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SOME PROPERTIES OF INFINITE, LUMPED SOLENOIDS

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ABSTRACT

The ripple characteristics of infinite, lumped solenoids are treated by a Fourier analysis. The transition from finite to infinite systems is pointed out. The coefficients in the resulting Fourier series for an infinite set of coaxial, equal current-bearing loops are solved for exactly. For loop radii, R, large compared to the spacing distance, λ , an approximate form for B_z on the axis is

$$B = B_{SOL} \left(1 + 2\pi \sqrt{\frac{R}{\lambda}}\, e^{-\frac{2\pi R}{\lambda}} \cos \frac{2\pi}{\lambda} z \right), \quad R \gg \lambda,$$

where B_{SOL} is the field due to an infinite, uniform solenoidal current sheet. The calculation is extended to points off the axis. An approximate solution is also found for an infinite set of rectangular coils.

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INTRODUCTION

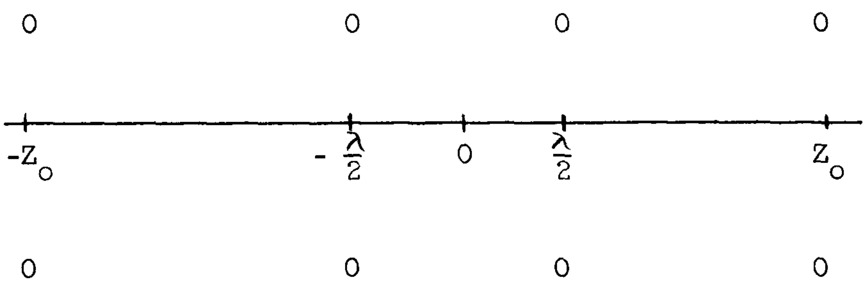
In the design of magnetic fields the problem of infinite, lumped solenoids often arises. In this treatment, certain important properties of these systems are studied. In particular, the ripples in the magnetic field due to lumped systems is of most interest. It is of importance to have simple formulas from which one may easily see the behavior of such systems. Most attacks are hampered by difficult summations and integrations. However, it was found here that several integrals could be evaluated giving the desired simple results.

It has been through the encouragement of Dr. R. J. Mackin, Jr., and Dr. W. F. Gauster that this work has been done.

A Note on Finite Sets of Loops

Before beginning the study of the case of an infinite set of loops it might be well to point out the effect of taking a finite set and increasing its number of elements. In seeing this transition, perhaps the results from the infinite sets will be more applicable to realizable situations.

Let us start with a pair of loops at the axial points $-\frac{\lambda}{2}$ and $\frac{\lambda}{2}$, carrying a current i. We wish to estimate the effect of adding another pair of loops at the points $-Z_o$ and Z_o. More specifically, we want to know the ripple produced by this addition.



We define the ripple, Δ, as the difference in the field at the point $\frac{\lambda}{2}$ and that at 0. Hence, for the outer pair we may write:

$$\Delta = B(z_o + \tfrac{1}{2}\lambda) + B(z_o - \tfrac{1}{2}\lambda) - 2\,B(z_o) \tag{1}$$

If we expand $B(z)$ in a Taylor series,

$$B(z_o + h) = B(z_o) + B'(z_o)h + B''(z_o)\,\frac{h^2}{2} + \ldots$$

then we obtain for Δ:

$$\Delta \cong B''(z_o) \frac{\lambda^2}{4} \qquad (2)$$

Now in MKS units we calculate for a circular loop

$$B''(z) = \frac{3}{2}\mu_o i R^2 \cdot \frac{4z^2 - R^2}{(z^2 + R^2)^{7/2}} \qquad (3)$$

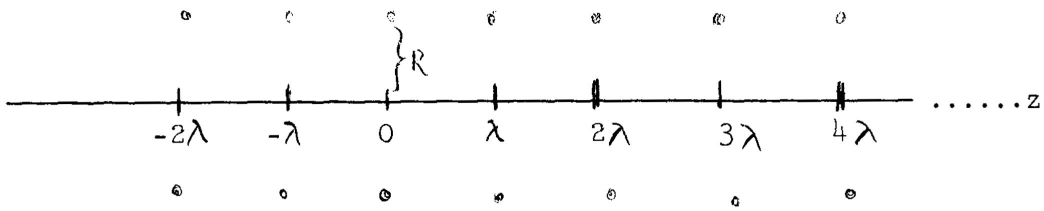
From equation (3) it clear that

$$B''(z) \gtrless 0, \text{ if } \quad z \gtrless R/2 \qquad (4)$$

Now when the equality of (4) holds we have the familiar Helmholtz coil pair with $\Delta = 0$. It is now clear that adding loops inside the point $z = R/2$ makes Δ more negative, while adding loops outside this interval makes Δ less negative. It may also be shown that when sufficiently many loops are added Δ becomes positive approaching a limit as the set becomes infinite.

Calculation of B_z Along the Axis for Loops

In this section we expand the field due to an infinite set of circular, coaxial current loops in a Fourier series. Each loop has the same dimensions and carries the same current, i. They are also equally spaced by a distance λ. It will be found that we have the field due to a uniform solenoidal current sheet with small harmonic perturbation terms.



The field produced at a point z on the axis due to a loop displaced $\nu\lambda$ meters from the origin is given by

$$B(z) = \frac{\mu_o i R^2}{2} \quad \frac{1}{\left[(z - \nu\lambda)^2 + R^2\right]^{3/2}} \qquad \text{(MKS units)}$$

Now to find the field due to all the loops we must sum over ν from $-\infty$ to $+\infty$:

$$B(z) = \frac{\mu_o i R^2}{2} \sum_{\nu = -\infty}^{\infty} \frac{1}{\left[(z - \nu\lambda)^2 + R^2\right]^{3/2}} \qquad (1)$$

The magnetic field of such an infinite array will be a periodic function of z. This is obvious since, as we advance along the z axis from one loop to another, we see the same physical structure. Our function then must have this distance, λ, as its period. Having such a periodic function, we shall perform a Fourier analysis. We would represent B(z) as

$$B(z) = \beta + \sum_{n=1}^{\infty} \gamma_n \cos\left(\frac{2\pi}{\lambda} n z\right) \tag{2}$$

To evaluate β, we multiply each side of equation (2) by dz and integrate from $-\frac{\lambda}{2}$ to $\frac{\lambda}{2}$. Note that the integral of the cosine series vanishes so that we have

$$\beta = \frac{\mu_o i R^2}{2\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \sum_{\nu=-\infty}^{\infty} \frac{dz}{\left[(z-\nu\lambda)^2 + R^2\right]^{3/2}} \tag{3}$$

To evaluate the integral let

$$u = z - \lambda\nu \,, \quad du = dz$$

Interchanging the order of summation and integration:

$$\beta = \frac{\mu_o i R^2}{2\lambda} \sum_{\nu=-\infty}^{\infty} \int_{-\frac{\lambda}{2}-\lambda\nu}^{\frac{\lambda}{2}-\lambda\nu} \frac{du}{\left(u^2 + R^2\right)^{3/2}} \tag{4}$$

Now note that equation (4) is of the form:

$$\beta = \ldots\ldots + \int_{-\frac{3}{2}\lambda}^{-\frac{1}{2}\lambda} + \int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda} + \int_{\frac{1}{2}\lambda}^{\frac{3}{2}\lambda} + \ldots\ldots\ldots$$

so that

$$\beta = \frac{\mu_o i R^2}{2\lambda} \int_{-\infty}^{\infty} \frac{du}{\left(u^2 + R^2\right)^{3/2}} = \frac{\mu_o i}{\lambda} \tag{5}$$

We observe that β is the field due to an infinite solenoidal current sheet with current per unit length equal to i/λ . Therefore, contributions from the rest of the series will be perturbations of this uniform field.

Now let us proceed to calculate the coefficients in the series. Multiply each side of equation (2) by $\cos k \frac{2\pi}{\lambda} z$ and integrate from $-\frac{\lambda}{2}$ to $\frac{\lambda}{2}$. Using the orthogonality relation the integral over the Fourier series gives $\frac{\gamma_n \lambda}{2}$. Thus,

$$\frac{\gamma_n \lambda}{2} = \frac{\mu_o i R^2}{2} \int_{-\lambda/2}^{\lambda/2} \frac{\cos\left(n \cdot \frac{2\pi}{\lambda} \cdot z\right) dz}{\left[(z - \lambda\nu)^2 + R^2\right]^{3/2}} \tag{6}$$

Proceeding as before in the evaluation of the integral,

let $z - \nu\lambda = u$, $dz = du$

whence $n \frac{2\pi}{\lambda} z = n \frac{2\pi}{\lambda} u + n\nu\, 2\pi$

Since $\cos\left(n\frac{2\pi}{\lambda}u + n\nu 2\pi\right) = \cos n\frac{2\pi}{\lambda}u$, we then have

$$\gamma_n = \frac{\mu_o iR^2}{\lambda}\sum_{\nu=-\infty}^{\infty}\int_{-\frac{\lambda}{2}-\lambda\nu}^{\frac{\lambda}{2}-\lambda\nu}\frac{\cos n\frac{2\pi}{\lambda}u}{\left(u^2+R^2\right)^{3/2}} \tag{7}$$

As before, the summation sign is incorporated in the integral:

$$\gamma_n = \frac{\mu_o iR^2}{\lambda}\int_{-\infty}^{\infty}\frac{\cos n\frac{2\pi}{\lambda}u\;du}{\left(u^2+R^2\right)^{3/2}} \tag{8}$$

This integral may be evaluated in the following way. It is known* that

$$\int_o^{\infty}\frac{\cos\alpha x dx}{(\beta^2+x^2)^{1/2}} = K_o(\alpha\beta) \tag{9}$$

where K_o is the modified Bessel function of the second kind. Differentiate each side of equation (9) with respect to β. This is permissible since the resulting integral is uniformly convergent.

$$\int_o^{\infty}\frac{\cos\alpha x dx}{(\beta^2+x^2)^{3/2}} = \frac{\alpha}{\beta}K_1(\alpha\beta) \tag{10}$$

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* Gröbner and Hofreiter, Integraltafel, Springer-Verlag, 1950, pp. 333-78a.

We have then that

$$\gamma_n = \left(\frac{\mu_o i}{\lambda}\right) \; n 4\pi \frac{R}{\lambda} \; K_1 \; \left(n 2\pi \frac{R}{\lambda}\right) \tag{11}$$

We may, therefore, write the Fourier series as

$$B(z) = \frac{\mu_o i}{\lambda} + \frac{\mu_o i}{\lambda} \; 4\pi \frac{R}{\lambda} \sum_{n=1}^{\infty} n \, K_1 (n 2\pi \frac{R}{\lambda}) \cos n \frac{2\pi}{\lambda} z \tag{12}$$

For large values of x, $K_1(x)$ may be expressed by the asymptotic relation

$$K_1(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}$$

In problems of interest this is a reasonable approximation, so we write:

$$\gamma_n \cong \left(\frac{\mu_o i}{\lambda}\right) 2\pi \sqrt{\frac{nR}{\lambda}} \; e^{-n \frac{2\pi R}{\lambda}} \quad , \quad n \frac{2\pi R}{\lambda} \gg 1$$

For large values of the argument we may drop all but the first term in the expansion, obtaining

$$B(z) \cong B_{solenoid} \left(1 + 2\pi \sqrt{\frac{R}{\lambda}} \; e^{-\frac{2\pi R}{\lambda}} \; . \; \cos \frac{2\pi}{\lambda} z \right) \tag{13}$$

Following is a table of the amplitude of the first harmonic vs. the quantity R/λ:

R/λ	Amplitude
1	1.07×10^{-2}
2	3.1×10^{-5}
3	6.1×10^{-8}
4	1.74×10^{-10}

Extension to Points off the Axis

For a current free region we have

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\overrightarrow{\nabla \times A}) = 0$$

or $$\vec{\nabla}^2 \vec{A} = 0$$

Using this property of the field it is a straight forward procedure to show that the magnetic quantities of an infinite set of loops off the axis are given (14)

by

$$A_\theta = B_{SOL} \left(\rho/2 + 4\pi \frac{R}{\lambda} \sum_{n=1}^{\infty} n\, K_1(n2\pi \frac{R}{\lambda})\, I_1 (n \frac{2\pi}{\lambda}\rho) \cos n \frac{2\pi}{\lambda} z \right)$$

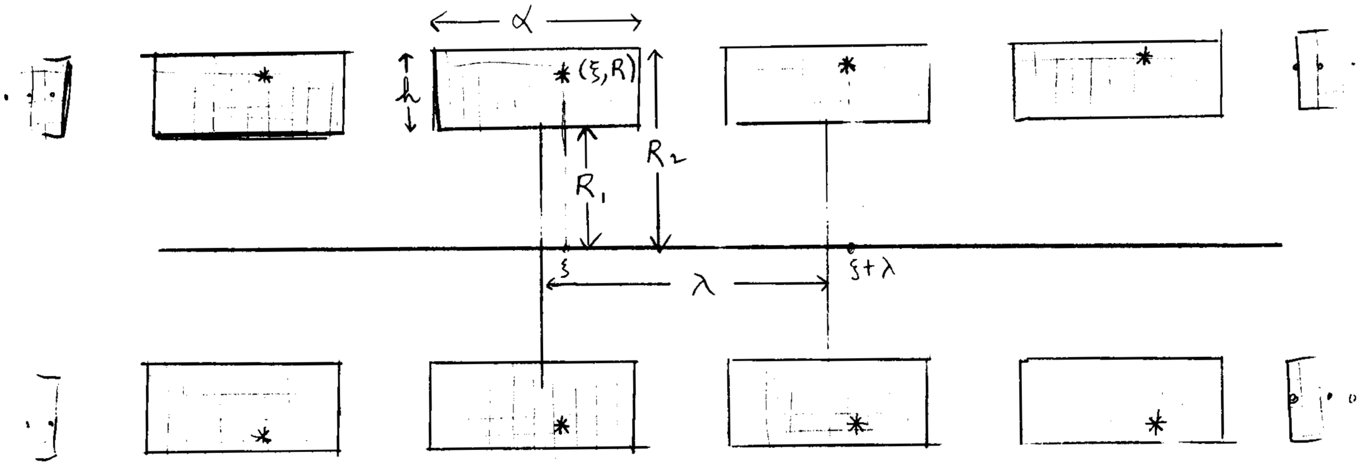
$$B_z = B_{SOL} \left(1 + 4\pi \frac{R}{\lambda} \sum_{n=1}^{\infty} n\, K_1 (n2\pi \frac{R}{\lambda})\, I_o (n \frac{2\pi}{\lambda}\rho) \cos n \frac{2\pi}{\lambda} z \right. \qquad (15)$$

$$B_r = - B_{SOL} \left(4\pi \frac{R}{\lambda} \sum_{n=1}^{\infty} n\, K_1(n2\pi \frac{R}{\lambda})\, I_1 (n \frac{2\pi}{\lambda}\rho) \sin n \frac{2\pi}{\lambda} z \right)$$

Here ρ is the distance from the axis to the point in question.

An Approximate Solution for Rectangular Coils

If we think of the current in a loop, i, as being a current density, J, we might sum a group of such loop sets to approximate a coil. In the limit this sum approaches the integral over the coil cross-section.



We might begin by integrating a function of the form of equation (1) over a coil cross-section; however, this method results in some distasteful integrals. We choose instead to integrate the approximate equation

$$dB = \frac{\mu_o J}{\lambda} \left(1 + 2\pi \sqrt{\frac{R}{\lambda}}\right) e^{-2\pi\frac{R}{\lambda}} \cos \frac{2\pi}{\lambda} (z-\xi)\, dR\, d\xi \qquad (16)$$

Let us first perform the ξ integration, which may be done directly. Using a trigonometric identity we have

$$dB(z) = dR \int_{\alpha/2}^{\alpha/2} \frac{\mu_o J}{\lambda} \left(1 + 2\pi\sqrt{\frac{R}{\lambda}}\right) e^{-\frac{2\pi R}{\lambda}} \cos \frac{2\pi}{\lambda}(z - \xi)\, d\xi$$

$$dB(z) = dR\ \mu_o J\ \frac{\alpha}{\lambda}\left(1 + \frac{2\lambda}{\alpha}\sqrt{\frac{R}{\lambda}}\ e^{\frac{-2\pi R}{\lambda}} \cdot \sin \pi \frac{\alpha}{\lambda}\ \cos \frac{2\pi}{\lambda} z\right) \qquad (17)$$

Now if J is thought of as a current per unit length we again have a solenoid field with a small harmonic correction term. Thus, equation (17) represents the first order field produced by an infinite set of current sheets of width α surrounding the axis.

We now proceed by integrating equation (17) with respect to R between the limits R_1 and R_2.

$$B(z) = \mu_o J \frac{\alpha}{\lambda} \int_{R_1}^{R_2} \left(1 + \frac{2\lambda}{\alpha}\sqrt{\frac{R}{\lambda}}\ e^{\frac{-2\pi R}{\lambda}} \cdot \sin\frac{\pi\alpha}{\lambda} \cdot \cos \frac{2\pi}{\lambda} z\right) dR \qquad (18)$$

The second part of equation (18) must be integrated by parts. After simplifications the result is

$$B(z) = \mu_o J \frac{\alpha}{\lambda} \left\{ h + \frac{\lambda^2}{\pi\alpha}\left(\frac{\rho_1}{\sqrt{2\pi}}\ e^{-\rho_1^2} - \frac{\rho_2}{\sqrt{2\pi}}\ e^{-\rho_2^2}\right.\right.$$

$$\left.\left. + \frac{1}{4\sqrt{2}}\ \mathrm{erf}(\rho_2) - \frac{1}{4\sqrt{2}}\ \mathrm{erf}(\rho_1)\right\} \sin \frac{\pi\alpha}{\lambda}\ \cos \frac{2\pi}{\lambda}\ z \qquad (19)$$

where $\rho_1^2 = 2\pi \frac{R_1}{\lambda}$, $\rho_2^2 = 2\pi \frac{R_2}{\lambda}$

In earlier parts of this development we have assumed ρ_1, and ρ_2 to be large compared to 1. We may then use the asymptotic form of erf(x):

$$\operatorname{erf}(x) \sim 1 - \frac{1}{x\sqrt{\pi}} e^{-x^2} \quad , \quad x \gg 1$$

Equation (19) then becomes

$$B(z) = \mu o J \frac{\alpha}{\lambda} \left\{ h + \frac{\lambda^2}{\pi\alpha} \left[\frac{\rho_1}{\sqrt{2\pi}} e^{-\rho_1^2} \left(1 - \frac{1}{4\rho_1{}^2}\right) - \frac{\rho_2 \; e^{-\rho_2^2}}{\sqrt{2\pi}} \left(1 - \frac{1}{4\rho_2{}^2}\right) \right] \sin \frac{\pi\alpha}{\lambda} \cos \frac{2\pi}{\lambda} z \right\} \tag{20}$$

We now note that $1 \gg \frac{1}{4\rho^2}$; so that substituting for ρ_1, ρ_2 we get

$$B(z) = B_{SOL} \frac{\alpha}{\lambda} \left\{ 1 + \frac{\lambda^2}{\pi\alpha h} \left[\sqrt{\frac{R_1}{\lambda}} \cdot e^{-2\pi\frac{R_1}{\lambda}} - \sqrt{\frac{R_2}{\lambda}} \cdot e^{-2\pi\frac{R_2}{\lambda}} \right] \sin \frac{\pi\alpha}{\lambda} \cos \frac{2\pi}{\lambda} z \right\} \tag{21}$$

where $B_{SOL} = \mu_0 J h$

A Numerical Example

Let us take

$$\frac{R_1}{\lambda} = 2 , \quad \frac{R_2}{\lambda} = 3 \quad , \quad \frac{\alpha}{\lambda} = \frac{1}{4} \quad , \quad \frac{h}{\lambda} = 1$$

With these values we calculate

$$\text{Amplitude of first harmonic} = .88 \times 10^{-5}$$

It is clear that with such realizable dimensions a very uniform field can be produced. Naturally, this is an approximate value; however, it is to within an order of magnitude. High speed computer codes are being written to better check this value with both infinite and finite coil arrangements.

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