}} \frac{1}{\sqrt{\left(\frac{t}{n}\right)\left(1 - \frac{t}{n}\right)}} .$$

Hence,

$$|S_n(t)| \leqq \frac{2B}{\sqrt{\pi n}} \frac{1}{\sqrt{\frac{t}{n}\left(1 - \frac{t}{n}\right)}} \quad \text{if} \quad 0 < t < n .$$

From Theorem 3.2 (special case),

$$\phi_{n+1}\left(\frac{t}{n+1}\right) - \phi_n\left(\frac{t}{n}\right) = -\frac{t+1}{n+1} S_{n+1}(t+1) ,$$

and

$$\phi_{n+1}\left(\frac{t+1}{n+1}\right) - \phi_n\left(\frac{t}{n}\right) = \frac{n-t}{n+1} S_{n+1}(t+1) .$$

Also, $\phi_n(0) = 0$ and $\phi_n(1) = c_0$, $n = 1, 2, \ldots$.

Suppose $x \in (0,1)$; $\phi_1(x)$, $\phi_2(x)$, ... is a bounded number sequence. Suppose $\phi_{n_0}(x)$, $\phi_{n_1}(x)$, ... converges to some number p. Note that

$$|\phi_{n_i+1}(x) - \phi_{n_i}(x)| \leqq |S_{n_i+1}(k)|, \text{ where } x \in \left[\frac{k}{n_i+1}, \frac{k+1}{n_i+1}\right) .$$

Hence,

$$|\phi_{n_i+1}(x) - \phi_{n_i}(x)| \leqq \frac{2B}{\sqrt{\pi(n_i+1)}} \; \frac{1}{\sqrt{\frac{k}{n_i+1}\left(1 - \frac{k}{n_i+1}\right)}} \text{ if } 0 < k < n_i + 1 .$$

Therefore,

$$|\phi_{n_i+1}(x) - \phi_{n_i}(x)| \leqq \frac{1}{\sqrt{n_i+1}} \left[\frac{2B}{\sqrt{\left(\frac{k}{n_i+1}\right)\left(1 - \frac{k}{n_i+1}\right)\pi}}\right] .$$

If $0 < d < x < 1 - d$, then there is an integer $N > 0$ so that, if $n > N$,

$$d < \frac{k}{n} \text{ and } 1 - d > 1 - \frac{k}{n} .$$

Let I be a positive integer such that, if $i > I$, $n_i > B$. Then,

$$\frac{2B}{\sqrt{\pi\frac{k}{n_i+1}\left(1 - \frac{k}{n_i+1}\right)}} \leqq \frac{2B}{\sqrt{\pi d(1-d)}} \text{ so, if } D = \frac{2B}{\sqrt{\pi d(1-d)}} ,$$

then $|\phi_{n_i+1}(x) - \phi_{n_i}(x)| < \frac{D}{\sqrt{n_i}}$. Hence, $\phi_{n_0+1}(x)$, $\phi_{n_1+1}(x)$, . . . also converges to p. By Theorem 4.8, C is a continuum.

5. LOCAL PROPERTIES OF A BOUNDED SEQUENCE OF ASSOCIATED STEP FUNCTIONS

Theorem 5.1. *Suppose* x *is in* [0,1] *and for each number* $\epsilon > 0$ *there is a number* $\delta > 0$ *and a number* $N > 0$ *such that, if* y *is in* [0,1], $|x - y| < \delta$ *and* n *is an integer greater than* N, *then* $|\phi_n(x) - \phi_n(y)| < \epsilon$. *Then* $\lim_{n\to\infty} \phi_n(x)$ *exists.*

Proof. Note that $\phi_n(0) = 0$ and $\phi_n(1) = c_0$ for $n = 1, 2 \ldots$. Suppose x is in (0,1) and $\epsilon_0 > 0$. Let B be a number such that $B \geqq |\phi_i(x)|$ for $x \in [0,1]$, $i = 1, 2, \ldots$. Let $\epsilon = \epsilon_0/(8B + 2)$. By hypothesis, there is a number δ, $0 < \delta < 1/2$, and a number $N > 0$ so that $y \in [0,1]$, $|x - y| < \delta$, and $n > N$ implies $|\phi_n(x) - \phi_n(y)| < \epsilon$. Let $d = \delta/3$. By Theorem 2.4, there is a number $W > 0$ such that, if $w > W$ and $z \in [d, 1 - d]$, then

$$1 - Q_w^z(t) < \epsilon \qquad \text{if } 0 \leqq t \leqq z - d,$$

and

$$Q_w^t(t) < \epsilon \qquad \text{if } z + d \leqq t \leqq 1 .$$

Let n and m be integers so that $m > n > W$. By Theorem 3.5,

$$|\phi_m(x) - \phi_n(x)| = \lim_{k\to\infty}\left|\int_0^1 \phi_k \, d[Q_m^x - Q_n^x]\right| .$$

If k is a positive integer, then

$$\left|\int_0^1 \phi_k \, d[Q_m^x - Q_n^x]\right| \leqq \left|\int_0^{x-d} \phi_k \, d[Q_m^x - Q_n^x]\right| + \left|\int_{x-d}^{x+d} \phi_k \, d[Q_m^x - Q_n^x]\right| + \left|\int_{x+d}^1 \phi_k \, d[Q_m^x - Q_n^x]\right| .$$

Note that

$$\left|\int_0^{x-d} \phi_k \, d[Q_m^x - Q_n^x]\right| \leqq \int_0^{x-d} |\psi_k| \, |d[Q_m^x - Q_n^x]| \leqq B V_0^{x-d}[Q_m^x - Q_n^x]$$

$$\leqq B[V_0^{x-d}Q_m^x + V_0^{x-d}Q_n^x] \leqq B[1 - Q_m^x(x-d) + 1 - Q_n^x(x-d)] \leqq B[\epsilon + \epsilon] = 2\epsilon B ,$$

using Theorem 2.6 and the fact that $1 - Q_m^x(x - d) < \epsilon$ and $1 - Q_n^x(x - d) < \epsilon$. Hence,

$$\left|\int_0^{x-d} \phi_k \, d[Q_m^x - Q_n^x]\right| < 2\epsilon B.$$

Similarly,

$$\left|\int_{x+d}^1 \phi_k \, d[Q_m^x - Q_n^x]\right| < 2\epsilon B.$$

For each positive integer w and each number t in $[x - d, x + d]$, let $\mu_w(t) = \phi_w(t) - \phi_w(x)$. Then,

$$\left|\int_{x-d}^{x+d} \phi_k \, d[Q_m^x - Q_n^x]\right| + \left|\int_{x-d}^{x+d} \mu_k + \phi_k(x) \, d[Q_m^x - Q_n^x]\right|$$

$$\leqq \left|\int_{x-d}^{x+d} \mu_k \, d[Q_m^x - Q_n^x]\right| + \left|\int_{x-d}^{x+d} \phi_k(x) \, d[Q_m^x - Q_n^x]\right|$$

$$\leqq \max_{t\in[x-d,x+d]} |\mu_k(t)| V_{x-d}^{x+d}[Q_m^x - Q_n^x] + \left|\phi_k(x)[Q_m^x - Q_n^x]\Big|_{x-d}^{x+d}\right| .$$

Because $V_0^1 Q_m^x = V_0^1 Q_n^x = 1$, $V_{x-d}^{x+d}[Q_m^x - Q_n^x] \leqq 2$. Also, $|\mu_k(t)| = |\phi_k(t) - \phi_k(x)|$. If $k > N$, $|\mu_k(t)| < \epsilon$ if $t \in [x-d, x+d]$. Since $m > n > W$, $x \in [d, 1-d]$, then

$$1 - Q_m^x(x-d) < \epsilon ,$$

$$1 - Q_n^x(x-d) < \epsilon ,$$

$$Q_m^x(x+d) < \epsilon ,$$

and

$$Q_n^x(x+d) < \epsilon ,$$

it follows that

$$|Q_m^x(x+d) - Q_n^x(x+d)| < 2\epsilon, \text{ and } |Q_m^x(x-d) - Q_n^x(x-d)| < 2\epsilon .$$

Hence,

$$\left|\int_{x-d}^{x+d} \phi_k \, d[Q_m^x - Q_n^x]\right| < 2\epsilon + B[2\epsilon + 2\epsilon] = 2\epsilon + 4\epsilon B ,$$

if $k > N$. Finally,

$$\left|\int_0^1 \phi_k \, d[Q_m^x - Q_n^x]\right| < 2\epsilon B + 2\epsilon + 4\epsilon B + 2\epsilon B = 2\epsilon + 8\epsilon B$$

if $k > N$, $m > n > W$ from which it follows that $|\phi_m(x) - \phi_n(x)| < \epsilon_0$ if $m > n > W$. Hence, $\phi_1(x), \phi_2(x), \ldots$ is a Cauchy sequence and therefore converges.

Suppose $[a,b]$ is a number interval and $f_0, f_1, \ldots$ is a sequence each term of which is a function on $[a,b]$.

Definition 5.1. The statement that $f_0, f_1, \ldots$ left slants at x in $(a,b]$ means that for each number $\epsilon > 0$ there is a number l, $a \leqq l < x$, so that, if $l < d < x$, there is a number $N > 0$ such that, if n is an integer greater than N and $l \leqq u \leqq v \leqq d$, then $|f_n(u) - f_n(v)| < \epsilon$.

Definition 5.2. The statement that $f_0, f_1, \ldots$ left slant converges at x in $(a,b]$ means there is a number L_x so that, if $\epsilon > 0$, there is a number l, $a \leqq l < x$ such that, if $l < d < x$, there is a number $N > 0$ so that, if n is an integer greater than N and $l \leqq b \leqq d$, $|f_n(b) - L_x| < \epsilon$.

There are entirely similar definitions for "right slant" and "right slant converges."

Theorem 5.2. *Suppose* x *is in* (0,1] *and* $\phi_1, \phi_2, \ldots$ *left slants at* x. *Then* $\phi_1, \phi_2, \ldots$ *left slant converges at* x.

Proof. If $\epsilon > 0$, then there is a number l, $0 \leqq l < x$, and a pair of increasing number sequences $d_0, d_1, \ldots$ and $N_0, N_1, \ldots$ so that

(1) $l < d_i < x$, $i = 0, 1, 2, \ldots$,

(2) $\lim_{i \to \infty} d_i = x$,

and

(3) if $l \leqq u \leqq v \leqq d_i$ and n is an integer greater than N_i, then $|\phi_n(u) - \phi_n(v)| < \epsilon$.

There is a number sequence $\left\{l_t\right\}_{t=1}^{\infty}$, a sequence of number sequences $\left\{\left\{d_i^t\right\}_{i=0}^{\infty}\right\}_{t=1}^{\infty}$, and a sequence of increasing positive sequences $\left\{\left\{N_i^t\right\}_{i=0}^{\infty}\right\}_{t=1}^{\infty}$ so that

(1) $0 < l_t < l_{t+1} \leqq d_0^t < d_0^{t+1}$, $\quad t = 1, 2, \ldots,$

(2) $d_i^t < d_{i+1}^t < x \quad i = 0, 1, 2, \ldots, \quad t = 1, 2, \ldots,$

(3) $\lim\limits_{i\to\infty} d_i^t = x \quad t = 1, 2, \ldots,$

(4) $\lim\limits_{t\to\infty} l_t = x$,

(5) $x - l_1 < 1/2$,

and

(6) for each positive integer t, if $l_t \leqq u \leqq v \leqq d_i^t$ and n is an integer greater than N_i^t, then $|\phi_n(u) - \phi_n(v)| < 1/t = \epsilon_t$.

For each positive integer t, let $y_t = \dfrac{l_t + d_0^t}{2}$ and $\delta_t = \min\left[\dfrac{d_0^t - y_t}{8}, \dfrac{x - y_t}{3}\right]$.

Now, $y_t - \delta_t \geqq \dfrac{l_t + d_0^t}{2} - \dfrac{d_0^t - l_t}{8} = \dfrac{5}{8}l_t + \dfrac{3}{8}d_0^t > 0$ and $(1 - \delta_t) - y_t \geqq 1 - \dfrac{x - y_t}{3} - y_t = 1 - \dfrac{x}{3} - \dfrac{2}{3}y_t > 0$, since $y_t < x$. Hence, $y_t \in [\delta_t, 1 - \delta_t]$ for $t = 1, 2, \ldots$. Also, $l_t - (y_t - \delta_t) \leqq l_t - \left(\dfrac{5}{8}l_t + \dfrac{3}{8}d_0^t\right) = \dfrac{3}{8}(l_t - d_0^t) < 0$ and $d_0^t - (y_t + \delta_t) \geqq d_0^t - \dfrac{l_t + d_0^t}{2} - \delta_t = \dfrac{d_0^t - l_t}{2} - \delta_t > 0$. Hence, $l_t < y_t - \delta_t < y_t + \delta_t < d_0^t$, for $t = 1, 2, \ldots$. It is clear that $\lim\limits_{t\to\infty} y_t = x$. By Theorem 2.4, there is for each positive integer t a number $W_t > 0$ such that, if $w > W_t$ and $z \in [\delta_t, 1 - \delta_t]$,

$$1 - Q_n^z(y) < \epsilon_t \qquad \text{if } 0 \leqq y \leqq z - \delta_t,$$

and

$$Q_n^z(y) < \epsilon_t \qquad \text{if } z + \delta_t \leqq y \leqq 1.$$

For each positive integer t, there is for each positive integer i less than t, a unique integer $r_{t,i}$ so that

$$d_{r_{t,i}-1}^i \leqq d_0^t < d_{r_{t,i}}^i.$$

For each number $d_{r_{t,i}}^i$, there is a corresponding integer $N_{r_{t,i}}^i$. For each positive integer t, let $K_t = 1 +$

$\max\left[W_1, W_2, \ldots, W_t, N^1_{r_{t,1}}, N^2_{r_{t,2}}, \ldots, N^{t-1}_{r_{t,t-1}}\right]$. Let B be a number such that $B \geqq |\phi_i(x)|$ if $x \in [0,1]$, $i = 1, 2, \ldots$.

Suppose $\epsilon > 0$. Let L be a positive integer so that $\epsilon > \dfrac{8B+4}{L}$. Consider the number sequence $\phi_{k_1}(y_1), \phi_{k_2}(y_2), \ldots$. Suppose each of m and n is an integer, $m > n > L$. Then, $|\phi_{k_m}(y_m) - \phi_{k_n}(y_n)| \leqq |\phi_{k_m}(y_m) - \phi_{k_m}(y_n)| + |\phi_{k_m}(y_n) - \phi_{k_n}(y_n)|$. Since $k_m > N^n_{r_{m,n}}$, $I_n < I_m < y_m < d^m_o < d^n_{r_{m,n}}$, and $I_n < y_n < d^n_0 \leqq d^n_{r_{m,n}}$, then $|\phi_{k_m}(y_m) - \phi_{k_m}(y_n)| < \epsilon_n$. By Theorem 3.5,

$$|\phi_{k_m}(y_n) - \phi_{k_n}(y_n)| = \lim_{n\to\infty} \left| \int_0^1 \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}] \right| .$$

For each positive integer a,

$$\left|\int_0^1 \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| \leqq \left|\int_0^{y_n - \delta_n} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right|$$

$$+ \left|\int_{y_n-\delta_n}^{y_n+\delta_n} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| + \left|\int_{y_n+\delta_n}^{1} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| .$$

Note that

$$\left|\int_0^{y_n-\delta_n} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| \leqq BV_0^{y_n-\delta_n}[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}] \leqq B[V_0^{y_n-\delta_n} Q^{y_n}_{k_m} + V_0^{y_n-\delta_n} Q^{y_n}_{k_n}] .$$

By Theorem 2.6, $V_0^{y_n-\delta_n} Q^{y_n}_{k_m} = 1 - Q^{y_n}_{k_m}(y_n - \delta_n)$, and $V_0^{y_n-\delta_n} Q^{y_n}_{k_n} = 1 - Q^{y_n}_{k_n}(y_n - \delta_n)$. Since $y_n \in [\delta_n, 1 - \delta_n]$ and $k_m \geqq k_n > W_n$, it follows that $1 - Q^{y_n}_{k_m}(y_n - \delta_n) < \epsilon_n$ and $1 - Q^{y_n}_{k_m}(y_n - \delta_n) < \epsilon_n$. It follows that

$$\left|\int_0^{y_n-\delta_n} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| < 2\epsilon_n B \qquad \text{for } a = 1, 2, \ldots .$$

Similarly,

$$\left|\int_{y_n+\delta_n}^{1} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| < 2\epsilon_n B \qquad \text{for } a = 1, 2, \ldots .$$

For each positive integer b, let $\mu_b(t) = \phi_b(t) - \phi_b(y_n)$ for $t \in [y_n - \delta_n, y_n + \delta_n]$. Then,

$$\left|\int_{y_n-\delta_n}^{y_n+\delta_n} \phi_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| = \left|\int_{y_n-\delta_n}^{y_n+\delta_n} \mu_a + \phi_a(y_n) \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right|$$

$$\leqq \left|\int_{y_n-\delta_n}^{y_n+\delta_n} \mu_a \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right| + \left|\int_{y_n-\delta_n}^{y_n+\delta_n} \phi_a(y_n) \, d[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\right|$$

$$\leqq \max_{t \in [y_n-\delta_n, y_n+\delta_n]} |\mu_a(t)| V_{y_n-\delta_n}^{y_n+\delta_n}[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}] + \left|\phi_a(y_n)[Q^{y_n}_{k_m} - Q^{y_n}_{k_n}]\Big|_{y_n-\delta_n}^{y_n+\delta_n}\right| .$$

Now, $V_0^1 Q_{k_m}^{y_n} = V_0^1 Q_{k_n}^{y_n} = 1$, and hence, $V_{y_n - \delta_n}^{y_n + \delta_n}[Q_{k_m}^{y_n} - Q_{k_n}^{y_n}] \leqq 2$. Since $l_n < y_n - \delta_n < y_n + \delta_n < d_0^n$,

$\max\limits_{t \in [y_n - \delta_n, y_n + \delta_n]} |\mu_a(t)| < \epsilon$ if $a > N_0^n$. Furthermore,

$$\left| [Q_{k_m}^{y_n} - Q_{k_n}^{y_n}] \Big|_{y_n - \delta_n}^{y_n + \delta_n} \right| \leqq \left| [Q_{k_m}^{y_n} - Q_{k_n}^{y_n}](y_n + \delta_n) \right| + \left| [Q_{k_m}^{y_n} - Q_{k_n}^{y_n}](y_n - \delta_n) \right|$$

$$\leqq Q_{k_m}^{y_n}(y_n + \delta_n) + Q_{k_n}^{y_n}(y_n + \delta_n) + 1 - Q_{k_m}^{y_n}(y_n - \delta_n) + 1 - Q_{k_n}^{y_n}(y_n - \delta_n) .$$

Since $k_m \geqq k_n > W_n$, $0 \leqq y_n - \delta_n$, and $y_n + \delta_n \leqq 1$, it follows that

$$Q_{k_m}^{y_n}(y_n + \delta_n) < \epsilon_n ,$$

$$Q_{k_n}^{y_n}(y_n + \delta_n) < \epsilon_n ,$$

$$1 - Q_{k_m}^{y_n}(y_n - \delta_n) < \epsilon_n ,$$

and

$$1 - Q_{k_n}^{y_n}(y_n - \delta_n) < \epsilon_n .$$

Hence,

$$\left| \int_{y_n - \delta_n}^{y_n + \delta_n} \phi_a \, d[Q_{k_m}^{y_n} - Q_{k_n}^{y_n}] \right| < 2\epsilon_n + B[\epsilon_n + \epsilon_n + \epsilon_n + \epsilon_n] = 2\epsilon_n + 4B\epsilon_n$$

if $a > N_0^n$, and so

$$\left| \int_0^1 \phi_a \, d[Q_{k_m}^{y_n} - Q_{k_n}^{y_n}] \right| < 2\epsilon_n B + 2\epsilon_n + 4\epsilon_n B + 2\epsilon_n B = 8\epsilon_n B + 2\epsilon_n$$

if $a > N_0^n$.

Finally,

$|\phi_{k_m}(y_m) - \phi_{k_n}(y_n)| < \epsilon_n + 8\epsilon_n B + 2\epsilon_n = (8B + 3)\epsilon_n = \dfrac{8B + 3}{n} < \epsilon$ as $n > L$. Hence, the sequence $\phi_{k_1}(y_1), \phi_{k_2}(y_2), \ldots$ converges to some number L_x.

It now will be shown that $\phi_1, \phi_2, \ldots$ left slant converges to L_x at x. Suppose $\epsilon > 0$; there is a number $M_0 > 0$ so that, if n is an integer greater than M_0, $|\phi_{k_n}(y_n) - L_x| < \epsilon/6$. There is a positive integer t so that, if each of b and c is a number $l_t \leqq b \leqq c \leqq d_i^t$ and $n > N_i^t$, $|\phi_n(b) - \phi_n(c)| < \epsilon/6$. There is a number $M > M_0$ such that, if m is an integer greater than M, $l_t < y_m - \delta_m$, and $1/m < \epsilon/24B$. If d is a number such that $l_t < d < x$, there is a positive integer h such that $d \leqq d_h^t$ and $\delta_m + y_m < d_h^t$. Let $N = N_h^t + W_m$, where W_m is as before. Suppose n is an integer greater than N and $l_t = l \leqq b \leqq d$. Then,

$$|\phi_n(b) - L_x| = |\phi_n(b) - \phi_n(y_m) + \phi_n(y_m) - \phi_{k_m}(y_m) + \phi_{k_m}(y_m) - L_x|$$

$$\leqq |\phi_n(b) - \phi_n(y_m)| + |\phi_n(y_m) - \phi_{k_m}(y_m)| + |\phi_{k_m}(y_m) - L_x| .$$

Since $l_t < y_m < d_h^t$, $l_t \leqq b \leqq d \leqq d_h^t$, and $n > N_h^t$, it follows that $|\phi_n(b) - \phi_n(y_m)| < \epsilon/6$. Also, $|\phi_{k_m}(y_m) - L_x| < \epsilon/6$ since $m > M$. By Theorem 3.5,

$$|\phi_{k_m}(y_m) - \phi_n(y_m)| = \lim_{a\to\infty} \left| \int_0^1 \phi_a \, [Q_{k_m}^{y_m} - Q_n^{y_m}] \right| .$$

For each positive integer a,

$$\left| \int_0^1 \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| \leqq \left| \int_0^{y_m - \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right|$$

$$+ \left| \int_{y_m - \delta_m}^{y_m + \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| + \left| \int_{y_m + \delta_m}^1 \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| .$$

Note that

$$\left| \int_0^{y_m - \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| \leqq BV_0^{y_m - \delta_m}[Q_{k_m}^{y_m} - Q_n^{y_m}] \leqq B[V_0^{y_m - \delta_m} Q_{k_m}^{y_m}$$

$$+ V_0^{y_m - \delta_m} Q_n^{y_m}] = [1 - Q_{k_m}^{y_m}(y_m - \delta_m) + 1 - Q_n^{y_m}(y_m - \delta_m)]$$

by Theorem 2.6. Now, $1 - Q_{k_m}^{y_m}(y_m - \delta_m) < \epsilon_m$, and $1 - Q_n^{y_m}(y_m - \delta_m) < \epsilon_m$ since $n > W_m$, $k_m > W_m$, and $y_m \in [\delta_m, 1 - \delta_m]$. Hence, $\left| \int_0^{y_m - \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| < 2\epsilon_m B$. Similarly, $\left| \int_{y_m + \delta_m}^1 \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| < 2\epsilon_m B$. If $\mu_a(t) = \phi_a(t) - \phi_a(y_m)$ for $t \in [y_m - \delta_m, y_m + \delta_m]$, $a = 1, 2, \ldots$, then

$$\left| \int_{y_m - \delta_m}^{y_m + \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| = \left| \int_{y_m - \delta_m}^{y_m + \delta_m} \mu_a + \phi_a(y_m) \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right|$$

$$\leqq \left| \int_{y_m - \delta_m}^{y_m + \delta_m} \mu_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| + \left| \int_{y_m - \delta_m}^{y_m + \delta_m} \phi_a(y_m) \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right|$$

$$\leqq \max_{t \in [y_m - \delta_m, y_m + \delta_m]} |\mu_a(t)| V_{y_m - \delta_m}^{y_m + \delta_m}[Q_{k_m}^{y_m} - Q_n^{y_m}] + \left| \phi_a(y_m)[Q_{k_m}^{y_m} - Q_n^{y_m}] \Big|_{y_m - \delta_m}^{y_m + \delta_m} \right| .$$

Note that

$$Q_{k_m}^{y_m}(y_m + \delta_m) < \epsilon_m ,$$

$$1 - Q_{k_m}^{y_m}(y_m - \delta_m) < \epsilon_m ,$$

$$Q_n^{y_m}(y_m + \delta_m) < \epsilon_m ,$$

and

$$1 - Q_n^{y_m}(y_m + \delta_m) < \epsilon_m ,$$

since $k_m > W_m$, $n > W_m$, and $y_m \in [\delta_m, 1 - \delta_m]$. Hence,

$$\left| [Q_{k_m}^{y_m} - Q_n^{y_m}] \Big|_{y_m - \delta_m}^{y_m + \delta_m} \right| \leqq Q_{k_m}^{y_m}(y_m + \delta_m) + Q_m^{y_m}(y_m + \delta_m) + 1 - Q_{k_m}^{y_m}(y_m - \delta_m)$$

$$+ 1 - Q_n^{y_m}(y_m - \delta_m) < \epsilon_m + \epsilon_m + \epsilon_m + \epsilon_m = 4\epsilon_m .$$

Also, $V_{y_m - \delta_m}^{y_m + \delta_m}[Q_{k_m}^{y_m} - Q_n^{y_m}] \leqq 2$ since $V_0^1 Q_{k_m}^{y_m} = V_0^1 Q_n^{y_m} = 1$, and $\max_{t \in [y_m - \delta_m, y_m + \delta_m]} |\mu_a(t)| < \epsilon/6$ if $a > N_h^t$ sinc

$l_t < y_m - \delta_m < y_m + \delta_m < d_h^t$. It follows that

$$\left| \int_{y_m - \delta_m}^{y_m + \delta_m} \phi_a \, d[Q_{k_m}^{y_m} - Q_n^{y_m}] \right| < 2B\epsilon_m + \frac{\epsilon}{3} + 4B\epsilon_m + 2B\epsilon_m = \frac{\epsilon}{3} + 8B\epsilon_m \text{ if } a > N_h^t .$$

Hence,

$$|\phi_n(b) - L_x| < \frac{\epsilon}{6} + \frac{\epsilon}{3} + 8B\epsilon_m + \frac{\epsilon}{6} = \frac{2\epsilon}{3} + 8B\epsilon_m = \frac{2\epsilon}{3} + \frac{8B}{m} < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

if $n > N$ and $l \leqq b \leqq d$. This completes the proof of the theorem.

There are entirely analogous results for the right side.

Theorem 5.3. *If* x *is in* (0,1) *and* $\phi_1, \phi_2, \ldots$ *both left and right slants at* x, *then there is a number* L_x *and a number* R_x *so that* $\lim_{n \to \infty} \phi_n(x) = \frac{1}{2}[L_x + R_x]$.

Proof. Since $\phi_1, \phi_2, \ldots$ both left and right slants, there is by Theorem 5.2 (and the entirely analogous results for the right side) a number L_x and a number R_x to which $\phi_1, \phi_2, \ldots$ left slant converges and right slant converges respectively. Let B be a number so that $B > |\phi_i(x)|$ for $x \in [0,1]$, $i = 1, 2, \ldots$. If $\epsilon > 0$, there is a number sequence $l_0, l_1, \ldots$, a number sequence $r_0, r_1, \ldots$, and a positive integer $N_0, N_1, \ldots$ so that

$$|l_0 - x| = |r_0 - x| ,$$

$$0 \leqq l_i < l_{i+1} < x < r_{i+1} < r_i \leqq 1 \qquad \text{for } i = 0, 1, 2, \ldots,$$

$$\lim_{n\to\infty} l_n = x = \lim_{n\to\infty} r_n ,$$

$$N_{i+1} > N_i \qquad \text{for } i = 0, 1, 2, \ldots ,$$

and

$$\text{if } m > N_i,\ l_0 \leqq a \leqq b \leqq l_i, \text{ and } r_i \leqq c \leqq d \leqq r_0 ,$$

then

$$|\phi_m(a) - \phi_m(b)| < \epsilon ,$$

$$|\phi_m(c) - \phi_m(d)| < \epsilon ,$$

$$|\phi_m(a) - L_x| < \epsilon ,$$

and

$$|\phi_m(c) - R_x| < \epsilon .$$

Let $d = \dfrac{|l_0 - x|}{16}$, $u = \dfrac{l_0 + x}{2}$, and $v = \dfrac{r_0 + x}{2}$. There is a positive integer t such that $d + u < l_t$ and $r_t < v - d$.

For each positive integer n, $|\phi_n(x) - \frac{1}{2}[L_x + R_x]| = \frac{1}{2}|2\phi_n(x) - L_x - R_x| \leqq \frac{1}{2}|\phi_m(u) - L_x + \phi_m(v) - R_x| + \frac{1}{2}|\phi_m(u) + \phi_m(v) - 2\phi_n(x)|$.

If $m > N_t$, then $|\phi_m(u) - L_x| < \epsilon$ and $|\phi_m(v) - R_x| < \epsilon$. Hence, $|\phi_n(x) - \frac{1}{2}[L_x + R_x]| < \epsilon + \frac{1}{2}|\phi_m(u) + \phi_m(v) - 2\phi_n(x)|$. By Theorem 3.4,

$$|\phi_m(u) + \phi_m(v) - 2\phi_n(x)| = \lim_{x\to\infty} \left| \int_0^1 \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| .$$

By Theorem 2.4, there is a number $W_0 > 0$ such that, if $w > W_0$ and $z \in [d, 1 - d]$,

$$1 - Q_w^z(t) < \epsilon \qquad \text{if } 0 \leqq t \leqq z - d$$

and

$$Q_w^z(t) < \epsilon \qquad \text{if } z + d \leqq t \leqq 1.$$

By Theorem 2.2, there is a number $W_1 > 0$ so that if $w > W_1$ then $|1/2 - Q_n^x(x)| < \dfrac{\epsilon}{2}$. Let $W = W_0 + W_1$. For each positive integer $t > W$, there is a number a_t and a number b_t such that

(1) $l_t < a_t < x < b_t < r_t$,

(2) $|Q_t^x(a_t) - 1/2| < \epsilon$,

and

$$(3)\quad |\, Q_t^x(b_t) - 1/2 \,| < \epsilon\,.$$

Let n be an integer greater than W. For each positive integer k,

$$\left|\int_0^1 \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| \leqq \left|\int_0^{u-d} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right|$$

$$+ \left|\int_{u-d}^{a_n} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| + \left|\int_{a_n}^{b_n} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right|$$

$$+ \left|\int_{b_n}^{v+d} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| + \left|\int_{v+d}^{1} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right|.$$

Note that

$$\left|\int_0^{u-d} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| \leqq BV_0^{u-d}[Q_m^u + Q_m^v - 2Q_m^x]$$

$$\leqq B[V_0^{u-d}Q_m^u + V_0^{u-d}Q_m^v + 2V_0^{u-d}Q_m^x]$$

$$= B[1 - Q_m^u(u-d) + 1 - Q_m^v(u-d) + 2(1 - Q_m^x(u-d))]$$

by Theorem 2.6.

If $m > W_0$, then $1 - Q_m^u(u-d) < \epsilon$ and $1 - Q_m^v(u-d) < \epsilon$. Since $n > W$, $1 - Q_n^x(u-d) < \epsilon$. It follows that

$$\left|\int_0^{u-d} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| < 4B\epsilon \qquad \text{for } k = 1, 2, \ldots.$$

Similarly,

$$\left|\int_{v+d}^{1} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| < 4B\epsilon \qquad \text{for } k = 1, 2, \ldots.$$

Let $\mu_k(t) = \phi_k(t) - \phi_k(u-d)$ for each positive integer k and $t \in [u-d, u+d]$. Then,

$$\left|\int_{u-d}^{a_n} \phi_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| = \left|\int_{u-d}^{a_n} [\mu_k + \phi_k(u-d)]\, d[Q_m^u + Q_m^v - 2Q_n^x]\right|$$

$$\leqq \left|\int_{u-d}^{a_n} \mu_k\, d[Q_m^u + Q_m^v - 2Q_n^x]\right| + \left|\int_{u-d}^{a_n} \phi_k(u-d)\, d[Q_m^u + Q_m^v - 2Q_n^x]\right|$$

$$\leqq \max_{t\in[u-d,a_n]} |\mu_k(t)|\, V_{u-d}^{a_n}[Q_m^u + Q_m^v - 2Q_n^x] + \left|\phi_k(u-d)[Q_m^u + Q_m^v - 2Q_n^x]\Big|_{u-d}^{a_n}\right|.$$

Now, $\left|[Q_m^u + Q_m^v - 2Q_n^x]\Big|_{u-d}^{a_n}\right| \leqq Q_m^u(a_n) + [1 - Q_m^v(a_n)] + [1 - 2Q_n^x(a_n)] + [1 - Q_m^u(u-d)] + [1 - Q_m^v(u-d)] +$ $[2 - 2Q_m^x(u-d)]$, $Q_m^u(a_n) < \epsilon$ since $u + d < l_n < a_n$, and $1 - Q_m^v(a_n) < \epsilon$, since $a_n < x < r_n < v - d$. Also, $2|1/2 - Q_n^x(a_n)| < 2\epsilon$ by the way a_n was chosen. Since $m > W_0$, $1 - Q_n^u(u-d) < \epsilon$ and $1 - Q_m^v(u-d) < \epsilon$.

Finally, $2|1 - Q_m^x(u-d)| < 2\epsilon$ since $n > W_0$. Hence,

$$\left| [Q_m^u + Q_m^v - 2Q_n^x] \Big|_{u-d}^{a_n} \right| < \epsilon + \epsilon + 2\epsilon + \epsilon + \epsilon + 2\epsilon = 8\epsilon .$$

Since $V_0^1 Q_m^u = V_0^1 Q_m^v = V_0^1 Q_n^x = 1$, $V_0^1[Q_m^u + Q_m^v - 2Q_n^x] \leqq 4$. Since $a_n < x$, there is a positive integer s so that $a_n < l_s$. Hence, if $k > N_s$, then $\max\limits_{t \in [u-d, a_n]} |\mu_k(t)| < \epsilon$. Finally, then,

$$\left| \int_{u-d}^{a_n} \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| < 8\epsilon B + 4\epsilon \qquad \text{if } k > N_s .$$

Similarly,

$$\left| \int_{b_n}^{v+d} \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| < 8\epsilon B + 4\epsilon \qquad \text{if } k > N_s .$$

Note that

$$\left| \int_{a_n}^{b_n} \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| \leqq B V_{a_n}^{b_n} [Q_m^u + Q_m^v - 2Q_n^x] \leqq B[V_{a_n}^{b_n} Q_m^u + V_{a_n}^{b_n} Q_n^v + 2V_{a_n}^{b_n} Q_n^x] .$$

By Theorem 2.6,

$$V_{a_n}^{b_n} Q_m^u = Q_m^u(a_n) - Q_m^u(b_n) < Q_m^u(a_n) < \epsilon,$$

and

$$V_{a_n}^{b_n} Q_m^v = Q_m^v(a_n) - Q_m^v(b_n) \leqq 1 - Q_m^v(a_n) + 1 - Q_m^v(b_n) < \epsilon + \epsilon + 2\epsilon$$

since $u + d < l_n < a_n < x < b_n < r_n < v - d$ and $m > W_0$. Also by Theorem 2.6,

$$V_{a_n}^{b_n} Q_n^x = Q_n^x(a_n) - Q_n^x(b_n) \leqq |1/2 - Q_n^x(a_n)| + |1/2 - Q_n^x(b_n)| \leqq \epsilon + \epsilon = 2\epsilon ,$$

by the way a_n and b_n were chosen. Finally then,

$$\left| \int_{a_n}^{b_n} \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| < B[\epsilon + 2\epsilon + 2\epsilon] = 5B\epsilon .$$

Hence,

$$\left| \int_0^1 \phi_k \, d[Q_m^u + Q_m^v - 2Q_n^x] \right| < 4B\epsilon + 8B\epsilon + 4\epsilon + 5B\epsilon + 8B\epsilon + 4\epsilon + 4B\epsilon = 8\epsilon + 29B\epsilon .$$

It follows that, if $n > W$,

$$|\phi_n(x) - \tfrac{1}{2}[L_x + R_x]| < \epsilon + \tfrac{1}{2}[8\epsilon + 29B\epsilon] = \frac{10\epsilon + 29B\epsilon}{2} = \epsilon_0 .$$

Hence, the theorem is established.

6. GLOBAL PROPERTIES OF A BOUNDED SEQUENCE OF ASSOCIATED STEP FUNCTIONS

Theorem 6.1. *If the hypothesis of Theorem* 5.1 *holds for each* x *in* $[0,1]$, *then there is a continuous function* ϕ *on* $[0,1]$ *such that* $c_n = \int_0^1 j^n\, d\phi$, $n = 1, 2, \ldots$.

Proof. By Theorem 5.1, $\lim_{n\to\infty} \phi_n(x)$ exists for each x in $[0,1]$. Let $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ for x in $[0,1]$. Suppose that $\epsilon > 0$ and that x is in $[0,1]$. By hypothesis, there is a number $\delta > 0$ and a number $N > 0$ so that, if $y \in [0,1]$, $|x - y| < \delta$, and $n > N$, then $|\phi_n(x) - \phi_n(y)| < \epsilon/3$. Suppose $y \in [0,1]$ and $|x - y| < \delta$. Then, there is a number $M > N$ so that, if $m > M$, then $|\phi(x) - \phi_m(x)| < \epsilon/3$, and $|\phi(y) - \phi_m(y)| < \epsilon/3$. Hence, $|\phi(x) - \phi(y)| \leqq |\phi(x) - \phi_m(x)| + |\phi_m(x) - \phi_m(y)| + |\phi_m(y) - \phi(y)| < (\epsilon/3) + (\epsilon/3) + (\epsilon/3) = \epsilon$. Therefore, ϕ is continuous at x.

By Theorem 3.4, $\lim_{x\to\infty} \int_0^1 j^n\, d\phi_k = c_n$, and hence, $\lim_{x\to\infty}\int_0^1 \phi_k\, dj^n = c_0 - c_n$.

Since $\lim_{k\to\infty} \phi_k(x) = \phi(x)$ for all x in $[0,1]$, $\lim_{k\to\infty}\int_0^1 (\phi_k - \phi)\, dj^n = 0$ for $n = 0, 1, 2, \ldots$ by the Bounded Convergence Theorem (Theorem 15.6, p. 71 of [8]). Hence, $\int_0^1 \phi\, dj^n = c_0 - c_n$ or $\int_0^1 j^n\, d\phi = c_n$, $n = 0, 1, 2, \ldots$.

Theorem 6.2. *If* $\phi_1, \phi_2, \ldots$ *either right slants or left slants except on at most a countable subset of* $[0,1]$, *then there is a function* ϕ *on* $[0,1]$ *such that* $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$.

Proof. Suppose that $d > 0$. Let M_d be the set to which x belongs if and only if for each $\delta > 0$ and each number $N > 0$ there is a y in $[0,1]$ with $|x - y| < \delta$ and an integer $n > N$ so that $|\phi_n(x) - \phi_n(y)| > d$. If there is a number x in $[0,1]$ such that $\phi_1, \phi_2, \ldots$ does not left slant or right slant, let K be the set of all such numbers x in $[0,1]$.

Suppose that M_d is uncountable. Let $M_d' = M_d \setminus K$. It follows from (Theorem 56′, p. 37 of [9]) that there is a number t in M_d' such that every segment containing t contains a point of M_d' both to the left of t and to the right of t.

Note that $t \notin K$. Suppose $\phi_1, \phi_2, \ldots$ left slants at t. Let $\epsilon = d/2$. There is a number l, $0 \leqq l < t$, so that, if $l < b < t$, there is a number $N > 0$ so that, if n is an integer greater than N and $l \leqq u \leqq v \leqq b$, $|\phi_n(u) - \phi_n(v)| < \epsilon$. Choose b so that M_d' intersects (l,b). Pick $g \in M_d' \cap (l,b)$. Then there is a number r in (l,b) so that $|\phi_n(q) - \phi_n(r)| > d$ for some $n > N$. This is a contradiction. One gets a similar contradiction if it is supposed that $\phi_1, \phi_2, \ldots$ right slants at t. Hence, M_d is countable. Let $M = \bigcup_{t=1}^{\infty} M_{1/t}$. If $x \in [0,1]\setminus[M \cup K]$, then the hypothesis of Theorem 5.1 holds, so $\lim_{x\to\infty} \phi_n(x)$ exists. Using a diagonal process, there is a subsequence $\phi_{n_1}, \phi_{n_2}, \ldots$ which converges pointwise on $[0,1]$. Let

$\phi(x) = \lim_{x\to\infty} \phi_{n_i}(x)$ for x in $[0,1]$.

As in the first part of the proof of Theorem 5.1, one sees that ϕ is continuous at each point of $[0,1]\setminus[M \cup K]$. Since $[M \cup K]$ is countable, it follows that ϕ is continuous except on, at most, a countable subset of $[0,1]$. Since ϕ is also bounded, it follows that $\int_0^1 \phi\, dj^n$, $n = 1, 2, \ldots$ exists.

Since $\lim_{i\to\infty} \phi_{n_i}(x) = \phi(x)$ for all x in [0,1], $\lim_{i\to\infty}\int_0^1 \phi_{n_i} - \phi\, dj^n = 0$, $n = 1, 2, \ldots$ by the Bounded Convergence Theorem (Theorem 15.6, p. 71 of [8]). Hence, $\int_0^1 \phi\, dj^n = c_0 - c_n$, $n = 0, 1, 2, \ldots$ since $\lim_{n\to\infty}\int_0^1 \phi_{n_i}\, dj^n = c_0 - c_n$, $n = 0, 1, 2, \ldots$. It follows (using integration by parts) that

$$\int_0^1 j^n\, d\phi = c_n, \qquad n = 0, 1, 2, \ldots .$$

7. CONVERSE THEOREMS OF SECTION 5

Theorem 7.1. *Suppose ϕ is a real-valued function on* [0,1] *such that* $\phi(0) = 0$ *and that* $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$. *If ϕ is continuous at a number* x *in* [0,1], *then the hypothesis of Theorem* 5.1 *holds at* x *and* $\lim_{n\to\infty} \phi_n(x) = \phi(x)$.

Proof. Since $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$, ϕ is bounded on [0,1], and hence, there is a number B so that $B \geqq |\phi(x)|$ if $x \in [0,1]$.

Suppose $\epsilon_0 > 0$. Suppose ϕ is continuous at 0. Let $\epsilon = \frac{\epsilon_0}{B+1}$. There is a number δ, $1/2 > \delta > 0$, so that, if $\delta \geqq y \geqq 0$, then $|\phi(y)| < \epsilon$. For each positive integer n,

$$\phi_n(t) = \int_0^1 Q_n^t\, d\phi \text{ if } t \in [0,1] .$$

Integrating by parts and recalling that $Q_n^y(1) = 0$ if $y \in [0,1)$ and $\phi(0) = 0$, one has

$$|\phi_n(t)| = \left|\int_0^1 \phi\, dQ_n^t\right|$$

for each $t \in [0,1)$, $n = 1, 2, \ldots$. Let $d = \delta/2$. By Theorem 2.4, there is a number $W > 0$ such that, if $w > W$ and $z \in [d, 1-d]$,

$$1 - Q_w^z(t) < \epsilon \text{ if } 0 \leqq t \leqq z - d ,$$

and

$$Q_w^z(t) < \epsilon \text{ if } z + d \leqq t \leqq 1 .$$

Let n be an integer greater than W and let $0 < y \leqq d$. Then,

$$|\phi_n(y)| \leqq \left|\int_0^\delta \phi\, dQ_n^y\right| + \left|\int_\delta^1 \phi\, dQ_n^y\right|$$

$$\leqq \max_{t \in [0,\delta]} |\phi(t)| V_0^\delta Q_n^y + B V_\delta^1 Q_n^y$$

$$\leqq \epsilon V_0^1 Q_n^y + B Q_n^y(\delta)$$

$$\leqq \epsilon + B Q_n^d(\delta)$$

$$\leqq \epsilon + B\epsilon = \epsilon_0$$

since by Theorems 2.5 and 2.6, if $0 \leqq y \leqq \delta$, then

$$V_0^\delta Q_n^y \leqq V_0^1 Q_n^y = 1 ,$$

and

$$V_\delta^1 Q_n^y = Q_n^y(\delta) - Q_n^y(1) = Q_n^y(\delta) \leqq Q_n^d(\delta) < \epsilon ,$$

and similarly for the case $x = 1$.

Suppose x is in $(0,1)$. Let $\epsilon = \dfrac{\epsilon_0}{8B+2}$. By Theorem 3.5, for each positive integer n,

$$|\phi_n(x) - \phi_n(y)| = \left| \int_0^1 \phi \, d[Q_n^x - Q_n^y] \right| .$$

There is a positive number δ so that $\delta < 1/2$, $\delta < x$, $x + \delta < 1$, and, if $|x - y| < \delta$, y in $[0,1]$, then $|\phi(x) - \phi(y)| < \epsilon$. Let $d = \delta/2$. There is, by Theorem 2.4, a number $W > 0$ so that, if $z \in [d, 1-d]$ and $w > W$, then

$$1 - Q_w^z(t) < \epsilon \text{ if } 0 \leqq t \leqq z - d ,$$

and

$$Q_w^z(t) < \epsilon \text{ if } z + d \leqq t \leqq 1 .$$

Suppose n is an integer greater than W, y is in $[0,1]$, and $|x - y| < d$. Then, by Theorem 3.5,

$$|\phi_n(x) - \phi_n(y)| \leqq \left| \int_0^{x-\delta} \phi \, d[Q_n^x - Q_n^y] \right| + \left| \int_{x-\delta}^{x+\delta} \phi \, d[Q_n^x - Q_n^y] \right| + \left| \int_{x+\delta}^1 \phi \, d[Q_n^x - Q_n^y] \right| .$$

Note that

$$\left| \int_0^{x-\delta} \phi \, d[Q_n^x - Q_n^y] \right| \leqq B V_0^{x-\delta}[Q_n^x - Q_n^y] \leqq B[V_0^{x-\delta} Q_n^x - V_0^{x-\delta} Q_n^y]$$

$$= B[1 - Q_n^x(x - \delta) + 1 - Q_n^y(x - \delta)]$$

$$< B[\epsilon + \epsilon] = 2B\epsilon ,$$

using Theorem 2.4 and the fact that $0 \leqq x - \delta \leqq x - d$, $0 \leqq x - \delta \leqq y - d$, and $n > W$. Similarly,

$$\left|\int_{x+\delta}^{1} \phi \, d[Q_n^x - Q_n^y]\right| < 2B\epsilon .$$

Let $\mu(t) = \phi(t) - \phi(x)$ for t in $[x - \delta, x + \delta]$. Then,

$$\left|\int_{x-\delta}^{x+\delta} \phi \, d[Q_n^x - Q_n^y]\right| \leqq \left|\int_{x-\delta}^{x+\delta} \mu \, d[Q_n^x - Q_n^y]\right| + \left|\int_{x-\delta}^{x+\delta} \phi(x) \, d[Q_n^x - Q_n^y]\right|$$

$$\leqq \max_{t\in[x-\delta, x+\delta]} |\mu(t)| V_{x-\delta}^{x+\delta}[Q_n^x - Q_n^y] + \left|\phi(x)[Q_n^x - Q_n^y]\Big|_{x-\delta}^{x+\delta}\right|$$

$$\leqq \max_{t\in[x-\delta, x+\delta]} |\mu(t)| \cdot 2 + B[Q_n^x(x + \delta) + Q_n^y(x + \delta)$$

$$+ 1 - Q_n^x(x - \delta) + 1 - Q_n^y(x - \delta)]$$

$$\leqq 2\epsilon + B[\epsilon + \epsilon + \epsilon + \epsilon] = 2\epsilon + 4B\epsilon$$

since

$$|\mu(t)| = |\phi(t) - \phi(x)| < \epsilon \text{ if } |t - x| < \delta ,$$

$$V_{x-\delta}^{x+\delta}[Q_n^x - Q_n^y] \leqq V_{x-\delta}^{x+\delta} Q_n^x + V_{x-\delta}^{x+\delta} Q_n^y \leqq V_0^1 Q_n^x + V_0^1 Q_n^y = 2 ,$$

and

$$Q_n^x(x + \delta) < \epsilon ,$$

$$Q_n^y(x + \delta) < \epsilon ,$$

$$1 - Q_n^x(x - \delta) < \epsilon ,$$

and

$$1 - Q_n^x(x - \delta) < \epsilon .$$

It follows that

$$|\phi_n(x) - \phi_n(y)| < 2B\epsilon + 2\epsilon + 4B\epsilon + 2B\epsilon = 8B\epsilon + 2\epsilon = \epsilon_0 .$$

It remains to be shown that $\lim_{n\to\infty} \phi_n(x) = \phi(x)$. In the cases $x = 0$ or $x = 1$ this is obvious since $\phi_n(0) = 0 = \phi(0)$ and $\phi_n(1) = c_0 = \phi(1)$, $n = 1, 2, \ldots$. For each positive integer n,

$$|\phi(x) - \phi_n(x)| = \left|\phi(x) - \int_0^1 Q_n^x \, d\phi\right| = \left|\phi(x) + \int_0^1 \phi \, dQ_n^x\right| = \left|\int_0^1 \phi \, dQ_n^x - \int_0^1 \phi \, dh\right| = \left|\int_0^1 \phi \, d[Q_n^x - h]\right|,$$

where

$$h(t) = \begin{cases} 1 \text{ if } 0 \leqq t \leqq x, \\ 0 \text{ if } x < t \leqq 1. \end{cases}$$

Also,

$$\left|\int_0^1 \phi \, d[Q_n^x - h]\right| \leqq \left|\int_0^{x-\delta} \phi \, d[Q_n^x - h]\right| + \left|\int_{x-\delta}^{x+\delta} \phi \, d[Q_n^x - h]\right| + \left|\int_{x-\delta}^1 \phi \, d[Q_n^x - h]\right|,$$

$$\left|\int_0^{x-\delta} \phi \, d[Q_n^x - h]\right| \leqq BV_0^{x-\delta}[Q_n^x - h] \leqq BV_0^{x-\delta} Q_n^x \leqq B[1 - Q_n^x(x-\delta)],$$

and

$$\left|\int_{x+\delta}^1 \phi \, d[Q_n^x - h]\right| \leqq BV_{x+\delta}^1 [Q_n^x - h] \leqq BV_{x+\delta}^1 Q_n^x \leqq BQ_n^x(x+\delta).$$

Let $\mu(t) = \phi(t) - \phi(x)$ for $t \in [x-\delta, x+\delta]$. Then,

$$\begin{aligned} \left|\int_{x-\delta}^{x+\delta} \phi \, d[Q_n^x - h]\right| &\leqq \left|\int_{x-\delta}^{x+\delta} \phi(x) \, d[Q_n^x - h]\right| + \left|\int_{x-\delta}^{x+\delta} \mu \, d[Q_n^x - h]\right| \\ &\leqq \left|\phi(x)[Q_n^x - h]\Big|_{x-\delta}^{x+\delta}\right| + \max_{t\in[x-\delta,x+\delta]} |\mu(t)| V_{x-\delta}^{x+\delta}[Q_n^x - h] \\ &\leqq B[Q_n^x(x+\delta) + 1 - Q_n^x(x-\delta)] + \max_{t\in[x-\delta,x+\delta]} |\mu(t)| \cdot 2 \\ &\leqq 2\epsilon + B[Q_n^x(x+\delta) + 1 - Q_n^x(x-\delta)], \end{aligned}$$

since $|\mu(t)| = |\phi(t) - \phi(x)| < \epsilon$ if $t \in [0,1]$ and $|x - t| < \delta$, and since $V_{x-\delta}^{x+\delta}[Q_n^x - h] \leqq V_0^1[Q_n^x - h] \leqq V_0^1 Q_n^x + V_0^1 h = 2$. Hence, if $n > W$, $\left|\int_0^1 \phi \, d[Q_n^x - h]\right| < B\epsilon + 2\epsilon + 2B\epsilon + B\epsilon = 4B\epsilon + 2\epsilon = \epsilon_0$. Hence, $|\phi(x) - \phi_n(x)| < \epsilon_0$ if $n > W$. That is, $\lim_{n\to\infty} \phi_n(x) = \phi(x)$.

Theorem 7.2. *Suppose ϕ is a real-valued function on $[0,1]$ such that $\phi(0) = 0$ and $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$. If $\phi(x-)$ exists, then $\phi_1, \phi_2, \ldots$ left slants at x, and $\phi_1, \phi_2, \ldots$ left slant converges at x to $\phi(x-)$.*

Proof. Since $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$, ϕ is bounded on $[0,1]$, and hence, there is a number B such that $B \geqq |\phi(x)|$ if $x \in [0,1]$. Suppose $\epsilon_0 > 0$ and let $\epsilon = \dfrac{\epsilon_0}{8B + 2}$. There is a number $b > 0$ such that, if $0 < x - b \leqq y < x$, then $|\phi(y) - \phi(x)| < \epsilon$. Let $\delta = \min\left(\dfrac{b}{4}, \dfrac{1-x}{4}\right)$. By Theorem 2.4, there is a number $W > 0$ such that, if $w > W$ and $z \in [\delta, 1 - \delta]$, then

$$1 - Q_w^z(t) < \epsilon \text{ if } 0 \leqq t \leqq z - \delta ,$$

and

$$Q_w^z(t) < \epsilon \text{ if } z + \delta \leqq t \leqq 1 .$$

Let $l = x - \dfrac{2b}{3}$, $d = x - \dfrac{b}{2}$, and $N = W$. Suppose that $l \leqq u \leqq v \leqq d$ and that n is an integer greater than N. By definition,

$$|\phi_n(v) - \phi_n(u)| = \left| \int_0^1 [Q_n^v - Q_n^u] \, d\phi \right| = \left| \int_0^1 \phi \, d[Q_n^v - Q_n^u] \right| .$$

Now,

$$\left| \int_0^1 \phi \, d[Q_n^v - Q_n^u] \right| \leqq \left| \int_0^{x-b} \phi \, d[Q_n^v - Q_n^u] \right| + \left| \int_{x-b}^{x-(b/4)} \phi \, d[Q_n^v - Q_n^u] \right| + \left| \int_{x-(b/4)}^1 \phi \, d[Q_n^v - Q_v^u] \right| .$$

Note that

$$\left| \int_0^{x-b} \phi \, d[Q_n^v - Q_n^u] \right| \leqq B V_0^{x-b}[Q_n^v - Q_n^u] \leqq B[V_0^{x-b} Q_n^v + V_0^{x-b} Q_n^u]$$

$$= B[1 - Q_n^v(x-b) + 1 - Q_n^u(x-b)]$$

$$< B[\epsilon + \epsilon] = 2B\epsilon ,$$

using Theorem 2.6 and the fact that $v \in [\delta, 1-\delta]$, $u \in [\delta, 1-\delta]$, $0 \leqq x - b \leqq v - \delta$ and $0 \leqq x - b \leqq u - \delta$. Similarly,

$$\left| \int_{x-(b/4)}^1 \phi \, d[Q_n^v - Q_n^u] \right| < 2B\epsilon .$$

Let $\mu(t) = \phi(t) - \phi(d)$ for $t \in [x - b, x - (b/4)]$. Then,

$$\left| \int_{x-b}^{x-(b/4)} \phi \, d[Q_n^v - Q_n^u] \right| \leqq \left| \int_{x-b}^{x-(b/4)} \mu \, d[Q_n^v - Q_n^u] \right| + \left| \int_{x-b}^{x-(b/4)} \phi(d) \, d[Q_n^v - Q_n^u] \right|$$

$$\leqq \max_{t \in [x-b, x-(b/4)]} |\mu(t)| V_{x-b}^{x-(b/4)}[Q_n^v - Q_n^u]$$

$$+ \left| \phi(d) [Q_n^v - Q_n^u] \Big|_{x-b}^{x-(b/4)} \right|$$

$$\leqq 2\epsilon + B[\epsilon + \epsilon + \epsilon + \epsilon] = 2\epsilon + 4B\epsilon$$

since

(1) $|\mu(t)| = |\phi(t) - \phi(d)| < \epsilon$ if $x - b < t < x$,

(2) $V_{x-b}^{x-(b/4)}[Q_n^v - Q_n^u] \leqq V_0^1[Q_n^v - Q_n^u] \leqq V_0^1 Q_n^v + V_0^1 Q_n^u = 1 + 1 = 2$,

(3) $\left|[Q_n^v - Q_n^u]\Big|_{x-b}^{x-(b/4)}\right| \leqq Q_n^v\left(x - \frac{b}{4}\right) + Q_n^u\left(x - \frac{b}{4}\right) + 1 - Q_n^v(x - b) + 1 - Q_n^u(x - b) < \epsilon + \epsilon + \epsilon + \epsilon$

inasmuch as $v + \delta \leqq x - \frac{b}{4} \leqq 1$, $u + \delta \leqq x - \frac{b}{4} \leqq 1$, $0 \leqq x - b \leqq x - \delta$, and $0 \leqq x - b \leqq u - \delta$. Hence,

$$|\phi_n(v) - \phi_n(u)| < 2B\epsilon + 2\epsilon + 4B\epsilon + 2B\epsilon = 2\epsilon + 8B\epsilon = \epsilon_0 .$$

Thus, the sequence $\phi_1, \phi_2, \ldots$ left slants at x.

It remains to be shown that $\phi_1, \phi_2, \ldots$ left slants at x to $\phi(x-)$. By Theorem 5.2, $\phi_1, \phi_2, \ldots$ left slant converges to a number L_x at x. Hence, it is now to be shown that $L_x = \phi(x-)$. Let $n > W$ and u in $[l, d]$; W, l, and d were previously defined. Then,

$$|\phi_n(u) - \phi(x-)| \leqq |\phi_n(u) - \phi(u)| + |\phi(u) - \phi(x-)| < \epsilon + |\phi_n(u) - \phi(u)|$$

since $l \leqq u \leqq d$. Also, by definition,

$$|\phi_n(u) - \phi(u)| = \left|\int_0^1 Q_n^u \, d\phi - \phi(u)\right| = \left|-\int_0^1 \phi \, dQ_n^u + \int_0^1 \phi \, dh\right| = \left|\int_0^1 \phi \, d[Q_n^u - h]\right| ,$$

where

$$h(t) = \begin{cases} 1 \text{ if } 0 \leqq t \leqq u , \\ 0 \text{ if } u < t \leqq 1 . \end{cases}$$

Note that

$$\left|\int_0^1 \phi \, d[Q_n^u - h]\right| \leqq \left|\int_0^{x-b} \phi \, d[Q_n^u - h]\right| + \left|\int_{x-b}^{x+b} \phi \, d[Q_n^u - h]\right| + \left|\int_{x+b}^1 \phi \, d[Q_n^u - h]\right| .$$

Furthermore,

$$\left|\int_0^{x-b} \phi \, d[Q_n^u - h]\right| \leqq B V_0^{x-b}[Q_n^u - h] = B[1 - Q_n^u(x - b)] \leqq B\epsilon$$

since $n > W$, $u \in [\delta, 1 - \delta]$ and $0 \leqq x - b \leqq u - \delta$. Similarly,

$$\left|\int_{x+b}^1 \phi \, d[Q_n^u - h]\right| \leqq B\epsilon .$$

Let $\mu(t) = \phi(t) - \phi(d)$ for $t \in [x - b, x - (b/4)]$. Then,

$$\left|\int_{x-b}^{x-(b/4)} \phi \, d[Q_n^u - h]\right| \leqq \left|\int_{x-b}^{x-(b/4)} \mu \, d[Q_n^u - h]\right| + \left|\int_{x-b}^{x-(b/4)} \phi(d) \, d[Q_n^u - h]\right|$$

$$\leqq \max_{t\in[x-b,x-(b/4)]} |\mu(t)| V_{x-b}^{x+b}[Q_n^u - h] + \left| \phi(d)[Q_n^u - h] \Big|_{x-b}^{x-(b/4)} \right|$$

$$\leqq 2\epsilon + B[\epsilon + \epsilon] = 2(\epsilon + B\epsilon)$$

since

$$(1)\quad |\mu(t)| = |\phi(t) - \phi(d)| < \epsilon ,$$

$$(2)\quad 1 - Q_n^u(x - b) < \epsilon ,$$

$$(3)\quad Q_n^u\left(x - \frac{b}{4}\right) < \epsilon ,$$

and

$$(4)\quad V_{x-b}^{x-(b/4)}[Q_n^u - h] \leqq V_0^1 Q_n^u + V_0^1 h = 1 + 1 = 2 .$$

Hence, $|\phi_n(u) - \phi(x-)| < \epsilon + B\epsilon + 2\epsilon + 2B\epsilon + B\epsilon = 3\epsilon + 4B\epsilon < \epsilon_0$. Thus, $\phi_1, \phi_2, \ldots$ left slant converges at x to $\phi(x-)$.

There is a similar theorem for right slant convergence.

Theorem 7.3. *Suppose ϕ is a real-valued function on* $[0,1]$ *such that* $\phi(0) = 0$ *and* $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$. *If each of* $\phi(x-)$ *and* $\phi(x+)$ *exists for some* x *in* $(0,1)$, *then* $\phi_1, \phi_2, \ldots$ *both left and right slants at* x *and* $\lim_{n\to\infty} \phi_n(x) = \frac{1}{2}[\phi(x-) + \phi(x+)]$.

Proof. By Theorem 7.2, $\phi_1, \phi_2, \ldots$ both left and right slants at x and both left and right slant converges at x to $L_x = \phi(x-)$ and $R_x = \phi(x+)$ respectively. By Theorem 5.3, $\lim_{n\to\infty} \phi_n(x) = \frac{1}{2}[L_x + R_x]$. It follows that $\lim_{n\to\infty} \phi_n(x) = \frac{1}{2}[\phi(x-) + \phi(x+)]$.

8. TWO ADDITIONAL THEOREMS

Two theorems are proved in this chapter. The first gives a derivation of a known result (see, for example, [3]) by means of the theory developed in this paper. The second theorem is new and gives a connection between Bernstein polynomials and slant convergences.

Theorem 8.1. *If* $c_0, c_1, \ldots$ *is a real number sequence such that the associated step function sequence* $\phi_1, \phi_2, \ldots$ *is of uniform bounded variation on* $[0,1]$, *then there is a function* ϕ *of bounded variation on* $[0,1]$ *such that* $\lim_{n\to\infty} \phi_n(x) = \phi(x)$ *for all* x *in* $[0,1]$ *and* $c_n = \int_0^1 j^n\, d\phi$, $n = 0, 1, 2, \ldots$.

Proof. Suppose $\phi_1, \phi_2, \ldots$ does not left slant for some number x in $(0,1]$. Then there is a number $\epsilon > 0$ so that for every number l, $0 \leqq l < x$, there is a number b, $0 \leqq l < b < x$, so that if N is a positive number then there is a triple of numbers u, v, and n, where $l \leqq u \leqq v \leqq b$ and n is an integer greater than N such that $|\phi_n(u) - \phi_n(v)| > \epsilon$. Let $l_0 = 0$. Then there is a number b_0, $l_0 < b_0 < x$, so that for every $N > 0$ there are numbers u, v, and n, $l_0 \leqq u \leqq v \leqq b_0$ and $n > N$ such that $|\phi_n(u) - \phi_n(v)| > \epsilon$. From

Theorem 4.6, it follows that there are number sequences $\{d_i\}_{i=0}^{\infty}$, $\{b_i\}_{i=0}^{\infty}$, $\{c_i\}_{i=0}^{\infty}$, $\{d_i\}_{i=0}^{\infty}$, and $\{M_i\}_{i=0}^{\infty}$ such that

(1) $a_0 = 0$,

(2) $a_i < b_i < x$, $i = 0, 1, 2, \ldots$,

(3) $d_i = x - b_i$, $i = 0, 1, 2, \ldots$,

(4) $c_i = \min[d_i/4, \epsilon/4]$, $i = 0, 1, 2, \ldots$,

(5) $a_{i+1} = x - (d_i/2)$, $i = 0, 1, 2, \ldots$,

(6) if N is a positive number, there is an integer $n > N$ and a pair of numbers u, v, $a_i \leqq u < v \leqq b_i$, so that $|\phi_n(u) - \phi_n(v)| > \epsilon$ if $i = 0, 1, 2, \ldots$,

and

(7) M_i is a positive number so that, if m is an integer greater than M_i, then

$$V_{a_i - c_i}^{b_i + c_i} \phi_m > \epsilon/2, \; i = 0, 1, 2, \ldots .$$

Now

$$a_1 - c_1 = \left(x - \frac{d_0}{2}\right) - c_1 \geqq x - \frac{x - b_0}{2} - \frac{d_1}{4} = \frac{x + b_0}{2} - \frac{x - b_1}{4} = \frac{x}{4} + \frac{b_0}{2} + \frac{b_1}{4} > 0 .$$

Hence, $0 < a_1 - c_1$. Also $(b_i + c_i) - (a_i - c_i) = b_i - a_i + 2c_i > 0$. Hence, $a_i - c_i < b_i + c_i$, $i = 0, 1, 2, \ldots$. Note that

$$(a_{i+1} - c_{i+1}) - (b_i + c_i) = x - \frac{d_i}{2} - c_{i+1} - b_i - c_i = x - \frac{x - b_i}{2} - b_i - c_{i+1} - c_i = \frac{x - b_i}{2} - c_{i+1} - c_i$$

$$\geqq \frac{d_i}{2} - \frac{d_{i+1}}{4} - \frac{d_i}{4} = \frac{d_i - d_{i+1}}{4} = \frac{b_{i+1} - b_i}{4} > \frac{a_{i+1} - b_i}{4} = \frac{x - (d_i/2) - b_i}{4}$$

$$= \frac{x - (x - b_i/2) - b_i}{4} = \frac{x - b_i}{8} > 0 .$$

Hence, $b_i + c_i < a_{i+1} - c_{i+1}$ if $i = 0, 1, 2, \ldots$. It follows that $0 < a_i - c_i < b_i + c_i < a_{i+1} - c_{i+1} < 1$ if $i = 1, 2, \ldots$. Let B be a number such that $B > V_0^1 \phi_k$, $k = 1, 2, \ldots$. Let L be a positive integer so that $(L-1)\epsilon/2 > B$. Let $M = M_0 + M_1 + \ldots + M_L$. If m is an integer greater than M, then

$$V_{a_i - c_i}^{b_i + c_i} \phi_m > \frac{\epsilon}{2}, \; i = 0, 1, 2, \ldots , L ,$$

and since $0 < a_i - c_i < b_i + c_i < a_{i+1} - c_{i+1} < 1$, $i = 1, 2, \ldots$ it follows that

$$\sum_{i=1}^{L} V_{a_i-c_i}^{b_i+c_i}\phi_m < V_0^1\phi_m < B\ .$$

But $\sum_{i=1}^{L} V_{a_i-c_i}^{b_i+c_i}\phi_m > (L-1)\dfrac{\epsilon}{2} > B$: a contradiction. Hence, $\phi_1, \phi_2, \ldots$ left slants at x for all x in (0,1]. Similarly, $\phi_1, \phi_2, \ldots$ right slants at x for all x in [0,1). By Theorem 5.3, $\lim_{n\to\infty} \phi_n(x)$ exists for all x in [0,1]. Let $\phi(x) = \lim_{n\to\infty} \phi_n(x)$. Then, by Theorem 6.2, $\int_0^1 j^n\, d\phi = c_n$, $n = 0, 1, 2, \ldots$. That ϕ is of bounded variation on [0,1] follows directly from the facts that $V_0^1\phi_k \leqq B$, $k = 1, 2, \ldots$, and $\lim_{n\to\infty} \phi_n(x) = \phi(x)$ if x is in [0,1].

Theorem 8.2. *Suppose* f *is a bounded real-valued function on* [0,1]. *If* f(x–) *exists for some number* x *in* (0,1], *then* $B_1^f, B_2^f, \ldots$ *left slant converges at* x *to* f(x–).

Proof. Let B be a number so that $B \geqq |f(x)|$ if $x \in [0,1]$. Suppose $\epsilon_0 > 0$ and let $\epsilon = \dfrac{\epsilon_0}{4B+1}$. There is a number L, $0 \leqq L < x$, so that, if y is in $[L,x)$, $|f(y) - f(x-)| < \epsilon$. Let $h = x - L$, $a = L + (h/4)$, and $l = L + (h/2)$. If $a < d < x$, let $k = x - d$ and $c = x - (k/2)$. (See Fig. 3.) There is by Theorem 2.4 a positive number W_0 so that, if $w > W_0$ and z is in $[k/8,\ 1-(k/8)]$, then

$$1 - Q_w^z(t) < \epsilon \text{ if } 0 \leqq t \leqq z - (k/8)\ ,$$

and

$$Q_w^z(t) < \epsilon \text{ if } z + (k/8) \leqq t \leqq 1\ .$$

Let W_1 be a positive number so that $4/W_1 < d - l$. Let $N = W_0 + W_1$. Suppose n is an integer greater than N and $y \in [l,d]$. Then

$$|f(x-) - B_n^f(y)| = \left| f(x-) - \sum_{t=0}^{n} \binom{n}{t} f\left(\frac{t}{n}\right) y^t(1-y)^{n-t} \right| = \left| \sum_{t=0}^{n} \binom{n}{t} \left[f(x-) - f\left(\frac{t}{n}\right)\right] y^t(1-y)^{n-t} \right|$$

$$\leqq \left| \sum_{t=0}^{n_a} \binom{n}{t} \left[f(x-) - f\left(\frac{t}{n}\right)\right] y^t(1-y)^{n-t} \right| + \left| \sum_{t=n_a+1}^{n_c} \binom{n}{t} \left[f(x-) - f\left(\frac{t}{n}\right)\right] y^t(1-y)^{n-t} \right|$$

$$+ \left| \sum_{t=n_c+1}^{n} \binom{n}{t} \left[f(x-) - f\left(\frac{t}{n}\right)\right] y^t(1-y)^{n-t} \right| \leqq 2B \sum_{t=0}^{n_a} \binom{n}{t} y^t(1-y)^{n-t} + \epsilon \sum_{t=n_a+1}^{n_c} \binom{n}{t} y^t(1-y)^{n-t}$$

$$+ 2B \sum_{t=n_c+1}^{n} \binom{n}{t} y^t(1-y)^{n-t} \leqq 2BQ_n^a(y) + \epsilon + 2B[1 - Q_n^c(y)]\ ,$$

where $a \in \left[\frac{n_a}{n}, \frac{n_a + 1}{n}\right)$ and $c \in \left[\frac{n_c}{n}, \frac{n_c + 1}{n}\right)$ since $Q_n^a = \sum_{t=0}^{n_a} \binom{n}{t} j^t (1-j)^{n-t}$ and $1 - Q_n^c = 1 - \sum_{t=0}^{n_c} \binom{n}{t} j^t (1-j)^{n-t} = \sum_{t=n_c+1}^{n} \binom{n}{t} j^t (1-j)^{n-t}$. But $Q_n^a(y) < \epsilon$ and $1 - Q_n^c(y) < \epsilon$ since $a + \frac{k}{8} \leqq y \leqq 1$ and $0 \leqq y \leqq c - \frac{k}{8}$. Hence,

$$\left|f(x-) - B_n^f(y)\right| < 2B\epsilon + \epsilon + 2B\epsilon = 4B\epsilon + \epsilon = \epsilon_0$$

if $y \in [l, d]$ and $n > N$. This establishes the theorem.

ORNL–DWG 65–5573

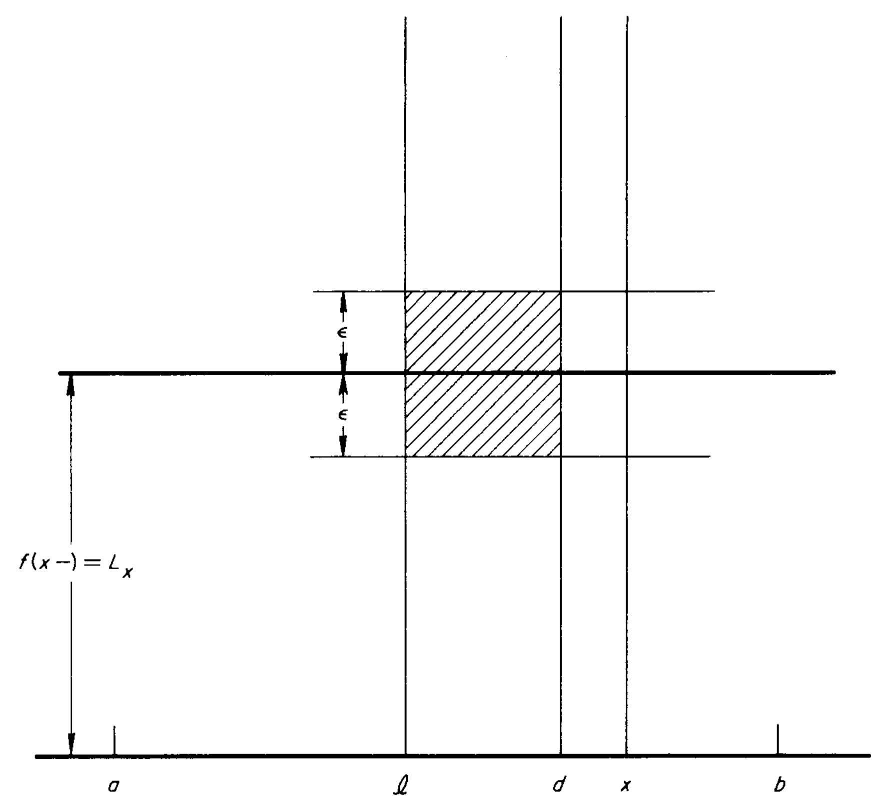

Fig. 3. Left Slant Convergence at x.

9. CONCLUSION

This work deals primarily with the moment problem for a bounded function on [0,1].

Given a number sequence $c_0, c_1, \ldots$ one constructs a certain sequence of step functions $\phi_1, \phi_2, \ldots$ on [0,1] which are called the associated step functions for $c_0, c_1, \ldots$. It seems natural to try to construct as some kind of limit of $\phi_1, \phi_2, \ldots$ a function ϕ which generates the sequence $c_0, c_1, \ldots$ in the sense that $c_n = \int_0^1 j^n \, d\phi$, $n = 0, 1, 2, \ldots$.

Using only the hypothesis of uniform boundedness (Sect. 4) of $\phi_1, \phi_2, \ldots$, it was found that the number sequence $\{V_0^1 \phi_n\}_{n=1}^{\infty}$ is a nondecreasing sequence with a rate of growth not exceeding $\sqrt{n}$ [i.e., $V_0^1\phi_n = 0(\sqrt{n})$, $n = 1, 2, \ldots$]. This is considered "small" in view of the following: For a sequence $c_0, c_1, \ldots$ bounded in magnitude by a number M one finds that $V_0^1\phi_n \leqq M3^n$, $n = 1, 2, \ldots$ since $V_0^1\phi_n = \sum_{t=0}^{n} \binom{n}{t} 2^{n-t} M = M3^n$, $n = 1, 2, \ldots$. A recent result of J. W. Neuberger, (Theorem B, p. 245 of [10]) states that no two continuous functions f which satisfy an inequality of the form

$$\sum_{t=0}^{n} \binom{n}{t} \left| \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i f\left(u + i\left(\frac{v-u}{n}\right)\right)\right| \leqq M(3-\epsilon)^n$$

for some $\epsilon > 0$ and all u, v in [0,1] have the property that they agree on any subsegment of [0,1]. The initial expectation that "$0((3-\epsilon)^n)$" would be a significant "rate" in the study of moment problems seems not to be realized.

It is to be noted that for each number x in [0,1] a continuum is produced as stated in Theorem 4.9. This indicates how closely the sequence $\phi_1, \phi_2, \ldots$ is "knit" together over the entire interval. Theorem 4.7 presents a tool for determining subintervals over which the variation is large. This leads one to examine the local behavior.

In Sect. 5 some positive results are presented for local behavior which results in convergence at a point in the ordinary sense or in a new sense. Using the "slant convergence" it was shown that, given a moment sequence generated by a quasi-continuous function, the following is true: the pointwise limit of the associated step function sequence converges to a normalized function which also generates the moment sequence. Hence, the usual procedure of normalization is unnecessary.

Having noted previously the strong dependence of this work on Bernstein polynomials, it is of interest to point out the strong similarity between certain theorems. Compare, in particular, Theorems 2.1 and 5.1, Theorems 2.2 and 5.3, and Theorems 5.2 and 8.2.

Theorem 8.2 indicates that slant convergence is applicable to problems other than the one settled by Theorem 5.2.

Section 7 along with Sect. 5 points out that if the moment function ϕ has a certain local property, this property is "carried" by the associated step functions produced by the moments generated by ϕ.

One can interpret the results of Sects. 4 and 5 in another sense. If the associated step functions do not converge at a number x in either the ordinary sense or the slantwise sense, there is extreme oscillation about x, and a continuum is produced at x. This leads one to suspect that if $d > 0$, the set M_d to which x belongs if and only if x is in [0,1] and the length of the continuum at x is greater than d is nowhere dense in [0,1] since

$$\lim_{k\to\infty} \int_0^1 j^n \, d\phi_k = c_n, \; n = 0, 1, 2, \ldots .$$

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