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ORNL-3040
UC-20 - Controlled Thermonuclear Processes

EXACT RELATIVISTIC FOKKER-PLANCK COEFFICIENTS FOR A PLASMA

A. Simon

OAK RIDGE NATIONAL LABORATORY
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Exact Relativistic Fokker-Planck Coefficients: II and III

Corrigendum

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Page 5 - Replace 8π by 4π in Eq. (2.1).

Page 6 - Replace 8π by 4π everywhere in Eqs. (2.2), (2.3), and (2.4).

Page 7 - Delete the factor 2 in Eq. (2.6) and (3 times) in Eq. (2.7).

Thereafter, explicitly delete all factors of 2 (not including those in superscripts or subscripts) which may appear in all subsequent equations through Eq. (4.14) [except for the $(2\pi)^{-3}$ in Eq. (4.11)].

Also delete the 2 in $(u^2 - c^2)/2$ which appears in Eqs. (6.5) and (6.6) and in the matrices on pages 40 and 41.

Page 14 - The first term on the second line should have a negative sign in front of it.

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EXACT RELATIVISTIC FOKKER-PLANCK COEFFICIENTS FOR A PLASMA

Albert Simon

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Exact Relativistic Fokker-Planck Coefficients for a Plasma

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ABSTRACT

We consider a plasma which, in zero order, is static, spatially uniform, and infinite in extent and with no external electric or magnetic fields. Exact relativistic Fokker-Planck coefficients are obtained by a solution of the first order coupled integral equations for the particle and oscillator pair correlation functions. The solution is by a generalization of the method of Lenard and Balescu. The resulting coefficients have contributions from both coulomb interactions and from interactions via transverse electromagnetic fields.

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*Operated by Union Carbide Corporation for the U.S. Atomic Energy Commission.

I. INTRODUCTION

In a previous paper,[1] we have generalized the expansion procedure of Rostoker and Rosenbluth[2] so as to include the radiation field. The starting point was the Liouville equation for the density in phase space of a system of charged particles and a denumerably infinite set of field oscillators. A chain of equations was then obtained by integrating out the coordinates of all but s particles and t oscillators for all possible values of s and t. A solution of the entire chain of equations would then be found in the form of an expansion in powers of e, m, n^{-1}, and V^{-1} where e is the particle charge, m its mass, n the particle density, and V the volume in which the system is enclosed.

To lowest order the solution of the s particle, t oscillator distribution function was of the form:

$$f_o^{s,t} = \prod_{k=1}^{s} f_o^1(k) \prod_{\lambda=1}^{t} f_o^1(\lambda) \tag{1.1}$$

where we have abbreviated the particle coordinates $\underset{\sim}{q}_k$, $\underset{\sim}{v}_k$ by k and the oscillator coordinates q_λ, v_λ by λ. The functions $f_o^1(k)$ and $f_o^1(\lambda)$ obeyed a coupled pair of equations which were seen to be equivalent to the usual Vlasov equations plus an added equation for the field oscillators. [These equations were denoted as (13a) and (13b) in I.]

To the next order the solution was of the form:

$$f_1^{s,t} = \sum_{j=1}^{s} f_1^1(j) \prod_{k=1}^{s}{}' f_o^1(k) \prod_{\nu=1}^{t} f_o^1(\nu) + \prod_{k=1}^{s} f_o^1(k) \sum_{\lambda=1}^{t} f_1^1(\lambda) \prod_{\nu=1}^{t}{}' f_o^1(\nu)$$

$$+ \frac{1}{2} \sum_{i\neq j=1}^{s} g(i,j) \prod_{k=1}^{s}{}'' f_o^1(k) \prod_{\nu=1}^{t} f_o^1(\nu) + \sum_{i=1}^{s} \sum_{\lambda=1}^{t} g(i,\lambda)$$

$$\times \prod_{k=1}^{s}{}' f_o^1(k) \prod_{\nu=1}^{t}{}' f_o^1(\nu) \tag{1.2}$$

where the primes denote omission from the product of those indices which are in the sums.

The pair correlation functions $g(i,j)$ and $g(i,\lambda)$ obeyed a coupled set of integral equations [denoted as (38) and (40) in I]. The first order particle distribution function $f_1^1(i)$ and the first order oscillator distribution function $f_1^1(\lambda)$ also obey a coupled set of integral equations [denoted as (41) and (43) in I] which also involve $g(i,j)$ and $g(i,\lambda)$. These equations, as they stand, are all time reversible. Irreversibility is achieved by substituting the <u>asymptotic</u> solutions of the $g(i,j)$ and $g(i,\lambda)$ equations (rather than the exact time-dependent values) into the equations for $f_1^1(i)$ and $f_1^1(\lambda)$. The resulting equations are then in the usual Fokker-Planck form.

This procedure seems entirely similar to the Bogoliubov[3] approach which assumes that the time development of the pair correlation function is very rapid compared to the time development of the distribution function and hence that the time asymptotic value of the correlation function should be used in

the equation for the distribution function. It is also seen from Bogoliubov's formal assumption that the many-particle distribution functions shall depend on time only through the time-dependence of the one-particle distribution function. Comparing with Eq. (1.2) we see that this implies that $g(i,j)$ and $g(i,\lambda)$ are to be used as time-independent quantities in the final equations for the particle distribution functions and hence that their time asymptotic values are to be used.

Our starting point then will be the coupled set of integral equations for $g(i,j)$ and $g(i,\lambda)$. We will then obtain the time asymptotic solutions of these equations for the case of a zero order particle distribution which is stationary in time and uniform in space and for a zero order oscillator distribution which is stationary in time. We will assume throughout that the particles are electrons, and that there are also infinite mass ions uniformly distributed so as to provide a neutralizing background.

A truly consistent treatment of plasma, including radiation, requires that the plasma be treated relativistically. Nevertheless, we shall derive our results by the use of non-relativistic equations. However, the final results will be made relativistic by use of the same prescription as in I [see the discussion preceding Eq. (4.24) in this paper].

II. THE PAIR CORRELATION EQUATIONS

From Eq. (13b) of I, we see that if $f_o^1(i)$ (note that we drop the superscripts hereafter) is space independent then the most general static solution of this equation is $f_o(v_\lambda^2 + w_\lambda^2 q_\lambda^2)$. Equation (13a) of I (in the absence of any external fields) is then satisfied by an arbitrary function of the electron

velocity, $f_o(\vec{v})$. Applying these results to Eq. (39) of I, we see that $\underset{\sim}{F}_o(1)$ vanishes and that the coefficients in Eqs. (38) and (40) are space-independent and hence that we may Fourier analyze in space. This Fourier transformation was already carried out in the last section of I; however, it was expressed entirely in terms of real sine and cosine transforms. It is simpler to do this in terms of the more usual complex exponentials and the results are:

$$\left[\frac{\partial}{\partial t} - i\underset{\sim}{k}\cdot(\underset{\sim}{v}_i - \underset{\sim}{v}_j) \right] g(\underset{\sim}{v}_i \underset{\sim}{v}_j) = - i \frac{\omega_p^2}{k^2} \underset{\sim}{k}\cdot \left[\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \int g^*(\underset{\sim}{v}_j \underset{\sim}{v})\, d^3v \right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \int g(\underset{\sim}{v}_i \underset{\sim}{v})\, d^3v \right] + \frac{e}{m} (4\pi V)^{\frac{1}{2}} \left\{ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \cdot \sum_K \int [q_K \underset{\sim}{v}_i \times (i\underset{\sim}{k} \times \underset{\sim}{u}_K) + v_K \underset{\sim}{u}_K] \right.$$

$$g^*(\underset{\sim}{v}_j q_K v_K \underset{\sim}{u}_K)\, dq_K\, dv_K - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \cdot \sum_K \int [q_K \underset{\sim}{v}_j \times (i\underset{\sim}{k} \times \underset{\sim}{u}_K) - v_K \underset{\sim}{u}_K]$$

$$\left. g(\underset{\sim}{v}_i,\ q_K v_K \underset{\sim}{u}_K)\, dq_K\, dv_K \right\} - i \frac{\omega_p^2}{nk^2} \vec{k} \cdot$$

$$\left[f_o(\underset{\sim}{v}_j) \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} - f_o(\underset{\sim}{v}_i) \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \right], \tag{2.1}$$

and:

$$\left(\frac{\partial}{\partial t} - i\mathbf{k}\cdot\mathbf{v} + v_\kappa \frac{\partial}{\partial q_\kappa} - \omega_k^2 q_\kappa \frac{\partial}{\partial v_\kappa}\right) g(\mathbf{v}, q_\kappa v_\kappa \mathbf{u}_\kappa) = - i \frac{\omega_p^2}{k^2} \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v})}{\partial \mathbf{v}} \int g(\mathbf{v}' q_\kappa v_\kappa \mathbf{u}_\kappa)\, d^3v'$$

$$- ne \left(\frac{4\pi}{V}\right)^{\frac{1}{2}} \frac{\partial f_o(\kappa)}{\partial v_\kappa} \int \mathbf{v}'\cdot\mathbf{u}_\kappa\, g(\mathbf{v}\mathbf{v}')\, d^3v' + \frac{e}{m}\left(\frac{4\pi}{V}\right)^{\frac{1}{2}} f_o(\kappa) \frac{\partial f_o(\mathbf{v})}{\partial \mathbf{v}}$$

$$\cdot\, [\mathbf{v} \times (i\mathbf{k} \times q_\kappa \mathbf{u}_\kappa) + v_\kappa \mathbf{u}_\kappa] - e\left(\frac{4\pi}{V}\right)^{\frac{1}{2}} \mathbf{v}\cdot\mathbf{u}_\kappa\, f_o(\mathbf{v}) \frac{\partial f_o(\kappa)}{\partial v_\kappa} \qquad (2.2)$$

In the above equations q_κ and v_κ refer to the coordinates of that oscillator with propagation vector $\mathbf{k}$ and with a unit transverse polarization vector $\mathbf{u}_\kappa$. The sum over κ then denotes the sum over the two orthogonal polarization directions. We have also suppressed the explicit dependence of the pair correlation functions on $\mathbf{k}$ and t. The usual plasma frequency $4\pi ne^2/m$ is denoted by ω_p^2 and $\omega_k^2 = c^2k^2$. Note that Eq. (2.1) is equivalent to Eqs. (61) and (62) of I, and that Eq. (2.2) is equivalent to Eqs. (63) and (64) of I. In this paper, of course, the superscript * denotes the usual complex conjugate.

Equations (2.1) and (2.2) may be simplified by defining the following moments:

$$\mathbf{E}(\mathbf{v}) = \frac{e}{m} \left(\frac{4\pi}{V}\right)^{\frac{1}{2}} \sum_\kappa \mathbf{u}_\kappa \int q_\kappa\, g(\mathbf{v}\, q_\kappa v_\kappa \mathbf{u}_\kappa)\, dq_\kappa\, dv_\kappa \qquad (2.3)$$

$$\mathbf{F}(\mathbf{v}) = \frac{e}{m} \left(\frac{4\pi}{V}\right)^{\frac{1}{2}} \sum_\kappa i\mathbf{u}_\kappa \int v_\kappa\, g(\mathbf{v}\, q_\kappa v_\kappa \mathbf{u}_\kappa)\, dq_\kappa\, dv_\kappa \qquad (2.4)$$

With this definition Eq. (2.1) becomes:

$$\left[\frac{\partial}{\partial t} - i\underset{\sim}{k}\cdot(\underset{\sim}{v}_i - \underset{\sim}{v}_j) \right] g(\underset{\sim}{v}_i \underset{\sim}{v}_j) = - i \frac{\omega_p^2}{k^2} \vec{k}\cdot \left[\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \int g^*(\underset{\sim}{v}_j \underset{\sim}{v})\, d^3v \right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \int g(\underset{\sim}{v}_i \underset{\sim}{v})\, d^3v \right] + i \left\{ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \cdot \left[\underset{\sim}{v}_i \times \left(\underset{\sim}{k} \times \underset{\sim}{E}^*(\underset{\sim}{v}_j) \right) + \underset{\sim}{F}^*(\underset{\sim}{v}_j) \right] \right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \cdot \left[\underset{\sim}{v}_j \times \left(\underset{\sim}{k} \times \underset{\sim}{E}(\underset{\sim}{v}_i) \right) + \underset{\sim}{F}(\underset{\sim}{v}_i) \right] \right\} - i \frac{\omega_p^2}{nk^2} \vec{k} \cdot$$

$$\left[f_o(\underset{\sim}{v}_j) \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} - f_o(\underset{\sim}{v}_i) \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \right] . \tag{2.5}$$

If we multiply Eq. (2.2) by either $q_\kappa \underset{\sim}{u}_\kappa$ or $i v_\kappa \underset{\sim}{u}_\kappa$, integrate over $dq_\kappa\, dv_\kappa$ and sum over κ, we obtain a closed pair of equations in $\underset{\sim}{E}$ and $\underset{\sim}{F}$. These are:

$$\left(\frac{\partial}{\partial t} - i\underset{\sim}{k}\cdot\underset{\sim}{v} \right) \underset{\sim}{E}(\underset{\sim}{v}) + i\, \underset{\sim}{F}(\underset{\sim}{v}) = - i \frac{\omega_p^2}{k^2} \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \int \underset{\sim}{E}(\underset{\sim}{v}')\, d^3v'$$

$$+ i \frac{2\omega_p^2}{nm} \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \left[\underset{\sim}{v} \times (\underset{\sim}{k} \times Q_o) \right] , \tag{2.6}$$

and

$$\left(\frac{\partial}{\partial t} - i\underset{\sim}{k}\cdot\underset{\sim}{v}_i \right) \underset{\sim}{F}(\underset{\sim}{v}) + i\omega_k^2\, \underset{\sim}{E}(\underset{\sim}{v}) = - i \frac{\omega_p^2}{k^2} \vec{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \int \underset{\sim}{F}(\underset{\sim}{v}')\, d^3v' + i\, 2\omega_p^2$$

$$\int \underset{\sim}{v}'\cdot I\, g(\underset{\sim}{v}\underset{\sim}{v}')\, d^3v' + i \frac{2\omega_p^2\, \omega_k^2}{nm} \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot Q_o + i \frac{2\omega_p^2}{n} f_o(\underset{\sim}{v})\, \underset{\sim}{v}\cdot I \tag{2.7}$$

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where the unit transverse dyadic I is defined as

$$I = \sum_{u_K} \underset{\sim}{u}_K \, \underset{\sim}{u}_K \tag{2.8}$$

and the dyadic Q_o is

$$Q_o = \sum_{u_K} \underset{\sim}{u}_K \, \underset{\sim}{u}_K \int q_K^2 \, f_o(K) \, dq_K \, dv_K \tag{2.9}$$

$$\equiv \quad \omega_k^{-2} \sum_{u_K} \underset{\sim}{u}_K \, \underset{\sim}{u}_K \int v_K^2 \, f_o(K) \, dq_K \, dv_K \tag{2.10}$$

Note that if there is no radiation present in zero order then Q_o is zero, and if the radiation is unpolarized then Q_o is a constant times I. Note also that I and Q_o are still functions of $\underset{\sim}{k}$. We further simplify by taking Laplace transforms in time and assuming that all pair correlations are zero at the initial time. Hence, we obtain:

$$\left(ip + k\cdot(\underset{\sim}{v}_i - \underset{\sim}{v}_j) \right) g(\underset{\sim}{v}_i \underset{\sim}{v}_j) = \frac{\omega_p^2}{k^2} \vec{k}\cdot \left[\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \int g^*(\underset{\sim}{v}_j \underset{\sim}{v}) \, d^3v \right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \int g(\underset{\sim}{v}_i \underset{\sim}{v}) \, d^3v \right] - \left\{ \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \cdot \left[\underset{\sim}{v}_i \times \left(\underset{\sim}{k} \times \underset{\sim}{E}^*(\underset{\sim}{v}_j) \right) + \underset{\sim}{F}^*(\underset{\sim}{v}_j) \right] \right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \cdot \left[\underset{\sim}{v}_j \times \left(\underset{\sim}{k} \times \underset{\sim}{E}(\underset{\sim}{v}_i) \right) + \underset{\sim}{F}(\underset{\sim}{v}_i) \right] \right\} + \frac{\omega_p^2}{pnk^2} \vec{k} \cdot$$

$$\left[f_o(\underset{\sim}{v}_j) \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} - f_o(\underset{\sim}{v}_i) \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \right] . \tag{2.11}$$

$$(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})\ \underset{\sim}{E}(\underset{\sim}{v}) - \underset{\sim}{F}(\underset{\sim}{v}) = \frac{\omega_p^2}{k^2}\,\underset{\sim}{k}\cdot\frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\int \underset{\sim}{E}(\underset{\sim}{v}')\ d^3v'$$

$$- \frac{2\omega_p^2}{nmp}\ \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\cdot\left[\ \underset{\sim}{v} \times (\underset{\sim}{k} \times Q_o)\ \right] \tag{2.12}$$

$$(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})\ \underset{\sim}{F}(\underset{\sim}{v}) - \omega_k^2\ \underset{\sim}{E}(\underset{\sim}{v}) = \frac{\omega_p^2}{k^2}\,\underset{\sim}{k}\cdot\frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\int \underset{\sim}{F}(\underset{\sim}{v}')\ d^3v' - 2\omega_p^2\int \underset{\sim}{v}'\cdot I\ g(\underset{\sim}{v}\underset{\sim}{v}')\ d^3v'$$

$$- \frac{2\omega_p^2\ \omega_k^2}{nmp}\ \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\cdot Q_o - \frac{2\omega_p^2}{pn}\ f_o(\underset{\sim}{v})\ \underset{\sim}{v}\cdot I \tag{2.13}$$

where the quantities g, $\underset{\sim}{E}$, and $\underset{\sim}{F}$ now denote the Laplace transforms and whose explicit dependence on $\underset{\sim}{k}$ and p has been suppressed.

Equations (2.12) and (2.13) may be solved immediately for $\underset{\sim}{E}$ and $\underset{\sim}{F}$ by noting that the integral terms are simply vector constants (more precisely, they are functions of $\underset{\sim}{k}$ and p) whose values can be determined from the solutions themselves. For the moment, we denote them as follows:

$$\int \underset{\sim}{E}(\underset{\sim}{v})\ d^3v = \underset{\sim}{C_1} \tag{2.14}$$

$$\int \underset{\sim}{F}(\underset{\sim}{v})\ d^3v = \underset{\sim}{C_2} \tag{2.15}$$

and solve immediately to obtain:

$$\underset{\sim}{E}(\underset{\sim}{v}) = \frac{2\,\omega_p^2}{(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})^2 - \omega_k^2} \left\{ \frac{1}{2k^2}\,\underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[\underset{\sim}{C}_2 + (ip + \underset{\sim}{k}\cdot\underset{\sim}{v})\underset{\sim}{C}_1 \right] - \frac{1}{nmp}\,(ip + \underset{\sim}{k}\cdot\underset{\sim}{v}) \right.$$

$$\frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \left[\underset{\sim}{v} \times (\underset{\sim}{k} \times Q_o) \right] - \int \underset{\sim}{v}'\cdot I\; g(\underset{\sim}{v}\underset{\sim}{v}')\; d^3v' - \frac{\omega_k^2}{nmp}\,\frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot Q_o$$

$$\left. - \frac{1}{np}\, f_o(\underset{\sim}{v})\; \underset{\sim}{v}\cdot I \right\} \tag{2.16}$$

$$\underset{\sim}{F}(\underset{\sim}{v}) = \frac{2\,\omega_p^2}{(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})^2 - \omega_p^2} \left\{ \frac{1}{2k^2}\,\underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})\underset{\sim}{C}_2 + \omega_k^2 \underset{\sim}{C}_1 \right] - \frac{\omega_k^2}{nmp}\,\frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right.$$

$$\cdot \left[\underset{\sim}{v} \times (\underset{\sim}{k} \times Q_o) \right] - (ip + \underset{\sim}{k}\cdot\underset{\sim}{v}) \int \underset{\sim}{v}'\cdot I\; g(\underset{\sim}{v}\underset{\sim}{v}')\; d^3v' - \frac{\omega_k^2}{nmp}\,(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})$$

$$\left. \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot Q_o - \frac{1}{np}\,(ip + \underset{\sim}{k}\cdot\underset{\sim}{v})\; f_o(\underset{\sim}{v})\; \underset{\sim}{v}\cdot I \right\} \tag{2.17}$$

One can now solve for $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ by integrating Eqs. (2.16) and (2.17) over d^3v, but we shall not do this at this point. Upon substitution of Eqs. (2.16) and (2.17) into Eq. (2.11), we finally obtain a simple integral equation for $g(\underset{\sim}{v}_i \underset{\sim}{v}_j)$.

As mentioned earlier, we shall consider only the time asymptotic solution in this paper. Now if we restrict ourselves to zero order distribution functions $f_o(i)$ and $f_o(\lambda)$ such that the Laplace transforms of the pair correlation functions have no poles in which p has a positive real part (i.e., no instabilities), then we may write

$$\lim_{t \to \infty} g(\underset{\sim}{v}_i, \underset{\sim}{v}_j, t) = \lim_{p \to o} p\ g(\underset{\sim}{v}_i \underset{\sim}{v}_j p).$$

At the risk of some confusion, we shall now let the notation $g(\underset{\sim}{v}_i \underset{\sim}{v}_j)$ denote this limit, with a similar understanding for the quantities $\underset{\sim}{E}(\underset{\sim}{v})$ and $\underset{\sim}{F}(\underset{\sim}{v})$. Thus

$$g(\underset{\sim}{v}_i \underset{\sim}{v}_j) = \lim_{p \to o} p\ g(\underset{\sim}{v}_i \underset{\sim}{v}_j p)$$

$$\underset{\sim}{E}(\underset{\sim}{v}) = \lim_{p \to o} p\ \underset{\sim}{E}(\underset{\sim}{v} p)$$

$$\underset{\sim}{F}(\underset{\sim}{v}) = \lim_{p \to o} p\ \underset{\sim}{F}(\underset{\sim}{v}, p)$$

$$\underset{\sim}{C}_1 = \lim_{p \to o} p\ \underset{\sim}{C}_1$$

$$\underset{\sim}{C}_2 = \lim_{p \to o} p\ \underset{\sim}{C}_2$$

and we now take this limit in Eqs. (2.16), (2.17), and (2.11). Equations (2.16) and (2.17) become:

$$\underset{\sim}{E}(\underset{\sim}{v}) = \frac{2\ \omega_p^2}{(\underset{\sim}{k}\cdot\underset{\sim}{v})^2 - k^2c^2} \left\{ \frac{1}{2k^2}\ \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[\underset{\sim}{C}_2 + (\underset{\sim}{k}\cdot\underset{\sim}{v})\underset{\sim}{C}_1 \right] - \frac{1}{nm}\ (\underset{\sim}{k}\cdot\underset{\sim}{v})\ \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ \underset{\sim}{v}\cdot Q_o \right.$$

$$\left. - \int \underset{\sim}{v}'\cdot\underset{\sim}{I}\ g(\underset{\sim}{v}\underset{\sim}{v}')\ d^3v' \qquad - \frac{1}{n}\ f_o(\underset{\sim}{v})\ \underset{\sim}{v}\cdot I \right\} + \frac{2\omega_p^2}{nm}\ \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot Q_o \qquad (2.18)$$

$$\underset{\sim}{F}(\underset{\sim}{v}) = \frac{2\omega_p^2}{(\underset{\sim}{k}\cdot\underset{\sim}{v})^2 - k^2c^2} \left\{ \frac{1}{2k^2}\, \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[(\underset{\sim}{k}\cdot\underset{\sim}{v})\underset{\sim}{C}_2 + \omega_k^2\, \underset{\sim}{C}_1 \right] - \frac{\omega_k^2}{nm}\, \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\, \underset{\sim}{v}\cdot Q_o \right.$$

$$- (\underset{\sim}{k}\cdot\underset{\sim}{v}) \int \underset{\sim}{v}'\cdot I\, g(\underset{\sim}{v}\underset{\sim}{v}')\, d^3v' \qquad - \frac{1}{n}\, (\underset{\sim}{k}\cdot\underset{\sim}{v})\, f_o(\underset{\sim}{v})\, \underset{\sim}{v}\cdot I \tag{2.19}$$

where we are free to take the limit in the denominator, since in a properly relativistic plasma there will be no electrons moving with the velocity of light. The denominator in Eq. (2.11) does not have this feature, however, and this equation becomes:

$$g(\underset{\sim}{v}_i\underset{\sim}{v}_j) = \lim_{p \to o} \frac{1}{ip + \underset{\sim}{k}(\underset{\sim}{v}_i - \underset{\sim}{v}_j)} \left\{ \frac{\omega_p^2}{k^2}\, \underset{\sim}{k}\cdot \left[\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \int g^*(\underset{\sim}{v}_j\underset{\sim}{v})\, d^3v \right.\right.$$

$$\left. - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \int g(\underset{\sim}{v}_i\underset{\sim}{v})\, d^3v \right] - \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \left[\underset{\sim}{v}_i \times \left(\underset{\sim}{k} \times \underset{\sim}{E}^*(\underset{\sim}{v}_j) \right) + \underset{\sim}{F}^*(\underset{\sim}{v}_j) \right]$$

$$+ \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \left[\underset{\sim}{v}_j \times \left(\underset{\sim}{k} \times \underset{\sim}{E}(\underset{\sim}{v}_i) \right) + \underset{\sim}{F}(\underset{\sim}{v}_i) \right] + \frac{\omega_p^2}{nk^2}\, \underset{\sim}{k}\, \cdot$$

$$\left.\left(\frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i}\, f_o(\underset{\sim}{v}_j) - \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j}\, f_o(\underset{\sim}{v}_i) \right) \right\} \tag{2.20}$$

We now substitute Eqs. (2.18) and (2.19) in Eq. (2.20) and, after some rearrangement, find that:

$$g(\mathbf{v}_i\mathbf{v}_j) = \lim_{p \to o} \frac{\omega_p^2}{ip + \mathbf{k}(\mathbf{v}_i - \mathbf{v}_j)} \left\{ \frac{1}{k^2}\, \mathbf{k}\cdot \left[\frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \int g^*(\mathbf{v}_j\mathbf{v})\, d^3v \right.\right.$$

$$\left. - \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \int g(\mathbf{v}_i\mathbf{v})\, d^3v \right] + \frac{1}{nk^2}\, \mathbf{k}\cdot \left[\frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} f_o(\mathbf{v}_j) - \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} f_o(\mathbf{v}_i) \right]$$

$$+ 2 \left[\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \frac{\mathbf{v}_i \cdot \int I\cdot\mathbf{v}\, g^*(\mathbf{v}_j\mathbf{v})\, d^3v}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} - \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \frac{\mathbf{v}_j \cdot \int I\cdot\mathbf{v}\, g(\mathbf{v}_i\mathbf{v})\, d^3v}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \right]$$

$$+ \frac{2}{n}\, \mathbf{v}_i\cdot I\cdot\mathbf{v}_j \left[\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \frac{f_o(\mathbf{v}_j)}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} - \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \frac{f_o(\mathbf{v}_i)}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - c^2} \right]$$

$$- \frac{1}{k^2}\, \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i}\, \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \left[\frac{\mathbf{v}_i \cdot \left\{ \mathbf{C}_2^* + (\mathbf{k}\cdot\mathbf{v}_j)\mathbf{C}_1^* \right\}}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} - \frac{\mathbf{v}_j \cdot \left\{ \mathbf{C}_2 + (\mathbf{k}\cdot\mathbf{v}_i)\mathbf{C}_1 \right\}}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \right]$$

$$+ \frac{1}{k^2} \left[(\mathbf{k}\cdot\mathbf{v}_i)(\mathbf{k}\cdot\mathbf{v}_j) - k^2c^2 \right] \left[\mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot \frac{\mathbf{C}_1^*}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} \right.$$

$$\left.\left. - \mathbf{k}\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \cdot \frac{\mathbf{C}_1}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \right] \right\}$$

$$- 2\omega_p^2 \left\{ \left[\frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} \cdot \frac{\int I\cdot\mathbf{v}\, g^*(\mathbf{v}_j\mathbf{v})\, d^3v}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2} + \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \cdot \frac{\int I\cdot\mathbf{v}\, g(\mathbf{v}_i\mathbf{v})\, d^3v}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2} \right] \right.$$

$$+ \frac{1}{n} \left[\frac{f_o(\mathbf{v}_j)}{(\mathbf{k}\cdot\mathbf{v}_j)^2 - k^2c^2}\, \mathbf{v}_j\cdot I\cdot \frac{\partial f_o(\mathbf{v}_i)}{\partial \mathbf{v}_i} + \frac{f_o(\mathbf{v}_i)}{(\mathbf{k}\cdot\mathbf{v}_i)^2 - k^2c^2}\, \mathbf{v}_i\cdot I\cdot \frac{\partial f_o(\mathbf{v}_j)}{\partial \mathbf{v}_j} \right]$$

(Equation continued on next page)

$$- \frac{1}{nm} \left[(\underset{\sim}{k}\cdot\underset{\sim}{v}_i)(\underset{\sim}{k}\cdot\underset{\sim}{v}_j) + k^2c^2 \right] \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \frac{\underset{\sim}{v}_j\cdot Q_o\cdot \underset{\sim}{v}_i}{[(\underset{\sim}{k}\cdot\underset{\sim}{v}_i)^2 - k^2c^2][(\underset{\sim}{k}\cdot\underset{\sim}{v}_j)^2 - k^2c^2]}$$

$$- \frac{1}{nm} \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \cdot Q_o \cdot \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} + \frac{1}{nm} \left[\frac{\underset{\sim}{k}\cdot\underset{\sim}{v}_i}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_i)^2 - k^2c^2} \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \underset{\sim}{v}_i\cdot Q_o\cdot \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \right.$$

$$\left. + \frac{\underset{\sim}{k}\cdot\underset{\sim}{v}_j}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_j)^2 - k^2c^2} \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \underset{\sim}{v}_j\cdot Q_o\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \right] - \frac{1}{2k^2}$$

$$\left. \left[\underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \cdot \frac{\underset{\sim}{C}_2^*}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_j)^2 - k^2c^2} + \underset{\sim}{k}\cdot \frac{\partial f_o(\underset{\sim}{v}_i)}{\partial \underset{\sim}{v}_i} \frac{\partial f_o(\underset{\sim}{v}_j)}{\partial \underset{\sim}{v}_j} \cdot \frac{\underset{\sim}{C}_2}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_i)^2 - k^2c^2} \right] \right\} \tag{2.21}$$

We have taken advantage of the fact that some terms were proportional to $\underset{\sim}{k}(\underset{\sim}{v}_i - \underset{\sim}{v}_j)$ to eliminate the denominator. These terms are in the second bracket of Eq. (2.21).

III. GENERALIZATION OF THE LENARD-BALESCU METHOD

Equation (2.21), despite its formidable appearance, may be solved exactly by a generalization of the method of Lenard and Balescu.[4] We define the new functions

$$h(\underset{\sim}{v}) = \int g(\underset{\sim}{v}\underset{\sim}{v}')\ d^3v' \tag{3.1}$$

$$H(u) = k \int h(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.2}$$

$$\underset{\sim}{p}(\underset{\sim}{v}) = \int I\cdot\underset{\sim}{v}'\ g(\underset{\sim}{v}\underset{\sim}{v}')\ d^3v' \tag{3.3}$$

$$\underset{\sim}{P}(u) = k \int \underset{\sim}{p}(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.4}$$

$$\underset{\sim}{S}(u) = k \int I\cdot\underset{\sim}{v}\ h(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.5}$$

$$T(u) = k \int \underset{\sim}{p}(\underset{\sim}{v})(I\cdot\underset{\sim}{v})\delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.6}$$

as well as the modified zero-order functions

$$F_o(u) = k \int f_o(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.7}$$

$$F_o'(u) = \underset{\sim}{k}\cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.8}$$

$$\underset{\sim}{J}_o'(u) = \underset{\sim}{k}\cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ \underset{\sim}{v}\cdot I\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.9}$$

$$\underset{\sim}{J}_o(u) = k \int f_o(\underset{\sim}{v})\ \underset{\sim}{v}\cdot I\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.10}$$

$$\underset{\sim}{K}_o(u) = k \int I\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.11}$$

$$M_o'(u) = \underset{\sim}{k}\cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ (I\cdot\underset{\sim}{v})(\underset{\sim}{v}\cdot I)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.12}$$

$$M_o(u) = k \int f_o(\underset{\sim}{v})(I\cdot\underset{\sim}{v})(\underset{\sim}{v}\cdot I)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.13}$$

$$L_o(u) = k \int (I\cdot\underset{\sim}{v}) \left(I\cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right) \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{3.14}$$

We resist the temptation to write F_o' as $\partial F_o/\partial u$, or to set $\underset{\sim}{K}_o(u)$ equal to zero. These and other simplifications are certainly true in the non-relativistic case. However, the special form in which we have written the definitions in Eqs. (3.1) to (3.14) allows us to convert them immediately to relativistic form. In that circumstance, most simplifying identities are no longer valid.

We now integrate Eq. (2.21) over all values of $\underset{\sim}{v}_j$ to obtain the first in a set of equations. The result is:

$$
\begin{aligned}
h(\underset{\sim}{v}) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \Bigg\{ & \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[H^*(u') + \frac{F_o(u')}{n} \right] - F_o'(u') \\
& \left[h(\underset{\sim}{v}) + \frac{f_o(\underset{\sim}{v})}{n} \right] + 2 \left[\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \frac{\underset{\sim}{v} \cdot \underset{\sim}{P}^*(u')}{(u')^2 - c^2} - \underset{\sim}{J}_o'(u') \cdot \frac{\underset{\sim}{p}(\underset{\sim}{v})}{u^2 - c^2} \right] \\
& + \frac{2}{n} \underset{\sim}{v} \cdot \left[\frac{\underset{\sim}{J}_o(u')}{(u')^2 - c^2} \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \underset{\sim}{J}_o'(u') \frac{f_o(\underset{\sim}{v})}{u^2 - c^2} \right] - \frac{1}{k^2} \underset{\sim}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \\
& \left[\frac{F_o'(u')}{(u')^2 - c^2} \underset{\sim}{v} \cdot \left\{ \underset{\sim}{C}_2^* + ku' \underset{\sim}{C}_1^* \right\} - \frac{\underset{\sim}{J}_o'(u')}{u^2 - c^2} \left\{ \underset{\sim}{C}_2 + hu \underset{\sim}{C}_1 \right\} \right] \\
& + (uu' - c^2) \left[\frac{F_o'(u')}{(u')^2 - c^2} \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{C}_1^* - \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \underset{\sim}{K}_o(u') \cdot \frac{\underset{\sim}{C}_1}{u^2 - c^2} \right] \Bigg\} \\
& - 2 \frac{\omega_p^2}{k^2} \Bigg\{ \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{C}_3^* + \frac{1}{n} \tilde{\underset{\sim}{J}}_o \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{k \tilde{\underset{\sim}{A}}_o \cdot \underset{\sim}{Q}_o \cdot \underset{\sim}{v}}{nm(u^2 - c^2)} \underset{\sim}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} u \\
& - \frac{c^2 k}{nm} \frac{\tilde{\underset{\sim}{J}}_o' \cdot \underset{\sim}{Q}_o \cdot \underset{\sim}{v}}{u^2 - c^2} \underset{\sim}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \frac{k^2 \tilde{\underset{\sim}{A}}_o \cdot \underset{\sim}{Q}_o}{nm} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{1}{2k} \tilde{F}_o' \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{C}_2^* \Bigg\}
\end{aligned}
\tag{3.15}
$$

In this equation we have set $\underset{\sim}{k}\cdot\underset{\sim}{v} = ku$ and have made frequent use of the properties of the unit (transverse) dyadic I. Thus we have $I\cdot I = I$ and $I\cdot R = R$ if R is a dyadic having transverse components only. Equation (3.15) is analogous to Eq. (8) in Lenard's paper. We have also introduced several constants above. They are:

$$\underset{\sim}{C}_3 = \int \frac{\underset{\sim}{P}(u)}{u^2 - c^2}\, du$$

$$\underset{\sim}{\tilde{J}}_0 = \int \frac{\underset{\sim}{J}_0(u)}{u^2 - c^2}\, du$$

$$\underset{\sim}{\tilde{J}}_0' = \int \frac{\underset{\sim}{J}_0'(u)}{u^2 - c^2}\, du$$

$$\tilde{F}_0' = \int \frac{F_0'(u)}{u^2 - c^2}\, du$$

$$\underset{\sim}{\tilde{A}}_0 = \int \frac{u\, \underset{\sim}{J}_0'(u)}{u^2 - c^2}\, du$$

and have used the fact that $\int F_0'(u)\, du = \int K_0(u)\, du = 0$. Note that $\underset{\sim}{C}_3$ is analogous to $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$ in that its value can only be determined after we have the solution.

A second equation results from multiplying Eq. (3.15) by $k\,\delta(ku - k\cdot\underset{\sim}{v})$ and integrating over d^3v.

$$H(u) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \Bigg\{ F_0'(u) \left[H^*(u') + \frac{F_0(u')}{n} \right] - F_0'(u') \left[H(u) + \frac{F_0(u)}{n} \right]$$

$$+ 2 \left[\frac{\underset{\sim}{J}_0'(u) \cdot \underset{\sim}{P}^*(u')}{(u')^2 - c^2} - \frac{\underset{\sim}{J}_0'(u') \cdot \underset{\sim}{P}(u)}{u^2 - c^2} \right] + \frac{2}{n} \left[\frac{\underset{\sim}{J}_0(u') \cdot \underset{\sim}{J}_0'(u)}{(u')^2 - c^2} - \frac{\underset{\sim}{J}_0'(u') \cdot \underset{\sim}{J}_0(u)}{u^2 - c^2} \right]$$

$$- \frac{1}{k} \left[\frac{F_0'(u')}{(u')^2 - c^2} \underset{\sim}{J}_0'(u) \cdot (\underset{\sim}{C}_2^* + ku' \underset{\sim}{C}_1^*) - \frac{F_0'(u)}{u^2 - c^2} \underset{\sim}{J}_0'(u') \cdot (\underset{\sim}{C}_2 + ku \underset{\sim}{C}_1) \right]$$

$$+ (uu' - c^2) \left[\frac{F_0'(u')}{(u')^2 - c^2} \underset{\sim}{K}_0(u) \cdot \underset{\sim}{C}_1^* - \frac{F_0'(u)}{u^2 - c^2} \underset{\sim}{K}_0(u') \cdot \underset{\sim}{C}_1 \right] \Bigg\}$$

$$- 2 \frac{\omega_p^2}{k^2} \Bigg\{ \underset{\sim}{K}_0(u) \cdot \underset{\sim}{C}_3^* + \frac{1}{n} \tilde{\underset{\sim}{J}}_0 \cdot \underset{\sim}{K}_0(u) - \frac{k^2 \tilde{\underset{\sim}{A}}_0 \cdot \underset{\sim}{Q}_0 \cdot \underset{\sim}{J}_0'(u)}{nm(u^2 - c^2)} u - \frac{c^2 k^2}{nm} \frac{\tilde{\underset{\sim}{J}}_0' \cdot \underset{\sim}{Q}_0 \cdot \underset{\sim}{J}_0'(u)}{u^2 - c^2}$$

$$+ \frac{k^2 \tilde{\underset{\sim}{A}}_0 \cdot \underset{\sim}{Q}_0 \cdot \underset{\sim}{K}_0(u)}{nm} - \frac{1}{2k} \tilde{F}_0' \underset{\sim}{K}_0(u) \cdot \underset{\sim}{C}_2^* \Bigg\} \qquad (3.16)$$

Equation (3.16) is analogous to Lenard's Eq. (7). We cannot use Lenard's trick in its original form at this point, however, since there are <u>three</u> functions $H(u')$ and $\underset{\sim}{P}(u')$ under the integral sign. To obtain a solution we require a third equation, which is obtained by multiplying Eq. (3.15) by $k(\underset{\sim}{v} \cdot I)\ \delta(ku - \underset{\sim}{k} \cdot \underset{\sim}{v})$ and integrating over d^3v. This equation (actually, of course, it is two equations) becomes:

$$\underset{\sim}{S}(u) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \left\{ \underset{\sim}{J}_0'(u) \left[H^*(u') + \frac{F_0(u')}{n} \right] - F_0'(u') \left[\underset{\sim}{S}(u) + \frac{\underset{\sim}{J}_0(u)}{n} \right] \right.$$

$$+ 2 \left[\frac{M_0'(u) \cdot \underset{\sim}{P}^*(u')}{(u')^2 - c^2} - \frac{\underset{\sim}{J}_0'(u') \cdot T(u)}{u^2 - c^2} \right] + \frac{2}{n} \left[\frac{M_0'(u) \cdot \underset{\sim}{J}_0(u')}{(u')^2 - c^2} - \frac{M_0(u) \cdot \underset{\sim}{J}_0'(u')}{u^2 - c^2} \right]$$

$$- \frac{1}{k} \left[\frac{F_0'(u') M_0'(u)}{(u')^2 - c^2} \cdot (\underset{\sim}{C}_2^* + ku' \underset{\sim}{C}_1^*) - \frac{\underset{\sim}{J}_0'(u)}{u^2 - c^2} \underset{\sim}{J}_0'(u') \cdot (\underset{\sim}{C}_2 + ku \underset{\sim}{C}_1) \right]$$

$$\left. + (uu' - c^2) \left[\frac{F_0'(u')}{(u')^2 - c^2} L_0(u) \cdot \underset{\sim}{C}_1^* - \underset{\sim}{J}_0'(u) \frac{\underset{\sim}{K}_0(u') \cdot \underset{\sim}{C}_1}{u^2 - c^2} \right] \right\}$$

$$- 2 \frac{\omega_p^2}{k^2} \left\{ L_0(u) \cdot \underset{\sim}{C}_3^* + \frac{L_0(u)}{n} \cdot \tilde{\underset{\sim}{J}}_0 - M_0'(u) \cdot Q_0 \cdot \tilde{\underset{\sim}{A}}_0 \frac{k^2 u}{nm(u^2 - c^2)} - \frac{c^2 k^2}{nm} \frac{M_0'(u) \cdot Q_0 \cdot \tilde{\underset{\sim}{J}}_0'}{u^2 - c^2} \right.$$

$$\left. + \frac{k^2 L_0(u) \cdot Q_0 \cdot \tilde{\underset{\sim}{A}}_0}{nm} - \frac{1}{2k} \tilde{F}_0' L_0(u) \cdot \underset{\sim}{C}_2^* \right\} \tag{3.17}$$

Equations (3.15), (3.16) and (3.17) represent four equations with three functions of u', $\left[H^*(u') \text{ and } \underset{\sim}{P}^*(u')/[(u')^2 - c^2] \right]$ under the integral sign. We can represent the coefficients of these functions as a matrix having four rows and three columns. From the general theory of determinants we know that if we multiply each row by the determinant of the 3 x 3 matrix found by the remaining rows and add, then the sum of each of the columns is zero. In this manner, it is not difficult to show that we must multiply Eq. (3.15) by the determinant

$$\mathcal{A} = \begin{vmatrix} F_0' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix} \tag{3.18}$$

where the subscripts 1 and 2 refer to the two directions of polarization perpendicular to $\underset{\sim}{k}$ and where we have omitted the subscript o on $\underset{\sim}{J}_0'$ and M_0'. Similarly, we must multiply Eq. (3.16) by:

$$\mathcal{B} = -\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \begin{vmatrix} 1 & v_1 & v_2 \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix} \tag{3.19}$$

and take the scalar product of Eq. (3.17) with the vector $\underset{\sim}{\mathcal{C}}$ whose two components are:

$$\mathcal{C}_1 = \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \begin{vmatrix} 1 & v_1 & v_2 \\ F_0' & J_1' & J_2' \\ J_2' & M_{21}' & M_{22}' \end{vmatrix}$$

$$\mathcal{C}_2 = -\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \begin{vmatrix} 1 & v_1 & v_2 \\ F_0' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \end{vmatrix} \tag{3.20}$$

After performing the above multiplications and then adding Eqs. (3.15), (3.16), and (3.17), we obtain a new equation which is no longer an integral equation. It is:

$$\left[1 + \frac{\omega_p^2}{k^2} \psi(u) \right] \left[\mathcal{A}\, h(\underset{\sim}{v}) + \mathcal{B} H(u) + \underset{\sim}{\mathcal{C}} \cdot \underset{\sim}{S}(u) \right] + \frac{2\, \omega_p^2\, \underset{\sim}{X}(u)}{k^2(u^2 - c^2)} \cdot \left[\mathcal{A}\, \underset{\sim}{p}(\underset{\sim}{v}) + \mathcal{B}\, \underset{\sim}{P}(u) \right.$$

$$\left. + T(u) \cdot \underset{\sim}{\mathcal{C}} \right] = - \frac{\omega_p^2\, \psi(u)}{k^2 n} \left[\mathcal{A}\, f_o(\underset{\sim}{v}) + \mathcal{B} F_o(u) + \underset{\sim}{\mathcal{C}} \cdot \underset{\sim}{J}_o(u) \right] - \frac{2\, \omega_p^2\, \underset{\sim}{X}(u)}{k^2 n(u^2 - c^2)}$$

$$\cdot \left[\mathcal{A}\, \underset{\sim}{v} \cdot f_o(\underset{\sim}{v}) + \mathcal{B} \underset{\sim}{J}_o(u) + M_o(u) \cdot \underset{\sim}{\mathcal{C}} \right] - \frac{2\, \omega_p^2}{k^2} \left[\mathcal{A}\, \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \mathcal{B} \underset{\sim}{K}_o(u) + \underset{\sim}{\mathcal{C}} \cdot L_o(u) \right]$$

$$X \left[- \frac{1}{2} \gamma(u) \underset{\sim}{C}_1^* + \underset{\sim}{C}_3^* + \frac{1}{n} \tilde{\underset{\sim}{J}}_o + \frac{k^2 Q_o \cdot \tilde{\underset{\sim}{A}}_o}{nm} - \frac{1}{2k} \tilde{F}_o' \underset{\sim}{C}_2^* \right] \tag{3.21}$$

where we have introduced the quantities

$$\psi(u) = \lim_{\epsilon \to o} \int \frac{F_o'(u')}{i\epsilon + u - u'}\, du' \tag{3.22}$$

$$\underset{\sim}{X}(u) = \lim_{\epsilon \to o} \int \frac{\underset{\sim}{J}_o'(u')}{i\epsilon + u - u'}\, du' \tag{3.23}$$

$$\gamma(u) = \lim_{\epsilon \to o} \int \frac{uu' - c^2}{[(u')^2 - c^2]} \frac{F_o'(u')}{i\epsilon + u - u'}\, du' \tag{3.24}$$

Equation (3.21) is somewhat analogous to Eq. (9) in Lenard's paper. However, we cannot simply divide both sides by $1 + \frac{\omega_p^2}{k^2}\psi(u)$ since there is also the term containing $\underset{\sim}{X}(u)$. To carry through the Lenard trick, we require a second equation relating $h(\underset{\sim}{v})$ and $\underset{\sim}{p}(\underset{\sim}{v})$.

Before obtaining this second set, we note some simplifications. We can rewrite Eq. (3.24) so that

$$\gamma(u) = \lim_{\epsilon \to 0} \int \frac{[(u')^2 - c^2 + u'(u - u')]\, F_0'(u')}{[(u')^2 - c^2][i\epsilon + u - u']}\, du'$$

$$= \psi(u) + \int \frac{u\, F_0'(u)}{u^2 - c^2}\, du \tag{3.25}$$

In addition, upon integrating Eq. (2.18) we obtain an equation relating $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2$, and $\underset{\sim}{C}_3$ which is:

$$\underset{\sim}{C}_1 = \frac{2\,\omega_p^2}{k^2}\left[\frac{\tilde{F}_0'}{2k}\underset{\sim}{C}_2 + \frac{1}{2}\int \frac{u\, F_0'(u)}{u^2 - c^2}\, du\, \underset{\sim}{C}_1 - \frac{k^2}{nm}\underset{\sim}{\tilde{A}}_0 \cdot Q_0 - \underset{\sim}{C}_3 - \frac{1}{n}\underset{\sim}{\tilde{J}}_0\right] \tag{3.26}$$

The last term on the right hand side of Eq. (3.21) simplifies greatly when Eqs. (3.25) and (3.26) are substituted therein and we rewrite Eq. (3.21) as:

$$\left[1 + \frac{\omega_p^2}{k^2} \psi(u) \right] \left[\mathcal{A}\, h(\underset{\sim}{v}) + \mathcal{B} H(u) + \underset{\sim}{\mathcal{C}} \cdot \underset{\sim}{S}(u) \right] + \frac{2\,\omega_p^2}{k^2(u^2 - c^2)} \underset{\sim}{\chi}(u)$$

$$\cdot \left[\mathcal{A}\, \underset{\sim}{p}(\underset{\sim}{v}) + \mathcal{B} \underset{\sim}{P}(u) + T(u) \cdot \underset{\sim}{\mathcal{C}} \right] = - \frac{\omega_p^2}{k^2 n} \psi(u) \left[\mathcal{A}\, f_o(\underset{\sim}{v}) + \mathcal{B} F_o(u) + \underset{\sim}{\mathcal{C}} \cdot \underset{\sim}{J}_o(u) \right]$$

$$- \frac{2\,\omega_p^2}{nk^2(u^2 - c^2)} \underset{\sim}{\chi}(u) \cdot \left[\mathcal{A}\, \underset{\sim}{v}\, f_o(\underset{\sim}{v}) + \mathcal{B} \underset{\sim}{J}_o(u) + M_o(u) \cdot \underset{\sim}{\mathcal{C}} \right] + \left[1 + \frac{\omega_p^2}{k^2} \psi(u) \right]$$

$$\times \left[\mathcal{A}\, \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \mathcal{B} \underset{\sim}{K}_o(u) + \underset{\sim}{\mathcal{C}} \cdot L_o(u) \right] \cdot \underset{\sim}{C}_1^* \tag{3.27}$$

Note that all dependence on Q_o has been eliminated.

We now obtain the first equation in our second set by multiplying Eq. (2.21) by $\underset{\sim}{v}_j \cdot I$ and integrating over $d^3 v_j$.

$$\underset{\sim}{p}(\underset{\sim}{v}) = \lim_{\epsilon \to o} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \left\{ \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \left[\underset{\sim}{S}^*(u') + \frac{\underset{\sim}{J}_o(u')}{n} \right] - \underset{\sim}{J}_o'(u') \right.$$

$$\left[h(\underset{\sim}{v}) + \frac{f_o(\underset{\sim}{v})}{n} \right] + 2 \left[\left(\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right) \frac{\underset{\sim}{v}}{(u')^2 - c^2} \cdot T^*(u') - M_o'(u') \cdot \frac{\underset{\sim}{p}(\underset{\sim}{v})}{u^2 - c^2} \right]$$

$$+ \frac{2}{n} \underset{\sim}{v} \cdot \left[\left(\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right) \frac{M_o(u')}{(u')^2 - c^2} - M_o'(u') \frac{f_o(\underset{\sim}{v})}{u^2 - c^2} \right] - \frac{1}{k} \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}$$

$$\left[\underset{\sim}{J}_o'(u') \underset{\sim}{v} \cdot \frac{(\underset{\sim}{C}_2^* + ku' \underset{\sim}{C}_1^*)}{(u')^2 - c^2} - M_o'(u') \cdot \frac{(\underset{\sim}{C}_2 + ku \underset{\sim}{C}_1)}{u^2 - c^2} \right] + (uu' - c^2)$$

Equation continued on next page

$$\left[\frac{\underset{\sim}{J}_o'(u')}{u'^2 - c^2} \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{C}_1^* - \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \frac{L_o(u') \cdot \underset{\sim}{C}_1}{u^2 - c^2} \right] \Bigg\}$$

$$- \frac{2\,\omega_p^2}{k^2} \Bigg\{ \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot C_4^* + \frac{\tilde{B}_o}{u^2 - c^2} \cdot \underset{\sim}{p}(\underset{\sim}{v}) + \frac{1}{n} \tilde{M}_o \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \frac{1}{n} \frac{f_o(\underset{\sim}{v})}{u^2 - c^2} \tilde{B}_o \cdot \underset{\sim}{v}$$

$$- \frac{uk^2}{nm} \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \frac{\tilde{D}_o \cdot Q_o \cdot \underset{\sim}{v}}{u^2 - c^2} - \frac{c^2 k^2}{nm} \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \frac{\tilde{M}_o' \cdot Q_o \cdot \underset{\sim}{v}}{u^2 - c^2} - \frac{k^2}{nm} \tilde{B}_o \cdot Q_o \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}$$

$$+ \frac{k^2}{nm} \tilde{B}_o \cdot Q_o \cdot \underset{\sim}{v} \frac{u}{u^2 - c^2} \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \frac{k^2}{nm} \tilde{D}_o \cdot Q_o \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{1}{2k}$$

$$\tilde{\underset{\sim}{J}}_o' \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{C}_2^* - \frac{1}{2k} \frac{\tilde{B}_o \cdot \underset{\sim}{C}_2}{u^2 - c^2} \left(\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right) \Bigg\} \tag{3.28}$$

where we have introduced several additional constants. These are:

$$C_4 = \int \frac{T(u)}{u^2 - c^2}\, du$$

$$\tilde{B}_o = \int L_o(u)\, du$$

$$\tilde{M}_o = \int \frac{M_o(u)}{u^2 - c^2}\, du$$

$$\tilde{D}_o = \int \frac{u\, M_o'(u)}{u^2 - c^2}\, du$$

$$\tilde{M}_o' = \int \frac{M_o'(u)}{u^2 - c^2}\, du$$

Note that C_4 is analagous to $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2$, and $\underset{\sim}{C}_3$ in that its value can only be determined after we have the solution.

A second equation results from multiplying Eq. (3.28) by $k\,\delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})$ and integrating over d^3v.

$$\underset{\sim}{P}(u) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \Bigg\{ F_0'(u) \left[\underset{\sim}{S}^*(u') + \frac{\underset{\sim}{J}_0(u')}{n} \right] - \underset{\sim}{J}_0(u') \left[H(u) + \frac{F_0(u)}{n} \right]$$

$$+ 2 \left[\frac{\underset{\sim}{J}_0'(u)}{(u')^2 - c^2} \cdot T^*(u') - \frac{M_0'(u')}{u^2 - c^2} \cdot \underset{\sim}{P}(u) \right] + \frac{2}{n} \left[\underset{\sim}{J}_0'(u) \cdot \frac{M_0(u')}{(u')^2 - c^2} \right.$$

$$\left. - \frac{M_0'(u')}{u^2 - c^2} \cdot \underset{\sim}{J}_0(u) \right] - \frac{1}{k} \left[\frac{\underset{\sim}{J}_0'(u')}{(u')^2 - c^2} \underset{\sim}{J}_0'(u) \cdot (\underset{\sim}{C}_2^* + ku' \underset{\sim}{C}_1^*) - \frac{F_0'(u')}{u^2 - c^2} M_0'(u') \right.$$

$$\left. \cdot (\underset{\sim}{C}_2 + ku\, \underset{\sim}{C}_1) \right] + (uu' - c^2) \left[\frac{\underset{\sim}{J}_0'(u')}{(u')^2 - c^2} \underset{\sim}{K}_0(u) \cdot \underset{\sim}{C}_1^* - \frac{F_0'(u)}{u^2 - c^2} L_0(u') \cdot \underset{\sim}{C}_1 \right] \Bigg\}$$

$$- 2 \frac{\omega_p^2}{k^2} \Bigg\{ \underset{\sim}{K}_0(u) \cdot C_4^* + \frac{\tilde{B}_0}{u^2 - c^2} \cdot \underset{\sim}{P}(u) + \frac{1}{n} \tilde{M}_0 \cdot \underset{\sim}{K}_0(u) + \frac{1}{n} \frac{\tilde{B}_0}{u^2 - c^2} \cdot \underset{\sim}{J}_0(u)$$

$$- \frac{uk^2}{nm} \frac{\tilde{D}_0 \cdot Q_0 \cdot \underset{\sim}{J}_0'(u)}{u^2 - c^2} - \frac{c^2 k^2}{nm} \frac{\tilde{M}_0' \cdot Q_0' \cdot \underset{\sim}{J}_0'(u)}{u^2 - c^2} - \frac{k^2}{nm} \tilde{B}_0 \cdot Q_0 \cdot \underset{\sim}{K}_0(u) + \frac{k^2}{nm} \frac{u}{u^2 - c^2} \tilde{B}_0 \cdot Q_0 \cdot \underset{\sim}{J}_0'(u)$$

$$+ \frac{k^2}{nm} \tilde{D}_0 \cdot Q_0 \cdot \underset{\sim}{K}_0(u) - \frac{1}{2k} \tilde{\underset{\sim}{J}}_0' \underset{\sim}{K}_0(u) \cdot \underset{\sim}{C}_2^* + \frac{1}{2k} \frac{F_0'(u)}{u^2 - c^2} \tilde{B}_0 \cdot \underset{\sim}{C}_2 \Bigg\} \qquad (3.29)$$

The third and final equation is obtained by multiplying Eq. (3.28) by $k(\underset{\sim}{y}\cdot I)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{y})$ and integrating over d^3v. This is a dyadic equation (and thus actually 4 equations) and is:

$$T(u) = \lim_{\epsilon \to 0} \int \frac{du'}{i\epsilon + u - u'} \frac{\omega_p^2}{k^2} \left\{ \left[\underset{\sim}{S}^*(u') + \frac{\underset{\sim}{J}_0(u')}{n} \right] \underset{\sim}{J}_0'(u) - \underset{\sim}{J}_0'(u') \right.$$

$$\left[\underset{\sim}{S}(u) + \frac{\underset{\sim}{J}_0(u)}{n} \right] + 2 \left[\hat{T}^*(u') \cdot \frac{M_0'(u)}{(u')^2 - c^2} - M_0'(u') \cdot \frac{T(u)}{u^2 - c^2} \right]$$

$$+ \frac{2}{n} \left[\frac{M_0(u')}{(u')^2 - c^2} \cdot M_0'(u) - \frac{M_0'(u')}{u^2 - c^2} \cdot M_0(u) \right] - \frac{1}{k} \left[\underset{\sim}{J}_0'(u') \frac{(\underset{\sim}{C}_2^* + ku'\ \underset{\sim}{C}_1^*)}{(u')^2 - c^2} \cdot M_0'(u) \right.$$

$$\left. - \frac{M_0'(u')}{u^2 - c^2} \cdot (\underset{\sim}{C}_2 + ku\ \underset{\sim}{C}_1)\ \underset{\sim}{J}_0'(u) \right] + (uu' - c^2) \left[\frac{\underset{\sim}{J}_0'(u')}{(u')^2 - c^2} L_0(u) \cdot C_1^* \right.$$

$$\left. \left. - \frac{L_0(u') \cdot \underset{\sim}{C}_1}{u^2 - c^2} \underset{\sim}{J}_0'(u) \right] \right\} - 2 \frac{\omega_p^2}{k^2} \left\{ \hat{C}_4^* \cdot \hat{L}_0(u) + \frac{\tilde{B}_0}{u^2 - c^2} \cdot T(u) + \frac{1}{n} \tilde{M}_0 \cdot \hat{L}_0(u) \right.$$

$$+ \frac{1}{n} \tilde{B}_0 \cdot \frac{M_0(u)}{u^2 - c^2} - \frac{uk^2}{nm} \frac{\tilde{D}_0 \cdot Q_0 \cdot M_0'(u)}{u^2 - c^2} - \frac{c^2 k^2}{nm} \frac{\tilde{M}_0' \cdot Q_0 \cdot M_0'(u)}{u^2 - c^2} - \frac{k^2}{nm} \tilde{B}_0 \cdot Q_0 \cdot \hat{L}_0(u)$$

$$+ \frac{k^2}{nm} \frac{u}{u^2 - c^2} \tilde{B}_0 \cdot Q_0 \cdot M_0'(u) + \frac{k^2}{nm} \tilde{D}_0 \cdot Q_0 \cdot \hat{L}_0(u) - \frac{1}{2k} \tilde{J}_0'\ L_0(u) \cdot \underset{\sim}{C}_2^*$$

$$\left. + \frac{1}{2k} \frac{\tilde{B}_0 \cdot \underset{\sim}{C}_2}{u^2 - c^2} \underset{\sim}{J}_0'(u) \right\} \qquad (3.30)$$

where the symbols $\hat{T}$ and $\hat{L}_o$ denote the transpose of the matrix.

Equations (3.28), (3.29), and (3.30) represent eight equations with six unknown functions under the integral sign. Actually they factor into two sets (corresponding to p_1, P_1, T_{11}, and T_{12} in one, and p_2, P_2, T_{21}, and T_{22} in the other) each having three unknown functions, and it is easy to show that one may eliminate the unknowns by use of precisely the same multiplicative factors as before. Thus we multiply Eq. (3.28) by the determinant $\mathcal{A}$ of Eq. (3.18), multiply Eq. (3.29) by the determinant $\mathcal{B}$ of Eq. (3.19) and take the scalar product (from the right) of the determinants $\underset{\sim}{\mathcal{C}}$ of Eq. (3.20) with Eq. (3.30). Upon addition, we obtain a new equation which is no longer an integral equation. It is:

$$\left[1 + \frac{2\,\omega_p^2}{k^2(u^2 - c^2)} \left\{ \Theta(u) + \tilde{B}_o \right\} \right] \cdot \left[\mathcal{A}\, \underset{\sim}{p}(\underset{\sim}{v}) + \mathcal{B}\, \underset{\sim}{P}(u) + T(u) \cdot \underset{\sim}{\mathcal{C}} \right] + \frac{\omega_p^2}{k^2}\, \underset{\sim}{\chi}(u)$$

$$\left[\mathcal{A}\, h(\underset{\sim}{v}) + \mathcal{B}\, H(u) + \underset{\sim}{S}(u) \cdot \underset{\sim}{\mathcal{C}} \right] = - \frac{\omega_p^2}{nk^2}\, \underset{\sim}{\chi}(u) \left[\mathcal{A}\, f_o(\underset{\sim}{v}) + \mathcal{B}\, F_o(u) + \underset{\sim}{J}_o(u) \cdot \underset{\sim}{\mathcal{C}} \right]$$

$$- \frac{2\,\omega_p^2}{nk^2(u^2 - c^2)} \left\{ \Theta(u) + \tilde{B}_o \right\} \cdot \left[\mathcal{A}\, \underset{\sim}{v}\, f_o(\underset{\sim}{v}) + \mathcal{B}\, \underset{\sim}{J}_o(u) + M_o(u) \cdot \underset{\sim}{\mathcal{C}} \right]$$

$$- 2\, \frac{\omega_p^2}{k^2} \left[- \frac{1}{2}\, \underset{\sim}{\xi}(u) \underset{\sim}{C}_1^* + \hat{C}_4^* + \frac{1}{n}\, \tilde{M}_o - \frac{k^2}{nm}\, \tilde{B}_o \cdot Q_o + \frac{k^2}{nm}\, \tilde{D}_o \cdot Q_o - \frac{1}{2k}\, \tilde{\underset{\sim}{J}}_o'\, \underset{\sim}{C}_2^* \right]$$

$$\cdot \left[\mathcal{A}\, \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \mathcal{B}\, \underset{\sim}{K}_o(u) + \hat{L}_o(u) \cdot \underset{\sim}{\mathcal{C}} \right] \tag{3.31}$$

We have introduced two new symbols above:

$$\underset{\sim}{\xi}(u) = \lim_{\epsilon \to 0} \int \frac{uu' - c^2}{u'^2 - c^2} \frac{\underset{\sim}{J}_o'(u')}{i\epsilon + u - u'} du' \tag{3.32}$$

$$\Theta(u) = \lim_{\epsilon \to 0} \int \frac{M_o'(u')}{i\epsilon + u - u'} du' \tag{3.33}$$

Once again we can simplify this last result. We can rewrite Eq. (3.22) so that

$$\underset{\sim}{\xi}(u) = \lim_{\epsilon \to 0} \int \frac{(u')^2 - c^2 + u'(u - u')}{[(u')^2 - c^2][i\epsilon + u - u']} \underset{\sim}{J}_o'(u') \, du'$$

$$= \underset{\sim}{\chi}(u) + \tilde{\underset{\sim}{A}}_o \tag{3.34}$$

where the symbols on the right hand side have been introduced previously. In addition, upon multiplying Eq. (2.18) by $I \cdot \underset{\sim}{v}$ and integrating over d^3v we obtain a new relation:

$$C_5 = \frac{2\,\omega_p^2}{k^2} \left[\frac{1}{2k} \tilde{\underset{\sim}{J}}_o' \underset{\sim}{C}_2 + \frac{1}{2} \tilde{\underset{\sim}{A}}_o \underset{\sim}{C} - \frac{k^2}{nm} \tilde{D}_o \cdot Q_o - \hat{C}_4 - \frac{1}{n} \tilde{M}_o + \frac{k^2}{nm} \tilde{B}_o \cdot Q_o \right] \tag{3.35}$$

where we have introduced the new constant dyadic

$$C_5 = \int (I \cdot \underset{\sim}{v}) \, \underset{\sim}{E}(\underset{\sim}{v}) \, d\underset{\sim}{v} \tag{3.36}$$

The last term on the right hand side of Eq. (3.31) simplifies greatly when Eqs. (3.34) and (3.35) are substituted therein and we rewrite Eq. (3.31) as:

$$\left[1 + \frac{2\,\omega_p^2}{k^2(u^2 - c^2)} \left\{ \Theta(u) + \tilde{B}_o \right\} \right] \cdot \left[\mathcal{A}\, \underset{\sim}{p}(\underset{\sim}{v}) + \mathcal{B}\, \underset{\sim}{P}(u) + T(u) \cdot \underset{\sim}{\mathcal{C}} \right] + \frac{\omega_p^2}{k^2} \underset{\sim}{\chi}(u)$$

$$\cdot \left[\mathcal{A}\, h(\underset{\sim}{v}) + \mathcal{B} H(u) + \underset{\sim}{S}(u) \cdot \underset{\sim}{\mathcal{C}} \right] = - \frac{\omega_p^2}{nk^2} \underset{\sim}{\chi}(u) \left[\mathcal{A}\, f_o(\underset{\sim}{v}) + \mathcal{B} F_o(u) + \underset{\sim}{J}_o(u) \cdot \underset{\sim}{\mathcal{C}} \right]$$

$$- \frac{2\,\omega_p^2}{nk^2(u^2 - c^2)} \left\{ \Theta(u) + \tilde{B}_o \right\} \cdot \left[\mathcal{A}\, \underset{\sim}{v}\, f_o(\underset{\sim}{v}) + \mathcal{B} \underset{\sim}{J}_o(u) + M_o(u) \cdot \underset{\sim}{\mathcal{C}} \right]$$

$$+ \left[\frac{\omega_p^2}{k^2} \underset{\sim}{\chi}(u) \underset{\sim}{C}_1^* + C_5^* \right] \cdot \left[\mathcal{A} \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} + \mathcal{B} \underset{\sim}{K}_o(u) + \hat{L}_o(u) \cdot \underset{\sim}{\mathcal{C}} \right]. \tag{3.37}$$

Equations (3.27) and (3.37) are now entirely analagous to Eq. (9) of Lenard. Before solving them, we note that the dyadic C_5 represents the self-interaction of a particle with its own vector field. Although the coulomb self-interaction is explicitly omitted from the initial Hamiltonian [see I, Eq. (4)], it is not possible to do this for the transverse vector field. Instead, we must remove this term at this point and set $C_5 = 0$.

IV. THE FOKKER-PLANCK COEFFICIENTS

Equations (3.27) and (3.37) can be written in a compact way if we introduce the 3 x 3 matrix,

$$Y(u) = \begin{pmatrix} 1 + \frac{\omega_p^2}{k^2}\psi & \frac{\omega_p^2}{k^2}\chi_1 & \frac{\omega_p^2}{k^2}\chi_2 \\ \frac{\omega_p^2}{k^2}\chi_1 & \frac{u^2 - c^2}{2} + \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11} & \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{12} \\ \frac{\omega_p^2}{k^2}\chi_2 & \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{21} & \frac{u^2 - c^2}{2} + \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{22} \end{pmatrix} \tag{4.1}$$

where the subscripts 1 and 2 refer to the two polarization directions perpendicular to $\vec{k}$. It should be noted that Y is a symmetric matrix (even in the relativistic case). We also define the 3-vectors:

$$\underset{\sim}{W} = \begin{bmatrix} \mathcal{A}h(\underset{\sim}{v}) + \mathcal{B}H + \underset{\sim}{\mathcal{C}}\cdot\underset{\sim}{S} - \left\{\mathcal{A}\frac{\partial f_0}{\partial \underset{\sim}{v}_i} + \mathcal{B}\underset{\sim}{K}_0 + \hat{L}_0\cdot\underset{\sim}{\mathcal{C}}\right\}\cdot\underset{\sim}{C}_i^* \\ \frac{2}{u^2 - c^2}\left\{\mathcal{A}\, p_1(\underset{\sim}{v}) + \mathcal{B}P_1 + T_{11}\mathcal{C}_1 + T_{12}\mathcal{C}_2\right\} \\ \frac{2}{u^2 - c^2}\left\{\mathcal{A}\, p_2(\underset{\sim}{v}) + \mathcal{B}P_2 + T_{21}\mathcal{C}_1 + T_{21}\mathcal{C}_2\right\} \end{bmatrix} \tag{4.2}$$

$$\underset{\sim}{V}_0 = \frac{1}{n}\begin{bmatrix} \mathcal{A}\, f_0(\underset{\sim}{v}) + \mathcal{B}F_0 + \mathcal{C}_1 J_1 + \mathcal{C}_2 J_2 \\ \frac{2}{u^2 - c^2}\left\{\mathcal{A}\, v_1\, f_0(v) + \mathcal{B}J_1 + M_{11}\mathcal{C}_1 + M_{12}\mathcal{C}_2\right\} \\ \frac{2}{u^2 - c^2}\left\{\mathcal{A}\, v_2\, f_0(\underset{\sim}{v}) + \mathcal{B}J_2 + M_{21}\mathcal{C}_1 + M_{22}\mathcal{C}_2\right\} \end{bmatrix} \tag{4.3}$$

and

$$\underset{\sim}{U}_0 = \frac{1}{n} \begin{bmatrix} \mathcal{A} f_0(\underset{\sim}{v}) + \mathcal{B} F_0 + \mathcal{C}_1 J_1 + \mathcal{C}_2 J_2 \\ \mathcal{A} v_1 F_0(\underset{\sim}{v}) + \mathcal{B} J_1 + M_{11}\mathcal{C}_1 + M_{12}\mathcal{C}_2 \\ \mathcal{A} v_2 f_0(\underset{\sim}{v}) + \mathcal{B} J_2 + M_{21}\mathcal{C}_1 + M_{22}\mathcal{C}_2 \end{bmatrix} \tag{4.4}$$

where we omitted the subscript 0 on $\underset{\sim}{J}_0$ and M_0. Hence, we can write Eqs. (3.27) and (3.37) as

$$Y(u) \cdot \left\{ \underset{\sim}{W} + \underset{\sim}{V}_0 \right\} = \underset{\sim}{U}_0 \tag{4.5}$$

and hence

$$\underset{\sim}{W} + \underset{\sim}{V}_0 = Y^{-1}(u) \cdot \underset{\sim}{U}_0 \tag{4.6}$$

We can now complete the Lenard procedure. Take the imaginary part of both sides of Eq. (4.6). The essential point is that the quantities H, $\underset{\sim}{P}$, $\underset{\sim}{S}$ and T, and their associated integrals $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2$, and $\underset{\sim}{C}_3$ have zero imaginary parts. This follows from the fact that they satisfy homogeneous equations. To see this, note that

$$\lim_{\epsilon \to 0} \int_{-} \frac{g(u')\, du'}{i\epsilon + u - u'} = P \int_{-\infty}^{\infty} \frac{g(u')\, du'}{u - u'} - i\pi g(u) \tag{4.7}$$

where the symbol P denotes the principle value. If we now take the imaginary part of both sides of Eqs. (3.16) and (3.30) as well as of the sum of Eqs. (3.17) and (3.29) we find that the imaginary parts of H, $\underset{\sim}{P}$, $\underset{\sim}{S}$, and T do satisfy a set of coupled homogeneous integral equations. Following Lenard, we conclude that all these quantities are zero. Thus Eq. (4.6) becomes

$$\mathrm{IM}\begin{bmatrix} h(\underset{\sim}{v}) \\ \dfrac{2\,p_1(\underset{\sim}{v})}{u^2 - c^2} \\ \\ \dfrac{2\,p_2(\underset{\sim}{v})}{u^2 - c^2} \end{bmatrix} = \frac{\mathrm{IM}\,[Y^{-1}(u)]}{\mathcal{A}(u)} \cdot \underset{\sim}{U_0} \tag{4.8}$$

where IM denotes the imaginary part.

As we shall see in a moment, we are interested only in the particular combination

$$\mathrm{IM}\left\{ h(\underset{\sim}{v}) + \frac{2}{(u^2 - c^2)}\,[v_1\,p_1(\underset{\sim}{v}) + v_2\,p_2(\underset{\sim}{v})] \right\}$$

Hence we take the scalar product of Eq. (4.8) and the row vector

$$\underset{\sim}{\alpha} = [1,\ v_1,\ v_2] \tag{4.9}$$

to obtain this. Thus

$$I_m \left\{ h(\underset{\sim}{v}) + \frac{2}{u^2 - c^2} \left[v_1\, p_1(\underset{\sim}{v}) + v_2\, p_2(\underset{\sim}{v}) \right] \right\} = \alpha \cdot (\mathrm{IM}\, Y^{-1}) \cdot \frac{\underset{\sim}{U_0}}{a} \tag{4.10}$$

The quantity calculated above is precisely the quantity which appears in the Fokker-Planck equation. This is seen by substituting the Fourier integral representations of $g(1,2)$ and $g(1,\nu)$ into the right hand side of Eq. (4.1) in I. The integrals are elementary and we obtain

$$\text{r.h.s.} = \frac{i}{(2\pi)^3} \int d^3k \frac{\partial}{\partial \underset{\sim}{v}} \left\{ \underset{\sim}{k} \frac{\omega_p^2}{k^2} h(\underset{\sim}{v}) - [\underset{\sim}{v} \times (\underset{\sim}{k} \times \underset{\sim}{E}) + \underset{\sim}{F}] \right\} \tag{4.11}$$

Since the equation is real, only the imaginary parts of h, $\underset{\sim}{E}$, and $\underset{\sim}{F}$ contribute. From Eqs. (2.18) and (2.19) we see that

$$\mathrm{IM}\, \underset{\sim}{E} = -2 \frac{\omega_p^2}{k^2} \frac{\mathrm{IM}\, \underset{\sim}{p}(\underset{\sim}{v})}{u^2 - c^2} \tag{4.12}$$

$$\mathrm{IM}\, \underset{\sim}{F} = -2 \frac{\omega_p^2 u}{k} \frac{\mathrm{IM}\, \underset{\sim}{p}(\underset{\sim}{v})}{u^2 - c^2} \tag{4.13}$$

Hence Eq. (4.11) may be written as:

$$\text{r.h.s.} = -\frac{1}{(2\pi)^3} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3k\, \underset{\sim}{k} \frac{\omega_p^2}{k^2} \mathrm{IM} \left\{ h(\underset{\sim}{v}_i) \right.$$

$$\left. + \frac{2}{u^2 - c^2} \left[v_1\, p_1(\underset{\sim}{v}) + v_2\, p_2(\underset{\sim}{v}) \right] \right\} \tag{4.14}$$

and by Eq. (4.10) this becomes

$$\text{r.h.s.} = -\frac{1}{(2\pi)^3}\,\frac{\partial}{\partial \underset{\sim}{v}} \cdot \int \underset{\sim}{k}\,\frac{\omega_p^2}{k^2}\,\underset{\sim}{\alpha} \cdot (\mathrm{IM}\ Y^{-1}) \cdot \frac{\underset{\sim}{U_0}}{\mathcal{A}}\, d^3k \tag{4.15}$$

This result can be expressed in a much simpler form by expanding $\underset{\sim}{U_0}$, as defined in Eq. (4.4), in the following manner:

$$\frac{\underset{\sim}{U_0}}{\mathcal{A}} = \frac{1}{n}\left[\underset{\sim}{f_0}(\underset{\sim}{v})\,\underset{\sim}{\alpha} + \begin{pmatrix} F_0 & J_1 & J_2 \\ J_1 & M_{11} & M_{12} \\ J_2 & M_{21} & M_{22} \end{pmatrix} \cdot \frac{1}{\mathcal{A}}\begin{pmatrix} \mathcal{B} \\ \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}\right]$$

From the definitions of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ in Eqs. (3.18) to (3.20), we can see almost at once that

$$\frac{1}{\mathcal{A}}\begin{pmatrix} \mathcal{B} \\ \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} = -\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \begin{pmatrix} F_0' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{pmatrix}^{-1} \cdot \underset{\sim}{\alpha}$$

Hence if we define the tensors

$$G = \begin{pmatrix} F_0 & J_1 & J_2 \\ J_1 & M_{11} & M_{12} \\ J_2 & M_{21} & M_{22} \end{pmatrix} \tag{4.16}$$

and

$$G' = \begin{pmatrix} F_0' & J_1' & J_2' \\ J_1' & M_{11}' & M_{12}' \\ J_2' & M_{21}' & M_{22}' \end{pmatrix} \tag{4.17}$$

then

$$\frac{\underset{\sim}{U_0}}{a} = \frac{1}{n}\left[f_0(\underset{\sim}{v}) - \frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} G\cdot(G')^{-1} \right] \cdot \underset{\sim}{\alpha} \tag{4.18}$$

We can now express the result in Eq. (4.15) in a standard form.

$$\text{r.h.s.} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left\{ \underset{\sim}{A}\, f_0(\underset{\sim}{v}) + B \cdot \frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \right\} \tag{4.19}$$

where

$$\underset{\sim}{A} = \frac{\omega_p^2}{(2\pi)^3 n} \int \frac{\underset{\sim}{k}}{k^2}\, \mathrm{IM}\, [\underset{\sim}{\alpha} \cdot \frac{1}{Y} \cdot \underset{\sim}{\alpha}]\, d^3k \tag{4.20}$$

$$B = -\frac{\omega_p^2}{(2\pi)^3 n} \int \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^3}\, \mathrm{IM}\, [\underset{\sim}{\alpha} \cdot \frac{1}{Y} \cdot \frac{G}{G'} \cdot \underset{\sim}{\alpha}]\, d^3k \tag{4.21}$$

Equation (4.19) represents our final result. Although the derivation has been carried through in a non-relativistic way, the results are made relativistic by simply replacing

$$\frac{\partial}{\partial \underset{\sim}{v}} \rightarrow m \frac{\partial}{\partial \underset{\sim}{\xi}}$$

and

$$d^3v \rightarrow d^3\xi$$

everywhere, and considering $f_o(\underset{\sim}{\xi})$ as normalized to unity for integration over $d^3\xi$. The quantity $\underset{\sim}{v}$ itself remains explicitly the relativistic velocity and is related to $\underset{\sim}{\xi}$ by

$$m \underset{\sim}{v} = \frac{\underset{\sim}{\xi}}{\left[1 + \left(\frac{\xi}{mc}\right)^2\right]^{1/2}} \tag{4.22}$$

For convenience in using Eq. (4.19), we repeat here the essential definitions which appear in the three matrices Y, G, and G'.

$$\psi(u) = \lim_{\epsilon \rightarrow o} \int \frac{F_o'(u')\, du'}{i\epsilon + u - u'} \tag{4.23}$$

$$\underset{\sim}{\chi}(u) = \lim_{\epsilon \rightarrow o} \int \frac{\underset{\sim}{J}_o'(u')\, du'}{i\epsilon + u - u'} \tag{4.24}$$

$$\textcircled{\mu}(u) = \lim_{\epsilon \rightarrow o} \int \frac{M_o'(u')\, du'}{i\epsilon + u - u'} \tag{4.25}$$

$$\tilde{B}_o = \int (I \cdot \underset{\sim}{v})\ I \cdot \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\, d^3v \tag{4.26}$$

$$F_o(u) = k \int f_o(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k} \cdot \underset{\sim}{v})\ d^3v \tag{4.27}$$

$$F_o'(u) = \underset{\sim}{k} \cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{4.28}$$

$$\underset{\sim}{J}_o(v) = k \int f_o(\underset{\sim}{v})\ \underset{\sim}{v}\cdot I\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{4.29}$$

$$\underset{\sim}{J}_o'(u) = \underset{\sim}{k} \cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ \underset{\sim}{v}\cdot I\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{4.30}$$

$$M_o(u) = k \int f_o(\underset{\sim}{v})\ (I\cdot\underset{\sim}{v})(I\cdot\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v \tag{4.31}$$

$$M_o'(u) = \underset{\sim}{k} \cdot \int \frac{\partial f_o(\underset{\sim}{v})}{\partial \underset{\sim}{v}}\ (I\cdot\underset{\sim}{v})(I\cdot\underset{\sim}{v})\ \delta(ku - k\cdot\underset{\sim}{v})\ d^3v \tag{4.32}$$

V. SELF-CONSISTENCY

It is easy to show that the final results are self-consistent in that the imaginary part of the self-interaction is actually zero, and in that the imaginary parts of H, $\underset{\sim}{P}$, $\underset{\sim}{S}$, and T are zero. From Eq. (4.12) we have

$$\mathrm{IM} \int (I\cdot\underset{\sim}{v})\ \underset{\sim}{E}(\underset{\sim}{v})\ d^3v \propto \int (I\cdot\underset{\sim}{v})\ \mathrm{IM}\ \underset{\sim}{p}(\underset{\sim}{v})\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v$$

and by Eq. (4.8) this becomes

$$\propto \int (I\cdot\underset{\sim}{v})\ \underset{\sim}{U}_o\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v$$

where the second and third components of the $\underset{\sim}{U}_o$ vector, as defined in Eq. (4.4), are involved. From our basic definitions, however, and by use of Eqs. (3.18) to (3.20) it is easy to see that

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_1 \mathcal{B}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.1}$$

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_2 \mathcal{B}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.2}$$

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_1 \mathcal{C}_1\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = -\mathcal{A} \tag{5.3}$$

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_2 \mathcal{C}_1\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.4}$$

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_1 \mathcal{C}_2\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.5}$$

$$\int (\mathrm{I}\cdot\underset{\sim}{v})_2 \mathcal{C}_2\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = -\mathcal{A} \tag{5.6}$$

Hence

$$\int (\mathrm{I}\cdot\underset{\sim}{v})\ \underset{\sim}{U}_0\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3\underset{\sim}{v} = 0 \tag{5.7}$$

where this holds for all three components of $\underset{\sim}{U}_0$. Hence the self-interaction vanishes.

Similarly, since

$$\int \mathcal{B}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = -\mathcal{A} \tag{5.8}$$

$$\int \mathcal{C}_1\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.9}$$

$$\int \mathcal{C}_2\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3v = 0 \tag{5.10}$$

we can show immediately from Eqs. (4.8) and (4.4) that IM H and IM $\underset{\sim}{P}$ are zero. Finally, since IM $\underset{\sim}{S}$ and IM T are also proportional to the quantity on the left hand side of Eq. (5.7), they too are zero.

VI. ISOTROPIC MOMENTUM DISTRIBUTION

The general result of Eq. (4.19) simplifies greatly if we assume that the zero order kinetic momentum distribution is isotropic. Thus, using the full relativistic notation, we assume that

$$f_o = f_o(p^2)$$

where p stands for the kinetic momentum (i.e., $p_{\text{canonical}} - \frac{e}{c} A$).
Now, by Eq. (4.29)

$$\underset{\sim}{J}_o(u) = k \int f_o(p^2)\ \underset{\sim}{v} \cdot I\ \delta(ku - \underset{\sim}{k} \cdot \underset{\sim}{v})\ d^3p$$

$$= 0$$

by symmetry. Similarly, by Eq. (4.30)

$$\underset{\sim}{J}_o'(u) = m\underset{\sim}{k} \cdot \int \frac{\partial f_o(p^2)}{\partial \underset{\sim}{p}}\ \underset{\sim}{v} \cdot I\ \delta(ku - \underset{\sim}{k} \cdot \underset{\sim}{v})\ d^3p$$

$$= 0$$

and by Eqs. (4.31), (4.32), and (4.26)

$$M_o(u) = I\ k \int f_o(p^2)\ v_1^2\ \delta(ku - \underset{\sim}{k} \cdot \underset{\sim}{v})\ d^3p \tag{6.1}$$

$$M_0'(u) = mI\underset{\sim}{k}\cdot\int \frac{\partial f_0(p^2)}{\partial \underset{\sim}{p}}\, v_1^2\, \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\, d^3p \tag{6.2}$$

$$\tilde{B}_0 = mI\int \frac{\partial f_0(p^2)}{\partial p_1}\, v_1\, d^3p \tag{6.3}$$

We see at once that the tensors G, G', and Y are all diagonal now. Hence

$$\frac{G}{G'} = \begin{pmatrix} F_0/F_0' & 0 & 0 \\ 0 & M_{11}/M_{11}' & 0 \\ 0 & 0 & M_{11}/M_{11}' \end{pmatrix} \tag{6.4}$$

and

$$Y^{-1}(u) = \begin{pmatrix} \left[1 + \frac{\omega_p^2}{k^2}\psi(u)\right]^{-1} & 0 & 0 \\ 0 & \left[\frac{u^2-c^2}{2} + \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right]^{-1} & 0 \\ 0 & 0 & \left[\frac{u^2-c^2}{2} + \frac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right]^{-1} \end{pmatrix}$$

$$\therefore \quad \mathrm{IM}\, Y^{-1} = \frac{\omega_p^2}{k^2} \begin{pmatrix} \dfrac{\mathrm{IM}\,\psi^*}{\left|1 + \dfrac{\omega_p^2}{k^2}\psi\right|^2} & 0 & 0 \\ 0 & \dfrac{\mathrm{IM}\,\Theta_{11}^*}{\left|\dfrac{u^2-c^2}{2} + \dfrac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right|^2} & 0 \\ 0 & 0 & \dfrac{\mathrm{IM}\,\Theta_{11}^*}{\left|\dfrac{u^2-c^2}{2} + \dfrac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right|^2} \end{pmatrix}$$

where the denominators represent the absolute squares. Now by Eqs. (4.7), (4.23), and (4.25),

$$\mathrm{IM}\ \psi^* = \pi\, F_0'(u)$$

$$\mathrm{IM}\ \Theta_{11}^* = \pi\, M_{11}'(u)$$

Hence, putting all these results together,

$$\underset{\sim}{A} = \frac{\pi\,\omega_p^4}{(2\pi)^3 n} \int \frac{\underset{\sim}{k}}{k^4} \left\{ \frac{F_0'(u)}{\left|1 + \dfrac{\omega_p^2}{k^2}\psi(u)\right|^2} + \frac{(v^2 - u^2)\, M_{11}'(u)}{\left|\dfrac{u^2 - c^2}{2} + \dfrac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right|^2} \right\} d^3k \qquad (6.5)$$

$$B = -\frac{\pi\,\omega_p^4}{(2\pi)^3 n} \int \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^5} \left\{ \frac{F_0(u)}{\left|1 + \dfrac{\omega_p^2}{k^2}\psi(u)\right|^2} + \frac{(v^2 - c^2)\, M_{11}(u)}{\left|\dfrac{u^2 - c^2}{2} + \dfrac{\omega_p^2}{k^2}(\Theta + \tilde{B}_0)_{11}\right|^2} \right\} d^3k \qquad (6.6)$$

For completeness, we rewrite the remaining (relativistic) definitions that enter in Eqs. (6.5) and (6.6)

$$F_o(u) = k \int f_o(p^2)\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3p \tag{6.7}$$

$$F_o'(u) = m\vec{k}\cdot\int \frac{\partial f_o(p^2)}{\partial \underset{\sim}{p}}\ \delta(ku - \underset{\sim}{k}\cdot\underset{\sim}{v})\ d^3p \tag{6.8}$$

$$\psi(u) = \lim_{\epsilon \to o} \int \frac{F_o'(u')}{i\epsilon + u - u'}\ du' \tag{6.9}$$

$$\Theta(u) = \lim_{\epsilon \to o} \int \frac{M_o'(u')}{i\epsilon + u - u'}\ du' \tag{6.10}$$

Note that there is now automatic Debye shielding for long wavelength (i.e., small k) perturbations and that the coefficients no longer diverge logarithmically in that limit. One must still introduce an artificial cutoff for short wavelengths (i.e., large k) in the usual manner. The first term in $\underset{\sim}{A}$ and B represents interaction between particles via coulomb forces only, and the long wavelength cutoff in this case is of the order of the Debye length. The second term in $\underset{\sim}{A}$ and B represents interaction between particles via the transverse electromagnetic field and its cutoff occurs at the distance

$$d = \left(\frac{mc^2}{4\pi ne^2} \right)^{1/2}$$

which is in accord with the usual magnetic screening depth. Note also that the transverse terms are smaller than the longitudinal terms by the factor $(v/c)^4$.

VII. MAXWELL-BOLTZMANN DISTRIBUTION

We expect that the Fokker-Planck terms should vanish when f_o is a Maxwell-Boltzmann distribution. This is readily verified. The relativistic Maxwell-Boltzmann distribution is

$$f_o = A\, e^{-c[p^2+(mc)^2]^{\frac{1}{2}}/\kappa T} \tag{7.1}$$

where A is a constant. Hence

$$\frac{\partial f_o}{\partial \underset{\sim}{p}} = -\frac{c}{\kappa T}\,\frac{\underset{\sim}{p}}{[p^2+(mc)^2]^{1/2}}\, f_o$$

which, by Eq. (4.22), is simply

$$\frac{\partial f_o}{\partial \underset{\sim}{p}} = -\frac{\underset{\sim}{v}}{\kappa T}\, f_o \tag{7.2}$$

Hence, by Eqs. (6.8) and (6.2)

$$F_o'(u) = -\frac{mu}{\kappa T}\, F_o(u)$$

$$M_o'(u) = -\frac{mu}{\kappa T}\, M_o(u)$$

Hence, by Eqs. (6.5) and (6.6)

$$\underset{\sim}{A}\,\frac{\underset{\sim}{k}}{k} = \frac{mu}{\kappa T}\,B$$

Substituting this in Eq. (4.19) (in relativistic form) we have

$$\text{r.h.s.} = -\,m\,\frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{A}\left\{ f_o + \frac{\kappa T}{u}\,\frac{\underset{\sim}{k}}{k} \cdot \frac{\partial f_o}{\partial \underset{\sim}{p}} \right\} = 0$$

by Eq. (7.2).

Finally, if we take the non-relativistic limit of Eqs. (6.5) and (6.6), the transverse terms vanish, and we obtain the result given in Lenard's Eqs. (12) and (13) except for normalization. Note that in this paper, and in I, the normalization follows unambiguously from Eq. (1.1) and the definition of the reduced distribution functions given in Eq. (8) of I. The normalization of f_o is to unity. In Lenard's paper, f_o is normalized to n.

In a subsequent paper, the results obtained above (which are for a plasma composed of electrons and infinite mass ions) will be generalized to the case of an arbitrary number of finite-mass ion species.

REFERENCES

1. A. Simon and E. G. Harris, Phys. Fluids 3, 245 (1960). This paper is referred to hereafter as I.
2. N. Rostoker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960).
3. N. N. Bogoliubov, Problems of Dynamic Theory in Statistical Physics, (USAEC translation - 3852, 1960).
4. A. Lenard, Ann. Phys. 10, 390 (1960); R. Balescu, Phys. Fluids 3, 57 (1960).

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