OAK RIDGE NATIONAL LABORATORY LIBRARY
3 4456 0022872 3

NTRAL RESEARCH LIBRARY
DOCUMENT COLLECTION

ORNL-3575
UC-32 – Mathematics and Computers
TID-4500 (38th ed.)

AN ELLIPTIC INTEGRAL COMPUTER PACKAGE FOR MAGNETIC FIELDS, FORCES, AND MUTUAL INDUCTANCES OF AXISYMMETRIC SYSTEMS, AND A VERSATILE LINE-TRACING ROUTINE

M. W. Garrett

OAK RIDGE NATIONAL LABORATORY
CENTRAL RESEARCH LIBRARY
CIRCULATION SECTION
4500N ROOM 175
LIBRARY LOAN COPY
DO NOT TRANSFER TO ANOTHER PERSON
If you wish someone else to see this report, send in name with report and the library will arrange a loan.

OAK RIDGE NATIONAL LABORATORY
operated by
UNION CARBIDE CORPORATION
for the
U.S. ATOMIC ENERGY COMMISSION

Printed in USA. Price $2.00. Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, U.S. Department of Commerce, Springfield, Virginia

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

ORNL-3575

Contract No. W-7405-eng-26

THERMONUCLEAR DIVISION

AN ELLIPTIC INTEGRAL COMPUTER PACKAGE FOR MAGNETIC FIELDS, FORCES, AND MUTUAL INDUCTANCES OF AXISYMMETRIC SYSTEMS, AND A VERSATILE LINE-TRACING ROUTINE

M. W. Garrett

APRIL 1965

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U. S. ATOMIC ENERGY COMMISSION

OAK RIDGE NATIONAL LABORATORY LIBRARY
3 4456 0022872 3

CONTENTS

ABSTRACT 1
INTRODUCTION 1
ACKNOWLEDGMENTS 2

1. MATHEMATICAL FORMULATION 2
   Outline 3
   Comparison with the Method of Zonal Harmonics 3
   Units and Notation. Signed Sources. The Gaussian Quadrature 4
   Differential Relations 5
   The Six Basic Equations. Geometrical Definitions 6
   The Axial Field $B_z(S)$. Solid Angle Subtended by a Circle 6
   Calculation of the Integrals 7
   Idealized Magnetostatic Elements 8
   Gaussian Quadrature in Practice 9
   Elliptic Integral Iterations 10

2. ELLIPTIC INTEGRAL PROGRAMS FOR THE IBM 7090 11
   Control and Sequence of Program and Subroutines 11
   Data Cards 12
   The Main Program (MP). START, NET, and MAP Cards 13
   Subroutine KCARDS 16
   Subroutine FORCE. The SEQ Card 17
   Subroutine FIELD 19
   Subroutine TABLES 19
   The FAP Section. GAUSS, KANDE, LOOP, SELF 20
   Subroutine JONES 20
   Subroutine TRACE 21

LIST OF TABLES

Table 1. Differential Relations 4
Table 2. Guide to COMMON Storage 24
Table 3. Outline of Data-Card Fields 25

AN ELLIPTIC INTEGRAL COMPUTER PACKAGE FOR MAGNETIC FIELDS, FORCES, AND MUTUAL INDUCTANCES OF AXISYMMETRIC SYSTEMS, AND A VERSATILE LINE-TRACING ROUTINE

M. W. Garrett[1]

ABSTRACT

This report describes a set of high-precision routines for the IBM 7090 that perform efficiently the calculations most often required on axially symmetric magnetic systems. These routines compute the vector potential, field components, and the mixed gradient $\partial B_\rho/\partial z$ for current systems and for certain idealized distributions of magnetized matter; they compute the force and mutual inductance between coaxial units of a system; and they can trace the lines of force, lines of constant total B, B_z, etc., for such systems. The tracing routine with slight changes should be able to find loci of constant values for any single-valued function of two variables, when a program exists to evaluate the function.

Coaxial units of any combination may be circular loops, thin solenoids, or thick cylindrical coils; also uniformly magnetized rings, cylindrical sheets, or thick-walled cylinders. The six types may be intermixed. The programs compute fields and trace lines accurately even within the windings, and calculate force and mutual inductance between coils in contact, as well as the self-inductance of ideal solenoids and of thick cylindrical coils.

The programs for current systems were developed from a set of six equations in elliptic integrals, which suffice when correlations are fully exploited for all the relevant properties of loops and solenoids. The axial solenoid field is calculated by a superior new method.

Thick coils are analyzed with the solenoid rather than the loop as the elementary unit. Magnetic properties are then integrated in radial depth by Gaussian numerical quadrature of variable order. The method can deal with coils whose sections are not rectangular and whose current densities are a function of the cylindrical radius, though the existing codes do not yet cover such cases.

The procedure is simpler and faster than the commonly used methods which require a more extensive use of numerical integration. Errors are routinely less than one part per million, in extreme cases less than one part per thousand.

The sequence of calculations is controlled by switch entries on the data cards so that a widely varying series of related or independent problems, which may require all the listed operations in any order, may be solved with a single set of data cards.

INTRODUCTION

The programs described here evolved in stages, so that the overall coding structure shows perhaps a certain affinity to that of a New England farmhouse. Nevertheless, the final package performs with high efficiency all calculations that ordinarily are required on axially symmetric systems. The speed and precision of the system and its relatively compact coding depend on a recodification of the available elliptic integral expressions, which is the subject of Sect. 1. This section concludes with some remarks on the use of the equations in practice.

Section 2 contains all the details needed to serve as a working guide to the use of the existing routines. It also describes the organization and interdependence of the various subroutines and discusses significant features of individual routines. It does not include a listing of actual source programs.

[1]Consultant, Swarthmore College, Swarthmore, Pa. This work was supported by Oak Ridge National Laboratory and in part by the U.S. Army Research Office (Durham). For the mathematical section, without details of the machine programs, see M. W. Garrett, *Rev. Sci. Instr.* **34,** 2567 (1963).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the continuing interest of W. F. Gauster in this work, and to thank Carl Parker for his invaluable and always cheerful help in expediting the computer programs through the machine and debugging the line-tracing routine. I am indebted to Arthur Downing for helpful discussions of the errors of Gaussian numerical quadrature.

1. MATHEMATICAL FORMULATION

The magnetic quantities that are computed by the programs reported here are the vector potential, the total scalar field magnitude B with its axial and radial components B_z and B_ρ, the mixed gradient $\partial B_\rho/\partial z$, and the force and mutual inductance between elements of a magnetic system, including self-inductance as a special case.

It is often desired to trace contours that are loci of the constant values of some scalar field function, for example, the contours of constant total field magnitude B or the lines of force which pass through points of constant ρA. The problem of finding such loci is essentially the same for any single-valued function of two orthogonal coordinates. The present package contains a highly developed tracing routine which, with slight changes of linkage, should be usable for any such function whose values can be supplied by some available program. Since its specific relation to the magnetic field is slight, a discussion of the tracing problem is deferred until Sect. 2.

Elements of the current systems with which these programs deal may be circular loops, cylindrical current sheets (ideal solenoids), or thick-walled cylindrical coils with uniform current density. Uniformly magnetized rings, cylindrical sheets, or thick cylindrical shells may also be included, and a single system may contain all these element types intermixed.

The elementary unit from which the properties of thick coils are derived is not the loop, but the cylindrical current sheet. The integration in radial depth over the elementary solenoids is carried out by Gaussian numerical quadrature. It is evident that a thick, axially symmetric coil whose current density is a function of ρ only and whose ends are conical or even curved may still be analyzed into ideal solenoid elements. In such cases, the desired magnetic properties may be integrated with little difficulty over the section by making appropriate changes in the weight factors or the contours that are assigned to the Gaussian quadrature routine, but this option is not included in the completed coding.

In spite of the long-standing use of elliptic integral formulas for loops and solenoids, a search of the scattered literature of fields, mutual inductance, and the "geometry" of particle counters[2,3] failed to reveal an efficiently ordered scheme to calculate economically the most essential magnetic properties of such elements. Special cases have been solved in various ways. Inadequate correlation multiplies the number of formulas that are needed, and some are inferior and should be replaced.

In particular, the axial field $B_z(S)$ of an ideal solenoid has been formulated in remarkably diverse ways, none of which offers a reasonably simple method of calculation. This quantity, which is proportional to the solid angle subtended by a circle at a point off the axis, is deceptively simple in appearance. It must be evaluated at each field point for both ends of each elementary Gaussian solenoid of a thick coil. Fortunately, a better method of solution is available in Eq. (12), below, derived in 1898 by Viriamu Jones,[4] to express the differential coefficient $\partial M/\partial\rho$ of the mutual inductance of a loop and solenoid! The companion formula (9) for M is well known,[4] but though several workers have applied Eq. (12) as originally intended to refine the calculation of absolute M standards, it does not seem to have been previously noticed that it also yields $B_z(S)$. For the relation between the two quantities see Eq. (6).

[2] The geometrical problem of calculating the efficiency with which a round window collects particles or quanta from a point source is the same as that of calculating the axial field of an ideal solenoid. An extensive literature has grown up around each application independently of the other (see ref. 3).

[3] M. W. Garrett, *Rev. Sci. Instr.* **25**, 1208 (1954).

[4] J. V. Jones, *Proc. Roy. Soc. (London)* **63**, 192 (1898).

Both of these formulas contain complete elliptic integrals of all three kinds. Most users of these and similar expressions have elected to use incomplete K and E integrals to avoid calculation of the Π integral. Such methods involve iterations through an arduous sequence, or interpolation in bivariate tables, which usually have an angle as one argument. At least when one uses machines, including desk calculators, it is easier to employ the iterative methods of King[5] and Bartky[6] in computing the three complete integrals. Equation (12) then proves to be far superior to other available expressions for $B_z(S)$.

Outline

After a comparison with the method of zonal harmonics and a section on notation and definitions, details of the elliptic integral method are developed. Correlations rather than derivations are presented, since most equations are drawn from the literature with some new interpretations and changes of form. Table 1 exhibits, in a two-dimensional array, the systematic relations expressed by Eqs. (1)–(7). It shows that only six distinct equations [(8)–(13)] are needed to compute 14 quantities of interest. The entire set of six may in principle be derived from its first member by successive differentiations. Equations (14)–(21) describe the definitions and iterations that are used to compute the complete elliptic integrals K, E, and Π. Working equations are then expressed in terms of the stable end values from the iterations. After listing some equations for the fields of magnetostatic elements, the section ends with a discussion of the use of the method in practice.

Comparison with the Method of Zonal Harmonics

The present attempt to codify the elliptic integral procedures grew out of the author's parallel work on zonal harmonics.[7,8] Recently completed zonal harmonic computer programs[9] are in some respects more versatile, in some respects less so, than are the elliptic integral programs. By carrying the harmonic series to the 33d order (even hand calculations may include the 17th, if the tables of ref. 8 are used), they extend the range of practical convergence into regions so close to the windings that most cases of experimental importance are covered. Exceptions are the calculation of field components within the windings or in the gaps between windings, internal forces, and the force or mutual inductance between coils close together or in actual contact. Precisely these cases are currently increasing in importance because of the large forces in modern high-current systems and the quenching of superconductivity by high fields. The use of elliptic integrals for such calculations removes the limitations on convergence.

The method of zonal harmonics, where it can be used, has advantages of its own that are more fully discussed in the references. The advantages depend on exploiting a distinctive feature of the method, namely, that it permits a clean separation of the field geometry from that of the generating system. Field coordinates and source coordinates can and should be restricted to separate equations used in separate steps of the calculations.

The zonal harmonic method is superior for design problems because it describes fields, forces, and mutual inductances as continuous scalar functions of the field coordinates, with coefficients that depend only on the source coordinates. Thus it is an efficient tool to design systems that generate a uniform field or gradient, search coils that measure field or gradient at a point, mutual inductors in which on rotation M varies accurately as $\cos\theta$, etc. The use of elliptic integrals develops no continuum, but only a series of spot checks at isolated points. Again, if many field points are to be computed, especially for a complicated system, the zonal harmonic method may be faster by a factor of 10.

The harmonic method is also more versatile in computing the rates of change of the magnetic properties, with respect to the field or the source coordinates. For example, it can be used to calculate the mutual inductance, hence also forces and torques, between systems whose axes are

[5] L. V. King, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge University Press, Cambridge, England, 1924.

[6] Walter Bartky, *Rev. Mod. Phys.* **10,** 264 (1938).

[7] M. W. Garrett, *J. Appl. Phys.* **22,** 1091 (1951).

[8] M. W. Garrett, *Tables of the Internal Magnetic Source Functions U_n for Thick Solenoids and Disk Coils*, Edwards Brothers, Inc., Ann Arbor, Mich., 1953. These tables are available upon request from the Thermonuclear Division of Oak Ridge National Laboratory.

[9] M. W. Garrett, *Computer Programs Using Zonal Harmonics for Magnetic Properties of Current Systems*, ORNL-3318 (1962).

separated but parallel, or which intersect at an angle.

The harmonic coefficients that relate the magnetic properties of loops and ideal solenoids to the system geometry, unlike the equations in elliptic integrals, can be directly integrated in radial depth to compute the corresponding coefficients of pancake coils and thick cylindrical coils. It has already been seen that this limitation of the elliptic integral expressions can be removed by resorting to Gaussian numerical quadrature. The same device, though it is not needed in the case of homogeneous cylindrical coils, can be used to find the zonal harmonic coefficients for coils that do not have plane ends or whose current density depends upon ρ. All the inherent advantages of the method are preserved by the use of coefficients so calculated.

Units and Notation. Signed Sources. The Gaussian Quadrature

Units are mks, but μ_0 is represented in all equations by $4\pi \times 10^{-7}$; hence, if it is desired to use the gauss, gauss-cm, etc., only decimal factors need be introduced. The equations refer to a field ring of cylindrical radius ρ passing through a field point, and a coaxial source ring of radius a, lying in planes whose axial separation is z. When the source is to the right, z is positive; that is, the origin lies in the plane of the field point. The simplest source is a circular current filament or loop.

When an equation for the field, force, mutual inductance, etc., of a current loop is integrated along the axis to find the corresponding property of an ideal solenoid, only the coordinates of the end circles of the solenoid appear in the definite integral, which thus contains two terms of opposite sign. It is convenient to ignore the physical distributed currents of the winding and to attribute the entire field to the two source rings, or more briefly, sources of identical absolute strength $N'i$ but with opposite signs. N' is the linear turn density, and $N'i$ is the surface current density. The positive sign belongs to the right end of a positive current sheet. The source strength of a loop is Ni, where N is the turn count and the sign depends only on the current i. The source concept is more fully treated in the references.[7–9]

All equations are written for single positive sources of unit turn-count N or N', with the tacit assumption that field components are to be computed by algebraic summation over all sources of a system, while forces and mutual inductances are to be summed over all possible pairs consisting of a primary and a secondary source. Each contribution to the sum is to be multiplied by the signed source strength or by the product of the two signed source strengths. In the case of force or mutual inductance the radius a is arbitrarily assigned to a primary and ρ to a secondary source, though it is obvious from the symmetry of the equations or from physical reasoning that one may interchange

Table 1. Differential Relations

Equation	Reference	Derivation	$\partial M/\partial z$ (1)	$M/2\pi\rho$ (3)	$-\partial A/\partial z$ (2)				$(1/2\pi\rho)(\partial M/\partial\rho)$ (6)	
(8)	10	$M(S,S)$								
(9)	3	$M(S,L)$	$F_z(S,S)$	$A(S)$	$\rightarrow$	(12)	$-V(L)$		$B_z(S)$	$\Omega(\rho,a)$
(10)	11	$M(L,L)$	$F_z(S,L)$	$A(L)$	$B_\rho(S)$		$\rightarrow$	(13)	$B_z(L)$	
(11)	11		$F_z(L,L)$		$B_\rho(L)$		$\partial B_\rho(S)/\partial z$			
		Derivation			$-F_z/2\pi\rho$ (4)		$B_\rho(L)$ (7)			

the complete set of ρ's with the corresponding α's for any pair, whether the paired elements are loops, solenoids, or loop and solenoid. This theorem is perhaps trivial for elliptic integral calculations, but it can often be used to accelerate the rate of convergence of a zonal harmonic series, by reducing the ratio R of maximum secondary polar radius to minimum primary radius.

The properties of single solenoid sources (circular end planes) may be visualized in terms of the equivalent disks uniformly charged with positive or negative magnetic pole distributions.

The constants of a zonal harmonic series can be integrated in closed form over the radial depth of the winding. This leads to the symbolic replacement of a disk winding, or of each end plane of a thick cylindrical coil, by two sources of the same strength but of positive sign at the outer edge of a positive end plane, negative at the inner edge. The corresponding integration in depth of the elliptic integral equations has not been accomplished, so that here only loop and solenoid sources occur, never disk or thick coil sources. The resort to Gaussian numerical quadrature simulates the disk by a set of concentric loop sources, and the thick coil end plane by a set of concentric solenoid source rings, whose strengths are proportional to the Gaussian weight coefficients. Calculations are completed as before by scalar summation over all the sources.

The weight factors may be scaled if necessary to take into account the variations with ρ alone of the current density, while the sources may be spaced in proportion to the Gaussian abscissas along a slant or curved line bounding an arbitrary axially symmetric coil section.

Differential Relations

In the following equations, the source type is designated by L for loop, S for solenoid, or G for an undefined generating source ring. G may include, besides L or S, the case of a disk or thick coil source, since the equations assume only axial symmetry and are not restricted to a formulation in elliptic integrals. B is written for the magnetic field, A for magnetic vector potential, V for scalar potential, M for mutual inductance, F for force, and Q for a generalized scalar function of the source coordinates and source strength. The first five equations are well known. Equation (6) follows from (3) and (5), while (7) is physically obvious, since the effect of a small increment dz to a solenoid is the same as if a loop source N_L were added, where $N_L = N'dz$.

$$F_z(G_1,G_2) = i_1 i_2 \partial M(G_1,G_2)/\partial z \ , \tag{1}$$

$$B_\rho(G) = -\partial A(G)/\partial z \ , \tag{2}$$

$$M(G,L) = 2\pi\rho_L A(G)/i_G \ , \tag{3}$$

$$F_z(G,L) = -2\pi i_L \rho_L B_\rho(G) \ , \tag{4}$$

$$B_z(G) = (1/\rho)(\partial/\partial\rho)[\rho A(G)] \ , \tag{5}$$

$$B_z(G) = (i_G/2\pi\rho_L)[\partial M(G,L)/\partial\rho_L] \ , \tag{6}$$

$$\partial Q(S)/\partial z = Q(L) \ . \tag{7}$$

With the aid of Eqs. (1) to (7), it is easy to visualize the relations among the quantities of interest by ordering them in the two-dimensional array of Table 1. Here the entries in each column may in principle be derived from the top down by successive z differentiations, according to Eq. (7). Quantities within each row are identical except for the inclusion of simple numerical factors that are constant for each column. (The second and third rows each count as two separate rows, as divided by the arrows.) The "derivation" at the head or foot of each column relates it to a preceding column by listing the relevant numerical factor, together with the number of the equation that defines the relation precisely. In the divided rows, quantities to the right of the arrows are derived from those to the left by ρ differentiation according to Eq. (6). At the left of each row are shown the number of the equation from which all quantities in the row may be calculated and the corresponding reference number. Only the six equations [(8)–(13)] are needed for the 15 quantities tabulated.[10]

[10] The history of these equations by no means parallels the idealized sequence of the table. Here the complete set is formally regarded as derivable from Chester Snow's formula [Eq. (8)], when the expression $M(S,L)$ has been replaced by the right side of Eq. (9). Snow's paper, in so far as it touches the present discussion, is concerned only with Eqs. (8) and (9) (see ref. 11). Snow credits both equations to Viriamu Jones, but the writer has not found Eq (8) in Jones's work.

The Six Basic Equations. Geometrical Definitions

(Snow[11]), $$M(S,S) = zM(S,L) + \frac{8}{3}\pi \times 10^{-7} a\rho r_1 \left[K - \left(\frac{2}{k^2} - 1\right)(K - E)\right], \tag{8}$$

(Jones[4]), $$M(S,L) = 2\pi \times 10^{-7} z \left[r_1(K - E) - \frac{(a-\rho)^2}{r_1}(\Pi - K)\right], \tag{9}$$

(Maxwell[12]), $$M(L,L) = 2\pi \times 10^{-7} r_1 [2(K - E) - k^2 K], \tag{10}$$

(Maxwell[12]), $$\frac{F_z(L,L)}{i_1 i_2} = -4\pi \times 10^{-7} \frac{z}{r_1}\left[(K - E) - \frac{2\rho a}{r_2^2} E\right]$$

$$= 2\pi \times 10^{-7} \frac{z}{k'^2 r_1}[k^2 K + (k^2 - 2)(K - E)], \tag{11}$$

(Jones[4]), $$\frac{\partial M}{\partial \rho}(S,L) = 2\pi \times 10^{-7} zck \left[K + \frac{a-\rho}{2a}(\Pi - K)\right] = 2\pi\rho B_z(S)/i_S, \tag{12}$$

$$\frac{\partial M}{\partial \rho}(L,L) = 4\pi \times 10^{-7} \frac{\rho}{r_1}\left[(K - E) + \frac{2a}{r_2^2}(a-\rho)E\right] = 2\pi\rho B_z(L)/i_L. \tag{13}$$

Table 1 and all equations apply to single positive sources of strength i (or of unit strength in M equations). That is, the loops are single turns; the solenoids have one turn per meter, and the algebraic sign must be changed for the left-hand sources. K, E, and Π are complete elliptic integrals to modulus k, of the first, second, and third kinds respectively. The parameter of Π is $-c^2$, or, in the notation of some writers, c. The lengths r_1 and r_2 are respectively the maximum and minimum distances from a point on the source ring of radius a to a point on the field ring of radius ρ. Definitions of the auxiliary geometrical quantities and some useful relations among them are given in Eqs. (14) to (17).

$$r_2^2 = (a-\rho)^2 + z^2, \quad r_1^2 = (a+\rho)^2 + z^2, \tag{14}$$

$$k^2 = 4a\rho/r_1^2, \qquad k'^2 = 1 - k^2 = (r_2/r_1)^2, \tag{15}$$

$$c^2 = 4a\rho/(a+\rho)^2,$$
$$c'^2 = 1 - c^2 = (a-\rho)^2/(a+\rho)^2, \tag{16}$$

$$2(\rho a)^{1/2} = kr_1 = c(a+\rho), \quad 4\rho a = k^2 r_2^2/k'^2. \tag{17}$$

[11] Chester Snow, *J. Res. Natl. Bur. Std.* **22**, 239 (1939), RP 1178.

[12] J. C. Maxwell, *Electricity and Magnetism*, Clarendon Press, Oxford, England, 1873; sect. 694 in 3d ed. (1891) reprinted by Dover Publications, Inc., New York, 1954.

The Axial Field $B_z(S)$. Solid Angle Subtended by a Circle

The relation between these two quantities has already been noted. Maxwell, who stated[12] that the solid-angle calculation requires elliptic integrals of all three kinds, preferred to derive a zonal harmonic series for it. Such series converge fast enough, over a fairly wide range, to be practical if recurrence formulas[7–9] or tables are used.

The many elliptic integral formulas that have been proposed are not reviewed here. They are so difficult to use when precision is demanded over a wide range that the principal tables[13,14] have been computed by methods of numerical integration.

None of these methods is so simple or efficient as the use of Jones's Eq. (12), when combined as it is here with the concept of summation over the individual signed sources of a system and computed by the iterative methods of King and Bartky. No special algebraic device is needed to introduce

[13] A. V. H. Masket, R. L. Macklin, and H. W. Schmitt, (a) *Tables of Solid Angles and Activatations*, ORNL-2170 (1956); TID-14975 (1962); (b) *Rev. Sci. Instr.* **28**, 189 (1957).

[14] N. B. Alexander and A. C. Downing, *Tables for a Semi-Infinite Current Sheet*, ORNL-2828 (1959). The tables include B_z, B_ρ, and the vector potential.

the discontinuity that $B_z(S)$ suffers in passing through the current sheet.[15] The computed values are correct on both sides.

Calculation of the Integrals

Quantities in the last two rows of Table 1 require only the integrals K and E. These may be computed within 2 or 3 parts in 10^8 from the Hastings approximations,[16] which require only the evaluation of $\ln k'$ and four five-term series in k'^2. If an iterative method is preferred, only Eqs. (18) and (19) are needed, and on the average four iterations suffice. For hand calculations, the second method is much better, but with a computer the series expansion is faster and simpler to program.

Quantities in the first and second rows of the table also require Π. For such cases no practicable expansion in series is available, and the method described by Thomas *et al.*[17] has been adopted with very little change. It requires on the average four passes through the six steps of Eqs. (19) and (20), followed by one or two more through Eqs. (20) alone. The number increases as $k' = r_2/r_1$ decreases, that is, as the field and source rings approach each other more closely, while eventually Π increases to very large values. The series expansion is never used if Π is required, since K and E then emerge as by-products.

Initial values of the parameters are defined by Eqs. (18), and the two iterative cycles are described by (19) and (20), while Eqs. (21) express the three complete integrals in terms of the final stable values. The procedure differs from that of ref. 17 mainly in the exclusion of k^2 from the summation S. Though only $k^2 + S$ occurs in Eqs. (21) and (23), the separate values are needed for other equations. Since the magnitudes are often quite different, precision may be lost if k^2 and S are added initially. The equations follow.

$$\alpha_0 = 1 \ , \quad \beta_0 = k' \ , \quad \delta_0 = c'^2/k' \ ,$$
$$\epsilon_0 = c^2/c'^2 \ , \quad \zeta_0 = 0 \ . \tag{18}$$

$$\alpha_{i+1} = \frac{1}{2}(\alpha_i + \beta_i) \ , \quad \beta_{i+1} = (\alpha_i\beta_i)^{1/2} \ ,$$
$$S_\infty = \sum_{i=0}^{\infty} 2^{i-1}(\alpha_i - \beta_i)^2 \ . \tag{19}$$

$$\delta_{i+1} = \frac{\beta_{i+1}}{4\alpha_{i+1}}(2 + \delta_i + \delta_i^{-1}), \quad \epsilon_{i-1} = \frac{\delta_i\epsilon_i + \zeta_i}{1 + \delta_i} \ ,$$
$$\zeta_{i+1} = \frac{1}{2}(\epsilon_i + \zeta_i) \ . \tag{20}$$

The iterations [(19), (20)] are continued until α is indistinguishable from β, and δ from unity. Then,

$$K = \pi/2\alpha_\infty, \qquad \Pi - K = K\zeta_\infty \ ,$$
$$K - E = \frac{1}{2}K(k^2 + S_\infty) \ . \tag{21}$$

The first five working equations can now be written with the integrals replaced by the end results of the iterations.

---

[15] The relation between $B_z(S)$ and the solid angle, including the discontinuity, is most simply visualized in terms of the following very general theorem. Consider a homogeneous cylindrical sheet whose currents circulate in planes normal to the generators (the section may be any arbitrary closed curve). Then B_z at any point in space is proportional to the total solid angle Ω subtended by the cylindrical surface, evaluated precisely as if the flux through the surface were to be computed from a positive charge at the point. That is, for a positive sheet the element $d\Omega$ is counted plus when the line of sight passes outward through the surface, minus if inward, and zero if it passes through an open end or cuts the surface twice.

[16] C. Hastings, *Approximations for Digital Computers*, Princeton University Press, Princeton, N.J., 1955, sheets 48 and 51. As k approaches zero, K and E approach the common limit $\pi/2$, and significant figures are lost from $K - E$. Since Hastings' error curves for K and E are nearly identical, while both sets of coefficients are given to 11 decimals, it was decided to compute $K - E$ directly, in addition to K, using the coefficients obtained by subtraction.

[17] J. L. Thomas *et al.*, *J. Res. Natl. Bur. Std.* **43**, 311 (1949), RP 2029. The method is that of the third example (p. 314). It combines procedures due to King[5] and to Bartky,[6] which in turn are based on Landen's transformation.

$$M(S,S) = zM(S,L) + 10^{-7} \times \frac{2\pi^2 \rho a r_1}{3\alpha_\infty}\left[2 + (k^2 - 2)\left(1 + \frac{S_\infty}{k^2}\right)\right], \tag{22}$$

$$F_z(S,S)/i_1 i_2 = M(S,L) = -2\pi\rho A(S)/i_S = 10^{-7}\frac{\pi^2 z}{\alpha_\infty}\left[\frac{1}{2}r_1(k^2 + S_\infty) - \frac{(a-\rho)^2}{r_1}\zeta_\infty\right], \tag{23}$$

$$F_z(S,L)/i_1 i_2 = M(L,L) = 2\pi\rho B_\rho(S)/i_S = -2\pi\rho A(L)/i_L = 10^{-7}\pi^2 r_1 S_\infty/\alpha_\infty, \tag{24}$$

$$F_z(L,L)/i_1 i_2 = 2\pi\rho B_\rho(L)/i_L = 2\pi\rho\partial B_\rho(S)/i_S\partial z = 10^{-7}\frac{\pi^2 z}{2r_1 k'^2\alpha_\infty}\left[k^4 + (k^2 - 2)S_\infty\right], \tag{25}$$

$$B_z(S)/i_S = 10^{-7}\frac{\pi z}{(\rho + a)r_1\alpha_\infty}\left[2a + (a-\rho)\zeta_\infty\right]. \tag{26}$$

The following relation is sometimes useful, in conjunction with (25) or (31). If j is the local volume current density,

$$\partial B_z/\partial\rho = \partial B_\rho/\partial z - 4\pi \times 10^{-7} j. \tag{27}$$

For systems that contain only loop elements, it is better to use the Hastings approximations for K and E, in the following slightly altered working equations:

$$B_z(L)/i_L = \frac{1}{2\pi\rho}\frac{\partial M(L,L)}{\partial\rho} = \frac{2 \times 10^{-7}}{r_1}\left[(K - E) + (a - \rho)\frac{2a}{r_2^2}E\right], \tag{28}$$

$$B_\rho(L)/i_L = \frac{2 \times 10^{-7} z}{\rho r_1}\left[(K - E) - \rho\frac{2a}{r_2^2}E\right] = \frac{10^{-7} z}{\rho r_1 k'^2}\left[(2 - k^2)(K - E) - k^2 K\right], \tag{29}$$

$$A(L)/i_L = 2 \times 10^{-7}\frac{r_1}{\rho}\left[2(K - E) - k^2 K\right], \tag{30}$$

$$\partial B_\rho(L)/i_L\partial z = \frac{z^2}{\rho r_1 r_2^2}\left[2(K - E) - k^2 K - \frac{8k^2 a\rho}{r_2^2}E\right] - \frac{B_\rho(L)}{z i_L}. \tag{31}$$

Equation (28) is derived with little change of form from (13), for close correlation with (29) and (30). It may be easily converted with the aid of (21) to use the end values that result from Eqs. (20), as do Eqs. (24) and (25). Equation (31) was derived from (29). Note that B_ρ/z is indeterminate for $z = 0$ if Eq. (29) is used. Hence if $\partial B_\rho/\partial z$ is needed, (29) should be programmed to compute B_ρ/z initially, then to find B_ρ by introducing z.

Idealized Magnetostatic Elements

It is not difficult to take into account the effect of small axially symmetric volumes of magnetized matter if depolarization is neglected, that is, if a known constant polarization vector is assumed. The basic element is a source ring of static magnetic charge of radius a and line density λ, whose scalar magnetic potential is[18]

$$V(Q) = 4\lambda a K/r_1 \qquad \text{(emu or Gaussian units)}. \tag{32}$$

The fields B_z and B_ρ of charge rings and of dipole rings can be found from this equation by differ-

[18] Ernst Weber, *Electromagnetic Fields*, vol. 1, Wiley, New York, 1950, p. 125. No search was made for an earlier occurrence of this equation. It seems simpler to switch at this point to Gaussian units, since two distinct conventions exist for the magnetic pole fiction in mks units (when it is used at all), and since the programs actually calculate fields in gauss. No distinction is made between the gauss and the oersted.

entiation. The following relations are listed for convenience:

$$\frac{\partial K}{\partial k} = -\frac{1}{k}\left(K - \frac{E}{1-k^2}\right), \quad \frac{\partial E}{\partial k} = -\frac{1}{k}\left(K - E\right), \tag{33}$$

$$\frac{\partial k}{\partial z} = -\frac{kz}{r_1^2}, \qquad \frac{\partial k}{\partial \rho} = -k\left(\frac{a+\rho}{r_1^2} - \frac{1}{2\rho}\right). \tag{34}$$

For charge rings,

$$B_z(Q) = -\frac{4\lambda a z}{r_1^3(1-k^2)} E \tag{35}$$

(here B_z is $+\ \partial V/\partial z$, since differentiation implies motion of the source), and

$$B\rho(Q) = \frac{4\lambda a}{r_1}\left[\frac{K}{2\rho} - \frac{E}{k(1-k^2)}\frac{\partial k}{\partial \rho}\right]. \tag{36}$$

The last two equations are again in Gaussian units.

For dipole rings with axial polarization vector (polar angle $\beta = 0$), change λ to $\lambda'(= d\rho/ad\phi$, the linear density of dipole moment), and differentiate with respect to z. If the polar angle β is 90°, differentiate with respect to a. But since the total charge Q is $2\pi\lambda z$ and the dipole ring is neutral, λa rather than λ must be held constant. Details of the differentiation are omitted. For a dipole ring whose polarization vector lies on a cone (β = constant),

$$B(p, \phi) = B(p, 0°)\cos\beta + B(p, 90°)\sin\beta\,. \tag{37}$$

Note that a cylindrical sheet with uniform axial magnetization is equivalent to a pair of charge rings of opposite sign at the ends of the sheet. Also, a thick cylindrical shell may be replaced by a pair of ideal current sheets (solenoids) coinciding with the cylindrical boundaries and carrying opposite currents. Finally, the vector potential $A(p)$ for a dipole ring is the same as $-B_\rho(Q)$ for a charge ring (see Table 1).

All the cases that have been discussed are included in the coding. The results of test cases run with this program and with the zonal harmonic package of ORNL-3318[19] agree to seven figures.

[19] See ref. 9. The option to include in any system idealized magnetostatic elements of the types discussed here has been added since that report was published.

Gaussian Quadrature in Practice

The writer reported the use of Gaussian numerical integration some years ago[7] in a similar context, to simulate a thick coil by a set of solenoid sources. The integration was done with a desk calculator to check the results of zonal harmonic field series in which the constants for true cylindrical coil sources were computed from the new source functions U_n. Arthur Downing recently used the same device to extend his method for computing solenoid fields[14] to cylindrical coils. The overall method takes six to eight times as long as that reported here, because an independent Gaussian integration (in azimuth) is used also for each field component of the elementary solenoids.

The method described here resorts to the Gaussian approximation for the final ρ integration only (in radial depth). The machine program provides a free choice of the order n between the limits 1 and 16.[20] The n-point Gaussian formula replaces each end plane of a cylindrical coil by a set of n solenoid sources. To calculate F or M for a pair of thick coils, assign a set of $2m$ j-subscripts to the m positive and m negative solenoid sources that replace one coil, and $2n$ k-subscripts to the other. Then sum the $4mn$ components $A_jA_kF_{jk}$ or $A_jA_kM_{jk}$, where the A's are the Gaussian weight factors. The proper value of N' for each coil is the total turn count per meter of the whole coil. To minimize the proximity errors discussed below, if the coils are in contact or very close together, m should not equal n, and it is best to have one even and the other odd.

With the same restrictions on m and n, self-inductance can be computed as the mutual inductance between coincident coils. In the limiting case of an ideal solenoid, set $m = 1$ for the primary and expand the coincident secondary into a thin coil with $2n$ Gaussian sources. It may be verified that the computed value of M approaches the correct value L as the order n increases and the thickness of the secondary decreases.

The self-inductance of an ideal solenoid is (Lorenz[21])

[20] For the Gaussian abscissas and weight factors, see A. N. Lowan, N. Davids, and A. Levenson, *Natl. Bur. Std. Appl. Math. Ser.* **37**, 185 (1954).

[21] L. V. Lorenz, *Wied. Annal.* **7**, 161 (1879).

$$L = \frac{4}{3}\pi \times 10^{-9} N^2 r_1 \left[K - E + \frac{k^2}{1-k^2}(E-k) \right] . \tag{38}$$

Here N is the total turn count, while r_1 and k are used as in Eqs. (14) and (15) with $z = l$, the length of the solenoid (i.e., the separation of its two end circles of equal radii ρ and a). That is, $r_1^2 = l^2 + 4a^2$, $k = 2a/r_1$. This equation was written into the program for the tests discussed in the last paragraph, and it remains in the completed code for routine use.

In the case of cylindrical coils, the use of Gaussian orders as low as 4 or 5 gives results for field components that are valid to within a few parts per million except at points near an end plane of any coil, and for force and mutual inductance even when the coil separation is moderately small. But as the field point approaches the end plane, whether from inside the coil or outside, very large errors may occur, especially in the case of B_ρ.

The errors are due to the fact that each Gaussian solenoid source introduces a fictitious singularity at which B_ρ is infinite, though the true integrated values are finite at every point. Increasing the order n reduces the error but not fast enough. The remedy is to break up the Gaussian integration into several ranges, subdividing the coil into parts by cylindrical walls. In the machine program, a point within any coil or near an end plane is automatically assigned to one of six categories, depending on the precise location of the point, and the coil is subdivided accordingly into two to four coaxial shells, to each of which the n-point formula is separately applied.

The coding also provides the option of selecting a series of orders, completing calculations for each order in turn, and printing out discrepancies that exceed a preset limit. The "normal" option is $n = 6$, without any check.

For minimum error the field point should lie in one of the cylindrical walls dividing the coil (or on its projection if the point is just outside the end plane), and the two immediately adjacent cylindrical shells making up the coil should be thin. This brings the nearest Gaussian sources very close to the field point and allows the high field values computed from such nearby sources to contribute appreciably to the summation. If the coil is so partitioned that the field point lies halfway between the inner and outer walls of a shell, only even Gaussian orders are permitted, the nearest Gaussian sources are more remote than in the previous case, and the convergence with increasing order is found to be much worse. The reason for this can be seen by drawing a rough graph representing the behavior of the integrand near a singularity and studying it in relation to a set of superposed Gaussian abscissas.

In practice, with $n = 10$ or 12 the uncertainties do not exceed one part in a thousand, even for fields in the end plane and for F or M between coils in contact, though in critical cases it is best to rely on one or more of the available checking options. Very few cases of experimental interest are in this critical range, while most are likely to lie within the range of high-order zonal harmonic series. In such cases the two methods check routinely to better than one part per million.

Elliptic Integral Iterations

When both a and ρ are much smaller than z, k approaches zero, and the number of iterations through Eqs. (19) and (20) increases. Rounding errors eventually reduce the number of reliable digits, particularly in the case of $M(S,S)$, which contains a term in z^2. This quantity cannot be reliably calculated with a standard eight-digit FORTRAN program when k^2 is less than about 10^{-7}.

As the field point comes close to a fictitious Gaussian singularity, the Π integral takes on large values, and the number of iterations through Eqs. (20) increases. For points very close to a coil, the average number of iterations for the sources in that coil may rise to 8 or 9, as compared to 3.5 or 4 for more remote coils. Evidently the count is higher still for the nearest individual sources.

Numerous tests, for example, comparisons of computed B_z, B_ρ, and A values very near to an ideal solenoid source with Arthur Downing's tables,[14] and of K and E values with the results of the Hastings series approximation,[16] confirm that the repeated iterations yield an accurate result. Also, with a modern computer the extra time required at near points for the iterations, and to compute separate contributions from two to four sections of a partitioned coil, is seldom significant. The overall procedure seems still to be the most efficient yet proposed.

Nevertheless, for very near points the method smacks of a brute-force approach. The author has failed to find an asymptotic series for Π, to replace the iterative method when k' is small. If such a series, once found, could be integrated with respect to ρ, the Gaussian quadrature also might be bypassed for the nearby section of the coil end plane.

2. ELLIPTIC INTEGRAL PROGRAMS FOR THE IBM 7090

This section assumes a minimal acquaintance with 7090 FORTRAN. The first part, ending with subroutine TABLES, contains the information needed to use the existing programs effectively. After a brief résumé of the general plan of the coding and of the data cards, each type of card is treated in a separate section that describes in detail the uses of the individual data fields.

The second part, beginning with the FAP section, presents a few details of the internal logic and procedures of certain subroutines. It may interest those who wish to understand the working method in greater detail, in order to be more confident (or more critical) of the results, to modify the programs to their own ends, or to write related codes for the same or other machines.

The completed elliptic integral package parallels in organization and scope the zonal harmonic programs of ORNL-3318,[9] with the addition of a line-tracing routine. It covers a wider variety of problems than other general-purpose programs that the author has seen, and it is several times as fast. Nevertheless, for many problems that fall within the range of satisfactory convergence of the zonal harmonic method, the latter is still faster, perhaps in about the same ratio. Extensive comparisons of the results of the two methods, over the wide range of magnetic quantities and of geometries for which they overlap, inspire confidence in the validity of both sets. Usually there is agreement to seven significant figures.

Control and Sequence of Program and Subroutines

The existing 7090 codes consist of one main program and seven subroutines, exclusive of the clock routine TAD which records the time and date. One subroutine package is written in the FAP language, using mainly floating-point operations, while all the rest use single-precision FORTRAN. All interchange of data between sections is effected through a COMMON storage section of 682 cells, which is mapped in Table 2.

The complete package can calculate magnetic-field components B_z, B_ρ, and total $|B|$, vector potential A, the flux function ρA, and $\partial B_\rho/\partial z$ for a complete magnetic system, at all points of a prescribed field net, and can compute axial force F and mutual inductance M between pairs of elements of the system, including as a special case the self-inductance of thick coils or ideal solenoids.

It includes also an efficient line-tracing routine, which was originally written to map the lines of force of a magnetic system, but which has been generalized to trace out lines that are the loci of constant values of any other of the magnetic quantities, in particular the contours of constant total scalar field $|B|$. This subroutine is self-contained, except that at each point it demands the local value of the selected field function. It should therefore be readily adaptable for use with any unrelated program that can compute a single-valued scalar function of two coordinates, to map the behavior of the function.

A magnetic system is defined as any set of MXK coaxial elements (the DIMENSION 40 is currently assigned to MXK). For F, M, or L computations the "system" need have no physical validity, but is often a mere list from which pairs are selected by one or more SEQ cards. Current elements may be circular loops, ideal solenoids, or thick cylindrical coils. Also, the uniformly polarized dipole rings, cylindrical sheets, and thick cylinders described in Sect. 1 may be introduced. [See Eqs. (32)–(37) and the section on entry IS in subroutine KCARDS, below.] Elements of all types may be intermixed in any system, except that subroutine FORCE does not presently deal with loops or with magnetostatic elements. Since the equations of Sect. 1 are valid inside as well as outside the windings, the current space (whose geometry is entered on K cards) and the field space (as specified on NET or MAP cards) may overlap without concern to the user.

A problem is defined as any series of calculations made with one set of K cards, that is, on a single

system of specified geometry. It may consist of a number of sections, each of which is controlled by one or two data cards, once the K cards have been entered. A section may call for the evaluation of magnetic quantities over a field net, for the tracing of a field map, or for one or more sets of calculations of force and mutual inductance between pairs of elements.

Each section of a problem begins at the START position of the main program (MP) by reading the clock and calling for a new card. Since an EXIT occurs only when the data card deck is exhausted, several unrelated problems, each with multiple sections, may be run at a single loading. Once only, on the first pass, MP causes subroutine GAUSS in the FAP section to transfer the 16 sets of Gaussian coefficients to COMMON storage. Cards read at START have 14 locations, or in computer jargon, "fields." Several of the integer fields are used for switch options that control the sequence of operations, including the selection of subroutines. Additional integer and decimal data fields (the net locations) can prescribe either a net of field points (on NET cards) or the interval and starting points for a set of lines of constant flux, constant $|B|$, etc. (on MAP cards). The card that begins the first section of any problem will be referred to as a START card. In every case but one (see TABLES), it first calls subroutine KCARDS to read into storage the system geometry. The first section is then executed under control of the switch options. Later sections of the same problem bypass KCARDS and are initiated by NET, MAP, or FORCE cards. On the FORCE card, which calls subroutine FORCE, the net locations are blank.

Though either a NET or FORCE card may be punched to call KCARDS (and thus may double as a START card to begin a new problem), the MAP card cannot do so. Otherwise, the number and sequence of sections in any problem are arbitrary.

To compute axial force F and mutual inductance M between selected pairs of solenoids or coils, a switch entry on the FORCE card calls subroutine FORCE. This reads in one or more sequence (SEQ) cards which control the selection of pairs. FORCE calls JONES for the contribution to F or M from each pair of elementary Gaussian solenoid sources. For thick coils, self-inductance is calculated as a special case of mutual inductance, while for ideal solenoids it is computed by SELF on a direct call from KCARDS.

A NET card causes MP to scan the field net, calling on FIELD at each point to sum the contributions to vector potentials and field components from all coils or loops of the complete system, and to print the results. FIELD, in turn, calls JONES or LOOP to compute the contributions from individual Gaussian solenoid or loop elements. If a field map is required, the MAP card defines the quantity to be constant for any one line and sets a switch to call TRACE. This subroutine follows the lines (e.g., ρA = constant for lines of force) from their prescribed starting points, calling FIELD for the values of ρA, $|B|$, etc., at each point.

TABLES is called by a START card that is blank except for the field MK. The required input data are read in by the subroutine itself, which is used to print tables of elliptic integrals, or of field components and vector potential, for single loop or ideal solenoid sources. If so desired, the subroutine may be used to check the agreement between independent values of certain quantities that may be computed in the FAP section, using the Hastings series, or in JONES by the iterations of Eqs. (19) and (20).

Data Cards

Table 3 is a working summary of the data-card fields. It is intended to be a self-contained reference for all information normally needed to assemble the data card deck for any set of problems. Though many entries in the table are self-explanatory, the data-card fields are discussed in detail below. The reader will need to refer frequently to Tables 2 and 3 when reading the remainder of this section.

To meet the varying needs of six functional types of data card, it was found necessary to replace the "normal" format[22] by two modified forms. These differ in the left-hand half, which provides ten integer fields in one case as compared to one integer and four decimal fields in the other.

[22]A data card with four three-digit integer fields and six nine-digit decimal fields is here considered "normal" in the sense that it conforms to the invariant card format of the author's parallel zonal harmonic programs (ref. 9), which have been used routinely in this laboratory to check the results computed by elliptic integrals or to supplement them, particularly for solving design problems.

The similarities between cards are best shown by listing them in the table according to format, rather than by ordering them in some typical sequence that they might logically follow in a working deck. Parentheses under each format number enclose the FORTRAN listing. Since each READ instruction is followed by an identical WRITE, input data always appear in the output as entered. The entry in column 1 of a NET, MAP, FORCE, or SEQ card controls paging or line spacing of the output. It is normally 1, to start a new page. The remaining five columns (2–6) of the A6 field on these cards and the entire A4 field (columns 1–4) on K or TAB cards are printed but do not affect the program. They may be used for some simple code to identify the particular problem, section, or system, or even the type of card (e.g., NET, MAP, SEQ, or TAB).

Integer fields (normally three columns) precede decimal data fields (normally nine columns) on each card. The table indexes for reference the column that carries the least significant integer digit. That is, the integral part of each entry is right adjusted to the index. In the case of decimal fields, this assumes that in normal practice the decimal point will not be punched; then the units digit must occupy the designated column. If the decimal point is punched, it may be placed anywhere in the field, but its "normal" position is in the designated column. (Other positions may be necessary to permit full 8-digit accuracy when the input magnitudes cannot be chosen for convenience.) Note that all fields end in columns whose numbers are multiples of 3, while for all decimal fields but one the column numbers are also multiples of 9. The units digit (or the normal decimal point when punched) always occupies the middle column of a group of three. Thus it is easier to check data that are punched on binary cards, since these are ruled in three-column fields. *Exception:* The first three fields of Format 126 have been compressed to two columns for the integer and six for the decimal. These fields end in columns 6, 12, and 18.

It has been found convenient to punch four master cards as templates for direct checking. These are labeled START, KCRD, SEQ, and TAB in the four or six available left-hand columns. The rest of each card lists the symbols for variables as in Table 3, each one right adjusted to the units column of the corresponding data field. It is also much easier to prepare data for a set of problems if special forms are prepared for them. A full line is used to end a field, and a dashed line to replace the decimal point (not punched). The two formats are identical after column 36. Since the guide lines are for reference only and not to be taken literally, the left-hand half of the form may be ruled to fit the two formats on alternate lines, or perhaps on alternate pairs of lines, without implying that the cards actually occur in any particular sequence.

Most integer and decimal fields fulfill normal integer and decimal data functions respectively, but in seven cases it was found expedient to depart from the normal mode. The change need not concern the user, since it is corrected by immediate conversion in the coding. Integers are normally used for counting and switching, decimals for magnitudes. The numerous options provide great flexibility, but they increase the number of data fields. To mitigate this hardship, every effort has been made to represent the most frequently used options by blank fields, even to the point where a completely blank SEQ card is not uncommon. Blank options exist for about 90% of all fields, namely, those which are enclosed in parentheses in Table 3. Here the decoding of the blank is indicated in each case, usually by equating the symbol ( ) to some equivalent positive value. A blank just over any symbol means that its use is sufficiently frequent to be considered "normal," while a dot over an integer symbol indicates that it is cast in a normally decimal role, and conversely.

For many integer fields the binary choice 0 or 1 is sufficient. In a few seldom-used cases the sign and magnitude are separately decoded, while in several cases the tens or hundreds digit of an integer field is separated from the units digit to control an independent option.

The following sections describe separate units of the coding, with the data cards they use, in greater detail. The fields of each card are discussed in order from left to right.

The Main Program (MP). START, NET, and MAP Cards

One data card is read at each pass through the START position of MP, but such a card will not be called a START card unless it begins a new problem.

MK.—This is the first data field. An entry of

−3, −2, or −1 calls TABLES (*q.v.*), while the rest of a card with negative MK is unused, since the subroutine then reads all remaining data. All other START cards must carry in MK the number of K cards (placed next in the deck) that describe the system geometry. MK > 0 reads in the geometry by calling subroutine KCARDS and returns to MP. Here, if IP is negative (see below), the START card doubles as a FORCE card, and the fields after IP are ignored. Otherwise, MP proceeds to scan the field net prescribed by the data fields NZ through A1. That is, the START card doubles as a NET card in all problems that begin by calling for either a field net or a field map. But if the net locations NZ-A1 are all blank, the field magnitudes are computed and printed at the origin only. This may be appropriate when one or more field maps are wanted without tabulating any net. Often in such cases ID through NN are not required, and the START card then needs only one or two entries. FORCE and NET cards placed after the K cards have MK blank. The distinction between the two is in IP. MAP cards cannot precede the K cards. They have MK = −4 when the lines of force (or lines for which $|B|$ or some other field function is constant) are to start in a common plane (z = Z1), at a small angle to the Z axis. MK = −5 when the lines are to start on the cylinder ρ = A1, more nearly parallel to a transverse plane. The second option is often used, for example, in the case of "cusp" geometries.

Maximum or Minimum Field Values. — MK provides also a special option for investigating fields near a point at which a selected field function passes through an absolute maximum or minimum value. For example, there is currently an active interest in so-called "magnetic wells," characterized by an absolute minimum value of the total scalar field $|B|$. The function of interest is selected by an entry in column 11 of the field IN. The special option is controlled by the hundreds digit (in column 7) of MK. When this is set on a NET card to 1 for a maximum or to 2 for a minimum, the program, while scanning the net, preserves the extreme value of the selected function and the coordinates of the point at which it occurs. At the end, all the current densities CD(K) of the system are so rescaled as to normalize the maximum or minimum value of the selected function to unity. A second NET card may now be entered, with a 3 in column 7 for maximum, or 4 for minimum. This card normally prescribes a net of finer mesh, including both negative and positive values of the coordinates. Before scanning the new net, the program shifts its origin to the point where the maximum or minimum was found and prints the newly normalized values. Also, the last column of print shows the deviation of the selected function, in parts per million, from unity. Evidently the process can be continued, if desired, with a third NET card, using a still finer mesh.

IN. — The sign of this field is plus when the coil dimensions are expressed in centimeters and the current densities in amp/cm or amp/cm², minus when the unit of length is the inch. This option does not affect the magnetic units; A is always computed in gauss-cm, $\partial B_\rho/\partial z$ in gauss/cm, etc.

If the absolute magnitude of the units digit of IN exceeds 2, it increases the normal ratio (DZ/DA = 2) of the end-zone parameters in subroutine FIELD (*q.v.*).

The absolute magnitude of the tens digit (column 11) is independently decoded, and it serves to select a field function: ( ) or 1 for B_z, 2 for B_ρ, 3 for A, 4 for $|B|$, 5 for ρA, and 6 for $\partial B_\rho/\partial z$. The function so designated on a NET card is used in routine normalization (see KCARDS) and in the extreme-value procedures described under MK. On a MAP card it selects the function to be held constant while tracing a set of lines, but in this case the flux function ρA is regarded as normal, and a blank in column 11 is decoded as 5.

IP. — A minus sign in this field identifies a FORCE card. If IP is minus and MK > 0, KCARDS is called, followed by FORCE. If MK is blank, FORCE is called directly. (The return to MP depends on the field IR of the SEQ card, *q.v.*)

Again, IP > 0 on a NET or MAP card causes the printing of optional monitoring information in MP or in TRACE. In MP this is a PDUMP that gives a compressed résumé of the previously printed field magnitudes. In TRACE it is a detailed record of successive corrections to the coordinates of individual points and of the local rates of change of the selected function on which the corrections depend. (See also the IP entry under "Subroutine JONES" in the latter part of Sect. 2.)

ID. — If ID > 2, it increases the normal ratio

(DA/A1 = 2%) that defines the radial spread of the end zones of coils in subroutine FIELD.

IE.—IE is an allowable error limit, multiplied by 10^6 on NET cards for MP calculations and by 10^4 on MAP cards for subroutine TRACE. It is used in MP only when comparisons are desired between the results of two or more orders of Gaussian quadrature, that is, when NN is nonzero. Calculations are then carried out at each field point with each value of the order n in succession. Only the last set of values (for maximum n) is printed. But when the sum of three absolute fractional differences in parts per million for the quantities B_z, B_ρ, and A, as computed for the highest order and a lower order n, exceeds the value of IE, the three differences are printed out, and the order n is identified. Such a check is reassuring in critical cases but is hardly necessary except for very precise calculations near the end face of any coil, when higher Gaussian orders may be needed. Normally the choice $n = 6$ or even $n = 4$ is sufficient for six-figure accuracy. In order to save machine time, it may be useful to set up limited nets for any highly critical regions, on separate NET cards that call for higher orders, with NN > 1.

On MAP cards, IE controls the precision with which the trial points must fit the line. It expresses the allowable magnitude in parts per 10^4 of the initial interval between lines (at the starting plane or cylinder) of the last correction made to the coordinate of a point.

MN, NDL, NN.—These three fields control the orders of Gaussian quadrature, which in special cases may range up to $n = 16$. The single order $n = 6$, however, has been chosen as normal. Any other desired single order is entered in MN (minimum), with NN (number of orders to be used in succession) and NDL (increment) blank. For a system of ideal solenoids alone, MN may be set to 1; but in fact the program supplies this value automatically for each solenoid element.

TRACE uses only one Gaussian order, which is determined by MN. This frees NDL and NN for special uses. NDL controls the spacing of lines in the field map. When NDL = 1, the lines intersect the common line of origin (z = constant or ρ = constant) at equal geometrical intervals, whereas when NDL = 2, they mark off equal increments of the selected field function that is constant along each line. NN controls the number of attempts to trace any given line. When convergence fails at any point on the line, two new tries are made with the interval halved each time if NN > 2, one try if NN = 2, none if NN < 2.

NZ, NA, DLZ, Z1, DLA, A1.—These are defining parameters for the field net or map. On NET cards NZ, DLZ, and Z1 are the number of values, the increment, and the initial value for z, with corresponding A symbols for ρ. If both Z1 and A1 are blank, the net begins at the origin; if all six are blank, it is restricted to the origin.

In the case of MAP cards, assume first that MK = −4, NDL = ( ) or 1. Then NA lines are traced, intersecting the plane z = Z1 at equal geometrical intervals DLA, beginning at A1. Each line is extended by a process of extrapolation and correction, with the points spaced at z-interval DLZ. It ends when the next extrapolated point falls outside a region bounded by axial limits Z1 − 0.6 DLZ, Z1 + (NZ + 3)DLZ, and by ρ limits of the same form, with A substituted for Z. A later section on subroutine TRACE discusses the devices that allow it to follow lines that may curve sharply, may take on infinite or negative values of slope, and ultimately close on themselves. In the above case, z is initially assigned as the independent variable x, which is stepped by increments DLZ while the dependent variable $y = \rho$ is computed. As the trace proceeds, the assignments $x = z$, $y = \rho$ are interchanged as often as necessary. The increment dx = DLZ remains constant. Also for convenience in manual plotting, the x-coordinate, whether z or ρ, is always an integral multiple of DLZ.

If MK = −5, A should be read for Z and conversely, in the preceding paragraph. Thus the initial assignment is $x = \rho$, $y = z$. If NDL = 2, the region to be mapped is defined in the same way as before, but the lines now correspond to equal increments of the selected field function.

The Output Print.—The field net output prints a heading for each value of ρ, while z appears in the second column of each line, followed by B_z, B_ρ, total magnitude $|B|$, the vector potential A, flux ρA, and $\partial B_\rho/\partial z$. The first column shows the maximum order MXN. The last three columns contain monitoring information. M records the maximum number of entries to subroutine JONES for any coil in the system. Normally $M = \Sigma n$, the sum of the Gaussian orders. But when the field point falls in the end zone of any coil or within the winding, though remote from the ends, the coil

is partitioned into from two to four ($= m$) coaxial shells in subroutine FIELD. The M count is then increased to $m\Sigma n$.

The I count is the maximum value, for any coil, of the number of iterations through Eqs. (20), averaged over all the $2m\Sigma n$ Gaussian sources assigned to that coil. At points in an end zone which lie very close to one or more of the equivalent sources, the I count usually shows an increase.

The last column shows the smallest value of c' of Eq. (16), multiplied by 10^6, that has entered the calculations. It measures the closest approach of the field point to any Gaussian source.

Subroutine KCARDS

A set of K cards may describe the geometry of a single system or of a number of subsystems or individual coils. The number of cards is defined by the (+) MK entry on the preceding START card. The first four columns of the K card are ignored by the program and may be used for indexing. Only one card is needed to define either a pair of coils with mirror symmetry or a symmetrical pair with opposed currents, but the program stores and counts the elements separately. Each coil is described by five parameters, to which a common subscript K is assigned, with dimension 40. The number of elements MXK is stored in COMMON as an amended K count.

IS.—IS is uniquely a two-digit field. The units digit is a symmetry indicator: blank for a mirror pair, 1 for a single coil, and 2 for an opposed pair. The tens digit (column 5) is normally blank, separating the A4 field from the numerical data, but it has two special uses.

The entry 1 is used only with thick cylindrical coils (B and D nonzero). It causes the coil to be replaced by the approximately equivalent Lyle filament or pair of filaments[23] in all NET and MAP calculations. This option provides a much faster way to map the fields of special systems, particularly in the design stages. It replaces the Gaussian quadrature of FIELD, which for order n requires at least $2n$ passes through JONES, with its long iterations, by one or two passes through the short series expansions of the FAP subroutine LOOP. The Lyle shortcut for remote coils may be combined with the full normal procedure for nearby coils by punching column 5 on selected K cards only. The precision may be checked in a typical case by computing a system that superposes on one or more thick coils the negative of their Lyle equivalents. When any Lyle replacement has been effected, the entire stored description of the magnetic system is reprinted with the changes that have been made.

The entry 2 in column 5 identifies a magnetostatic dipole ring (p ring), while 3 is used for a magnetic charge or Q ring. In either case, B = D = ( ). There is no special provision for cylinders, but a thin p cylinder with uniform axial polarization is equivalent to a pair of opposed Q rings. If the cylinder is centered on the midplane, it may be entered as such a pair on a single K card with Z0 blank. A symmetric pair of dipole cylinders reduces to four Q rings and requires two cards (Z0 $\neq$ 0.). In either case, IS = 32. In other cases two cards are required for each p cylinder (IS = 31). An idealized thick p cylinder is entered as a current system, since it may be replaced by a pair of ideal solenoids (D = 0.) with the same Z0 and B but with opposed currents. For dipole rings with conical polarization, the polar angle β in degrees is entered in BZ1 (*q.v.*).

Magnetostatic elements of all types, together with current elements, may be included in a single system if the entries in the respective CD data fields for magnetic charge, magnetic moment, and current are expressed in compatible units (see below). Contributions to B_z or B_ρ will then be correctly summed in gauss for the complete system. The vector potential A for axially polarized p rings only (BZ1 = 0.) and the gradient $\partial B_\rho/\partial z$ for Q rings only are also computed and are

[23] T. R. Lyle, *Phil. Mag.* **3**(6), 310 (1902). Lyle's principle may be restated as follows. The magnetic field of a cylindrical coil of moderately small axial breadth B and radial depth D is nearly equal, except at points quite close to the winding, to that of a pair of ideal circular loops, each carrying half the total current. The loops cut the section at points displaced from its effective central filament, in the direction of the longer section dimension, by distances $\pm\delta$, where $\delta^2 = |B^2 - D^2|/12$. The radius of the effective central filament is larger than the geometrical mean radius $A0$, in the ratio $[1 + (1/12)(L/A0)^2]^{1/2}$, where L is the smaller of the dimensions B, D. If $B \geqq D$, the effective radius is just the root-mean-square radius. Evidently, if the section is square, the two filaments coalesce.

This is a second-order principle. That is, if the field is represented by a series expansion, terms containing the factors $(B/A0)^4$ and $(D/A0)^4$ are neglected. The principal error is due to the longer section dimension, and if the ratio B/D or D/B approaches or exceeds 2, the coil should be subdivided.

included in the summation for the system. For all other types of magnetostatic element, however, these two quantities are arbitrarily set to zero.

ZB, AB.—Both fields are blank except on a K card that is the last of the set for a given system, and on which BZ1 is nonzero. In such a case, all the magnetic quantities computed for field nets or maps for that system are scaled in such proportion that the function defined by column 11 of the START card takes on the value BZ1 at the point $z = \mathrm{ZB}$, $\rho = \mathrm{AB}$, for example, at the origin if both fields are blank. Note that these are six-digit fields (FORMAT F6.4).

Z0, A0, B, D.—These parameters, all with subscript K, are respectively the z and ρ coordinates of the central filament of a thick coil, its axial breadth, and radial depth. The combination B = ( ), D = ( ) identifies a circular filament or loop, and B $\neq$ 0, D = ( ) an ideal solenoid. B = (−), D = 0 is a special code that calls for the self-inductance of an ideal solenoid. The program sets B = $|B|$, calls SELF in the FAP section, prints the self-inductance, and then stores the data for the solenoid normally as part of the system.

CD.—Enter the volume current density of a coil, the surface current density of a solenoid, or the total current of a loop. The unit for current is the ampere, but all magnetostatic quantities use absolute electromagnetic units (emu). List in CD the linear magnetic charge density for a Q ring, or the linear, surface, or volume density of magnetic moment for a dipole ring, cylindrical sheet, or thick cylinder. The units correspond, in the case of a thin ring of ideal constant permeability, to the relation CD = $\mu_e B dS$, where B is the total local field, dS is the area of cross section, and μ_e is the effective permeability, taking into account demagnetization. The angle β of the polarization vector is nearly the same as that of the total local field.

The unit of length for all parameters, including linear, surface, or volume densities, is the centimeter if IN is ( ) or (+), the inch if it is (−). Subroutine FORCE does not deal with magnetostatic elements. For the designated pairs of current elements, it prints both force and mutual inductance, but since the first quantity depends on current density and the second on turn density, there is an extraneous factor in one or the other unless the current is unity.

BZ1.—BZ1 has two distinct functions. For normalization of the field values, it is ignored except on the K card whose serial number is MK, that is, on the last card of a set. Here, if there is a nonzero entry in BZ1, it is equated to the value of the particular field function that is defined in column 11 of the START card at the point ZB, AB. This is done by adjusting the current densities so that the values of all field quantities throughout the net are normalized in proportion. Only the first line of print, which gives absolute field values at the reference point, is computed with the original set of current densities.

On any K card (identified by the entry 3 in column 5) that describes a magnetic dipole ring, BZ1 is read as β, the polar angle (in degrees) of the magnetic moment. Only the *exact* entry +1.000° is forbidden, since it would be decoded as a factor for normalization if punched on the last K card of a set, and ignored on any other card. It follows that if a set of K cards must end with a dipole ring, the field values can be normalized only to the value unity at the reference point. To circumvent this trivial limitation, normalize to any value by using a dummy end card which is included in the MK count, but on which CD = ( ).

When MK cards have been processed, the program returns to MP and proceeds as determined by the sign of IP.

Subroutine FORCE. The SEQ Card

FORCE is called before or after KCARDS by a card in MP with IP negative. It reads in one or more sequence (SEQ) cards, calls JONES to compute force and mutual inductance for each of the indicated pairs of elements, and prints the results before returning to MP. This subroutine does not presently accept loop or magnetostatic elements. The SEQ card permits a varied selection of coil pairs or the inclusion of all possible pairs; it prescribes the order or orders of Gaussian quadrature to be used; and it defines the required precision. The 14 fields provide a very flexible set of options, especially if more than one card is used. But the code is so arranged that a completely blank card calls for calculations that include without duplication every possible pairing of coils. If the system has symmetry, some of these combinations are redundant.

IR.—If this field is blank, there is a return to MP when the sequence is completed; an entry in column 9 (units digit) indicates that a new SEQ card is to be read. The tens and hundreds digits are normally blank. They can be used to control print options that have been retained from the testing stage of the program. A nonzero in the tens digit calls out a line of print in FORCE which was used to test the error resulting from very small values of the modulus k. Details are omitted, since they can be read from a listing of the FORTRAN program. From the test it was concluded that k should not be less than about 0.001. This is not normally a significant restriction, since k is twice the ratio of the geometric mean of the radii ρ and a of the field and source rings to the maximum distance r_1 between any two points on them.

Any entry from 1 to 4 in the hundreds digit (column 7) is transferred to IP in COMMON, where it produces a potential avalanche of print in subroutine JONES (*q.v.*). It must be used with caution, usually with a single pair of ideal solenoids and with low Gaussian orders. This digit has been placed for safety in a conspicuous and otherwise unused column.

ND, MN, MXN.—These three fields, in which blanks are read as 1, 3, and 16, set respectively the increment, the minimum, and the maximum for the order of Gaussian quadrature. Calculations for each pair start with order MN and are repeated through order MXN or until the fractional changes of F and M (in parts per million) from one order to the next are so small that the sum of their absolute values is less than the entry in the field PPM. Only the end values are printed, but to monitor the convergence there is a running record of the changes in both F and M in parts per million. The limit MXN is usually redundant, since unless the coils are very close together the orders 4, 5, or even 3, 4 will agree within the "normal" limit PPM = ( ) = 100. If either the K or J element of a pair is an ideal solenoid, the program reduces the Gaussian order to 1 for that element, while ND, MN, and MXN retain their defined values.

NK, KD, K1, and NJ, JD, J1.—These are twin sets used to designate subscripts K and J, both properly K subscripts, that index the locations of stored parameters describing the two coils of a pair. Members of each set are respectively the number of coils in the set, the increment to K or J, and its initial value. Blanks are read as MXK-1, 1, 1 in the K set and MXK-1, 1, 2 in the J set. The J sequence is stepped through for each successive value of K, except that the coding enforces the restrictions K < MXK, J < MXK + 1, and K < J, regardless of what may be punched on the cards. The maximum number of pairs is the product of NK and NJ. It may be reduced by the enforced restrictions, especially when J is stepped by an entry in FJ (*q.v.*).

DJK.—A blank here calls for the matching of the Gaussian orders, at each stage in the calculations, for the K and J sets. That is, if the K coil of any pair is replaced by n fictitious Gaussian solenoids and the J coil by m solenoids, $m = n +$ DJK. It may happen that identical end planes of two coils approach each other closely or may even coincide, as in the calculation of self-inductance or of the mutual inductance and force between two sections of a coil cut by a transverse plane. Such problems cannot be solved with matched orders, since the case of coincident Gaussian solenoids is indeterminate. The entry in DJK should then be (+) and preferably odd, perhaps 3. If it is even, all orders must also be even, since every odd order generates one Gaussian source at $a =$ A0. In case of an oversight, "Singularity" is printed as a warning. A moderate error results and convergence is slow except in the case of coincident ideal solenoids, for which only one order is tested and the result has no meaning. Apart from this necessary use, variation of DJK provides another option for checking critical cases.

FJ.—FJ, if blank, has no effect. If it is (+), the initial subscript J1 for each successive J set is stepped by an increment equal to FJ. For example, with MXK = 8, let NK = 3, KD = 2, K1 = ( ), NJ = 2, JD = ( ), J1 = 6, and FJ = ( ). The KJ combinations are 16, 17, 36, 37, 56, and 57. If FJ = 1., the pairs are changed to 16, 17, 37, 38, and 58.

It can be seen that a blank SEQ card (conveniently labeled SEQ in columns 2—6) calls for the calculation of all the $1/2m(m-1)$ possible pairs in a system of m coils, but without duplication. The entry FJ = 1. would not change the sequence, since it merely repeats the restriction K > J.

Again, assume a system of $2m$ coils with a plane of symmetry or antisymmetry. The number of distinct combinations is reduced from $m(2m-1)$ to m^2. Arrange the coils in memory in the order

1,1′, 2,2′, ... , m,m', where the prime denotes a mirror image and any two pairs such as 13′, 1′3 are equivalent. This order results automatically from a set of m cards, each with IS = ( ) or IS = 2. Evidently the true geometrical sequence is irrelevant. A study of the enforced restrictions and of the conventions for blanks will show that the m^2 distinct combinations are defined without redundancy by a card with the single entry KD = 2.

These two examples illustrate the versatility of even a single SEQ card, when the many options on the card are supplemented by the freedom of ordering the K cards.

PPM.—PPM, if blank, is reset to 100. It is an allowable error limit whose use has already been discussed.

EP.—This is a dummy field.

The Output Print.—Mutual inductance is printed in henries, force in newtons and in tons (2000 lb), together with their fractional changes from each order to the next, the initial order MN, and values of the highest K order n and J order m that were used.

Subroutine FIELD

This subroutine computes B_z, B_ρ, $|B|$ (total), A, ρA, and $\partial B_\rho/\partial z$ for a complete system, at a single field point whose coordinates are z = Z(5), ρ = A1. Elements of the system may be cylindrical coils, ideal solenoids, or loops including Lyle filaments and magnetostatic charge or dipole rings. All types of element may be freely intermixed, with correct summation in the case of B_z and B_ρ. However, the following quantities are computed as zero and do not contribute to the summation over the elements of a system: A and ρA for rings of magnetic charge, $\partial B_\rho/\partial z$ for dipole rings, and all three quantities for dipole rings whose magnetic moment vector is not parallel to the axis. Contributions from loops are computed by calling LOOP in the FAP section, those from other elements by calling JONES. While computing the contributions from an ideal solenoid, the program drops the Gaussian order automatically to 1, ignoring the current value of the order n. Contributions from thick coils normally use the single order MN. If a succession of orders is prescribed by entries in the fields NDL and NN of the MP data cards, any significant changes with increasing order are reported as has been noted.

Either NET or MAP data cards for MP may be punched without regard to whether the windings intrude into the field region.

The coding assigns a field point, with respect to each cylindrical coil in turn, to one of six classes, depending on its location. If it lies outside the winding and the end zones, the coil is treated as a single unit. If it is inside but not in an end zone, the coil is divided by a cylindrical wall passing through the field point. If it lies in an end zone, the coil is divided into 2, 3, or 4 cylindrical shells, depending on the position of the field point with respect to the cylindrical coil boundaries, and each shell is replaced by a separate set of fictitious Gaussian solenoids.

The considerations that require this complication are discussed at the end of Sect. 1. The coding defines an end zone as a cylindrical shell of axial length 2DZ, with inner and outer radii respectively decreased and increased from those of the coil by the increment 2DA, that is bisected by the end plane. The increments may be set on the data cards for MP, through the relations DA = $0.01\rho \times$ ID, DZ = DA $\times$ |IN|. But blanks in ID and IN are read as 2, and punched values less than 2 are ignored. It will rarely be necessary to increase these limits. Though tests revealed small errors at points near an end plane, which were reduced well below one part in a thousand when either ID or |IN| was reset to 5, no detailed analysis of these limits was made. For each coil pair, the subscripts K and J are printed with the mean z-coordinates Z0(K) and Z0(J) of the coils. The average number of iterations through Eqs. (20) is also computed and printed (as I Ave.) for each order, together with the smallest value of the ratio c' of Eq. (16) that was found for any pair of fictitious Gaussian solenoid sources, multiplied by 10^6.

Subroutine TABLES

This short routine was written to test the iterations in JONES and the series expansions in the FAP section by tabulating the elliptic integrals to argument k'^2 and the field components generated by single loop or solenoid sources at points of a net. Comparisons were made with published tables and between values of the same functions as computed by the two subroutines. TABLES is called by a START card with MK = −1, −2, or −3.

It reads in one or more TAB cards of the same format as K cards, but most of the fields are used more nearly as in a NET card for MP.

IR.—A blank causes a return to MP for a new START card after completing the assignment on the current TAB card; IR = (+) calls for a new TAB card.

FNZ, FNA.—These fields show respectively the z and ρ counts, and their product is the number of points in the net. These entries are functionally integers.

Z(2).—Z(2) in COMMON is a loop current or the surface current density of a single ideal solenoid source. If (−), it is decoded as $1/2\pi$, if ( ) as unity, while (+) values are taken literally.

A2.—This entry, which need not be punched if it is unity, is the loop or solenoid radius.

DLZ, Z1, DLA, A1.—These complete the description of the field net, as in a NET card.

MK = −1 on the START card calls JONES, followed by LOOP, and prints two lines of field values and elliptic integrals for each point (one computed by each subroutine). Both routines compute the integrals K and K − E and the solenoid fields B_ρ and $\partial B_\rho/\partial z$. One value is printed in each case directly below the other. MK = −2 calls LOOP only, and it prints a single line. MK = −3 generates arguments for a table of K, K − E, and E by decoding the field net in a special way. The value of z/ρ is computed and is interpreted as k'^2, where k' is the complementary modulus. Here ρ has usually the single value A1 = 100, 1000, etc., while z is stepped by small increments. The entry point to the FAP section is KANDE.

The FAP Section. GAUSS, KANDE, LOOP, SELF

The title lists four entry points to a single FAP subroutine. At the first entry to any of the four, for each loading of the machine, complete sets of abscissas and weight coefficients for Gaussian numerical integration of orders N = 1 through N = 16 are transferred to COMMON as the one-dimensional array G. A range of 2N subscripts, beginning with NG(N), is assigned to order N. The index NG is a separate array. Each abscissa has an odd-numbered subscript and is followed immediately by the corresponding even-numbered weight factor. The first pass through the FAP section sets an internal switch, and GAUSS operates thereafter as a dummy subroutine.

KANDE (K and E) computes the elliptic integrals K, K − E, and E, which it stores in the consecutive cells F(4,5,6). It is entered after supplying the quantity $k'^2 = (r_2/r_1)^2$ to F(7). The values are computed in floating point from the Hastings series approximations.[16] The standard series is used for K, but K − E is computed directly rather than E by using coefficients obtained from the standard K and E series by subtraction. This gives more accurate values of K − E when k^2 and K − E are small, since the errors of the truncated series for K and for E oscillate in phase with each other. E is calculated as K − (K − E).

LOOP computes B_z, B_ρ, A, and $\partial B_\rho/\partial z$ for a field point at $0,\rho$, and a circular loop at $z.a$. The field components are stored consecutively in F(1,2,3), with the elliptic integrals K, K − E, and E in F(4,5,6) as before, the maximum radius r_1 in Z(3), and $\partial B_\rho/\partial z$ in F(8). Before entry, z must be placed in Z(1), ρ in A1, a in A2, and the loop current in Z(2).

SELF calculates the self-inductance of an ideal solenoid. It is entered with the length of the solenoid in Z(1), a in A2, and the square of the linear turn density (CD^2, multiplied by 2.54 when the inch is used) in Z(2). The output value, expressed in henries, is in Z(4).

LOOP and KANDE are completely independent of JONES. LOOP is much faster than JONES for computing any of the eight quantities in the last two rows of Table 1, provided that no quantity in the first two rows is needed. Subroutine FIELD uses LOOP instead of JONES in NET or MAP operations to compute all contributions from loops or from Lyle equivalent loops. This substitution is controlled by the switch B(K) = ( ). The existing code does not compute the limiting case of a pancake coil (B = 0, D > 0) directly, but only by setting B ≪ D. It could of course be done directly by Gaussian integration of field magnitudes computed by LOOP.

Subroutine JONES

This subroutine carries out the iterations (19) and (20) and computes from the end values all the magnetic quantities listed in Table 1, with the exception of B_z(L). To control the number of iterations, one more pass through both (19) and (20) is programmed when α and β differ by less than 10^{-7}, and one more pass through (20)

alone when δ differs from unity by the same amount. The adequacy of this criterion was established algebraically and confirmed by tests. The iterations are counted in M5, from which the average number of iterations is computed in subroutines FIELD and FORCE and printed out for monitoring purposes.

JONES is called once by FIELD for each elementary Gaussian solenoid in NET or MAP operations, and by FORCE for each elementary K − J pair. Input parameters supplied through COMMON are: A1 = ρ, A2 = a, Z(L) = z_L, L1 = 1, 2, or 4. JONES uses Eqs. (14) through (26), of which the last five express the magnetic properties of individual positive sources of unit strength. L1 is the number of passes through JONES at each entry; that is, L1 = 1 for a loop or single solenoid end plane as in TABLES, L1 = 2 for a complete solenoid element as in FIELD, and L1 = 4 in FORCE, where there are four combinations of the end planes for each pair of solenoids. JONES computes the quantities given by Eqs. (22) to (26) for each source or source pair, and stores them in COMMON with L subscripts whose range is L1. The subscripts are as follows: F(S,S), 1 to 4; M(S,S), 5 to 8; F(S,L), 9, 10; $B_\rho(S)$, 11, 12; A(S), 13, 14; $B_z(S)$, 15, 16; $\partial B_\rho/\partial z(S)$, 17, 18.

These values must be multiplied by a current density or by the product of two current densities, by Gaussian weight factors or by a product of two such factors, by constants that appear in the equations, and by others that depend on the chosen units. All these operations are performed in MP or in the calling subroutines.

There is no direct print from this routine unless IP has been set to a positive value (call it L2). The value +L2 calls for the printing of a, ρ, z, r_1, Π − K, k^2, K − E, K, B_ρ, A, B_z, and $\partial B_\rho/\partial z$ (all for solenoid sources), from L = 1 through L = (L1 or L2, the lesser). Since a normal program may require tens of thousands of passes through JONES, the incautious user of this monitoring print option will find himself in the same plight as the Sorcerer's Apprentice. For this reason, though IP is entered by one field of the MP data card, it is used only for other print options and is reset to zero immediately. Only column 7 of the SEQ card, which is well protected from inadvertent use, can call forth this flood; for test purposes, however, it is easy to set IP in any short exploratory program.

Subroutine TRACE

As was stated at the beginning of the section, TRACE exists in a restricted version that maps only lines of force (ρA = constant) and also in a more general form to plot loci of the constant values of B_z, total $|B|$, etc. TRACE is called and supplied with data by a MAP card in MP on which MK is punched −4 or −5. Either routine traces the lines of force if column 11 is blank, but if one of the digits 2–6 is entered here, the general routine maps a different function, as has been previously listed.

At each trial point, TRACE calls FIELD for the local value of the field function. The restricted version calls also for B_z and B_ρ, whose ratio defines the slope of the line and whose values readily yield the derivatives of ρA. The general version is estimated to require from 50 to 70% more time, since it must find the rates of change by actual trial or by extrapolation and interpolation procedures.

Since the calculation of magnetic-field quantities at closely spaced points, even in a simple system, takes far more time than any set of interpolations or extrapolations that may permit the use of a larger interval, a reasonable degree of refinement in the trial-and-error tracing procedure is justified. Moreover, a line is easily lost if it curves sharply or if its slope passes through infinity and reverses sign. Finally, for manual plotting it is a great convenience if one coordinate is always an integral multiple of some simple unit.

TRACE uses x as the independent and y as the dependent variable. Lines are started from points x_0, y_0, where x_0 is constant and y_0 is stepped, and are continued with an interval Δx between points. Usually MK = −4, which sets $x = y$, $y = \rho$ initially, but if MK = −5, the variables are interchanged. This is necessary, for example, when it is desired to map lines of force near a plane of antisymmetry, since $\rho A = 0$ at all points in the plane, and the nearby lines run at nearly 90° to the axis.

In the restricted version, y_1 for the second point on any line of force is first estimated from y_0 and the initial slope B_z/B_ρ. To improve this value, and in all calculations for later points, the derivative $\partial(\rho A)/\partial y$, as computed from B_z or B_ρ, is used. The

general version must use instead a differential quotient of the selected function f, as computed directly from Δy and Δf. Here, for the first trial made on the second point, y_1 is equated to y_0 and Δy to EPS. For subsequent operations an estimate of Δy is always available, and EPS serves only as an error limit. It is determined by IE on the MAP card and is normally 0.001 of the initial interval between lines.

In both programs the first guess is refined by successive approximations, computing each new Δy from the error Δf and the last known derivative or differential quotient. Actually, in the restricted version the rate of convergence is significantly improved by using both rates of change. This gives an indication of curvature. In other words, Δy is estimated at each step from two values each of y and f and a derivative, which means that a parabolic rather than a linear inverse interpolation is made. The iteration count is limited to 6; that is, six tries are allowed to reach a value of Δy that is less than EPS.

As successive firm points are established, the first guess for the next point is made to depend on a 2-, 3-, and eventually a 4-point Lagrangian extrapolation. In the general version only, the values of $\Delta f/\Delta y$ at the latest three points are similarly used to extrapolate an estimate of the derivative. In this case it seems to be necessary to ensure at least one firm value of $\Delta f/\Delta y$ at each point by imposing a lower limit of 2 on the iteration count. Also, both routines enforce an upper limit to the magnitude of Δy equal to the initial value of Δx.

A detailed record is kept of the successive corrections, and these are usually printed for inspection by making a (+) entry in IP. When the interval Δx is not excessive, the average iteration count is only slightly greater than 2 after the first four points, when the general routine is used, and it often falls nearly to 1 in the restricted version. In either case it may rise to 3 or 4 if the interval is large, or in any event when the line curves sharply or crosses a coil boundary. When the count passes 6, if NN = ( ) or 1, the firm points are listed, and tracing of the next line starts. If NN = 2 or 3, one or two new tries are made on the same line, but with the interval Δx halved each time. The line is not retraced from the starting point if there are as many as three firm points. Two new points are first interpolated and improved, midway in the last two intervals, and tracing is resumed with a normal four-point extrapolation by using the new points and the last two from the old set.

Besides the extrapolations that have been described, from the fourth point onward, there is also a three-point extrapolation to estimate the line slope $\partial y/\partial x$. A predicted slope greater than 2 is read as a warning to interchange z and ρ with respect to x and y and to continue by stepping the previously dependent coordinate. The new series must overlap the old, and its independent coordinates are required to be integral multiples of Δx. Its first three points are estimated by three-point inverse interpolation, and they are refined as before by successive approximations. For the refinement process, the general routine requires also an initial estimate of $\partial f/\partial x$ at each of the three points. The notation here preserves the original meanings of x and y, before the turnover. The derivatives are estimated from the relation

$$\partial f/\partial x = -(\partial f/\partial y) \times (\partial y/\partial x).$$

The two factors on the right are found from the inversely interpolated estimate of x, by using respectively a four-point direct function interpolation with remembered values of $\Delta f/\Delta y$ and a four-point direct slope interpolation with remembered values of y. The normal tracing procedure is now resumed, starting with a three-point extrapolation from the three new firm points.

The option NDL = 2 causes lines to be traced at a constant interval of the selected function, rather than with uniform z or ρ spacing. In this case the starting points, NA or NZ in number, are first set up exactly as before, and the function is computed at each point. These figures are used to locate NA-2 or NZ-2 new points that mark off equal increments between the extreme values of the function. These intermediate starting points are established as usual by successive refinements, and NA or NZ lines are then traced through the full set. There is currently no provision for tracing lines with preset values of the function or with prescribed increments to it.

Although methods of interpolation are well known, the appropriate formula is not always at hand. The following equations are used for interpolation or extrapolation, to find the function f_θ at the point $y_\theta = y_0 + \theta h$, where h is the constant

interval, and θ is a proper fraction for interpolation in the positive unit range y_0 to y_1, a negative or improper fraction for extrapolation outside this range.

For four-point full-interval function extrapolation ($\theta = 2$),

$$f_2 = 4f_1 - 6f_0 + 4f_{-1} - f_{-2} \ .$$

Formulas for other orders are obvious, since they use only the binomial coefficients.

For three-point full-interval slope extrapolation ($\theta = 2$),

$$s_2 = {}^1/_2(5f_1 - 8f_0 + 3f_{-1}) \ .$$

For three-point inverse interpolation,[24]

$$\begin{aligned} r &= 2(f_\theta - f_0)/(f_1 - f_{-1}) \ , \\ s &= (2f_0 - f_1 - f_{-1})/(f_1 - f_{-1}) \ , \\ \theta &= r(((5rs + 2)rs + 1)rs + 1) \ . \end{aligned}$$

The last equation has been arranged in the standard form for computing a power series, with the parentheses grouped as in FORTRAN. This short-range series approximation is more convenient than an exact parabolic inverse interpolation for the routine improvement of estimated y-coordinates. But for the initial estimate of the y-intercepts that mark off equal increments of the function to be mapped, it is better to solve the quadratic equation

$$a\theta^2 + b\theta + c = 0 \ ,$$

where

$$\begin{aligned} a &= {}^1/_2(f_0 - 2f_1 + f_2) \ , \\ b &= {}^1/_2(4f_1 - 3f_0 - f_2) \ , \\ c &= f_0 - f_\theta \ . \end{aligned}$$

This procedure works well over a wider range than the preceding method if the higher derivatives of the curve are not too large. The algebraic sign before the radical in the quadratic formula may have to be reversed from one point to the next, to select the physically valid root. In the absence of an obvious test, the sign is arbitrarily set (+). If the resulting estimate places y outside the prescribed range, it is diminished by twice the magnitude of the square root.

For four-point interpolation at fractional interval θ, let $\theta_1 = 1 + \theta$, $\theta_2 = 2 + \theta$, and $\theta_3 = 1 - \theta$. Then

$$6f_\theta = \theta\theta_1\theta_2 f_1 + 3\theta_1\theta_2\theta_3 f_0 - 3\theta\theta_2\theta_3 f_{-1} + \theta\theta_1\theta_3 f_{-2}.$$

When $\theta = 2$, this formula reduces to the full-interval extrapolation above.

For four-point interpolation of the slope f' at fractional interval θ,

$$\begin{aligned} 6hf'_\theta = {} & (3\theta_1^2 - 1)f_1 - (9\theta^2 + 12\theta - 3)f_0 \\ & + (9\theta_1^2 - 12\theta_1 - 3)f_{-1} - (3\theta^2 - 1)f_{-2} \ . \end{aligned}$$

[24] H. E. Salzer, *Natl. Bur. Std. Appl. Math. Ser.* **30** (1949).

Table 2. Guide to COMMON Storage

Dimensions	Begin	Symbol	End	Use[a]
Open end		Q or DY	76 207	Storage for optional print
40	76 210	PH	257	Polar angle of moment vector
40	260	IQ	327	Indicator for magnetostatic element
5	330	L	334	Not presently used
		NDL	335	Switch location for TRACE
		NZ	336	Number of z values in field net
		NA	337	Number of ρ values (or flux lines)
		MXK	340	Number of coils in system
		MXN	341	Maximum Gaussian order
		MN	342	Minimum Gaussian order
		M6	343	Various count and switch operations
		M4	344	
		M3	345	
		M2	346	
		I1	76 347	
40	76 350	CD	417	Current density or turn density
40	420	D	467	Radial depth of coil
40	470	B	537	Axial breadth of coil
40	540	A0	607	Mean radius of coil
40	610	Z0	657	Mean axial coordinate of coil
36	660	FR	723	Mutual inductance values
8	76 724	Z	733	Axial coordinates (relative)
		L1	734	Number of z values per entry to JONES
		M5	735	Various counts
		A2	736	Solenoid radius a
		A1	737	Field point radius ρ
24	76 740	FS	76 767	Force or field values
24	76 770	F	77 017	F, M, or field values, for single source pairs or single sources
272	77 020	G	437	Gaussian coefficients (16 orders)
16	440	NG	457	Indexing for Gaussian coefficients
		IP	460	Print options
		N	77 461	Gaussian order

[a]A number of locations have alternative uses. In particular, results computed in subroutine LOOP are stored in the range Z through F for use in FIELD.

Format, Code, and Card Names	Columns 7–18				
FORMAT 194 (A6, 10I3, 4F9.4)	(9)	(12)	(15)	(18)	
FORCE	( ) (IR)	( ) (ND)	( ) (MN)	( ) (MXN)	
SEQ Card	( ) Return to MP (+) Read new SEQ card (Or print options) (+, column 7) JONES (+, column 8) FORCE	( ) = 1 (+) Increment to Gaussian order	( ) = 3 (+) Minimum Gaussian order	( ) = 16 (+) Maximum Gaussian order	( ) (+)
MAIN PROGRAM	(MK)	(·) (IN)	( ) (IP)	(·) (ID)	
START, NET, MAP, or FORCE Card	−5, −4 MAP card −3, −2, −1 TAB card ( ) FORCE or new NET (+) FORCE or NET (number of K cards) 1 to 4 (column 7)[a]	( ) = +2 (−) Inch unit (+) Centimeter unit |IN| (in FIELD) Ratio DZ/DA 1 to 6 (column 11)[b]	(−) FORCE card (+) Optional print (in MP or TRACE)	( ) = 2 (+) (In FIELD) Ratio DA/A1 in percent	((+
FORMAT 126 (A4, I2, 2F6.4, 6F9.4)	(6)	(8)	(14)		
TABLES	( ) (IR)	(·) FNZ	(·) FNA		
TAB Card	( ) Return to MP (+) New TAB card	(+) Number of z values	(+) Number of ρ values		
KCARDS	( ) (IS)	( ) (ZB)	( ) (AB)		
	( ) Mirror pair (1) Single coil (2) Opposed pair (+) (column 5)[c]	Reference value of z (if BZ1 $\neq$ 0.)	Reference value of ρ (if BZ1 $\neq$ 0.)		

[a]Normalize NET to maximum value of field function if column 7 is 1 or 3, minimum value if it is 2 or 4. Shift NET origin to the point of maxi 7 is 3 or 4.

[b]Column 11 (NET) selects field function for normalization: ( ) or 1,B_z; 2,B_ρ; 3,A; 4,$|B|$; 5,ρA; 6,$\partial B_\rho/\partial z$. [MAP selects same function, except (

[c](1) in column 5 causes a cylindrical coil to be replaced by the "equivalent" Lyle filaments. (2) in column 5 identifies a magnetostatic dipole Q ring. For a dipole ring, enter polar angle in degrees in BZ1.

Table 3. Outline of Data-Card Fields

Format, Code, and Card Names	Columns 7–18			Columns 19–27					Columns 28–36		Columns 37–45	Columns 46–54	Columns 55–63	Columns 64–72
FORMAT 194 (A6, 10I3, 4F9.4)	(9)	(12)	(15)	(18)	(21)	(24)	(27)	(30)	(33)	(36)	(41)	(50)	(59)	(68)
FORCE	( ) (IR)	( ) (ND)	( ) (MN)	( ) (MXN)	(NK)	( ) (KD)	( ) (K1)	(NJ)	( ) (JD)	(J1)	(·) (DJK)	(·) (FJ)	( ) (PPM)	( ) ((EP))
SEQ Card	( ) Return to MP (+) Read new SEQ card (Or print options) (+, column 7) JONES (+, column 8) FORCE	( ) = 1 (+) Increment to Gaussian order	( ) = 3 (+) Minimum Gaussian order	( ) = 16 (+) Maximum Gaussian order	( ) = MXK − 1 (+) Number of coils in K set	( ) = 1 (+) Increment to K	( ) = 1 (+) Minimum value of K	( ) = MXK − 1 (+) Number of coils in J set	( ) = 1 (+) Increment to J	( ) = 2 (+) Minimum value of J	( ) Match Gaussian orders for K and J (+) Mismatch orders	( ) J1 constant (+) Step J1 with K	( ) = 100. (+) Error limit $\times 10^4$	( ) Dummy
MAIN PROGRAM	(MK)	(·) (IN)	( ) (IP)	(·) (ID)	(·) (IE)	( ) (MN)	( ) (NDL)	( ) (NN)	(NZ)	(NA)	(DLZ)	( ) (Z1)	(DLA)	( ) (A1)
START, NET, MAP, or FORCE Card	−5, −4 MAP card −3, −2, −1 TAB card ( ) FORCE or new NET (+) FORCE or NET (number of K cards) 1 to 4 (column 7)[a]	( ) = +2 (−) Inch unit (+) Centimeter unit |IN| (in FIELD) Ratio DZ/DA 1 to 6 (column 11)[b]	(−) FORCE card (+) Optional print (in MP or TRACE)	( ) = 2 (+) (In FIELD) Ratio DA/A1 in percent	( ) = 10 (+) Error limit $\times 10^4$ (NET) $\times 10^4$ (MAP)	( ) = 6 (+) Minimum Gaussian order	( ) = 1 (+) Increment to order (NET) (2) MAP at equal intervals of field function[b]	( ) = 1 (+) Number of orders (NET) Number of tries (MAP)	( ) = 1 (+) Number of z values (points or MAP lines)	( ) = 1 (+) Number of ρ values (points or MAP lines)	(+) Increment to z ( ) Only if NZ = ( )	Initial value of z	(+) Increment to ρ ( ) Only if NA = ( )	(+) Initial value of ρ
FORMAT 126 (A4, I2, 2F6.4, 6F9.4)	(6)	(8)	(14)			(23)			(32)		(41)	(50)	(59)	(68)
TABLES	( ) (IR)	(·) FNZ	(·) FNA			( ) (Z(2))			( ) (A2)		DLZ	( ) (Z1)	DLA	( ) (A1)
TAB Card	( ) Return to MP (+) New TAB card	(+) Number of z values	(+) Number of ρ values			(−) = $1/2\pi$ ( ) = 1. (+) Current density			( ) = 1. (+) Radius of loop		As above	As above	As above	As above
KCARDS	( ) (IS)	( ) (ZB)	( ) (AB)			(Z0)			A0		(B)	(D)	(CD)	( ) (BZ1)
	( ) Mirror pair (1) Single coil (2) Opposed pair (+) (column 5)[c]	Reference value of z (if BZ1 ≠ 0.)	Reference value of ρ (if BZ1 ≠ 0.)			Mean z of section			Mean ρ of section		Axial breadth of coil section B = (−), D = 0. call SELF	Radial depth of coil section	( ) = 1. (±) Magnitude or density of current, turn count, magnetic charge, moment	( ) Do not normalize (±) Set field function to this value at ZB, AB[b] See[c]

[a]Normalize NET to maximum value of field function if column 7 is 1 or 3, minimum value if it is 2 or 4. Shift NET origin to the point of maximum or minimum value if column 7 is 3 or 4.

[b]Column 11 (NET) selects field function for normalization: ( ) or 1, B_z; 2, B_ρ; 3, A; 4, $|B|$; 5, ρA; 6, $\partial B_\rho/\partial z$. [MAP selects same function, except ( ) is decoded as 5].

[c](1) in column 5 causes a cylindrical coil to be replaced by the "equivalent" Lyle filaments. (2) in column 5 identifies a magnetostatic dipole or p ring, (3) a magnetic charge or Q ring. For a dipole ring, enter polar angle in degrees in BZ1.

ORNL-3575
UC-32 – Mathematics and Computers
TID-4500 (38th ed.)

INTERNAL DISTRIBUTION

1. Biology Library
2–4. Central Research Library
5. Reactor Division Library
6–7. ORNL – Y-12 Technical Library
Document Reference Section
8–27. Laboratory Records Department
28. Laboratory Records, ORNL R.C.
29. J. A. Auxier
30. P. R. Bell
31. E. G. Bohlmann
32. W. R. Busing
33. D. L. Coffey
34. C. V. Dodd
35–44. W. F. Gauster
45. R. P. Jernigan
46. Troyce Jones
47. C. E. Larson
48. R. S. Lord
49. Gordon R. Love
50. J. A. Martin
51. H. G. MacPherson
52. S. W. Mosko
53–57. C. E. Parker
58–59. R. B. Parker
60. A. H. Snell
61. M. J. Skinner
62. A. M. Weinberg
63–137. M. W. Garrett (consultant)
138. H. Grad (consultant)
139. H. Heckrotte (consultant)
140. D. J. Rose (consultant)

EXTERNAL DISTRIBUTION

141. R. J. Mackin, Jr., Jet Propulsion Laboratory
142. E. P. Gray, Applied Physics Laboratory, Johns Hopkins University, 8621 Georgia Avenue, Silver Spring, Maryland
143. A. E. Ruark, U.S. Atomic Energy Commission, Washington
144. W. C. Gough, U.S. Atomic Energy Commission, Washington
145. Research and Development Division, AEC, ORO
146–759. Given distribution as shown in TID-4500 (38th ed.) under Mathematics and Computers category (75 copies – CFSTI)