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# ABSTRACT

The location of possibly unstable roots to the dispersion equation for the electrostatic waves in an infinite plasma situated in a constant magnetic field is carefully examined.

NOTE ON ELECTROSTATIC INSTABILITIES IN A PLASMA WITH ANISOTROPIC VELOCITY DISTRIBUTION*

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In this note we wish to extend previous work[1] to delimit as much as possible the location of possibly unstable roots to the dispersion equation for the electrostatic waves in an infinite plasma situated in a constant magnetic field.

The dispersion equation may be written in the form

$$1 + \sum_j \omega_{\rho j}^2 \, \Omega_j^{-2} \, F_j = 0$$

where $F_j$ is given in Ref. 1 for the bi-Maxwellian distribution function.

The theorems given in the previous article, which were based on an examination of Im $(F_i)$, are here extended to show that marginally unstable solutions (Im $(\omega) = 0^+$) of the dispersion equation can occur only if both

$$\ell + \tfrac{1}{2} < \omega < \ell + 1 - \frac{T_{\parallel}}{T_{\perp}} \quad \text{and} \quad \omega < \frac{T_{\perp}}{T_{\parallel}} - 1 \tag{1}$$

where $\ell$ is an integer.

Hence, in particular the system is stable if $\dfrac{T_{\parallel}}{T_{\perp}} > \tfrac{1}{2}$ .

Following the same arguments as in Ref. 1, it is sufficient to show that Im (F) is non-negative in the regime of interest located in the first quadrant of the complex $\omega$-plane, where

$$F = n_{||}^2 F_c + n_\perp^2 F_B = \int_0^\infty dx \; (n_{||}^2 \; x + n_\perp^2 \sin x) \exp\left[i\omega x - \mu x^2 - \lambda (1 - \cos x)\right]$$

and $$\mathrm{Im}\,(F_B) \geq 0 \; .$$

when $\omega = \ell + \epsilon$ is real, $0 \leq \epsilon < 1$,

$$\mathrm{Im}\,(F_c) = A \; e^{-\lambda} \sum_{k=-\infty}^{\infty} g\,(k + \omega) \; I_k\,(\lambda)$$

where $A = \frac{1}{2}\sqrt{\frac{\pi}{\mu}} \; \frac{1}{\lambda} \; \frac{T_{||}}{T_\perp} > 0$ and $g\,(x) = x \exp\,(-\frac{x^2}{4\mu})$ .

Thus, since $I_{k-\ell}\,(\lambda) > I_{k+\ell}\,(\lambda) > 0$ for $k,\; \ell > 0$,

$$\mathrm{Im}\,(F_c) > A \; e^{-\lambda} \left\{ g\,(\epsilon) \; I_\ell\,(\lambda) + \sum_{k=1}^{\infty} \left[ g\,(k + \epsilon) - g\,(k - \epsilon)\right] \; I_{k+\ell}\,(\lambda) \right\} \; .$$

Notice that, when $x > a > 0$, $g\,(x) - g\,(x - a)$ is positive for $x > x_a$, and negative thereafter. Therefore, because $I_k(\lambda)$ is a positive, decreasing function of k, there is an $m \geq 0$ such that, for all k,

$$\left[g\,(k + \epsilon) - g\,(k - \epsilon)\right] \left[I_{k+\ell}\,(\lambda) - I_{m+\ell}\,(\lambda)\right] \geq 0 \; .$$

Therefore

$$\begin{aligned} \mathrm{Im}\,(F_c) &> A \; e^{-\lambda} \left\{ g\,(\epsilon) \; I_\ell\,(\lambda) + \sum_{k=1}^{\infty} \left[g\,(k + \epsilon) - g\,(k - \epsilon)\right] \; I_{m+\ell}\,(\lambda) \right\} \\ &> A \; e^{-\lambda} \; I_{m+\ell}\,(\lambda) \sum_{k=-\infty}^{\infty} g\,(k + \epsilon) \\ &= -\,2\mu \; A \; e^{-\lambda} \; I_{m+\ell}\,(\lambda) \; \frac{d}{d\omega} \left\{ \sum_{k=-\infty}^{\infty} \exp\left[-\,\frac{(k + \omega)^2}{4\mu}\right] \right\} \end{aligned}$$

The infinite sum is just

$$e^{-\omega^2/4\mu}\theta_3\left[-i\omega/4\mu\ ,\ e^{-\frac{1}{4}\mu}\right] = \sqrt{4\pi\mu}\ \ \theta_3\ (\pi\omega,\ e^{-4\pi^2\mu})$$

where $\theta_3\ (z,\ q)$ is a theta-function.[2]

One can show[3]

$$\frac{d}{dz}\left\{\log\ \theta_3\ (z,\ q)\right\} = -\ B\ (z,\ q)\ \sin\ (2\ z)$$

where $B\ (z,\ q)$ is positive if $z,\ q > 0$.

In our case, $z = \pi\omega$, $q = e^{-4\pi^2\mu}$ and $\theta_3$ is positive. Therefore

$$\mathrm{Im}\ (F_c) > D\ \sin\ (2\pi\omega)\ ,\qquad D > 0 \tag{2}$$

which is non-negative whenever $\ell \leq \omega \leq \ell + \frac{1}{2}$ .

Now, it was shown in Ref. 1 that $\mathrm{Im}\ (F_c) \geq 0$ when $\omega = \ell + i\sigma$, and by a completely analogous argument it is also easy to show that $\mathrm{Im}\ (F_c) \geq 0$ when $\omega = \ell + \frac{1}{2} + i\sigma$. Therefore, by Cauchy's theorem $\mathrm{Im}\ (F_c) \geq 0$ whenever $\ell \leq \mathrm{Re}(\omega) \leq \ell + \frac{1}{2}$ .

In order to complete the proof of (1), we write (for real $\omega$)

$$\mathrm{Im}\ (F) = K\frac{T_\perp}{T_{||}}\sum_{k=-\infty}^{\infty}\left[\omega + k\ (1 - \frac{T_{||}}{T_\perp})\right]\ \exp\left[-\ \frac{(\omega + k)^2}{4\mu}\right]\ I_k(\lambda)$$

and restrict our attention to the case $\frac{T_{||}}{T_\perp} < 1$, since $\mathrm{Im}\ (F)$ is obviously positive otherwise. Let $\underset{\sim}{\left[\ x\ \right]}$ denote the largest integer contained in x. Thus, if

$$n = \underset{\sim}{\left[\ \omega\ /\ (1 - \frac{T_{||}}{T_\perp})\ \right]},$$

then $\omega = (1 - \frac{T_{||}}{T_\perp})\ (n + \delta)$, where $0 \leq \delta < 1$ . Hence,

$$\mathrm{Im}\ (F) > K\left(\frac{T_\perp}{T_{||}} - 1\right) \sum_{\ell=1}^{\infty} \ell \left\{ I_{\ell-n}(\lambda) \exp\left\{ - \left[ \ell - n\frac{T_{||}}{T_\perp} + \delta\left(1 - \frac{T_{||}}{T_\perp}\right)\right]^2 / 4\mu \right\} - I_{\ell+n}(\lambda) \exp\left\{ - \left[ \ell + n\frac{T_{||}}{T_\perp} - \delta\left(1 - \frac{T_{||}}{T_\perp}\right)\right]^2 / 4\mu \right\} \right\}$$

and so $\mathrm{Im}\ (F) > 0$ if $n\frac{T_{||}}{T_\perp} > \delta\left(1 - \frac{T_{||}}{T_\perp}\right)$ . Rewriting, this becomes

$$\underset{\sim}{\Big[}\omega / \left(1 - \frac{T_{||}}{T_\perp}\right)\underset{\sim}{\Big]} \geq \omega, \text{ stable.} \tag{3}$$

Now if $\omega = \ell + \epsilon$, then for $1 - \frac{T_{||}}{T_\perp} < \epsilon < 1$

$$\underset{\sim}{\Big[}\omega / \left(1 - \frac{T_{||}}{T_\perp}\right)\underset{\sim}{\Big]} \geq \underset{\sim}{\Big[}\ell / \left(1 - \frac{T_{||}}{T_\perp}\right)\underset{\sim}{\Big]} + 1 \geq \ell + 1 > \omega$$

so that we have stability if $\ell + \left(1 - \frac{T_{||}}{T_\perp}\right) < \omega < \ell + 1$ .

Moreover,

$$0 \leq \underset{\sim}{\Big[}\omega / \left(\frac{T_\perp}{T_{||}} - 1\right)\underset{\sim}{\Big]} \leq \underset{\sim}{\Big[}\omega / \left(1 - \frac{T_{||}}{T_\perp}\right)\underset{\sim}{\Big]} - \underset{\sim}{\Big[}\omega\underset{\sim}{\Big]} < \underset{\sim}{\Big[}\omega / \left(1 - \frac{T_{||}}{T_\perp}\right)\underset{\sim}{\Big]} + 1 - \omega$$

so that when $\underset{\sim}{\Big[}\omega / \left(\frac{T_\perp}{T_{||}} - 1\right)\underset{\sim}{\Big]} \geq 1$, inequality (3) is satisfied, which completes the proof of (1).

In summary, if we refer to Fig. 1, no roots to the dispersion equation can be found for $\omega$ lying in the shaded portion of the complex $\omega$-plane. In addition, the condition of marginal instability $\left[\mathrm{Im}\ (\omega) = 0^+\right]$ is possible only if both $\ell - \frac{1}{2} < \omega < \ell - \frac{T_{||}}{T_\perp}$ and $\omega < \frac{T_\perp}{T_{||}} - 1$

It may be mentioned that these theorems which have been proved explicitly above for the case of bi-Maxwellian distribution of particle velocities,

$$f\,(v_{\parallel}\,,\;v_{\perp}) \sim \exp\left(-\frac{mv_{\parallel}^{2}}{2KT_{\parallel}} - \frac{mv_{\perp}^{2}}{2KT_{\perp}}\right)$$

also hold when the distribution of parallel velocities is Lorenzian, viz.

$$f\,(v_{\parallel}\,,\;v_{\perp}) \sim \left(1 + \frac{mv_{\parallel}^{2}}{2KT_{\parallel}}\right)^{-1} \exp\left(-\frac{mv_{\perp}^{2}}{2KT_{\perp}}\right) \quad .$$

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[1]L. S. Hall and W. Heckrotte, Phys. Rev. A1474, Vol. 134 (1964).

[2]E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge Univ. Press (Cambridge, 1950), 4th ed., pp. 464 and 474.

[3]ibid., p. 489.

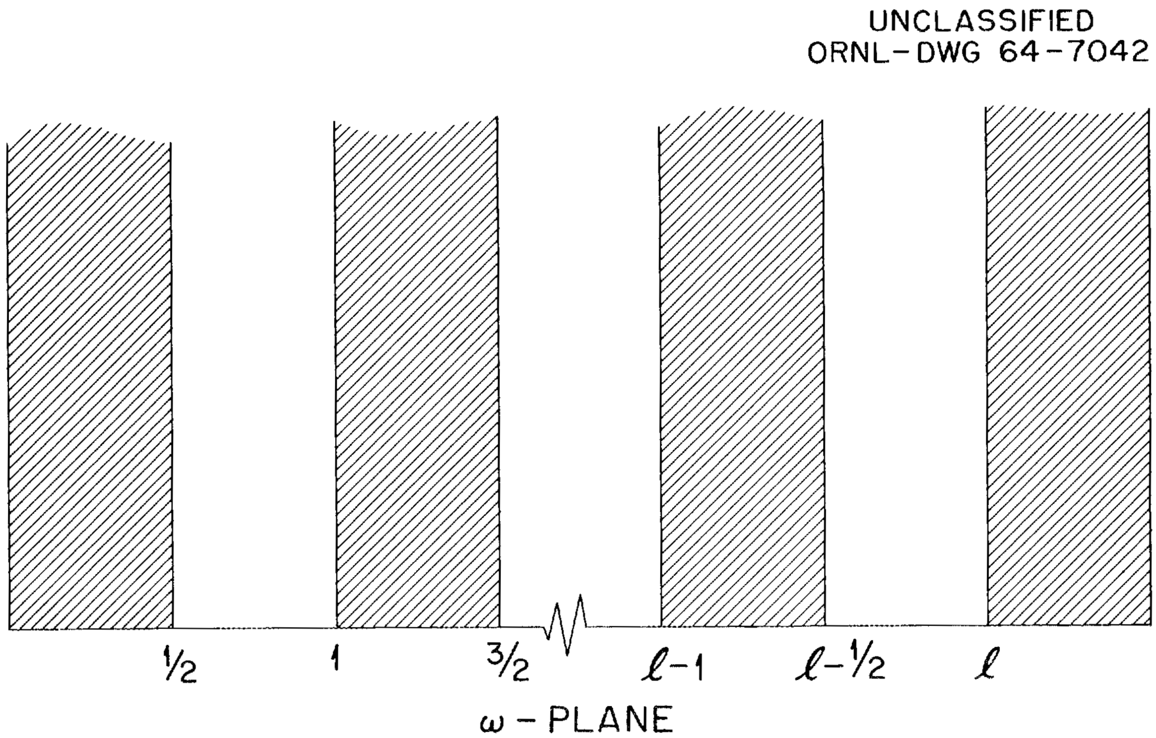


Fig. 1. Regions of Stability (Shaded) and Instability

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