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ANALYSIS OF SIMPLY SUPPORTED GRIDS OF PERPENDICULAR BEAMS

F. J. Witt

OAK RIDGE NATIONAL LABORATORY
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ABSTRACT

An analysis of a rectangular grid of perpendicular beams simply supported at the ends is presented. Bending, shearing, and torsional effects of beams with arbitrary cross sections are taken into account, and the beams may be subjected to uniform or concentrated loads. An energy method is used in the analysis, and a matrix formulation for determining the unknown coefficients appearing in the Fourier series expression for the lateral deflection is given. This formulation is especially suited for computer application, and a computer program for performing the analysis is described. Data are presented for use in the design and evaluation of certain grid configurations. Applications to nonrectangular grids are also considered.

NOMENCLATURE*

A	Cross-sectional area of a beam
a_{mn}	Series coefficients, where m, n = 1, 2, ...
B	External work of uniform loads
C	External work of concentrated loads
E	Young's modulus of elasticity
V_T	Total energy of the grid
G	Shear modulus of the material
g	Number of beams perpendicular to the y axis of the grid
h	Number of beams perpendicular to the x axis of the grid
I	Moment of inertia
J	Torsional factor dependent on the cross section (polar moment of inertia if the cross section is circular)
L_I	Common length of the beams perpendicular to the x axis
L_J	Common length of the beams perpendicular to the y axis
M	Bending moment
P_k	Concentrated load (positive downward) located at (x_{ik}, y_{jk})
q	Uniform load (positive downward)
T	Torsional strain energy
$t = \frac{GJ}{EI}$	Torsional constant
$S = \frac{\alpha}{2GA}$	Shearing constant
$s = \frac{\alpha EI}{2GA}$	Shearing constant
U	Shearing force
R	Bending strain energy
w	Deflection, positive downward
x_i	x coordinate of ith beam perpendicular to x axis
y_j	y coordinate of jth beam perpendicular to y axis
Z	Shearing strain energy in combination with bending strain energy

*Additional definitions are given as needed in the discussion of the matrix formulation.

α	Cross-sectional constant
θ	Angle of twist of a segment

Subscripts

xi, i	ith beam perpendicular to x axis
yj, j	jth beam perpendicular to y axis
T	Total strain energy
yij	Segment $[(x_i, y_j), (x_{i+1}, y_j)]$
xij	Segment $[(x_i, y_j), (x_i, y_{j+1})]$

1. INTRODUCTION

With the advent of gas-cooled, graphite-moderated reactors another important use was found for the already widely used beam-grillage support system. This type of supporting structure is useful in reactors with vertical fuel channels because it does not adversely affect the flow of coolant and it covers a minimum of the lower surface of the active core. A beam-grillage arrangement which is supported at its outer edge and covered by a plate for transmitting loads to the beams is commonly used. Such a structure offers the advantages of a low depth requirement and increased accessibility to the reactor face, as compared with a truss arrangement, for example. These considerations are of great importance when the fuel is loaded and unloaded from the bottom of the reactor. Because of the exacting nature of the design analysis for this application, new impetus was given to developing methods for analyzing these structures.

An analysis for an array of perpendicular beams arranged to form a structure that is rectangular in plane form is presented in this report. Only beams with simply supported ends are considered, but the beam cross sections may be of any shape. The bending, shearing, and torsional effects are taken into account. The beams may be subjected to distributed or concentrated loads or both. An energy method is used in the analysis and the lateral deflection is represented by a Fourier series with unknown coefficients. The values of these coefficients depend upon the geometry of the grid, the properties of the material, and the loadings to which the grid is subjected.

An expression is derived for the total energy in a grid system as a result of combinations of bending, shearing, and torsional effects and of external loadings. Upon minimizing the total energy of the grid, a linear simultaneous system of equations results; a formulation is given for determining the matrix elements of this system. A simplified analysis is given for rectangular grids with identical and equally spaced beams. The analysis may also be applied to grid systems in which the beams in a given direction are unequal in length.

An IBM 704 computer program for making the analysis outlined in this report was written. It is described briefly in Appendix A. Through the use of this program, almost any rectangular grid of perpendicular beams may be analyzed.

Generalized analyses were made for square structures with equally spaced beams that are subjected to uniformly distributed (or simply uniform) loads, and data for use in the design of this type of structure are presented. The data given are for bending alone, but the shearing and torsional effects may be included through the use of specially prepared curves, which give multiplicative constants to apply to the bending results. Finally, one section of this report is devoted to nonrectangular grids and other grid applications.

2. TOTAL ENERGY FORMULAS

A typical structure to which the equations derived herein apply consists of the rectangular grid of perpendicular beams shown in Fig. 1. The strain energy method used in the analysis presented consists of deriving expressions for the total energy of the system in terms of unknown coefficients. These coefficients are then determined by minimizing the total energy [1].* The unknown coefficients, a_{mn}, to be evaluated appear in the expression for the lateral deflection, w, which is taken to have the following series form:

$$w = \sum_{n}\sum_{m} a_{mn} \sin \frac{n\pi x}{L_J} \sin \frac{m\pi y}{L_I} , \tag{1}$$

where the coordinates and L_J and L_I are defined in Fig. 1. This expression yields deflections and moments that are zero at the end, and, hence, it satisfies the boundary conditions for simply supported beams. The series can be made to represent any deflection curve to which this analysis may be applied with a degree of accuracy that depends upon the number of terms of the series taken. Only odd values of m and n are required for a rectangular grid possessing load and geometrical symmetry

*Brackets refer to numbered references in the Bibliography.

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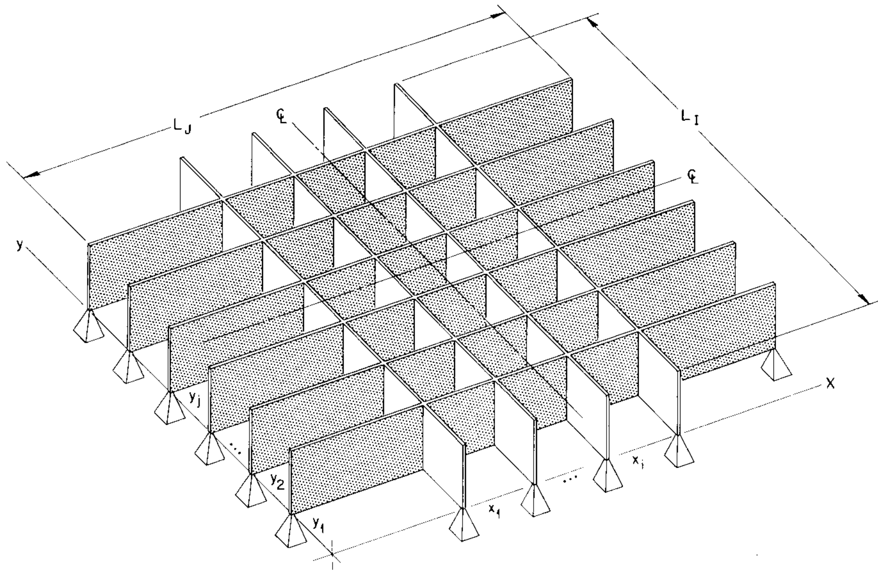

Fig. 1. Coordinate System of Typical Grid Structure.

about the center lines, while both odd and even values are required for the nonsymmetrical case.

The total energy of the system is given by

$$V_T = Z_T + T_T + B_T + C_T \ , \tag{2}$$

where

Z_T = total shearing strain energy in combination with bending strain energy,

T_T = total torsional strain energy,

B_T = total external work of uniform loads,

C_T = total external work of concentrated loads.

If the shearing strain energy is neglected, Z_T in Eq. 2 is replaced by R_T, the total bending strain energy.

Each term on the right-hand side of Eq. 2 may be expressed in terms of the deflection as given by Eq. 1. Thus, Eq. 2 gives an expression for the total energy of the structure in terms of the coefficients, a_{mn}. The expression for the total energy is minimized by taking the partial derivative with respect to each a_{mn} and setting each expression thus obtained equal to zero. This gives a set of simultaneous linear equations in the coefficients, a_{mn}, the solution of which provides an explicit expression for the deflection. The expressions for the terms on the right-hand side of Eq. 2 are listed below, and the derivations are given in the next section.

1. Total Bending Strain Energy.

$$R_T = \frac{\pi^4}{4}\left[\frac{1}{L_J^3}\sum_j (EI)_{yj}\sum_n\left(\sum_m a_{mn}\sin\frac{m\pi y_j}{L_I}\right)^2 n^4 + \frac{1}{L_I^3}\sum_i (EI)_{xi}\sum_m\left(\sum_n a_{mn}\sin\frac{n\pi x_i}{L_J}\right)^2 m^4\right]. \quad (3)$$

2. Total Shearing Strain Energy in Combination With Bending Strain Energy.

$$Z_T = \frac{\pi^4}{4}\left[\frac{1}{L_J}\sum_j (EI)_{yj}\sum_n \frac{n^4}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2}\left(\sum_m a_{mn}\sin\frac{m\pi y_j}{L_I}\right)^2 + \frac{1}{L_I}\sum_i (EI)_{xi}\sum_m \frac{m^4}{2S_{xi}(EI)_{xi}m^2\pi^2 + L_I^2}\left(\sum_n a_{mn}\sin\frac{n\pi x_i}{L_J}\right)^2\right]. \quad (4)$$

3. Total Torsional Strain Energy.

$$T_T = \frac{\pi^2}{2}\left[\frac{1}{L_I^2}\sum_j (GJ)_{yj}\sum_{i=0}^{h}\frac{1}{x_{i+1} - x_i}\left\{\sum_m m\cos\frac{m\pi y_j}{L_I}\left[\sum_n a_{mn}\left(\sin\frac{n\pi x_{i+1}}{L_J} - \sin\frac{n\pi x_i}{L_J}\right)\right]\right\}^2 + \frac{1}{L_J^2}\sum_i (GJ)_{xi}\sum_{j=0}^{g}\frac{1}{y_{j+1} - y_j}\left\{\sum_n n\cos\frac{n\pi x_i}{L_J}\left[\sum_m a_{mn}\left(\sin\frac{m\pi y_{j+1}}{L_I} - \sin\frac{m\pi y_j}{L_I}\right)\right]\right\}^2\right]. \quad (5)$$

4. Total External Work of Uniform Loads.

$$B_T = -\frac{2}{\pi}\left[L_J \sum_j q_{yj} \sum_m \left(\sum_n a_{mn}\frac{1}{n}\right) \sin\frac{m\pi y_j}{L_I}\right.$$

$$\left. + L_I \sum_i q_{xi} \sum_n \left(\sum_m a_{mn}\frac{1}{m}\right) \sin\frac{n\pi x_i}{L_J}\right] . \qquad (6)$$

5. Total External Work of Concentrated Loads.

$$C_T = -\sum_k P_k \sum_n\sum_m a_{mn} \sin\frac{n\pi x_{ik}}{L_J} \sin\frac{m\pi y_{jk}}{L_I} . \qquad (7)$$

3. ENERGY TERMS

The derivations for the terms R_T, Z_T, T_T, B_T, and C_T are given in this section. The coordinate system and nomenclature shown in Fig. 1 are used in the analysis.

First, the expression for the total bending strain energy, R_T, is derived. If R_{yj} and R_{xi} are the bending strain energies along the jth and ith beams, respectively, then

$$R_{yj} = \frac{(EI)_{yj}}{2}\int_0^{L_J}\left(\frac{d^2w}{dx^2}\right)^2 dx \qquad \text{at } y = y_j , \qquad (8)$$

and

$$R_{xi} = \frac{(EI)_{xi}}{2}\int_0^{L_I}\left(\frac{d^2w}{dy^2}\right)^2 dy \qquad \text{at } x = x_i . \qquad (9)$$

The lateral deflection, w, is given by Eq. 1, and it follows that

$$\frac{dw}{dx} = \frac{\pi}{L_J}\left[\sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y}{L_I}\right) n \cos \frac{n\pi x}{L_J}\right] , \tag{10}$$

and

$$\frac{d^2w}{dx^2} = \frac{-\pi^2}{L_J^2}\left[\sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y}{L_I}\right) n^2 \sin \frac{n\pi x}{L_J}\right]. \tag{11}$$

Similarly,

$$\frac{d^2w}{dy^2} = \frac{-\pi^2}{L_I^2}\left[\sum_m \left(\sum_n a_{mn} \sin \frac{n\pi x}{L_J}\right) m^2 \sin \frac{m\pi y}{L_I}\right]. \tag{12}$$

Thus,

$$\left(\frac{d^2w}{dx^2}\right)^2 = \frac{\pi^4}{L_J^4}\left[\sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y}{L_I}\right)^2 n^4 \sin^2 \frac{n\pi x}{L_J} + (\text{cross product terms})\right] , \tag{13}$$

and

$$\left(\frac{d^2w}{dy^2}\right)^2 = \frac{\pi^4}{L_I^4}\left[\sum_m \left(\sum_n a_{mn} \sin \frac{n\pi x}{L_J}\right)^2 m^4 \sin^2 \frac{m\pi y}{L_I} + (\text{cross product terms})\right] . \tag{14}$$

The cross product terms of Eq. 13 are of the form

$$k \sin \frac{n\pi x}{L_J} \sin \frac{m\pi x}{L_I} ,$$

where k is a function of the summation containing y, m, or n; also $m \neq n$. Since

$$\int_0^{L_J} \sin^2 ax \, dx = \frac{L_J}{2} , \tag{15}$$

and

$$\int_0^{L_J} \sin ax \sin bx \, dx = 0, \qquad a \neq b , \tag{16}$$

then

$$R_{yj} = \frac{(EI)_{yj}\pi^4}{2L_J^4} \int_0^{L_J} \left[\sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y_j}{L_I} \right)^2 n^4 \sin \frac{n\pi x}{L_J} \right.$$

$$\left. + \text{(cross product terms)} \right] dx$$

$$= \frac{(EI)_{yj}\pi^4}{4L_J^3} \sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y_j}{L_I} \right)^2 n^4 . \tag{17}$$

Similarly,

$$R_{xi} = \frac{(EI)_{xi}\pi^4}{4L_I^3} \sum_m \left(\sum_n a_{mn} \sin \frac{n\pi x_i}{L_J} \right)^2 m^4 . \tag{18}$$

Hence, the strain energy from bending is

$$R_T = \sum_j R_{yj} + \sum_i R_{xi} , \tag{19}$$

where the summations $\sum_j$ and $\sum_i$ are summed over all the beams perpendicular to the y and x axes, respectively. Thus,

$$R_T = \frac{\pi^4}{4} \left[\frac{1}{L_J^3} \sum_j (EI)_{yj} \sum_n \left(\sum_m a_{mn} \sin \frac{m\pi y_j}{L_I} \right)^2 n^4 \right.$$

$$\left. + \frac{1}{L_I^3} \sum_i (EI)_{xi} \sum_m \left(\sum_n a_{mn} \sin \frac{n\pi x_i}{L_J} \right)^2 m^4 \right] . \tag{20}$$

If both bending and shearing are taken into account, then the following relations must hold for the jth beam [2]:

$$\frac{d^2w}{dx^2} = -\frac{M_{yj}}{(EI)_{yj}} + 2S_{yj}\frac{dU_{yj}}{dx}, \tag{21}$$

and

$$\frac{dM_{yj}}{dx} = U_{yj}, \tag{22}$$

where S_{yj} is a constant, the value of which is discussed later.

Similar conditions hold for the beams in the y direction. Eliminating U_{yj} from Eq. 21 and substituting for w yields

$$2S_{yj}\frac{d^2M_{yj}}{dx^2} - \frac{M_{yj}}{(EI)_{yj}} = -\frac{\pi^2}{L_J^2}\sum_n\sum_m n^2 a_{mn} \sin\frac{n\pi x}{L_J} \sin\frac{m\pi y}{L_I}, \tag{23}$$

which has for a general solution

$$M_{yj} = C_1 \exp\frac{x}{\sqrt{2S_{yj}(EI)_{yj}}} + C_2 \exp -\frac{x}{\sqrt{2S_{yj}(EI)_{yj}}}$$

$$+ \pi^2(EI)_{yj}\sum_m\sum_n \frac{n^2}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2} a_{mn} \sin\frac{n\pi x}{L_J} \sin\frac{m\pi y_j}{L_I}. \tag{24}$$

The simple-support boundary conditions require $C_1 = C_2 = 0$. Thus,

$$M_{yj} = \pi^2(EI)_{yj}\sum_m\sum_n \frac{n^2}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2} a_{mn} \sin\frac{n\pi x}{L_J} \sin\frac{m\pi y_j}{L_I}, \tag{25}$$

and

$$U_{yj} = \frac{\pi^3(EI)_{yj}}{L_J} \sum_m \sum_n \frac{n^3}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2} a_{mn} \cos \frac{n\pi x}{L_J} \sin \frac{m\pi y_j}{L_I} . \tag{26}$$

The strain energy, Z_{yj}, in the jth beam is

$$Z_{yj} = \frac{1}{2(EI)_{yj}} \int_0^{L_J} M_{yj}^2 \, dx + S_{yj} \int_0^{L_J} U_{yj}^2 \, dx , \tag{27}$$

which, proceeding as for the bending analysis, yields

$$Z_{yj} = \frac{\pi^4(EI)_{yj}}{4L_J} \sum_n \frac{n^4}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2} \left(\sum_m a_{mn} \sin \frac{m\pi y_j}{L_I} \right)^2 . \tag{28}$$

Similarly,

$$Z_{xi} = \frac{\pi^4(EI)_{xi}}{4L_I} \sum_m \frac{m^4}{2S_{xi}(EI)_{xi}m^2\pi^2 + L_I^2} \left(\sum_n a_{mn} \sin \frac{n\pi x_i}{L_J} \right)^2 . \tag{29}$$

The strain energy, Z_T, resulting from bending and shear is

$$Z_T = \sum_j Z_{yj} + \sum_i Z_{xi} , \tag{30}$$

which yields

$$Z_T = \frac{\pi^4}{4} \left[\frac{1}{L_J} \sum_j (EI)_{yj} \sum_n \frac{n^4}{2S_{yj}(EI)_{yj}n^2\pi^2 + L_J^2} \left(\sum_m a_{mn} \sin \frac{m\pi y_j}{L_I} \right)^2 \right.$$

$$\left. + \frac{1}{L_I} \sum_i (EI)_{xi} \sum_m \frac{m^4}{2S_{xi}(EI)_{xi}m^2\pi^2 + L_I^2} \left(\sum_n a_{mn} \sin \frac{n\pi x_i}{L_J} \right)^2 \right] . \tag{31}$$

For each beam the shearing constant, S, is of the form

$$S = \frac{\alpha}{2GA} , \tag{32}$$

where α depends on the cross section [2]. The constant, S, which may vary from beam to beam, can be expressed as a function of EI; however, it is more convenient to replace S by $\frac{s}{EI}$ so that

$$s = \frac{\alpha EI}{2GA} . \tag{33}$$

It is this expression for the shearing constant that is an input parameter to the computer program described in Appendix A.

In the following discussion, $x_o = y_o = 0$, $x_{h+1} = L_J$, $y_{g+1} = L_I$. The torsional strain energy, T_{yij}, of the segment $[(x_i, y_j), (x_{i+1}, y_j)]$ of the jth beam, as shown in Fig. 2, is [3]

$$T_{yij} = \frac{(GJ)_{yj}\theta^2_{yij}}{2(x_{i+1} - x_i)} , \tag{34}$$

where

$$\theta_{yij} = \frac{dw(x_{i+1}, y_j)}{dy} - \frac{dw(x_i, y_j)}{dy} . \tag{35}$$

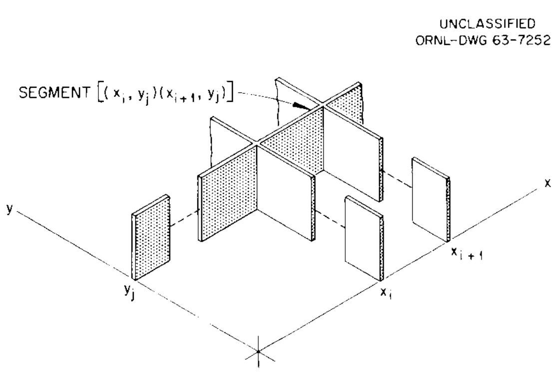

Fig. 2. Typical Segment Used in Torsional Analysis.

Similarly, the torsional strain energy, T_{xij}, of the segment $[(x_i,y_j),(x_i,y_{j+1})]$ of the ith beam is

$$T_{xij} = \frac{(GJ)_{xi}\theta^2_{xij}}{2(y_{j+1} - y_j)} , \tag{36}$$

where

$$\theta_{xij} = \frac{dw(x_i,y_{j+1})}{dx} - \frac{dw(x_i,y_j)}{dx} . \tag{37}$$

Thus, the torsional strain energy, T_j, of the jth beam is

$$T_j = \frac{(GJ)_{yj}}{2} \sum_{i=0}^{h} \frac{\left[\frac{dw(x_{i+1},y_j)}{dy} - \frac{dw(x_i,y_j)}{dy}\right]^2}{x_{i+1} - x_i} . \tag{38}$$

Similarly, for the ith beam,

$$T_i = \frac{(GJ)_{xi}}{2} \sum_{j=0}^{g} \frac{\left[\frac{dw(x_i,y_{j+1})}{dx} - \frac{dw(x_i,y_j)}{dx}\right]^2}{y_{j+1} - y_j} . \tag{39}$$

Hence, the total torsional strain energy of the system is

$$T_T = \frac{1}{2} \left\{ \sum_j (GJ)_{yj} \sum_{i=0}^{h} \frac{\left[\frac{dw(x_{i+1},y_j)}{dy} - \frac{dw(x_i,y_j)}{dy}\right]^2}{x_{i+1} - x_i} + \sum_i (GJ)_{xi} \sum_{j=0}^{g} \frac{\left[\frac{dw(x_i,y_{j+1})}{dx} - \frac{dw(x_i,y_j)}{dx}\right]^2}{y_{j+1} - y_j} \right\} . \tag{40}$$

By substituting for $\frac{dw}{dx}$ and $\frac{dw}{dy}$ it follows that

$$T_T = \frac{\pi^2}{2}\left[\frac{1}{L_I^2}\sum_j (GJ)_{yj}\sum_{i=0}^{h}\frac{1}{x_{i+1}-x_i}\left\{\sum_m m\cos\frac{m\pi y_j}{L_I}\left[\sum_n a_{mn}\left(\sin\frac{n\pi x_{i+1}}{L_J}-\sin\frac{n\pi x_i}{L_J}\right)\right]\right\}^2 + \frac{1}{L_J^2}\sum_i (GJ)_{xi}\sum_{j=0}^{g}\frac{1}{y_{j+1}-y_j}\left\{\sum_n n\cos\frac{n\pi x_i}{L_J}\left[\sum_m a_{mn}\left(\sin\frac{m\pi y_{j+1}}{L_I}-\sin\frac{m\pi y_j}{L_I}\right)\right]\right\}^2\right] . \tag{41}$$

As for the shearing analysis, a constant, t, defined by

$$t = \frac{GJ}{EI} , \tag{42}$$

is introduced as an input parameter for the aforementioned computer program.

The external work, B_{yj} and B_{xi}, caused by uniform loads of q_{yj} and q_{xi} on the jth and ith beams, respectively, are

$$B_{yj} = -q_{yj}\int_0^{L_J} w\, dx \tag{43}$$

and

$$B_{xi} = -q_{xi}\int_0^{L_I} w\, dy . \tag{44}$$

Thus,

$$B_{yj} = -q_{yj}\int_0^{L_J}\sum_m\sum_n a_{mn}\sin\frac{m\pi y_j}{L_I}\sin\frac{n\pi x}{L_J}\, dx$$

$$= -\frac{2q_{yj}L_J}{\pi}\sum_m\left(\sum_n a_{mn}\frac{1}{n}\right)\sin\frac{m\pi y_j}{L_I} , \tag{45}$$

for n an odd integer only; the integral is zero for even integer values of n. Similarly,

$$B_{xi} = -\frac{2q_{xi}L_I}{\pi}\sum_n\left(\sum_m a_{mn}\frac{1}{m}\right)\sin\frac{n\pi x_i}{L_J}\ , \tag{46}$$

for m an odd integer only. Thus, the external work, B_T, for all beams is

$$B_T = \sum_j B_{yj} + \sum_i B_{xi} = -\frac{2}{\pi}\left[L_J\sum_j q_{yj}\sum_m\left(\sum_n a_{mn}\frac{1}{n}\right)\sin\frac{m\pi y_j}{L_I}\right.$$

$$\left.+ L_I\sum_i q_{xi}\sum_n\left(\sum_m a_{mn}\frac{1}{m}\right)\sin\frac{n\pi x_i}{L_J}\right], \tag{47}$$

where n takes on odd values only in the first summation and m takes on odd values only in the second.

If P_k is a concentrated load at (x_{ik},y_{jk}) on a beam, then the external work, C_k, developed by this load is

$$C_k = -P_k w \qquad \text{at } (x_{ik},y_{jk})\ . \tag{48}$$

Thus,

$$C_k = -P_k\sum_n\sum_m a_{mn}\sin\frac{n\pi x_{ik}}{L_J}\sin\frac{m\pi y_{jk}}{L_I}\ . \tag{49}$$

Hence, the external work, C_T, resulting from concentrated loads is

$$C_T = \sum_k C_k\ , \tag{50}$$

or

$$C_T = -\sum_k P_k\sum_n\sum_m a_{mn}\sin\frac{n\pi x_{ik}}{L_J}\sin\frac{m\pi y_{jk}}{L_I}\ . \tag{51}$$

4. MATRIX FORMULATIONS OF THE TOTAL ENERGY FORMULAS

In this section, procedures are described for obtaining the system of equations derived by minimizing the total energy [1]. From Eq. 2, if

$$\frac{\partial V_T}{\partial a_{ij}} = 0 \ , \tag{52}$$

a linear simultaneous system of equations of order pq is obtained, where p is the maximum value of m and q is the maximum value of n for the series coefficients. The strain energies give rise to the coefficients of the unknowns, while the external work energies produce the constants. The system of equations is given in matrix form and the matrix or vector formulation for each energy is considered separately, the exception being that the combined energies of bending and shearing are considered as one. The sequence in taking partial derivatives and arranging the unknowns is a_{11}, a_{21}, ..., a_{p1}, a_{12}, a_{22}, ..., a_{p2}, ..., a_{1q}, ..., a_{pq}, and each equation (i.e., matrix and vector) is divided by $\pi^4/2$.

The final matrix equation is of the form

$$Ax = b \ , \tag{53}$$

where A is the total energy matrix depending on the form of Eq. 2 for the case under consideration, x is the coefficient vector of the a_{mn}'s, and b is the loading vector. The arrangements of the elements of A, x, and b are determined as indicated above. The arrangements for the individual energies are discussed below.

If the total energy due to bending alone is considered, then from Eq. 3,

$$\frac{2}{\pi^4}\frac{\partial R_T}{\partial a_{cd}} = \frac{d^4}{L_J^3}\sum_j (EI)_{yj} \sin\frac{c\pi y_j}{L_I}\sum_m a_{md} \sin\frac{m\pi y_j}{L_I}$$

$$+ \frac{c^4}{L_I^3}\sum_i (EI)_{xi} \sin\frac{d\pi x_i}{L_J}\sum_n a_{cn} \sin\frac{n\pi x_i}{L_J} \ , \tag{54}$$

where c and d are particular values for m and n. If R is the matrix derived from the bending analysis, it is of order pq and its elements may be determined from Eq. 54. A simple way to determine R is to partition it into submatrices in the following manner:

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1q} \\ R_{21} & \cdots & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & R_{qq} \end{bmatrix} \tag{55}$$

and then determine the elements of the submatrices. Each submatrix is of the form

$$R_{k\ell} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & \cdots & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & r_{pp} \end{bmatrix} . \tag{56}$$

The elements r_{rs} of submatrix $R_{k\ell}$ may be determined by careful inspection of the sequence of taking partial derivatives and arranging the unknowns described above and then relating this analysis to Eq. 54. It is to be noted that in Eq. 55 the subscripts run from 1 to q; in Eq. 56, from 1 to p. One result of this inspection is that for any k, the elements, r_{rs}, of $R_{k\ell}$, ℓ going from 1 to q, are the coefficients determined by taking the partial derivative of a_{rk}, r going from 1 to p. From Eq. 54,

$$\frac{2}{\pi^4}\frac{\partial R_T}{\partial a_{rk}} = \frac{k^4}{L_J^3}\sum_j (EI)_{yj} \sin\frac{r\pi y_j}{L_I}\sum_m a_{mk} \sin\frac{m\pi y_j}{L_I}$$

$$+ \frac{r^4}{L_I^3}\sum_i (EI)_{xi} \sin\frac{k\pi x_i}{L_J}\sum_n a_{rn} \sin\frac{n\pi x_i}{L_J} . \tag{57}$$

The coefficients in the right-hand side of Eq. 57 may now be related to the matrix elements. All the coefficients of the first summation are elements of $R_{k\ell}$ where $k = \ell$, while only one coefficient of the second summation is an element of $R_{k\ell}$ where $k = \ell$ (this is the case of $n = k$). For a fixed r the coefficient of a_{rk} is an element on the main diagonal of $R_{k\ell}$ where $k = \ell$. This means that if $r = s$ and $k = \ell$, then r_{rs} is the coefficient of $a_{r\ell}$ for a fixed r. Hence, from Eq. 57,

$$r_{rs} = \frac{k^4}{L_J^3} \sum_j (EI)_{yj} \sin^2 \frac{r\pi y_j}{L_I} + \frac{r^4}{L_I^3} \sum_i (EI)_{xi} \sin^2 \frac{k\pi x_i}{L_J} . \tag{58}$$

If $r \neq s$, $k = \ell$, and r is fixed, then r_{rs} is the coefficient of a_{sk}; thus, from Eq. 57,

$$r_{rs} = \frac{k^4}{L_J^3} \sum_j (EI)_{yj} \sin \frac{r\pi y_j}{L_I} \sin \frac{s\pi y_j}{L_I} . \tag{59}$$

As r and s go from 1 to p and k and ℓ from 1 to q, all the elements of $R_{k\ell}$ for $k = \ell$ are determined by Eqs. 58 and 59.

It remains to determine the elements of $R_{k\ell}$ for k and ℓ going from 1 to q, except $k \neq \ell$. The only nonzero elements of $R_{k\ell}$, $k \neq \ell$, arise from the terms which are unaccounted for above in the second summation of Eq. 57. Only these will be given further consideration. As before, r_{rs} is taken as an element of $R_{k\ell}$ for ℓ going from 1 to q, except $k \neq \ell$. The coefficient of a_{rn} is an element of the main diagonal of R_{kn} for n going from 1 to q. Thus, from Eq. 57 for $\ell = n$ and $r = s$,

$$r_{rs} = \frac{r^4}{L_I^3} \sum_i (EI)_{xi} \sin \frac{k\pi x_i}{L_J} \sin \frac{\ell\pi x_i}{L_J} . \tag{60}$$

This accounts for all the terms in Eq. 57. Thus, for the case $k \neq \ell$ and $r \neq s$,

$$r_{rs} = 0 . \tag{61}$$

In summary the elements, r_{rs}, of $R_{k\ell}$ may be determined as follows: If $k = \ell$, then, for $r = s$,

$$r_{rs} = \frac{k^4}{L_J^3} \sum_j (EI)_{yj} \sin^2 \frac{r\pi y_j}{L_I} + \frac{r^4}{L_I^3} \sum_i (EI)_{xi} \sin^2 \frac{k\pi x_i}{L_J} \; ; \tag{62}$$

for $r \neq s$,

$$r_{rs} = \frac{k^4}{L_J^3} \sum_j (EI)_{yj} \sin \frac{r\pi y_j}{L_I} \sin \frac{s\pi y_j}{L_I} \; . \tag{63}$$

If $k \neq \ell$, then, for $r = s$,

$$r_{rs} = \frac{r^4}{L_I^3} \sum_i (EI)_{xi} \sin \frac{k\pi x_i}{L_J} \sin \frac{\ell\pi x_i}{L_J} \; ; \tag{64}$$

for $r \neq s$,

$$r_{rs} = 0 \; . \tag{65}$$

If both bending and shearing strain energies are considered, then, from Eq. 4,

$$\frac{2}{\pi^4} \frac{\partial Z_T}{\partial a_{cd}} = \frac{d^4}{L_J} \sum_j \frac{(EI)_{yj}}{2s_{yj}\pi^2 d^2 + L_J^2} \sin \frac{c\pi y_j}{L_I} \sum_m a_{md} \sin \frac{m\pi y_j}{L_I}$$

$$+ \frac{c^4}{L_I} \sum_i \frac{(EI)_{xi}}{2s_{xi}\pi^2 c^2 + L_I^2} \sin \frac{d\pi x}{L_J} \sum_n a_{cn} \sin \frac{n\pi x_i}{L_J} \; , \tag{66}$$

where $S_{yj}(EI)_{yj} = s_{yj}$, etc. If the bending and shearing matrix, Z, is set up similar to the bending matrix, then the element z_{rs} of matrix $Z_{k\ell}$ is found as follows: If $k = \ell$, then, for $r = s$,

$$z_{rs} = \frac{k^4}{L_J} \sum_j \frac{(EI)_{yj}}{2s_{yj}\pi^2 k^2 + L_J^2} \sin^2 \frac{r\pi y_j}{L_I} + \frac{r^4}{L_I} \sum_j \frac{(EI)_{xi}}{2s_{xi}\pi^2 r^2 + L_I^2} \sin^2 \frac{k\pi x_i}{L_J} \; ; \tag{67}$$

for $r \neq s$,

$$z_{rs} = \frac{k^4}{L_J} \sum_j \frac{(EI)_{yj}}{2s_{yj}\pi^2 k^2 + L_J^2} \sin \frac{r\pi y_j}{L_I} \sin \frac{s\pi y_j}{L_I} \; . \tag{68}$$

If $k \neq \ell$, then, for $r = s$,

$$z_{rs} = \frac{r^4}{L_I} \sum_i \frac{(EI)_{xi}}{2s_{xi}\pi^2 r^2 + L_I^2} \sin \frac{k\pi x_i}{L_J} \sin \frac{\ell\pi x_i}{L_J} \; ; \tag{69}$$

for $r \neq s$,

$$z_{rs} = 0 \; . \tag{70}$$

If the torsional strain energy alone is considered, then, from Eq. 5,

$$\frac{2}{\pi^4} \frac{\partial T_T}{\partial a_{cd}} = \frac{2}{\pi^2} \left\{ \frac{c}{L_I^2} \sum_j (GJ)_{yj} \cos \frac{c\pi y_j}{L_I} \left[\sum_{i=0}^{h} \frac{1}{x_{i+1} - x_i} \left(\sin \frac{d\pi x_{i+1}}{L_J} - \sin \frac{d\pi x_i}{L_J} \right) \sum_m m \cos \frac{m\pi y_j}{L_I} \sum_n a_{mn} \left(\sin \frac{n\pi x_{i+1}}{L_J} - \sin \frac{n\pi x_i}{L_J} \right) \right] \right.$$
$$\left. + \frac{d}{L_J^2} \sum_i (GJ)_{xi} \cos \frac{d\pi x_i}{L_J} \left[\sum_{j=0}^{g} \frac{1}{y_{j+1} - y_j} \left(\sin \frac{c\pi y_{j+1}}{L_I} - \sin \frac{c\pi y_j}{L_I} \right) \sum_n n \cos \frac{n\pi x_i}{L_J} \sum_m a_{mn} \left(\sin \frac{m\pi y_{j+1}}{L_I} - \sin \frac{m\pi y_j}{L_I} \right) \right] \right\} \; . \tag{71}$$

If the torsional matrix, T, is set up similar to the bending matrix, then the element t_{rs} of matrix $T_{k\ell}$ is

$$t_{rs} = \frac{2}{\pi^2} \left\{ \frac{sr}{L_I^2} \sum_j (GJ)_{yj} \cos \frac{r\pi y_j}{L_I} \cos \frac{s\pi y_j}{L_I} \left[\sum_{i=0}^{h} \frac{1}{x_{i+1} - x_i} \left(\sin \frac{k\pi x_{i+1}}{L_J} - \sin \frac{k\pi x_i}{L_J} \right) \left(\sin \frac{\ell\pi x_{i+1}}{L_J} - \sin \frac{\ell\pi x_i}{L_J} \right) \right] + \frac{\ell k}{L_J^2} \sum_i (GJ)_{xi} \cos \frac{k\pi x_i}{L_J} \cos \frac{\ell\pi x_i}{L_J} \left[\sum_{j=0}^{g} \frac{1}{y_{j+1} - y_j} \left(\sin \frac{r\pi y_{j+1}}{L_I} - \sin \frac{r\pi y_j}{L_I} \right) \left(\sin \frac{s\pi y_{j+1}}{L_I} - \sin \frac{s\pi y_j}{L_I} \right) \right] \right\} . \qquad (72)$$

The uniform load vector may also be determined in a manner similar to that for the bending matrix. From Eq. 6

$$-\frac{2}{\pi^4} \frac{\partial B_T}{\partial a_{cd}} = \frac{4}{\pi^5} \left(\frac{L_J}{d} \sum_j q_{yj} \sin \frac{c\pi y_j}{L_I} + \frac{L_I}{c} \sum_i q_{xi} \sin \frac{d\pi x_i}{L_J} \right) . \qquad (73)$$

If B is the uniform load vector and

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_q \end{bmatrix} , \qquad (74)$$

where the elements B_ℓ of B are

$$B_\ell = \begin{bmatrix} b_{1\ell} \\ b_{2\ell} \\ \cdot \\ \cdot \\ \cdot \\ b_{p\ell} \end{bmatrix} , \tag{75}$$

then

$$b_{r\ell} = \frac{4}{\pi^5} \left(\frac{L_J}{\ell} \sum_j q_{yj} \sin \frac{r\pi y_j}{L_I} + \frac{L_I}{r} \sum_i q_{xi} \sin \frac{\ell \pi x_i}{L_J} \right) , \tag{76}$$

where the first summation is zero if ℓ is even and the second is zero if r is even.

It follows from Eq. 7 that for concentrated loads,

$$- \frac{2}{\pi^4} \frac{\partial C_T}{\partial a_{cd}} = \frac{2}{\pi^4} \sum_k P_k \sin \frac{d\pi x_{ik}}{L_J} \sin \frac{c\pi y_{jk}}{L_I} . \tag{77}$$

If C is the concentrated load vector and is set up similar to the uniform load vector, then $c_{r\ell}$ of C_ℓ is

$$c_{r\ell} = \frac{2}{\pi^4} \sum_k P_k \sin \frac{\ell \pi x_{ik}}{L_J} \sin \frac{r\pi y_{jk}}{L_I} . \tag{78}$$

The above analysis holds for both symmetrical and nonsymmetrical deflections about the center lines. For geometrical and load symmetry, however, only the odd-numbered values of m and n are necessary. If the sequence in taking partial derivatives is a_{11}, a_{31}, a_{51}, ..., a_{31}, a_{33}, ... and each equation is divided by $\pi^4/2$, the formulas presented are valid when k is replaced by $2k - 1$, ℓ by $2\ell - 1$, r by $2r - 1$, and s by $2s - 1$, with the row-column matrix element designation remaining unchanged.

5. ANALYSES OF EQUALLY SPACED SYMMETRICALLY LOADED GRID SYSTEMS

One of the more common configurations of symmetrical, rectangular beam grillages is one in which the beams in a given direction are equally spaced and uniformly loaded (hence, only odd values of m and n are required). It is also assumed that all the beams are identical, except for length. For structures of this type the analysis given in the previous section can be greatly simplified. The following trignometric relationships are used for this purpose. The expression to be evaluated is:

$$\sum_{i=1}^{p} \sin \frac{a\pi i}{p+1} \sin \frac{b\pi i}{p+1} , \tag{79}$$

where a and b are odd integers. If S is the value of expression 79, then the desired relationships are as follows [4]:

1. If $a \neq b$, and $a + b$ and $|a - b|$ are simultaneously multiples or nonmultiples of $2(p + 1)$, then

$$S = 0 . \tag{80}$$

2. If $a = b$ and a is a multiple of $p + 1$, then

$$S = 0 . \tag{81}$$

3. If $a = b$ and a is not a multiple of $p + 1$, then

$$S = \frac{p+1}{2} . \tag{82}$$

The matrix elements derived in the previous section may now be reformalized; in particular, Eq. 57 may be written as

$$\frac{2}{\pi^4} \frac{\partial R_T}{\partial a_{rk}} = EI \left(\frac{k^4}{L_J^3} \sum_{j=1}^{h} \sin \frac{r\pi j}{h+1} \sum_{m} a_{mk} \sin \frac{n\pi j}{h+1} \right.$$

$$\left. + \frac{r^4}{L_I^3} \sum_{i=1}^{g} \sin \frac{k\pi i}{g+1} \sum_{n} a_{rn} \sin \frac{n\pi i}{g+1} \right) . \tag{83}$$

From Eqs. 80 through 82 the elements, r_{rs}, of $R_{k\ell}$ may be determined.

Since only a few of the a_{ij}'s are sufficient for most applications and the number of beams is relatively great, $(m + n)$ is usually less than $2(g + 1)$; hence, Eqs. 63 and 64 are zero (i.e., for most applications the only nonzero elements of the bending matrix are on the main diagonal). Also, m and n are usually less than $g + 1$. In these cases, if $k = \ell$ and $r = s$ and r and k are not multiples of $h + 1$ and $g + 1$, respectively, Eq. 62 becomes

$$r_{rs} = \frac{EI}{2L_I^3}\left[\left(\frac{L_I}{L_J}\right)^3 k^4(1 + h) + r^4(1 + g)\right] . \tag{84}$$

If r and k are simultaneously multiples of $h + 1$ and $g + 1$, respectively, then Eq. 62 is also zero, and a singular system of equations results. Should such a case occur, an acceptable analysis may be obtained either by reducing the number of coefficients to be determined or, at least, omitting that particular coefficient. For uniform loading on the beams $(q_{yj} = q_{xi} = q)$, Eq. 76 may be reduced to

$$b_{rk} = \frac{4L_I q}{\pi^5}\left(\frac{1 + \cos\frac{k\pi}{1 + g}}{r \sin\frac{k\pi}{1 + g}} + \frac{L_J}{L_I}\,\frac{1 + \cos\frac{r\pi}{1 + h}}{k \sin\frac{r\pi}{1 + h}}\right) . \tag{85}$$

Thus from Eqs. 84 and 85 the coefficients may be found directly. It is of note that the factors in parentheses in Eqs. 84 and 85 are dimensionless. Also by the method of arranging the unknowns previously described, r_{rr} of R_{kk} is the matrix coefficient of a_{rk}. Thus,

$$\frac{a_{rk}EI}{qL_I^4} = \frac{8}{\pi^5}\left[\frac{\dfrac{1 + \cos\frac{k\pi}{1 + g}}{r \sin\frac{k\pi}{1 + g}} + \dfrac{L_J}{L_I}\,\dfrac{1 + \cos\frac{r\pi}{1 + h}}{k \sin\frac{r\pi}{1 + h}}}{\left(\dfrac{L_I}{L_J}\right)^3 k^4(1 + h) + r^4(1 + g)}\right] . \tag{86}$$

The combined bending and shearing coefficients may also be determined as above. If $k = \ell$, then for $r = s$ and r and k not simultaneously multiples of $p + 1$, Eq. 67 becomes

$$z_{rs} = \frac{EI}{2L_I^3}\left[\frac{L_I^3 k^4(1 + h)}{L_J(2s\pi^2k^2 + L_J^2} + \frac{L_I^2 r^4(1 + g)}{2s\pi^2r^2 + L_I^2}\right] . \tag{87}$$

The other conditions are the same as those given for the bending analysis above. From Eqs. 85 and 87 the coefficients in dimensionless form for a bending and shearing analysis are

$$\frac{a_{rk}EI}{qL_I^4} = \frac{8}{\pi^5}\left[\frac{\dfrac{1 + \cos k\pi}{r \sin \dfrac{k\pi}{1 + g}} + \dfrac{L_J}{L_I}\,\dfrac{1 + \cos \dfrac{r\pi}{1 + h}}{k \sin \dfrac{r\pi}{1 + h}}}{\dfrac{L_I^3 k^4(1 + h)}{L_J(2s\pi^2k^2 + L_J^2)} + \dfrac{L_I^2 r^4(1 + g)}{2s\pi^2r^2 + L_I^2}}\right] . \tag{88}$$

In the case of square grids ($L_I = L_J$), the above equations become even simpler. The elements of the torsional matrix may also be simplified considerably; however, there are nonzero elements off the main diagonal, and, as a result, the exact solution of the system of equations is not readily available.

Since shear and torsional effects are often negligible, it is usually only necessary to make a bending analysis. However, the secondary shearing and torsional effects may become significant in certain configurations. For square grids under uniform load the first coefficient, a_{11}, is dominant, and the first term alone may be used to determine these effects. For rectangular grids in general, other coefficients may become significant; the investigations of these secondary effects are more difficult.

Using the computer program described in Appendix A, analyses were made for square grids ($L_I = L_J = L$) with beams equally spaced and alike in both directions. The grids were assumed to be uniformly loaded, and results were obtained for up to 20 beams in a given direction. Four terms in the series were used, except where g equals 1 or 2.* The systems for these cases are given in Appendix B. Results in dimensionless form for bending analyses are given in Table 1. Both the deflections and the coefficients were normalized by the factor qL^4/EI; the normalizing factor for the moments was qL^2. Only the maximum deflections and moments are given, but the series coefficients may be used to determine these quantities at any location.

The normalized maximum deflections and moments from Table 1 are plotted versus the number of beams in Fig. 3, and Fig. 4 gives the dimensionless variation of the maximum deflections and of the maximum moments with the number of beams for the case in which the total load on each grid is 1 lb.

If shear is taken into account, the ratio of the first bending and shearing coefficient to the first bending coefficient is

$$\frac{a_{11}(\text{bending and shearing})}{a_{11}(\text{bending})} = 1 + \frac{2\pi^2 s}{L^2} . \tag{89}$$

Since for square grids the first coefficient is the dominating one, the effect of shear should vary little from that given by this ratio. A graph of s/L^2 versus the ratio of the maximum bending and shearing deflection to the maximum bending deflection is given in Fig. 5. The right-hand side of Eq. 89 is also plotted on Fig. 5. It may be seen that the simple expression given above for the shearing effect is adequate. For the cases considered there is less than 4% variation of the bending and shearing moment from the bending moment (that is, the consideration of shearing affects only the deflection).

It is of note that the results from page 172 of ref. 2 for a single beam may be compared directly with the bending and shearing results of

*Since the maximum value for m or n is then 3, the only case in which the bending or bending and shearing matrices have nonzero off-diagonal elements is when $g = 1$. This case is equivalent to the case for $g = 2$. For $g > 2$, the diagonal elements of the matrix taken in order are $g + 1$, $41(g + 1)$, $41(g + 1)$, and $81(g + 1)$.

Table 1. Dimensionless Results from Bending Analyses of Uniformly Loaded Square Grids

g	Normalized Vector Elements			Normalized Coefficients			Normalized Maximum Deflection	Normalized Maximum Moment
	b_{11}	$b_{13} = b_{31}$	b_{33}	a_{11}	$a_{13} = a_{31}$	a_{33}		
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-4}$	$\times 10^{-5}$	$\times 10^{-2}$	$\times 10^{-1}$
1	2.6142	−0.8714		1.3125	0.5379		1.3017	1.2423
2	4.5279	0.7547		1.5093	0.6211		1.3017	1.2423
3	6.3113	1.5933	0.3609	1.5778	0.9715	1.1140	1.5595	1.4713
4	8.0457	2.2906	0.6331	1.6091	1.1174	1.5632	1.5141	1.4177
5	9.7564	2.9333	0.8714	1.6261	1.1923	1.7928	1.6040	1.5031
6	11.454	3.5481	1.0928	1.6362	1.2362	1.9270	1.5750	1.4712
7	13.143	4.1466	1.3041	1.6428	1.2642	2.0126	1.6195	1.5145
8	14.826	4.7351	1.5094	1.6473	1.2832	2.0703	1.6003	1.4939
9	16.505	5.3163	1.7102	1.6505	1.2966	2.1114	1.6267	1.5198
10	18.182	5.8924	1.9080	1.6529	1.3065	2.1417	1.6132	1.5055
11	19.857	6.4651	2.1037	1.6547	1.3140	2.1643	1.6306	1.5227
12	21.530	7.0348	2.2976	1.6561	1.3198	2.1821	1.6207	1.5122
13	23.202	7.6026	2.4904	1.6573	1.3245	2.1959	1.6330	1.5244
14	24.873	8.1683	2.6819	1.6582	1.3282	2.2074	1.6253	1.5164
15	26.543	8.7327	2.8726	1.6589	1.3312	2.2165	1.6345	1.5256
16	28.212	9.2961	3.0627	1.6595	1.3337	2.2241	1.6285	1.5193
17	29.881	9.8584	3.2522	1.6600	1.3358	2.2305	1.6355	1.5264
18	31.549	10.420	3.4411	1.6605	1.3376	2.2359	1.6307	1.5213
19	33.217	10.981	3.6297	1.6608	1.3391	2.2405	1.6363	1.5269
20	34.884	11.541	3.8174	1.6611	1.3404	2.2445	1.6323	1.5228

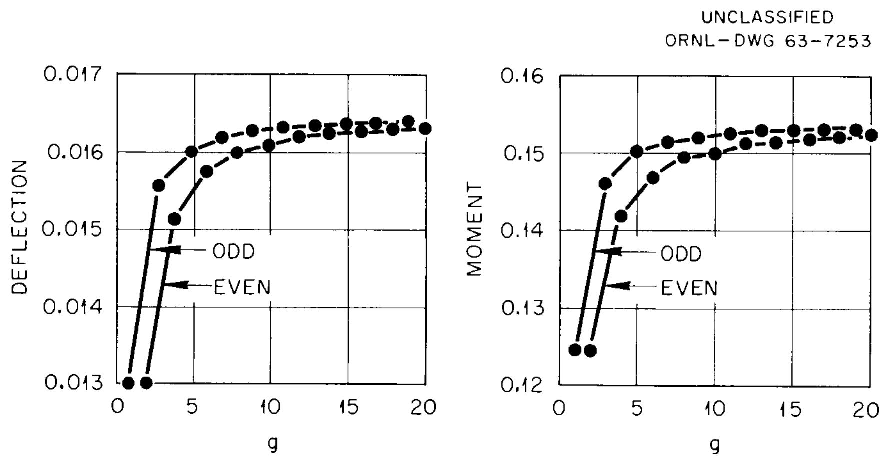

Fig. 3. Dimensionless Variation of Deflections and Moments with the Number of Beams for Uniformly Loaded Square Grids.

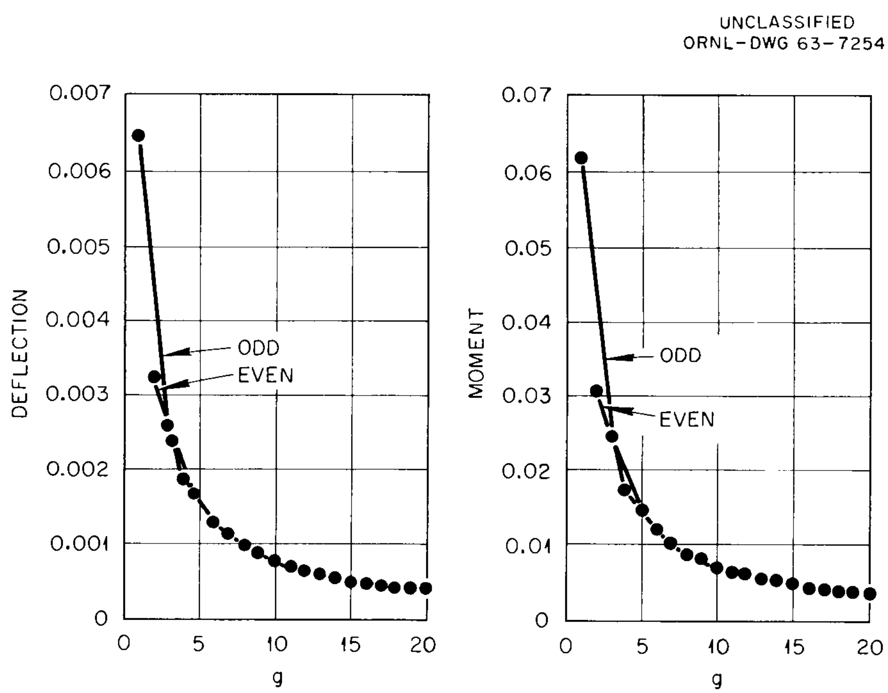

Fig. 4. Dimensionless Variation of the Deflections and Moments with the Number of Beams for Grids Uniformly Loaded to 1 lb.

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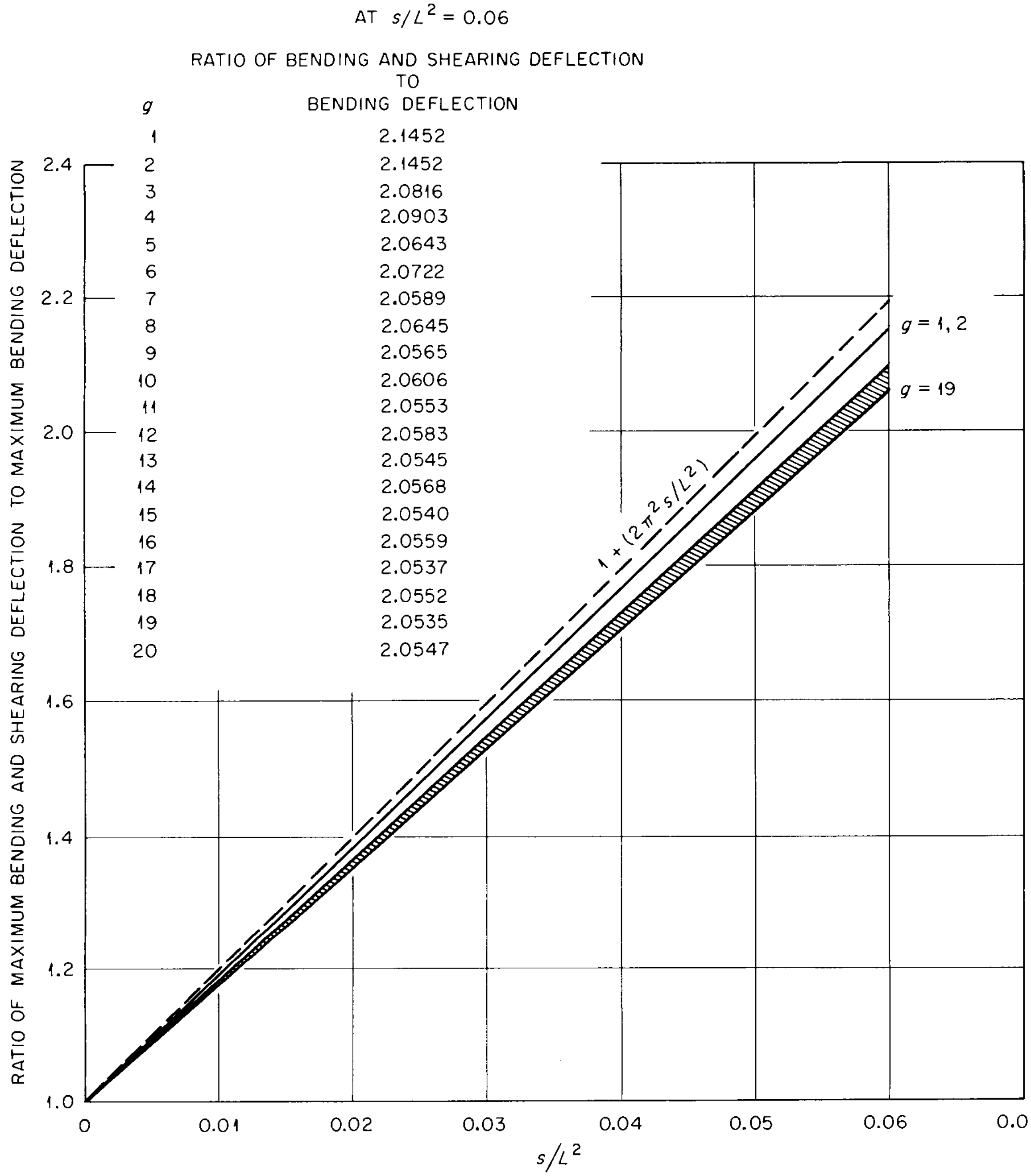

Fig. 5. Shearing Effects on Deflections of Uniformly Loaded Square Grids.

the analysis of this report where g = 1. The bending results for the deflection from the reference is 5/384, which is within 0.03% of the result (0.013017) for g = 1 in Table 1. If $h^2/L^2 = 0.615385$, then s = 0.1, and the deflection from the combined bending and shearing analysis from the reference is 2.92, as compared with 2.9087 from the analysis presented here. These results agree within 0.4%.

In considering the torsional effects, the expression

$$\frac{1}{x_{i+1} - x_i}$$

is a constant and

$$\frac{1}{x_{i+1} - x_i} = \frac{1}{L/(g + 1)} = \frac{g + 1}{L} \ . \tag{90}$$

If the torsional matrix is multiplied by L^3/EI, the constant factor $t(g + 1)$ remains. It follows that the ratio of bending and torsional deflection to bending deflection is a function of $t(g + 1)$. This function for the maximum deflection is plotted in Fig. 6 with g as a parameter. In Fig. 7 the corresponding curves for the maximum moment are given.

6. ANALYSES OF NONRECTANGULAR GRIDS AND OTHER APPLICATIONS

Nonrectangular grids of perpendicular beams may also be analyzed by the method described previously. This is done by dividing the grid of interest into a series of rectangular grids and analyzing each grid for its particular loading. The grids may then be "fastened together" by imposing concentrated loads at the intersections common to two grids. The concentrated load at a point on one grid will be in a direction opposite to that of the corresponding point on the other grid. Since the concentrated loads are unknown, the deflection for a concentrated load of 1 lb (or −1 lb, as the case may be) at each common intersection may be determined. This in turn gives the deflection at any point of the grid in terms of the unknown concentrated loads. After this is done for

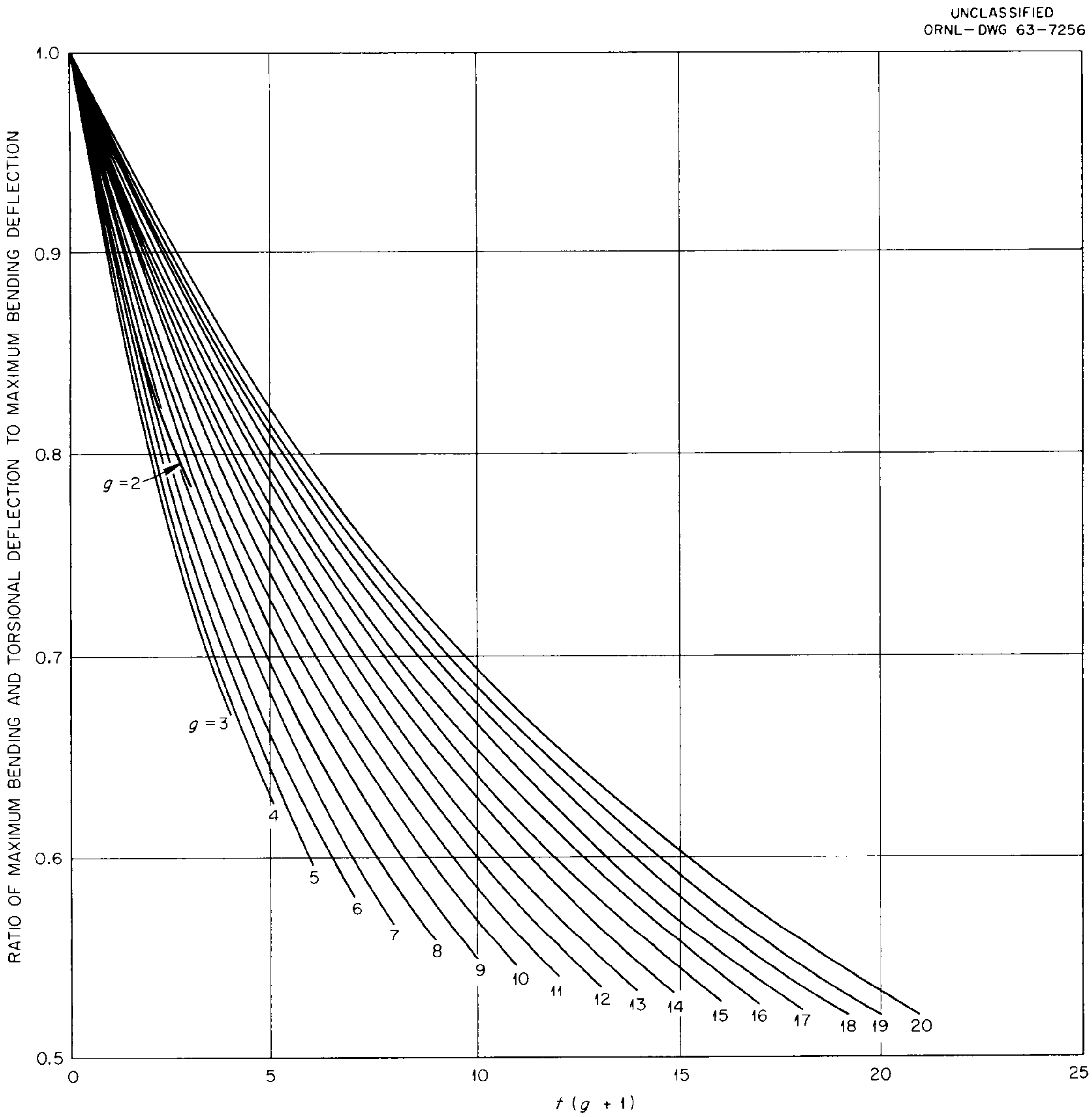

Fig. 6. Torsional Effects on Deflections of Uniformly Loaded Square Grids.

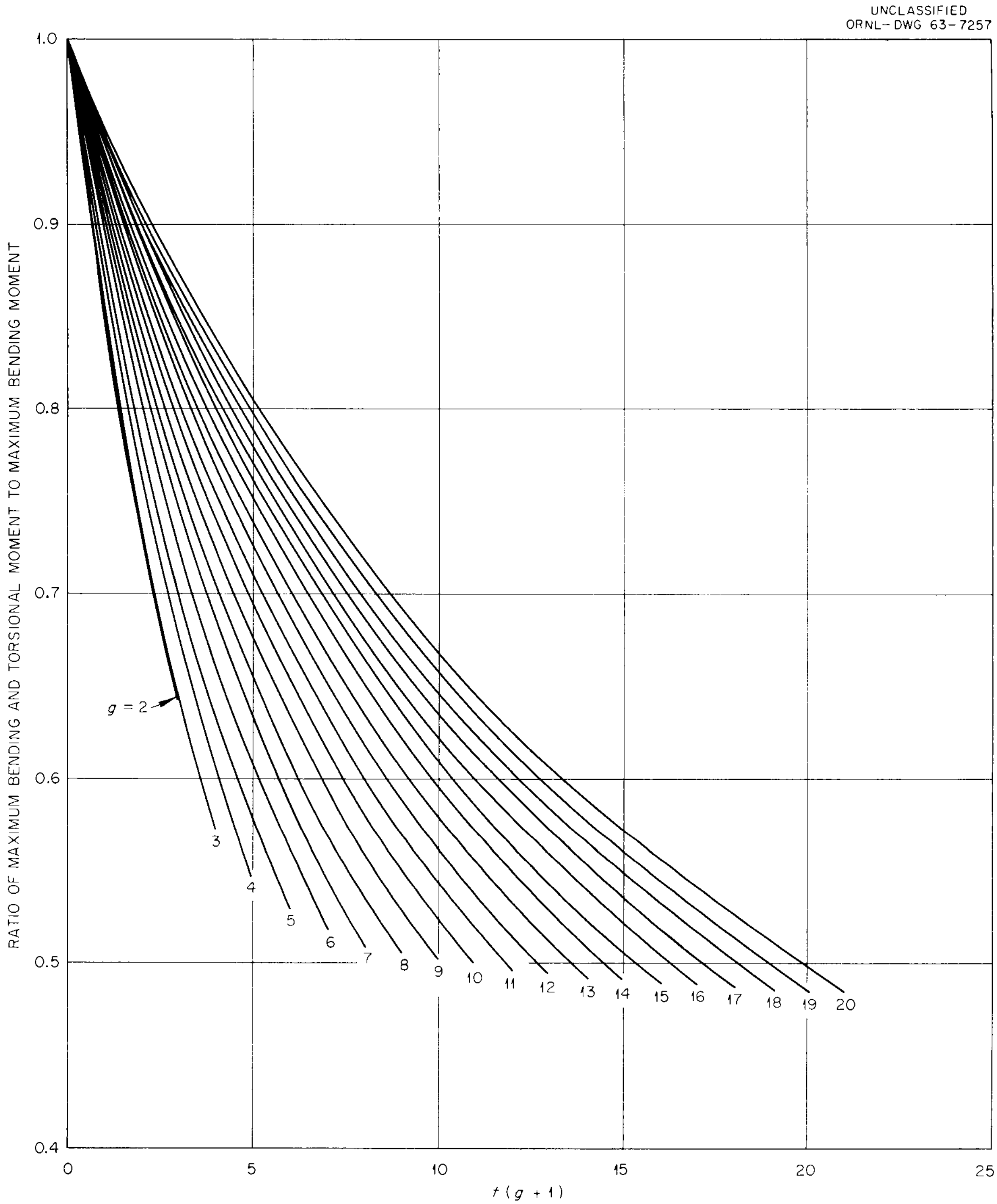

Fig. 7. Torsional Effects on Moments of Uniformly Loaded Square Grids.

each intersection, the total deflections at a point of intersection of the grids resulting not only from their respective loadings but also from the imposed concentrated loads at all intersections are set equal. By doing this at each intersection a linear system of equations in the unknown concentrated loads results. Having solved the system, the total loading of each individual rectangular grid is available, and the analysis previously presented may be applied. Also symmetry of the grids often allows consideration of several points of intersection at the same time. In particular, if a nonrectangular symmetrical grid is divided into rectangular grids, the placing of the unknown concentrated loads at a set of symmetrically located points reduces the computations to one for each grid and set. Also, if shearing is taken into account on any one grid, it must of course be taken into account throughout the entire analysis.

As previously indicated an analysis for distributed loadings in general may be performed by the method presented. This may be done by approximating distributed loads by concentrated loads applied at the centroids of small portions of the beams, the concentrated loads being the total load on the various portions.

It is to be noted that the assumption of perpendicularity of the beams does not enter into the analysis of grids. It enters only in considering the adequacy of the analysis. Thus grids of nonperpendicular beams may be analyzed the same as grids of perpendicular beams, but additional analysis must be made to assess the twisting and warping resulting from the nonperpendicularity.

Another point of interest is that the analysis provides only that the deflection at an intersection (or a point of contact before loading) is the same for the beams in question. Thus for an analysis including bending and shearing, it is often immaterial whether the beams are actually connected or a set in one direction is resting on a set in the other direction. The beams must actually intersect, or be connected, however, if a torsional analysis is to be made.

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APPENDIX A. THE GRID ANALYSIS COMPUTER PROGRAM

Two codes were written for the IBM-704 [5], one for solving the symmetrical case and the other for solving the nonsymmetrical case. The codes have the following features:

1. Any rectangular grid may be analyzed which has 90 beams or less in a given direction.

2. Any n^2 coefficients may be found; $n = 1, \ldots 5$.

3. The effect of shearing and torsion in any combination with bending may be considered.

4. The cross section of individual beams and beam spacing are arbitrary.

5. A uniform load is associated with each beam while concentrated loads may be placed at will.

6. Deflections and moments may be calculated along any desired beam.

7. Studies of the variation of the number of coefficients and in the variation of shearing or torsional effects or both may be made without running separate cases.

The analysis for bending alone, based on a 20×20 array of beams, usually requires only a few seconds, while the bending and shearing computation may require up to 30 sec. In contrast, the torsional computation may require 4 or 5 min.

APPENDIX B. SQUARE GRIDS OF ONE AND TWO BEAMS IN A GIVEN DIRECTION

The bending analyses of these two cases are equivalent to the analysis of a single beam with a uniform load (see Table 1). The nonsingular matrix equation given by the formulas of this report for the one-by-one grid is

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 82 & 0 \\ -1 & 0 & 82 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{31} \\ a_{13} \end{bmatrix} = \begin{bmatrix} 0.026142109 \\ -0.0087140364 \\ -0.0087140364 \end{bmatrix} ,$$

which has the solution

$$a_{11} = 0.13125 \times 10^{-1} ,$$

$$a_{31} = a_{13} = 0.53790 \times 10^{-4} .$$

There is no torsional effect on the one-by-one grid as indicated in Figs. 6 and 7.

A three-by-three nonsingular matrix exists for the two-by-two grid. The diagonal elements for this case are 3, 121.5, and 121.5, respectively.

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